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RANDOM FIELDS AND THE GEOMETRY OF WIENER SPACE[1]

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In this work we consider infinite dimensional extensions of some finite dimensional Gaussian geometric functionals called the Gaussian Minkowski functionals. These functionals appear as coefficients in the probability content of a tube around a convex set $D \subset \mathbb{R}^k$ under the standard Gaussian law $N(0, I_{k\times k})$. Using these infinite dimensional extensions, we consider geometric properties of some smooth random fields in the spirit of [*Random Fields and Geometry* (2007) Springer] that can be expressed in terms of reasonably smooth Wiener functionals.

1. Introduction and motivation. We start with a description of a certain class of set functionals determined by the canonical Gaussian measure on $\mathbb{R}^k$. By canonical, we shall mean centered and having covariance $I_{k\times k}$. Its density with respect to the Lebesgue measure on $\mathbb{R}^k$ is therefore given by $(2\pi)^{-k/2}e^{-\|x\|^2/2}$. For this measure, we consider computing the probability content of a tube around M, leading us to a *Gaussian tube formula* which we state as

$$\gamma_k(M + \rho B_k) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_k}(M), \tag{1.1}$$

where $\mathcal{M}_j^{\gamma_k}(M)$ is the jth *Gaussian Minkowski Functional* (GMF) of the set M. If M is compact and convex, that is, if M is a convex body, then we can take the right-hand side (1.1) to be a power series expansion for the left-hand side. For certain M, this expansion must be taken to be a formal expansion, in the sense that up to terms of some order, the left and right-hand side above agree. For example, if M is a centrally-symmetric cone such as the rejection region for a T or F statistic, then M has a singularity at the origin in the sense that the geometric structure of the cone around 0 is nonconvex and the expansion above is accurate only up to terms of size $O(\rho^{n-1})$.

Our interest in this tube formula lies in the appearance of these coefficients in the expected Euler characteristic heuristic [1, 17–19].

1.1. *Expected Euler characteristic heuristic.* The Euler characteristic heuristic was developed by Robert Adler and Keith Worsley (cf., e.g., [1, 17–19]) to

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approximate the probability

$$P\Big(\sup_{x\in M} f(x) \geq u\Big)$$

with $E(\chi(A_t(f;M)))$, where $A_t(f;M) = \{x \in M : f(x) \geq t\} \subset M$, and χ is the Euler–Poincaré characteristic.

Let M be an m-dimensional reasonably smooth manifold, with $(\xi_1, \dots, \xi_k)$ identically and independently distributed copies of a Gaussian random field defined on M. Subsequently, for any $F : \mathbb{R}^k \to \mathbb{R}$, with two continuous derivatives, we can define a new random field on M given by $f(x) = F(\xi_1(x), \dots, \xi_k(x))$, for each $x \in M$.

Using the above Euler characteristic heuristic for approximating the P-value for appropriately large values of u, and Theorem 15.9.5 of [1], we have

$$\begin{aligned} P\Big(\max_{x\in M} f(x) \geq u\Big) &\approx E(\chi(A_u(f;M))) \\ &= \sum_{j=0}^{m} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_k}(F^{-1}[u,\infty)), \end{aligned} \tag{1.2}$$

where $\mathcal{M}_j^{\gamma_k}(F^{-1}[u,\infty))$ for $j = 0, 1, \dots$ are the GMFs of the set $F^{-1}[u,\infty) \subset \mathbb{R}^k$ that appear in (1.1), and $\mathcal{L}_j(M)$ for $j = 0, 1, \dots, m$ are the Lipschitz–Killing curvatures (LKCs) of the manifold M defined with respect to the Riemannian metric given by $g(X,Y) = E(X\xi_1 Y\xi_1)$, where X and Y are two vector fields on M, with $X\xi_1$ and $Y\xi_1$ representing the directional derivatives of ξ_1.

1.2. *Curvature measures.* The LKCs for a large class of subsets of any finite dimensional Euclidean space can be defined via a *Euclidean tube formula*. In particular, let $M \subset \mathbb{R}^k$ be an m-dimensional set with convex support cone, then writing λ_k as the standard k-dimensional Euclidean measure, B_k as the k-dimensional unit ball centered at origin, for small enough values of ρ, we have

$$\begin{aligned} \lambda_k(M + \rho B_k) &= \sum_{j=0}^{m} \frac{\pi^{(n-j)/2}}{\Gamma((n-j)/2+1)} \rho^{n-j} \mathcal{L}_j(M) \\ &= \sum_{j=0}^{m} \frac{\rho^{n-j}}{(n-j)!} \theta_{n-j}(M), \end{aligned} \tag{1.3}$$

where $\mathcal{L}_j(M)$ is the jth LKC of the set M with respect to the usual Euclidean metric, and $\theta_j(M)$'s are called the Minkowski functionals of the set M.

Geometrically, $\mathcal{L}_{k-1}(M)$ for a smooth $(k-1)$-dimensional manifold embedded in $\mathbb{R}^l$ is the $(k-1)$-dimensional Lebesgue measure of the set M, and the other LKCs can be defined as

$$\mathcal{L}_j(M) = \frac{1}{s_{k-j}(k-1-j)!} \int_{\partial M} P_{k-1-j}(\lambda_1(x), \dots, \lambda_{k-1}(x)) \mathcal{H}_{k-1}(dx),$$

where s_j is the surface area of a unit ball in $\mathbb{R}^j$, $(\lambda_1(x), \dots, \lambda_{k-1}(x))$ are the principal curvatures at $x \in \partial M$, and $P_i(\lambda_1(x), \dots, \lambda_{k-1}(x))$ is the ith symmetric polynomial in $(k-1)$ indices. In the case when the set M is not unit codimensional, then the definition involves another integral over the normal bundle.

The (generalized) curvature measures defined this way are therefore signed measures induced by the Lebesgue measure of the ambient space. By replacing the Lebesgue measure in (1.3) with an appropriate Gaussian measure, we can define a parallel Gaussian theory. The GMFs in (1.1) play the role of Minkowski functionals in the Gaussian theory. In particular,

$$\mathcal{M}_j^{\gamma_k}(M) \triangleq (2\pi)^{-k/2} \sum_{m=0}^{j-1} \binom{j-1}{m} \Theta_{m+1}\big(M, H_{j-1-m}(\langle \eta, x\rangle) e^{-|x|^2/2}\big), \tag{1.4}$$

where $\Theta_{m+1}(M, H_{j-1-m}(\langle \eta, x\rangle) e^{-|x|^2/2})$ is the integral of $H_{j-1-m}(\langle \eta, x\rangle) e^{-|x|^2/2}$ with respect to the $(m+1)$th generalized Minkowski curvature measure, and $H_k(y)$ is the kth Hermite polynomial in y (cf. [1]).

1.3. *Our object of study*: *A richer class of random fields*. In this paper we intend to extend (1.2) to a larger class of random fields f, which can be expressed using $F : C_0[0,1] \to \mathbb{R}$, where $C_0[0,1]$ is the space of continuous functions $f : [0,1] \to \mathbb{R}$, such that $f(0) = 0$, also referred to as the *classical Wiener space*, when equipped with the standard Wiener measure on this sample space. In other words, we shall consider random fields which can be expressed as some smooth *Wiener functional*. For instance, let us start with a smooth manifold M together with a Gaussian field $\{B^x(t) : t \in \mathbb{R}_+, x \in M\}$ defined on it, such that its covariance function is given by

$$E(B^x(t)B^y(s)) = s \wedge t C(x, y), \tag{1.5}$$

where $C : M \times M \to \mathbb{R}$ is assumed to be a smooth function, with more details appearing in Section 6, where we actually prove an extension of (1.2). This infinite dimensional random field can be used to construct many more random fields on M, for instance, the following:

EXAMPLE 1.1 (Stochastic integrals). Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function, and consider the following random field,

$$f(x) = \int_0^1 V(B^x(s))\, dB^x(s) = F(B^x(\cdot)), \tag{1.6}$$

where $F : C_0 \to \mathbb{R}$ is the Wiener functional

$$F(\omega) = \left(\int_0^1 V(B(t))\, dB(t)\right)(\omega).$$

This is clearly an extension of the random fields in (1.2). As a consequence of our extension of the Gaussian Minkowski functionals to smooth Wiener functionals, we prove that, under suitable smoothness conditions on V,

$$\mathbb{E}(\chi(A_u(f;M))) = \sum_{j=0}^{\dim(M)} (2\pi)^{-j/2}\mathcal{L}_j(M)\mathcal{M}_j^{\mu}(F^{-1}[u,+\infty)).$$

Our smoothness conditions are rather strong in this paper: we assume V is C^4 with essentially polynomial growth. We need such strict assumptions to ensure regularity of various conditional densities derived from the random field (1.6) and its first two derivatives at a point $x \in M$.

A quick look at (1.2) reveals that in order to extend it to the case when $F : C_0[0,1] \to \mathbb{R}$, we must be able to define GMFs for infinite dimensional subsets of $C_0[0,1]$, as $F^{-1}[u,\infty) \subset C_0[0,1]$. In the present form, that is, (1.4), the definition of GMFs appears to depend on the summability of the principal curvatures of the set $\partial(F^{-1}[u,\infty))$ at each point $x \in \partial(F^{-1}[u,\infty))$ as well as the integrability of these sums. In infinite dimensions this summability requirement is equivalent to an operator being trace class. This is quite a strong requirement, and may be very hard to check. Indeed, the natural summability requirements of operators in the natural infinite dimensional calculus on C_0, the Malliavin calculus, is the Hilbert–Schmidt class rather than the trace class.

Therefore, we shall first modify the definition of GMFs, from (1.4) to one which is more amenable for an extension to the infinite dimensional case. This will be done in Section 2.

After setting up the notation and some technical background on the Wiener space in Section 3, the all important step, that of extending the appropriate definition of GMFs to the case of codimension one, *smooth* subsets of the Wiener space, is accomplished in Section 5. The characterization of GMFs in the infinite dimensional case will be done precisely the same way as in the case of finite dimensions, where, as noted earlier, the GMFs are identified as the coefficients appearing in the Gaussian tube formula.

Finally, in Section 6, we use the infinite dimensional extension of the GMFs to obtain an extension of (1.2), for random fields which can be expressed as stochastic integrals driven by $B^x(\cdot)$ as defined in Example 1.1, and discuss other possible implications of the extension. Most of our methods in Section 6 are invariant to the formulation of the random field as a stochastic integral. Hence, should a random field satisfy all the regularity conditions appearing in Section 6, we expect our methods to work, modulo a few changes.

2. Preliminaries I: The finite dimensional theory. In this section we shall use the standard finite dimensional theory of transformation of measure for Gaussian spaces to modify the definition (1.4) of the GMFs to one which is more suited to extension to the infinite dimensional case.

We begin by recalling some well-known facts about analysis on finite dimensional Gaussian spaces from Section 6.6.3 of Chapter II of [4], and Chapter 3 of [15]. Let γ_k be the Gaussian measure on $\mathbb{R}^k$ given by $(2\pi)^{-k/2}e^{-\|x\|^2/2}\,dx$, and T a mapping from $\mathbb{R}^k$ into itself, given by $T(x) = x + u(x)$, where $u : \mathbb{R}^k \to \mathbb{R}^k$ is Sobolev differentiable and $|u(x) - u(y)| \le c(\rho)|x - y|$ for any $x, y \in \mathbb{R}^k$ with $|x - y| < \rho$. Then, the Radon–Nikodym derivative of $\gamma_k \circ T$ with respect to the measure γ_k is given by

$$\frac{d\gamma_k \circ T}{d\gamma_k} = |\det{}_2(I_{\mathbb{R}^k} + \nabla u)| \exp\biggl(-\delta(u) - \frac{1}{2}\|u\|^2\biggr), \tag{2.1}$$

where $\|\cdot\|$ is the usual Euclidean norm, and $\det_2$ is the generalized Carleman–Fredholm determinant.

Subsequently, for a smooth, unit codimensional, convex set $A \subset \mathbb{R}^k$, let us define the tube $\mathrm{Tube}(A, \rho)$ of width ρ around the set A as the set $(A \oplus B(0, \rho))$, where $B(0, \rho)$ is the k-dimensional ball of radius ρ centered at the origin. Next, we shall define a signed distance function given by

$$d_{\partial A}(x) = \begin{cases} \inf\limits_{y \in \partial A} \|y - x\|, & \text{for } x \notin A, \\ -\inf\limits_{y \in \partial A} \|y - x\|, & \text{for } x \in \mathrm{Int}(A), \end{cases}$$

where $\mathrm{Int}(A)$ denotes the interior of the set A.

Applying the co-area formula, and using the fact that $\|\nabla d_{\partial A}\| = 1$, we get

$$\gamma_k(\mathrm{Tube}(A, \rho)) = \gamma_k(A) + \int_0^\rho \int_{d_A^{-1}(r)} \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}}\,dx\,dr. \tag{2.2}$$

For $r < \rho$ fixed, we can now use equation (2.1) with any suitable transformation $T_r : \mathbb{R}^k \to \mathbb{R}^k$ that agrees with $x \mapsto x + r\eta_x$ on $\{y : d_A(y) \in (-\nu, \rho)\}$ for some small positive ν. Any such transformation maps $\mathrm{Tube}(d_A^{-1}(r), \varepsilon)$ to $\mathrm{Tube}(\partial A, \varepsilon)$ for $r < \rho$ and any $\varepsilon < \nu$. Two further applications of the co-area formula yield

$$\begin{aligned} &\int_{d_A^{-1}(r)} \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}}\,dx \\ &\quad = \int_{\partial A} |\det{}_2(I_{\mathbb{R}^k} + r\nabla^2 d_{\partial A})| \exp\biggl(-r\delta(\nabla d_{\partial A}) - \frac{1}{2}r^2\biggr) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}}\,dx. \end{aligned}$$

Therefore, equation (2.2) simplifies to

$$\begin{aligned} &\gamma_k(\mathrm{Tube}(A, \rho)) \\ &\quad = \gamma_k(A) + \int_0^\rho \int_{\partial A} |\det{}_2(I_{\mathbb{R}^k} + r\nabla^2 d_{\partial A})| \exp\biggl(-r\delta(\nabla d_{\partial A}) - \frac{1}{2}r^2\biggr) \\ &\qquad \times \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}}\,dx\,dr. \end{aligned} \tag{2.3}$$

Using a yet-to-be justified Taylor series expansion of the integrand appearing in the above integral, we can finally rewrite the GMFs as

$$\mathcal{M}_{j+1}^{\gamma_k}(A) = \int_{\partial A} \frac{d^j}{d\rho^j}\left(\det_2(I + \rho\nabla\eta)\exp\left(-\rho\delta(\eta) - \rho^2/2\right)\right)\Big|_{\rho=0} da^{\partial A}(x), \tag{2.4}$$

where $\eta = \nabla d_{\partial A}$ is the outward unit normal vector field to the set ∂A, and $da^{\partial A}$ is the surface measure of the set ∂A. Note that in the above expression we have removed the modulus around the $\det_2$ part, which can be justified by taking reasonably small values of ρ. This new definition of GMFs involves terms which have obvious extensions in the infinite dimensional case.

3. Preliminaries II: The infinite dimensional theory. In this section we recall some established concepts in Malliavin calculus which we shall need in later sections. We begin with an abstract Wiener space (X, H, μ), where H, equipped with the inner product $\langle\cdot,\cdot\rangle_H$, is a separable Hilbert space, called the Cameron–Martin space, X is a Banach space into which H is injected continuously and densely, and, finally, μ is the standard cylindrical Gaussian measure on H. For the sake of simplicity, one can appeal to the classical case when we have H as the space of real-valued, absolutely continuous functions on [0, 1] with $L^2([0,1])$ derivatives, which is continuously embedded in $X = C_0([0,1])$ the space of real-valued continuous functions f on [0, 1], such that $f(0) = 0$.

Sobolev spaces on Wiener space

Following the notation used in [4, 5, 16], Sobolev spaces $D_\alpha^p(X; E)$ for $p > 1$ and $\alpha > 0$ are defined as the class of E -valued functions $f \in L^p(X; E)$ such that

$$\|f\|_{p,\alpha} \triangleq \|(I - L)^{\alpha/2} f\|_{L^p(X;E)} < \infty,$$

where L is the Ornstein–Uhlenbeck operator defined on the Wiener space. Writing D as the Gross–Sobolev derivative and δ as its dual under the Wiener measure, $L = -\delta D$. The Sobolev spaces $D_{-\alpha}^p(X;\mathbb{R})$ for $\alpha > 0$ are the spaces of distributions, defined as the dual of $D_\alpha^q(X;\mathbb{R})$, where, as usual, $p^{-1} + q^{-1} = 1$. Throughout this paper, whenever appropriate, we will adopt this convention.

The space of infinitely integrable, α-smooth Wiener functionals is given by

$$D_\alpha^{\infty-}(X;\mathbb{R}) \triangleq \bigcap_{1<p<\infty} D_\alpha^p(X;\mathbb{R}).$$

Consequently, let us define the analogous *infinitely integrable* random variables as $L^{\infty-}(X;\mathbb{R}) \triangleq D_0^{\infty-}(X;\mathbb{R})$. Finally, we shall end this section with another definition which translates to the regularity of Wiener functionals.

DEFINITION 3.1. For an $\mathbb{R}^k$-valued Wiener functional $F = (F_1, \dots, F_k)$, the Malliavin covariance (matrix) $\sigma^F = (\langle DF_i, DF_j\rangle_H)_{ij}$, and the functional F itself, is called nondegenerate in the sense of Malliavin if $(\det\sigma^F)^{-1} \in L^{\infty-}$, whenever $\det\sigma^F$ is well defined.

H-Convexity

In order to characterize the class of subsets of the Wiener space for which we shall define the GMFs, we shall recall the notion of H-convexity.

DEFINITION 3.2. An H-convex functional is defined as a measurable functional $F : X \to : \mathbb{R} \cup \{\infty\}$ such that for any $h, k \in H$, $\alpha \in [0, 1]$

$$F\big(\omega + \alpha h + (1-\alpha)k\big) \le \alpha F(\omega + h) + (1-\alpha)F(\omega + k) \qquad \text{a.s.} \tag{3.1}$$

One of the properties of H-convex functionals which will be used in later sections is that a necessary and sufficient condition for a Wiener functional $F \in L^p$ for some $p > 1$ to be H-convex is that the corresponding D^2F must be a positive and symmetric Hilbert–Schmidt operator valued distribution on X (cf. [15]).

3.1. *Quasi-sure analysis*. In this section, most of which is based upon [4, 13], we shall resolve some technical aspects of defining integrals of Wiener functionals with respect to measures concentrated on μ-zero sets. Since all Wiener functionals are *de facto* defined up to μ-zero sets, thus, in order to be able to define the integral of Wiener functionals with respect to measures which are concentrated on μ-zero sets, we must resort to what is referred to as *quasi-sure analysis*, which in turn relies on the concept of *capacities* on the Wiener space.

DEFINITION 3.3. Let $1 < p < \infty$ and $\alpha > 0$. For an open set O of X, we define its (p, α)-capacity $C_\alpha^p(O)$ by

$$C_\alpha^p(O) = \inf\{\|U\|_{p,\alpha} : U \in D_\alpha^p(X; \mathbb{R}), U \ge 1\ \mu\text{-a.e. on } O\}.$$

For each subset of A of X, we define its (p, α)-capacity C_α^p by

$$C_\alpha^p(A) = \inf\{C_\alpha^p(O) : O \text{ is open and } O \supset A\}.$$

These capacities are finer scales to estimate the size of sets in X than μ. In particular, a set of (p, α)-capacity zero is always a μ-zero set, but the converse is not true in general.

A property π is said to be true (p, α)-quasi-everywhere (q.e.) if

$$C_\alpha^p(\pi \text{ is not satisfied}) = 0.$$

One of the most crucial steps in obtaining the *co-area* formula in the Wiener space, which in turn is a necessary step to obtain the *tube-formula* in the Wiener space, is to be able to extend ordinary Wiener functionals to sets of μ-zero measure. Quasi-sure analysis lets us do precisely that and much more.

DEFINITION 3.4. A measurable functional F is said to have a (p, α)-redefinition F^*, satisfying $F^* = F$ μ-almost surely, and F^* is (p, α)-quasi-continuous, if for all $\varepsilon > 0$, there exists an open set O_ε of X, such that $C_\alpha^p(O_\varepsilon) < \varepsilon$ and the restriction of F^* to the complement set O_ε^c is continuous under the norm of uniform convergence on X.

It can easily be seen that two redefinitions of the same functional differ only on a set of (p,α)-capacity zero, thereby implying the uniqueness of a (p,α)-redefinition up to (p,α)-capacity zero sets. According to Theorem 2.3.3 of [4], every functional $F \in D_\alpha^p(X;\mathbb{R}^k)$ has a (p,α)-quasi-continuous redefinition, which can be taken to be in the first Baire class.

In what follows in the remainder of this section, we recall some facts from the Malliavin calculus that will be helpful in our description of a tube below. If $\alpha > 1$, one can make a statement similar to Theorem 2.3.3 of [4] related to the differentiability of $F \in D_\alpha^p(X;\mathbb{R})$, essentially a form of Taylor's theorem with remainder.

LEMMA 3.5. *Suppose $F \in D_\alpha^p(X;\mathbb{R}), \alpha > 1$. Then, for each $h \in H$*

$$\frac{1}{\varepsilon}\big(F(x+\varepsilon h) - F(x)\big) - \langle DF(x), h\rangle_H \overset{D_{\alpha-1}^{p_1}(X;\mathbb{R})}{\longrightarrow} 0$$

for any $p_1 < p$.

PROOF. Define

$$\begin{aligned} X_{n,h} &= n\big(F(x+h/n) - F(x)\big) \in D_{\alpha-1}^{p_1} \\ &= \Lambda^{\alpha-1} Y_{n,h}, \qquad Y_{n,h} \in L^{p_1}, \end{aligned}$$

where $\Lambda = (I-L)^{-1/2}$ is the inverse of the Cauchy operator [4]. For each $h \in H$, $X_{n,h}$ converges in L^{p_1}, so the Kree–Meyer inequalities imply that $Y_{n,h}$ also converges in L^{p_1}. A second application of the Kree–Meyer inequalities implies that

$$\|Y_{n,h} - Y_{m,h}\|_{L^p} \approx \|X_{n,h} - X_{m,h}\|_{D_{\alpha-1}^{p_1}}.$$

Or, $X_{n,h}$ is Cauchy in $D_{\alpha-1}^{p_1}$, hence, its limit $\langle DF(x), h\rangle_H \in D_{\alpha-1}^{p_1}$. □

Hence, by the Borel–Cantelli property for the capacities $C_\alpha^{p_1}$ (Corollary IV.1.2.4 of [4]), for each $h \in H$ we can extract a sequence $\varepsilon_n(h)$ such that

$$C_{\alpha-1}^{p_1}\left(\left\{x : \lim_{n\to\infty} \frac{1}{\varepsilon_n(h)}\big(F\big(x+\varepsilon_n(h)h\big) - F(x)\big) = \langle DF(x), h\rangle\right\}^c\right) = 0. \tag{3.2}$$

COROLLARY 3.6. *Suppose $F \in D_\alpha^p(X;\mathbb{R}), \alpha > 1$ is nondegenerate and $H_\infty \subset H$ is a countable dense subset. Then,*

$$\begin{aligned} C_{\alpha-1}^{p_1}\Bigg(\Bigg\{x : DF(x) \neq 0\ \forall h \in H_\infty\ \exists \varepsilon_n(h) \to 0 \text{ such that} \\ \lim_{n\to\infty}\left(\frac{1}{\varepsilon_n(h)}\big(F^*\big(x+\varepsilon_n(h)h\big) - F^*(x)\big) - \langle DF^*(x), h\rangle_H\right) = 0\Bigg\}^c\Bigg) = 0. \end{aligned} \tag{3.3}$$

PROOF. The only thing that needs verifying beyond what was pointed out above is that $C^{p_1}_{\alpha-1}(\{x : DF(x) \neq 0\}^c) = 0$. This follows from the Tchebycheff inequality (Theorem IV.2.2 of [4]) applied to $\|DF\|_H \in D^{p_1}_{\alpha-1}$ and a Borel–Cantelli argument. □

There is an obvious higher order Taylor expansion of $F^*(x + \varepsilon_n(h)h)$, which we will use upto the second order term in our description of the tube below. If we are willing to sacrifice some moments, we can further specify in Corollary 3.6 that the existence of the partial derivatives of F as a limit at x implies their existence as limits at $x + h$ for all $h \in H_\infty$, $\|h\| \le K$ for some fixed, large K.

COROLLARY 3.7. *Suppose $F \in D^p_\alpha(X;\mathbb{R})$ and $H_\infty \subset H$ is a countable dense subset. Then, for all $p_1 < p$*

$$C^{p_1}_{\alpha-1}\Biggl(\Biggl\{x : \forall h_1, h_2 \in H_\infty, \|h_1\| \le K \ \exists \varepsilon_n(h_1,h_2) \to 0 \text{ such that}$$

$$\lim_{n\to\infty}\biggl(\frac{1}{\varepsilon_n(h_1,h_2)}\bigl(F^*\bigl(x + h_1 + \varepsilon_n(h_1,h_2)h_2\bigr) - F^*(x+h_1)\bigr) - \langle DF^*(x+h_1), h_2\rangle_H\biggr) = 0\Biggr\}\Biggr) \tag{3.4}$$

$$= 0.$$

PROOF. This follows from the fact that the translation operator $f(\cdot) \overset{T_h}{\mapsto} f(\cdot + h)$ is a continuous map from D^p_α to $D^{p_1}_\alpha$ for any $p_1 < p$ which follows directly from the Cameron–Martin theorem. □

REMARK 3.8. Finally, we note that we can, by choosing H_∞ appropriately, choose the set, say, A in Corollary 3.7, in such a way that $y \in A$ and $y + \varepsilon h \in A$ for all $h \in H_\infty$ and for all ε in some countable dense subset of $\mathbb{R}$.

4. Key ingredients for a tube formula. In this section we shall adopt a stepwise approach to reach our first goal, that of obtaining a (Gaussian) volume of the tube formula, for reasonably smooth subsets of the Wiener space. The three main steps are as follows: (i) characterizing subsets of the Wiener space via Wiener functionals, for which tubes, and thus GMFs, are well defined; (ii) assurance that the surface measures are well defined for the sets defined via the Wiener functionals; and finally, (iii) a change of measure formula for surface area measures corresponding to the lower dimensional surfaces of the Wiener space.

We shall first characterize the functionals for which the surfaces measures are well defined, subsequently, we shall prove a change of measure formula for the surfaces defined via such functionals. Finally, we shall define the class of sets for which the tube formula and GMFs are well defined by imposing more regularity conditions on the Wiener functionals.

4.1. *The Wiener surface measures.* Let us start with a reasonably smooth, $\mathbb{R}^k$ valued Wiener functional $F = (F_1, \dots, F_k)$. For $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$, we write $Z_{\mathbf{u}} = \bigcap_{i=1}^k F_i^{-1}(u_i)$. The sets $\{Z_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{R}^k}$ define a foliation of hypersurfaces imbedded in X.

The surface measures of these foliations $Z_{\mathbf{u}}$ are closely related to the density p_F of the push-forward measure $F_*(\mu)$ on $\mathbb{R}^k$ with respect to the Lebesgue measure on $\mathbb{R}^k$.

Heuristically, writing $\delta_{\mathbf{u}}$ for the Dirac delta at $\mathbf{u} \in \mathbb{R}^k$, the density p_F can be defined as

$$p_F(\mathbf{u}) = E(\delta_{\mathbf{u}} \circ F) \tag{4.1}$$

as long as we can make sense of the composition $\delta_{\mathbf{u}} \circ F$. For a smooth, real-valued Wiener functional G, we also expect the following relation to hold:

$$E[G\delta_{\mathbf{u}} \circ F] = E^{F=\mathbf{u}}(G) \times p_F(\mathbf{u}), \tag{4.2}$$

where $E^{F=\mathbf{u}}(G)$ is the conditional expectation of G given $F = \mathbf{u}$, assuming the composition $\delta_{\mathbf{u}} \circ F$ is well defined.

Making this heuristic calculation rigorous leads us back to the Sobolev spaces of Section 3, where the object $(\delta_{\mathbf{u}} \circ F)$ is related to a *generalized Wiener functional*, that is, an element of some $D^p_{-\alpha}$ for $p > 1, \alpha > 0$ through the pairing

$$\langle G, \delta_{\mathbf{u}} \circ F\rangle_{D^q_\alpha, D^p_{-\alpha}} = E[G\delta_{\mathbf{u}} \circ F],$$

representing conditional expectation given $F = \mathbf{u}$ for any $G \in D^q_\alpha$. What is left to determine is, for a given F, which Sobolev spaces contain $\delta_{\mathbf{u}} \circ F$.

The following theorem, the proof of which can be found in [16], provides the answer, taking us one step closer to defining the surface measure corresponding to the conditional expectation.

THEOREM 4.1. *Let F be an $\mathbb{R}^k$-valued, nondegenerate Wiener functional such that $F \in D^{\infty-}_{1+\varepsilon}(X; \mathbb{R}^k)$ for $\varepsilon > 0$, and the density p_F of the law of F is bounded. Also, let $0 \le \beta < \min(\varepsilon, \alpha)$ and $1 < p < \infty$ satisfy*

$$1 < p < \frac{k}{\max\{(k + \beta - \min(\alpha, \varepsilon)), 0\}}, \tag{4.3}$$

and, finally, $\mathcal{O} = \{z \in \mathbb{R}^k : p_F(z) > 0\}$. Then for $G \in D^q_\alpha(X; \mathbb{R})$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\zeta(u) = E(G\delta_u \circ F) \in W^q_\beta(\mathcal{O}), \tag{4.4}$$

where $W^q_\beta(\mathcal{O})$ is the Sobolev space of real-valued, weak β-differentiable functions which are q-integrable.

Recall that for $F = (F_1, \dots, F_k) \in D^{\infty-}_{1+\varepsilon}$, the density $p_F \in W^{\infty-}_{\varepsilon}(\mathcal{O})$ (cf. [5]). Now using the differentiability of the density p_F together with equations (4.2), (4.4) and the algebraic structure of the Sobolev spaces, we have $E^F(G) \in W^q_\beta(\mathcal{O})$, for any $G \in D^q_\alpha(X; \mathbb{R})$, where $(E^F(G))(\mathbf{u}) = E^{F=\mathbf{u}}(G)$.

That is, for each $F \in D^{\infty-}_{1+\varepsilon}(X; \mathbb{R})$, there exists a continuous mapping $E^F: D^q_\alpha(X; \mathbb{R}) \to W^q_\beta(\mathcal{O})$. This, in turn, induces a dual map $(E^F)^*: W^p_{-\beta}(\mathcal{O}) \to D^p_{-\alpha}(X; \mathbb{R})$ defined via the dual relationship

$$\langle E^F(G), v\rangle_{W^q_\beta(\mathcal{O}), W^p_{-\beta}(\mathcal{O})} = \langle G, (E^F)^* v\rangle_{D^q_\alpha(X;\mathbb{R}), D^p_{-\alpha}(X;\mathbb{R})}. \tag{4.5}$$

Informally, this map, sometimes referred to as the Watanabe map (see Section 6 of Chapter III of [4]), is just composition, that is, $(E^F)^* v = v \circ F$.

The object $(E^F)^*\delta_{\mathbf{u}}$ is almost the surface measure needed in (2.4), but it is just a generalized Wiener functional, that is, distribution on X, at this point. If we are to justify our Taylor series expansion via a dominated convergence argument, we need to know that it has a representation as a measure on X.

Clearly, for positive $G \in D^q_\alpha(X; \mathbb{R})$, we shall have

$$\langle G, (E^F)^*\delta_{\mathbf{u}}\rangle_{D^q_\alpha(X;\mathbb{R}), D^p_{-\alpha}(X;\mathbb{R})} = E^{F=\mathbf{u}}(G) > 0.$$

Therefore, $(E^F)^*\delta_{\mathbf{u}} \in D^p_{-\alpha}(X; \mathbb{R})$ defines a *positive generalized Wiener functional*. Next, Theorem 4.3 of [12] together with the conditions stated in Theorem 4.1 implies that for each $u \in \mathcal{O}$, there exists a finite positive Borel measure $\nu^{F,\mathbf{u}}$ defined on Borel subsets of the Wiener space X, supported on $F^{-1}(\mathbf{u})$, such that

$$E^{F=\mathbf{u}}(G) = \int_X G^*(x)\nu^{F,\mathbf{u}}(dx)$$

for all $G \in D^q_\alpha(X; \mathbb{R})$, with G^* its (q, α)-quasi continuous redefinition.

The measure $\nu^{F,\mathbf{u}}$ defined is a probability measure on the set $F^{-1}(\mathbf{u})$. Using Airault and Malliavin's arguments in [2], an appropriate area measure $da^{Z_{\mathbf{u}}}$, corresponding to the measure $\nu^{F,\mathbf{u}}$, can be defined as

$$\int_X G^*(x)\, da^{Z_{\mathbf{u}}}(x) \stackrel{\Delta}{=} p_F(\mathbf{u}) \int G^*(x)(\det(\sigma_F))^{1/2}\nu^{F,\mathbf{u}}(dx), \tag{4.6}$$

where σ_F is the Malliavin covariance matrix. Note that the surface measure depends only on the geometry of the set $Z_{\mathbf{u}}$, whereas the conditional probability measure depends on the functional from which the set is derived, thus the superscripts on the respective measures. We are now in a position to justify at least part of (2.4).

THEOREM 4.2. *Let F be a $\mathbb{R}$-valued nondegenerate Wiener functional such that $F \in D^{\infty-}_{2+\varepsilon}(X; \mathbb{R}^k)$ and the density p_F of the law of F is bounded. Define the unit normal vector field $\eta = DF/\|DF\|_H$. Furthermore, suppose that*:

- $E(\exp(\rho\delta(\eta))) < \infty$ *for* ρ *in some neighborhood of* 0;
- $E(\exp(\rho^2\|D\eta\|^2_{\otimes^2 H})) < \infty$ *for* ρ *in some neighborhood of* 0.

Then, for $0 \le \rho < \rho_c$ *for some nonzero critical radius*

$$\begin{aligned}&\int_0^\rho \int_{F^{-1}(\mathbf{u})} \det{}_2(I_H + rD\eta)\exp\left(-r\delta(\eta) - \frac{1}{2}r^2\right) da^{Z_\mathbf{u}}\, dr \\ &\quad = \sum_{j\ge1} \frac{\rho^j}{j!} \int_{F^{-1}(\mathbf{u})} \frac{d^{j-1}}{dr^{j-1}}\left(\det{}_2(I_H + rD\eta)\exp(-r\delta(\eta) - r^2/2)\right)\Big|_{r=0} da^{Z_\mathbf{u}}.\end{aligned} \tag{4.7}$$

Before proving the above theorem, we shall state a few results concerning the regularity of functions of smooth Wiener functionals.

PROPOSITION 4.3. *Let* $\alpha > 0$ *and* $U \in D^{\infty-}_\alpha(X;\mathbb{R})$.

- *If* $\exp(U) \in L^p(X;\mathbb{R})$, *then* $\exp(U) \in D^{p'}_{\alpha'}$ *where* $p' = p^2(\alpha - \alpha')$ *for* $\alpha' < \alpha$.
- *If* $U > 0$ μ *almost surely and* $1/U \in L^p$, *then* $1/U \in D^{p'}_{\alpha'}$ *where* $p' = p^2(\alpha - \alpha')$ *for* $\alpha' < \alpha$.

We shall skip the proofs of the above, as these can be proved by replicating the proofs of Theorems 1.4 and 1.5 of Watanabe [16].

PROOF OF THEOREM 4.2. This is just dominated convergence combined with the nondegeneracy of F as well as the following bound (cf. Theorem 9.2 of [11]):

$$|\det{}_2(I + A)| \le \exp(C\|A\|^2_{\otimes^2 H})$$

for some fixed $C > 0$.

Note that while using the dominated convergence, we are inherently assuming the *well definedness* of integrals of $\exp(\rho\delta(\eta))$ and $\exp(\rho^2\|D\eta\|^2_{\otimes^2 H})$ with respect to the surface measure $da^{Z_\mathbf{u}}$, which requires

$$\exp(\rho\delta(\eta)) \in D^q_\alpha(X;\mathbb{R}) \qquad \text{such that } q > 1/(\min\{\alpha, 1+\varepsilon\}), \tag{4.8}$$

$$\exp(\rho^2\|D\eta\|^2_{\otimes^2 H}) \in D^q_\alpha(X;\mathbb{R}) \qquad \text{such that } q > 1/(\min\{\alpha, 1+\varepsilon\}). \tag{4.9}$$

Now, using Theorem 1.5 of [16], we have $\eta \in D^{\infty-}_{1+\varepsilon'}$ for all $\varepsilon' < \varepsilon$. Subsequently, using the above proposition together with the assumption involving the existence of exponential moments, we have $\exp(\rho\delta(\eta)), \exp(\rho^2\|D\eta\|^2_{\otimes^2 H}) \in D^p_{\varepsilon''}(X;\mathbb{R})$, such that $p = (\frac{\rho_c}{\rho})^2(\varepsilon' - \varepsilon'')$, where $\varepsilon'' < \varepsilon'$. In order to satisfy (4.8) and (4.9), we must choose ε' and ε'' such that $\rho < \rho_c^2\varepsilon''(\varepsilon' - \varepsilon'')$. □

REMARK 4.4. Note that Theorem 4.2 does not say that the Gaussian measure of the tube is given by the power series in (4.7). Rather, it gives conditions on the sets $Z_{\mathbf{u}} = F^{-1}(\mathbf{u})$ for which the coefficients in the power series are well defined. These conditions allow us to define GMFs for level sets of functions that are not necessarily H-convex. However, for such functions we will lose the interpretation of the power series in (4.7) as an expansion for the Gaussian measure of the tube. This is similar to the distinction between the formal and exact versions of the Weyl/Steiner tube formulae [14].

4.2. *Change of measure formula*: *A Ramer type formula for surface measures*. After assuring ourselves of the existence of the surface Wiener measures, we shall now move onto proving a change of measure formula for the surface measures given by equation (4.6).

To begin with, let $F \in D^{\infty -}_{1+\varepsilon}(X; \mathbb{R}^k)$ so that we can define the surface measure using Theorem 4.1. In order to obtain a change of measure formula for the lower-dimensional subspaces of the Wiener space, we shall start with the standard change of measure formula on the Wiener space X. Let us define a mapping $T_\eta : X \to X$ given by $T_\eta(x) = x + \eta_x$, for some smooth $\eta : X \to H$. Moreover, let U be an open subset of X, and:

(1) T_η is a homeomorphism of U onto an open subset of X,

(2) η is an H-valued C^1 map and its H derivative at each $x \in U$ is a Hilbert–Schimdt operator on H.

This transformation induces two types of changes on the initial measure μ defined on X. These two induced measures can be expressed as

$$P(A) = \mu(T_\eta^{-1}(A)) = T_\eta^* \mu(A),$$

$$Q(A) = \mu(T_\eta(A)) = (T_\eta^{-1})^* \mu(A)$$

for A a Borel set of X.

Ramer's formula for change of measure on X, induced by a transformation defined on X and satisfying the above conditions, gives an expression for the Radon–Nikodym derivative of $\mu \circ T_\eta$ with respect to μ and can be stated as follows:

$$\frac{dQ}{d\mu} = \left|\det{}_2\big(I_H + \nabla\eta(x)\big)\right| \exp\left(-\delta(\eta) - \frac{1}{2}\|\eta(x)\|_H^2\right) \triangleq Y_\eta(x), \tag{4.10}$$

where $\delta(\eta)$ denotes the Malliavin divergence of an H-valued vector field η in X. The proof of this result can be found in [7, 15]. It is to be noted here that, for appropriately smooth transformations, a similar result for $d(\mu \circ T_\eta^{-1})/d\mu$ can be obtained by using the relationship between $d(\mu \circ T_\eta)/d\mu$ and $d(\mu \circ T_\eta^{-1})/d\mu$ given by

$$\frac{d\mu \circ T_\eta}{d\mu}(x) = \left(\frac{d\mu \circ T_\eta^{-1}}{d\mu}(T_\eta x)\right)^{-1}.$$

The following theorem is the first step toward obtaining similar formulae for change of measure on lower-dimensional subsets of the Wiener space.

THEOREM 4.5. *Let $F \in D^{\infty-}_{1+\varepsilon}(X;\mathbb{R}^k)$ satisfy the conditions from Theorem 4.1, and α, β and p be as given in (4.3). Then, there exists a sequence of probability measures $\{\nu_n^{F,\mathbf{u}}\}_{n\geq 1}$ defined on Borel subsets of X such that the measures $\{\nu_n^{F,\mathbf{u}}\}_{n\geq 1}$ are absolutely continuous with respect to the Wiener measure μ and the sequence $\{\nu_n^{F,\mathbf{u}}\}_{n\geq 1}$ converges weakly to $\nu^{F,\mathbf{u}}$.*

PROOF. Let us choose a sequence of positive distributions on $\mathcal{O}$ given by $\{v_n\}_{n\geq 1} \subset W^p_{-\beta}(\mathcal{O})$, such that it converges to $\delta_{\mathbf{u}}$ weakly in $W^p_{-\beta}(\mathcal{O})$ and that $\int v_n(\xi)\,d\xi = 1$ for all $n \geq 1$. Then define the measures $\nu_n^{F,\mathbf{u}}$ as

$$\int G(x)\nu_n^{F,\mathbf{u}}(dx) = \frac{1}{p_F(\mathbf{u})}\int G(x)v_n(F(x))\mu(dx)$$

for all measurable G on (X,μ). In view of (4.5), we can clearly identify the restriction of the measures $\nu_n^{F,\mathbf{u}}$ to $D^q_\alpha(X;\mathbb{R})$ with $(E^F)^* v_n$. Now, by construction, $\{(E^F)^* v_n\}_{n\geq 1}$ converges to $(E^F)^*\delta_{\mathbf{u}}$ in $D^p_{-\alpha}(X;\mathbb{R})$ and their limit is a nonnegative generalized Wiener functional. Therefore, using Lemma 4.1 of [12], we see that the measures $\nu_n^{F,\mathbf{u}}$ converge weakly to $\nu^{F,\mathbf{u}}$. □

Thus, the surface (probability) measure of $Z_{\mathbf{u}}$, or the conditional probability measure corresponding to $\{F = \mathbf{u}\}$, for any $\mathbf{u} \in \mathcal{O}$ can also be defined as

$$\begin{aligned}\int G^*(x)\nu^{F,\mathbf{u}}(dx) &= \lim_n \int G^*(x)\nu_n^{F,\mathbf{u}}(dx)\\ &= \lim_n \frac{1}{p_F(\mathbf{u})}\int G^*(x)v_n(F(x))\mu(dx)\end{aligned} \tag{4.11}$$

for the appropriate class of Wiener functionals G, which, as noted earlier, depends on the regularity of F.

Let us now define a mapping $T_{\rho,\eta}: X \to X$ given by $T_{\rho,\eta}(x) = x + \rho\eta_x$, for some $\eta \in D^{\infty-}_{1+\varepsilon}(X;H)$. We shall study the change that the mapping $T_{\rho,\eta}$ induces on the surface measure of $F^{-1}(\mathbf{u})$. Note that

$$T_{\rho,\eta}(F^{-1}(\mathbf{u})) = \{x + \rho\eta_x : x \in F^{-1}(\mathbf{u})\}.$$

Set $F_{\rho,\eta} = F \circ T^{-1}_{\rho,\eta} \in D^{\infty-}_{1+\varepsilon}$, so that

$$F^{-1}_{\rho,\eta}(\mathbf{u}) = T_{\rho,\eta}(F^{-1}(\mathbf{u})) \triangleq Z^{\eta,\rho}_{\mathbf{u}}.$$

Using the above theorem and the relationship (4.6), the area measure for $Z^{\eta,\rho}_{\mathbf{u}}$ can now be identified as

$$\int_{Z^{\eta,\rho}_{\mathbf{u}}} G^*\,da^{Z^{\eta,\rho}_{\mathbf{u}}} = \lim_n \int_X G^*(y)[\det\sigma_{F_{\rho,\eta}}(y)]^{1/2} v_n(F_{\rho,\eta}(y))\mu(dy).$$

Now using the transformation $y = T_{\rho,\eta}(x)$ and replacing the function $G^*(\cdot)$ by $G^*(T_{\rho,\eta}^{-1}(\cdot))$ and, finally, using the standard Ramer's formula from equation (4.10), we get

$$\begin{aligned}
&\int_{Z_{\mathbf{u}}^{\eta,\rho}} G^*(T_{\rho,\eta}^{-1}y)\, da^{Z_{\mathbf{u}}^{\eta,\rho}} \\
&\quad = \lim_n \int_X G^*(x)[\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} \upsilon_n(F_{\rho,\eta}(T_{\rho,\eta}x)) Y_\rho^\eta(x)\mu(dx) \qquad (4.12)\\
&\quad = \lim_n \int_X G^*(x)[\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} \upsilon_n(F(x)) Y_\rho^\eta(x)\mu(dx),
\end{aligned}$$

where $Y^{\rho,\eta}(x)$ is the Radon–Nikodym derivative of the measure $\mu \circ T_{\rho,\eta}$ with respect to the measure μ and, as a result of (4.10), can be expressed as

$$Y_\rho^\eta(x) = \left|\det_2\big(I_H + \rho \nabla \eta(x)\big)\right| \exp\big(-\rho\delta(\eta) - \tfrac{1}{2}\rho^2 \|\eta\|_H^2\big). \qquad (4.13)$$

Using the definitions of F_ρ, the surface (probability and area) measures and, finally, rearranging the terms, we can rewrite (4.12) as

$$\begin{aligned}
&\int_{Z_{\mathbf{u}}^{\eta,\rho}} G^*(T_{\rho,\eta}^{-1}y)\, da^{Z_{\mathbf{u}}^{\eta,\rho}} \\
&\quad = \lim_n \int_X G^*(x)[\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} Y_\rho^\eta(x) \upsilon_n(F(x))\mu(dx) \\
&\quad = \lim_n p_F(u) \int_X G^*(x) \left(\frac{\det \sigma_{F_\rho}(T_{\rho,\eta}x)}{\det \sigma_F(x)}\right)^{1/2} Y_\rho^\eta(x)[\det \sigma_F(x)]^{1/2} \nu_n^{F,\mathbf{u}}(dx) \qquad (4.14)\\
&\quad = p_F(u) \int_{Z_{\mathbf{u}}} G^*(x) \left(\frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)}\right)^{1/2} Y_\rho^\eta(x)[\det \sigma_F(x)]^{1/2} \nu^{F,\mathbf{u}}(dx) \\
&\quad = \int_{Z_{\mathbf{u}}} G^*(x) \left(\frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)}\right)^{1/2} Y_\rho^\eta(x)\, da^{Z_{\mathbf{u}}},
\end{aligned}$$

which proves the following theorem.

THEOREM 4.6. *Let $F \in D_{1+\varepsilon}^{\infty-}$ satisfy the conditions of Theorem 4.1 and $\eta \in D_{1+\varepsilon}^{\infty-}(X; H)$ be such that*:

- *$(I + \rho\eta)$ is one-to-one and onto when restricted to a domain B_η with complement having C_ε^p capacity 0 for all p*;
- *$(I_H + \rho\nabla\eta)$ is an invertible operator on H, when restricted to B_η.*

Then,

$$\begin{aligned}
\frac{da^{Z_{\mathbf{u}}} \circ T_{\rho,\eta}}{da^{Z_{\mathbf{u}}}}(x) &= \left(\frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)}\right)^{1/2} Y_\rho^\eta(x) \qquad (4.15)\\
&\triangleq J_{\rho,\eta}^{Z_{\mathbf{u}}}.
\end{aligned}$$

REMARK 4.7. (1) In order to better understand the above theorem, we shall now try to simplify the expression involved in (4.15), for the simple case where $F = \delta(h)$, for some $h \in H$, and $\eta = \nabla F = h$, is a constant vector field. Clearly, $\nabla \eta \equiv 0$, implying $DT_\rho = DT_\rho^{-1} = I_H$. Then the whole expression boils down to

$$\begin{aligned} J_{\rho,\eta}^{Z_{\mathbf{u}}} &= Y_\rho^\eta(x) \\ &= \exp\bigl(-\rho\delta(\eta) - \rho^2 \|\eta\|_H^2/2\bigr). \end{aligned}$$

(2) The above expression in (4.15) can be rewritten as

$$J_{\rho,\eta}^{Z_{\mathbf{u}}} = \left(\frac{\det(\langle DT_{\rho,\eta}^{-1}(x)\nabla F_i(x), DT_{\rho,\eta}^{-1}(x)\nabla F_j(x)\rangle_H)_{ij}}{\det(\langle \nabla F_i(x), \nabla F_j(x)\rangle_H)_{ij}} \right)^{1/2} Y_\rho^\eta(x),$$

where DT_ρ^{-1} is the operator given by $(I_H + \rho\nabla\eta)^{-1}$.

(3) From the definition of $Y_\rho^\eta(x)$, note that the expression in (4.15) is well defined as long as $\nabla\eta$ is a Hilbert–Schmidt class-valued operator acting on $H \times H$. Next, in order to be able to use the formula in (4.15), we need $J_{\rho,\eta}^{Z_{\mathbf{u}}}$ to be integrable with respect to the surface measure $da^{Z_{\mathbf{u}}}$.

(4) Note that the submanifold $Z_{\mathbf{u}}$ and the measure $da^{Z_{\mathbf{u}}}$ are not dependent on F. Therefore, to make the above calculation simpler, we can choose an appropriate functional F', such that $\{F'(x) = \mathbf{v}\} = Z_{\mathbf{u}}$, for some $\mathbf{v} \in \mathbb{R}^k$, and that $\{\nabla F_i'\}$ form an orthonormal basis of the normal space of $Z_{\mathbf{u}}$.

(5) Finally, we note that $a^{Z_u} \circ T_{\rho,\eta}$ is defined only up to capacity C_ε^p sets for any p. That is, $C_\varepsilon^p(A) = 0$ implies $a^{Z_u} \circ T_{\rho,\eta}(A) = 0$. Hence, the image of the discontinuities of Z_u under $T_{\rho,\eta}$ has C_ε^p capacity 0.

4.3. *The set and its tube.* Finally, we shall define the class of sets for which we shall prove a tube formula, and, therefore, define the GMFs. In our bid to keep the calculations much easier to handle, we shall restrict our attention to the unit codimensional case.

Continuing the way we have been defining subsets of the Wiener space via Wiener functionals, we shall start with a nondegenerate Wiener functional $F \in D_{2+\varepsilon}^{\infty-}(X;\mathbb{R})$, such that F is an H-convex functional. We shall write $A_{\mathbf{u}} = F^{-1}(-\infty, \mathbf{u}]$ for $\mathbf{u} \in \mathcal{O}$. This is an H-convex set and its boundary $\partial A_{\mathbf{u}}$ is a smooth unit codimensional submanifold of the Wiener space.

For each $x \in A_{\mathbf{u}}$ define the support cone

$$\mathcal{S}_x(A_{\mathbf{u}}) = \{h \in H : \text{for any } \delta > 0,\ \exists 0 < \varepsilon < \delta \text{ such that } x + \varepsilon h \in A_{\mathbf{u}}\}$$

and its dual, the (convex) normal cone

$$N_x(A_{\mathbf{u}}) = \{h \in H : \langle h, h'\rangle \le 0\ \forall h' \in \overline{\mathcal{S}_x(A_{\mathbf{u}})}\},$$

where $\overline{\mathcal{S}_x(A_{\mathbf{u}})}$ is the closure of $\mathcal{S}_x(A_{\mathbf{u}})$ in H.

The following lemmas describe some of the properties of the tube around $A_{\mathbf{u}}$. For clarity we state the results only for the case of unit codimension, though similar statements hold for k-codimension.

Define the smooth points of F,

$$\operatorname{Sm}(F) \stackrel{\Delta}{=} \Big\{x : \nabla F(x) \neq 0 \ \forall h_1, h_2 \in H_\infty \ \exists \varepsilon_n(h_1, h_2), \downarrow 0 \quad \text{such that} \lim_{n\to\infty} T_2(F, x, h_1, h_2, \varepsilon_n) = 0\Big\}, \tag{4.16}$$

where

$$T_2(F, x, h_1, h_2, \varepsilon) = F(x+h_1) + \varepsilon\langle \nabla F(x+h_1), h_2\rangle_H + \varepsilon^2 \langle \nabla^2 F(x+h_1), h_2 \otimes h_2\rangle_{H\otimes H} - F(x+h_1+\varepsilon h_2)$$

is the difference between the second-order Taylor expansion of $F(x+h_1+\varepsilon h_2)$ evaluated at $x+h_1$ and its true value.

LEMMA 4.8. *Suppose* $F \in D^{\infty-}_{2+\varepsilon}(X;\mathbb{R})$. *Then*, $C^p_\varepsilon(\operatorname{Sm}(F)^c) = 0 \ \forall p > 1$. *At every* $x \in \partial A_{\mathbf{u}} \cap \operatorname{Sm}(F)$,

$$N_x(A_{\mathbf{u}}) = \{c\nabla F(x) : c \geq 0\}. \tag{4.17}$$

PROOF. The first conclusion follows essentially directly from Corollary 3.6. Suppose now that $x \in \operatorname{Sm}(F)$. Then, for any $h \perp \nabla F(x)$, $\|h\| \leq K$ we can find a sequence $H_\infty \ni h_n \stackrel{n\to\infty}{\to} h$ satisfying $\langle h_n, \nabla F(x)\rangle_H < -1/n$. Because the 2nd order Taylor expansion holds at x, we see that $h_n \in \mathcal{S}_x(A_{\mathbf{u}})$ for all n. Hence, $h \in \overline{\mathcal{S}_x(A_{\mathbf{u}})}$. This is enough to conclude that any $\eta \in N_x(A_{\mathbf{u}})$ is parallel to $\nabla F(x)$. It is not hard to see that it must therefore be a positive multiple of $\nabla F(x)$. $\square$

LEMMA 4.9. *Suppose* $F \in D^{\infty-}_{2+\varepsilon}$ *is* H*-convex. Then, for each* $r > 0$, *the restriction of*

$$x \mapsto x + r\nabla F(x)/\|\nabla F(x)\| \stackrel{\Delta}{=} x + r\eta_x$$

to $\operatorname{Sm}(F) \cap \partial A_{\mathbf{u}}$ *is one-to-one in the sense that for each* $x \in \operatorname{Sm}(F) \cap \partial A_{\mathbf{u}}$

$$\{y \in \operatorname{Sm}(F) \cap \partial A_{\mathbf{u}} : \|y - (x + r\eta_x)\|_H \leq r\} = \varnothing.$$

PROOF. Given $x \in \operatorname{Sm}(F) \cap \partial A_{\mathbf{u}}$, suppose such a y exists with $\|x + r\eta_x - y\| < r$. As y is a smooth point of F, we can find some $h \in H_\infty$ such that $\|x + r\eta_x - (y+h)\|_H < r$ and $F(y+h) < u$ with F continuous at $y+h$. Choose $\nu(x,y) \in H_\infty$ such that $x + \nu(x,y)$ is arbitrarily close to $y+h$. Then, by continuity of F on $\operatorname{Sm}(F)$, $F(x+\nu(x,y)) < u$ and $\|x + r\eta_x - (x+\nu(x,y))\|_H = \|r\eta_x - \nu(x,y)\|_H < r$. Note that this implies $\langle \nu(x,y), \eta_x\rangle_H > 0$, or, alternatively, $\nu(x,y) \notin \mathcal{S}_x(A_{\mathbf{u}})$.

Now consider the restriction of F to the line segment joining $[x, x + \nu(x, y)]$, denoted by

$$f(t) = F\big(x + t\big(x - \nu(x, y)\big)\big), \qquad 0 \le t \le 1,$$

which, by Remark 3.8, is continuous, twice-differentiable and convex on a dense subset of $t \in [0, 1]$, hence, we can find a continuous, twice-differentiable convex function $\tilde{f}$ on all of $[0, 1]$ that agrees with f on this dense subset.

There are two possibilities, the first being that $\tilde{f}(t) \le u$ for all $t \in [0, 1]$. This would imply $\eta(x, y) \in \mathcal{S}_x(A_{\mathbf{u}})$, contradicting our previous observation. The second alternative is that there exists t such that $\tilde{f}(t) > u$. However, $\tilde{f}(0) = u$, $\tilde{f}(1) < u$ and this would violate convexity. By contradiction, there can be no such y. This proves the assertion that there are no points y of distance strictly less than r to $x + r\eta_x$. Now, suppose there exists a smooth point $y \ne x$ of distance exactly r from $x + r\eta_x$. Then, for any $\delta > 0$ it is not hard to show that

$$\big\| y - \big(x + (\delta + r)\eta_x\big)\big\|_H < \delta + r,$$

but we just proved that there can be no such y. $\square$

We are now in a position to define the tube

$$\mathrm{Tube}(A_{\mathbf{u}}, \rho) = \{y \in X : \exists x \in A_{\mathbf{u}}, \|y - x\|_H \le \rho\} = \{y \in X : d_H(y, A_{\mathbf{u}}) \le \rho\},$$

where the distance function is defined as

$$d_H(y, A_{\mathbf{u}}) = \inf_{h \in H_\infty : y - h \in A_{\mathbf{u}}} \|h\|_H. \tag{4.18}$$

The level sets of the distance function are hypersurfaces at distance r,

$$\partial A^r_{\mathbf{u}} = \{y \in X : d_H(y, A_{\mathbf{u}}) = r\}. \tag{4.19}$$

Lemma 4.9 asserts that the restriction of $x \mapsto x + r\eta_x$ to $A_{\mathbf{u}} \cap \mathrm{Sm}(F)$ is one-to-one. On the image of $\mathrm{Sm}(F)$, its inverse is easily defined as $x + r\eta_x \mapsto (x, \eta_x)$, and, as noted in the remarks following Theorem 4.6, the image of $\partial A_{\mathbf{u}} \cap \mathrm{Sm}(F)$ has C^p_ε-capacity 0. Hence, up to a set of C^p_ε-capacity 0, it is a bijection and Theorem 4.6 can be applied to study the surface measure of $\partial A^r_{\mathbf{u}}$.

Moreover, the following theorem further corroborates the fact that the change of measure formula established in Theorem 4.6 is the appropriate result to use in order to obtain a tube formula, as will be seen later.

THEOREM 4.10. *Let $C^{\infty-}_\varepsilon(A) = 0$, then under hypotheses* (H2) *and* (H3) *of* [9], *for $\varepsilon_1 < \varepsilon$,*

$$C^{\infty-}_{\varepsilon_1}\big(A \oplus B_H(0, r)\big) = 0,$$

where $B_H(0, r)$ is a ball in H centered at 0 *with radius r.*

Since capacities are continuous from below, it suffices to prove that $C_{\varepsilon_1}^{\infty-}(A \oplus B_{E_n}(0,r)) = 0$, for each n, whenever $C_{\varepsilon}^{\infty-}(A) = 0$, where $B_{E_n}(0,r)$ is a ball of radius r, centered at 0, in the vector space $E_n = \operatorname{span}(h_1, \dots, h_n)$, where $\{h_i\}_{i\geq 1}$ is the orthonormal basis of H. Also, note that the proof is given for an open subset A of the Wiener space X, but, using the arguments of [9], we can extend it to general subsets of the Wiener space.

Before proving the above theorem, we shall, first, obtain some estimates on functionals derived from the Wiener functionals. Note that

$$A \oplus B_{E_n}(0,r) = \{(A + \langle s, h^{(n)}\rangle) : s \in B_{\mathbb{R}^n}(0,r)\},$$

where $\langle s, h^{(n)}\rangle = \sum_{i=1}^n s_i h_i$. Further, for the later part, we shall denote $I_n \subset B_{\mathbb{R}^n}(0,r)$ as the set of all rationals in the set $B_{\mathbb{R}^n}(0,r)$. The following result is, essentially, an extension of Theorem 2.1 of [8].

THEOREM 4.11. *Let $f \in D_\alpha^p(X)$ for $\alpha \in (1/p, 1)$, and $\mathbb{R}^n \ni t \mapsto \xi(t,\cdot) = f(\cdot + \langle t, h^{(n)}\rangle)$, such that $|t| \leq T$, for some fixed T, that is, t belongs to some large enough cube. Then for all $p' \in (1/\alpha, p)$ there exists a $C = C(p, p', \alpha, T)$, such that*

$$\|\xi(t) - \xi(s)\|_{p'} \leq C \|f\|_{p,\alpha} |t - s|^\alpha.$$

PROOF. Before we shall start proving the above result, we shall recall that the estimates of Lemma 4.1 of [8] remain unchanged in our setup. Now we need an estimate analogous to the one obtained in Lemma 4.2 of [8], for which we recall the Ramer's change of measure formula,

$$\begin{aligned}
&\|G(\cdot + \langle t_2, h^{(n)}\rangle) - G(\cdot + \langle t_1, h^{(n)}\rangle)\|_{p'} \\
&\quad = \left\| G\left(\cdot + \frac{1}{2}\langle t_1 + t_2, h^{(n)}\rangle + \frac{1}{2}\langle t_2 - t_1, h^{(n)}\rangle\right)\right. \\
&\qquad \left. - G\left(\cdot + \frac{1}{2}\langle t_1 + t_2, h^{(n)}\rangle - \frac{1}{2}\langle t_2 - t_1, h^{(n)}\rangle\right)\right\|_{p'} \qquad (4.20)\\
&\quad = \left(\int_X \left| G\left(x + \frac{1}{2}\langle t_2 - t_1, h^{(n)}\rangle\right) - G\left(x - \frac{1}{2}\langle t_2 - t_1, h^{(n)}\rangle\right)\right|^{p'}\right. \\
&\qquad \left. \times \exp\left(-\frac{1}{2}\left\|\frac{1}{2}\langle t_1 + t_2, h^{(n)}\rangle\right\|_H^2 - \delta\left(\frac{1}{2}\langle t_1 + t_2, h^{(n)}\rangle\right)\right)\mu(dx)\right)^{1/p'}.
\end{aligned}$$

Now writing $h_1 = |t_2 - t_1|^{-1}\langle t_2 - t_1, h^{(n)}\rangle$, $h_2 = |t_1 + t_2|^{-1}\langle t_1 + t_2, h^{(n)}\rangle$, and $Y_{|t_1+t_2|/2}^{h_2} = \exp[-\||t_1 + t_2|h_2/8\|_H^2 - \delta(|t_1 + t_2|h_2/2)]$, we can rewrite the above as

$$\|G(\cdot+\langle t_2,h^{(n)}\rangle)-G(\cdot+\langle t_1,h^{(n)}\rangle)\|_{p'} \tag{4.21}$$
$$=\left(\int_X\left|G\left(x+\frac{1}{2}|t_2-t_1|h_1\right)-G\left(x-\frac{1}{2}|t_2-t_1|h_1\right)\right|^{p'} \times Y^{h_2}_{|t_1+t_2|/2}(x)\mu(dx)\right)^{1/p'}.$$

This reduces the above expression to the case dealt in [8]. Therefore, using the rest of the calculations of Lemma 4.2 of [8], and writing $G=T_a f$, where $\{T_a\}_{a\geq 0}$ is the semigroup associated with the Ornstein–Uhlenbeck operator L, we get the desired estimate expressed as

$$\|T_a f(\cdot+\langle t_2,h^{(n)}\rangle)-T_a f(\cdot+\langle t_1,h^{(n)}\rangle)\|_{p'}\leq C(p,p',\alpha)\|T_a f\|_{p,1}|t_2-t_1|.$$

Thereafter, we can mimic the proof of Theorem 2.1 of [8] and get the desired estimate. □

Now coming back to our case, let e_A be the potential equilibrium of A (cf. [10]) and $e_A\in D^{\infty-}_{\varepsilon}$, therefore, there exists a $v_A\in L^{\infty-}$, such that

$$e_A=(I-L)^{-\varepsilon/2}v_A\stackrel{\Delta}{=}(I-L)^{-\varepsilon_1/2}v_{(\varepsilon-\varepsilon_1),A},$$

where ε_1 is some number strictly smaller than ε and L is the Ornstein–Uhlenbeck operator. Then, clearly,

$$e_A(\cdot+\langle t,h^{(n)}\rangle)=(I-L)^{-\varepsilon/2}v_A(\cdot+\langle t,h^{(n)}\rangle)\stackrel{\Delta}{=}(I-L)^{-\varepsilon_1/2}v_{(\varepsilon-\varepsilon_1),A}(\cdot+\langle t,h^{(n)}\rangle).$$

Now, writing $\xi(t)\stackrel{\Delta}{=}e_A(\cdot+\langle t,h^{(n)}\rangle)$ and $\xi_{(\varepsilon-\varepsilon_1)}(t)\stackrel{\Delta}{=}v_{(\varepsilon-\varepsilon_1),A}(\cdot+\langle t,h^{(n)}\rangle)$, and also, in the process, choosing the appropriate quasi-continuous redefinitions of the processes ξ and $\xi_{(\varepsilon-\varepsilon_1)}$, and choosing a large p' (conditions on p' will appear later), such that by Kree–Meyer inequalities, we have

$$\|\xi(t)-\xi(s)\|_{p',\varepsilon_1}\leq C\|\xi_{(\varepsilon-\varepsilon_1)}(t)-\xi_{(\varepsilon-\varepsilon_1)}(s)\|_{p'}. \tag{4.22}$$

Now using the above theorem with f replaced by $v_{(\varepsilon-\varepsilon_1),A}$, we get

$$\begin{aligned}\|\xi_{(\varepsilon-\varepsilon_1)}(t)-\xi_{(\varepsilon-\varepsilon_1)}(s)\|_{p'}&\\ &=\|v_{(\varepsilon-\varepsilon_1),A}(\cdot+\langle t,h^{(n)}\rangle)-v_{(\varepsilon-\varepsilon_1),A}(\cdot+\langle t,h^{(n)}\rangle)\|_{p'}\\ &\leq C\|v_{(\varepsilon-\varepsilon_1),A}\|_{p,(\varepsilon-\varepsilon_1)}|t-s|^{(\varepsilon-\varepsilon_1)}\\ &=C\|e_A\|_{p,\varepsilon}|t-s|^{(\varepsilon-\varepsilon_1)},\end{aligned} \tag{4.23}$$

where $p'\in(2/\varepsilon,p)$. Combining (4.22) and (4.23), we get

$$\|\xi(t)-\xi(s)\|_{p',\varepsilon_1}\leq C\|e_A\|_{p,\varepsilon}|t-s|^{(\varepsilon-\varepsilon_1)}, \tag{4.24}$$

which can be rewritten as

$$\sup_{s\neq t}\frac{\|\xi(t)-\xi(s)\|_{p',\varepsilon_1}^{p'}}{|t-s|^{p'(\varepsilon-\varepsilon_1)}}\le C\|e_A\|_{p,\varepsilon}^{p'}. \tag{4.25}$$

Now we can list the assumptions on the various indices as follows: we start with any fixed $\varepsilon_1<\varepsilon$, then choose a large enough p such that $(\varepsilon-\varepsilon_1)\in(1/p,1)$, and then we choose p' such that $p'\in(1/(\varepsilon-\varepsilon_1),p)$ and $p'(\varepsilon-\varepsilon_1)>n$. This can be achieved by choosing p and p' of the order of n, in particular, choosing $p=an/(\varepsilon-\varepsilon_1)$ and $p'=bn/(\varepsilon-\varepsilon_1)$, for $a>b>0$ will do. Then, using Theorem 3.4 of [10], we get

$$C_{\varepsilon_1}^p\Big(\sup_{s\neq t}|\xi(t)-\xi(s)|\Big)\le C\|e_A\|_{p,\varepsilon}^{p'}. \tag{4.26}$$

PROOF OF THEOREM 4.10. Now let us consider

$$\begin{aligned}
C_{\varepsilon_1}^p\big(A\oplus B_{E_n}(0,r)\big)&=C_{\varepsilon_1}^p\Big(\sup_{s\in B_{\mathbb{R}^n}(0,r)}1_A\big(\cdot+\langle s,h^{(n)}\rangle\big)\Big)\\
&\le C_{\varepsilon_1}^p\Big(\sup_{s\in I_n}e_A\big(\cdot+\langle s,h^{(n)}\rangle\big)\Big)\qquad\text{as } e_A\ge 1_A\\
&\le C_{\varepsilon_1}^p\Big(\sup_{s\in B_{\mathbb{R}^n}(0,r)}\xi(s)\Big)\\
&\le C_{\varepsilon_1}^p\Big(\sup_{s\in B_{\mathbb{R}^n}(0,r)}|\xi(s)-\xi(0)|+|\xi(0)|\Big).
\end{aligned} \tag{4.27}$$

Now using (4.26), we shall get

$$C_{\varepsilon_1}^p\big(A\oplus B_{E_n}(0,r)\big)\le(C+1)\|e_A\|_{p,\varepsilon}^{p'}=(C+1)(C_\varepsilon^p(A))^{p'/p},$$

which proves that $C_{\varepsilon_1}^{\infty-}(A\oplus B_{E_n}(0,r))=0$, for all $p>n(\varepsilon-\varepsilon_1)^{-1}$. Now by the definition of the capacities and the hierarchy of the Sobolev spaces, we shall have $C_{\varepsilon_1}^{\infty-}(A\oplus B_{E_n}(0,r))=0$, thereby proving the result. □

Using the definition of the smooth points $\mathrm{Sm}(F)$ and $\mathrm{Tube}(A_{\mathbf{u}},\rho)$, we can conclude that

$$\begin{aligned}
\mathrm{Tube}(A_{\mathbf{u}},\rho)=&\big[\big(A_{\mathbf{u}}\cap\mathrm{Sm}(F)\big)\oplus B_H(0,\rho)\big]\\
&\cup\big[\big(A_{\mathbf{u}}\cap\{\mathrm{Sm}(F)\}^c\big)\oplus B_H(0,\rho)\big].
\end{aligned} \tag{4.28}$$

Using the above calculations, we have

$$\mu(\mathrm{Tube}(A_{\mathbf{u}},\rho))=\mu\big(\big(A_{\mathbf{u}}\cap\mathrm{Sm}(F)\big)\oplus B_H(0,\rho)\big), \tag{4.29}$$

since $C_{\varepsilon_1}^p((A_{\mathbf{u}}\cap\{\mathrm{Sm}(F)\}^c)\oplus B_H(0,\rho))=0$, implying that the μ-measure of the set is zero. Therefore, it is enough, for the tube formula, to consider the set $((A_{\mathbf{u}}\cap\mathrm{Sm}(F))\oplus B_H(0,\rho))$, on which the transformation $x\mapsto x+\eta_x$ is well defined up to C_ε^p-zero sets, and, hence, we can use the change of measure formula for the surface areas given in Theorem 4.6.

5. A Wiener tube formula. After setting up the basics, definitions and the conditions, concerning a tube formula in the Wiener space, we shall finally prove one of the main results of this paper, which can be stated in the form of the following theorem.

THEOREM 5.1. *Let* $F \in \mathbb{D}_{2+\delta}^{\infty-}(X;\mathbb{R})$ *be an* H*-convex Wiener functional such that it satisfies all the regularity conditions of Theorem* 4.2, *and* $A_{\mathbf{u}} = F^{-1}(-\infty, \mathbf{u}]$, *then*

$$\mu(\text{Tube}(A_{\mathbf{u}}, \rho)) = \mathcal{M}_0^{\mu}(A_{\mathbf{u}}) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\mu}(A_{\mathbf{u}}),$$

where $\mathcal{M}_j^{\mu}(A_{\mathbf{u}})$ *are the infinite dimensional versions of Gaussian Minkowski functionals and, as usual,* $\mathcal{M}_0^{\mu}(A_{\mathbf{u}}) = \mu(A_{\mathbf{u}})$.

PROOF. Let us start with recalling the definition of the *outward* pointing normal space $N_x(A_{\mathbf{u}})$ from (4.17) and writing $N(A_{\mathbf{u}}) = \bigcup_{x \in A_{\mathbf{u}}} N_x(A_{\mathbf{u}})$. Then, let us define a distance function, $d_{A_{\mathbf{u}}} : \text{Tube}(A_{\mathbf{u}}, \rho) \to \mathbb{R}$, such that for $x \in \text{Tube}(A_{\mathbf{u}}, \rho)$ writing the "residual" as

$$\hat{r}_x = \operatorname*{argmin}_{r \in \mathbb{R}; \eta \in N(A_{\mathbf{u}})} d(x - r\eta, A_{\mathbf{u}}),$$

the distance function $d_{A_{\mathbf{u}}}$ is given by $d_{A_{\mathbf{u}}}(x) = \|\hat{r}_x\|$. Clearly, from the above definition, $d_{A_{\mathbf{u}}}^{-1}(0) = A_{\mathbf{u}}$. Also, we can further express $\text{Tube}(A_{\mathbf{u}}, \rho)$ as the disjoint union of $A_{\mathbf{u}}$ and $\text{Tube}^+(\partial A_{\mathbf{u}}, \rho)$, where $\text{Tube}^+(\partial A_{\mathbf{u}}, \rho) = \text{Tube}(\partial A_{\mathbf{u}}, \rho) \cap A_{\mathbf{u}}^c$. Thus,

$$\mu(\text{Tube}(A_{\mathbf{u}}, \rho)) = \mu(A_{\mathbf{u}}) + \mu(\text{Tube}^+(\partial A_{\mathbf{u}}, \rho)). \tag{5.1}$$

Now using the Wiener space version of Federer's co-area formula as it appears in [2], we shall obtain

$$\mu(\text{Tube}^+(\partial A_{\mathbf{u}}, \rho)) = \int_0^{\rho} \int_{d_{A_{\mathbf{u}}}^{-1}(r)} (\sigma_{d_{A_{\mathbf{u}}}}(x))^{-1} \, da^{\partial^+ A_{\mathbf{u}}^r} \, dr, \tag{5.2}$$

where $\partial^+ A_{\mathbf{u}}^r = d_{A_{\mathbf{u}}}^{-1}(r) \cap A_{\mathbf{u}}^c$ are the level sets of the distance function $d_{A_{\mathbf{u}}}$ in the outward direction. Now note that $\nabla d_{A_{\mathbf{u}}} = \eta$, hence, $\sigma_{d_{A_{\mathbf{u}}}}(x) = 1$. Then let us define the transformation $T_{r,\eta} : X \to X$, such that its restriction to $A_{\mathbf{u}}$ is given by $T_{r,\eta}(x) = x + r\eta$. Clearly, $T_{r,\eta}(\partial A_{\mathbf{u}}) = \partial A_{\mathbf{u}}^r$. Then, we shall use our change of measure formula for surfaces on $\int_{d_{A_{\mathbf{u}}}^{-1}(r)}$ to further simplify the expression in (5.2) to obtain

$$\mu(\text{Tube}^+(\partial A_{\mathbf{u}}, \rho)) = \int_0^{\rho} \int_{A_{\mathbf{u}}} J_{r,\eta}^{\partial A_{\mathbf{u}}} \, da^{\partial A_{\mathbf{u}}} \, dr = \int_0^{\rho} \int_{A_{\mathbf{u}}} Y_r^{\eta} \, da^{\partial A_{\mathbf{u}}} \, dr,$$

where terms $J_{r,\eta}^{\partial A_{\mathbf{u}}}$ and Y_r^{η} are as they appear in Theorem 4.6.

Now using a Taylor series expansion for Y_r^η with respect to r, we can rewrite the above expression as

$$\mu(\mathrm{Tube}^+(\partial A_{\mathbf{u}}, \rho)) = \sum_{j=0}^{\infty} \frac{\rho^{j+1}}{(j+1)!} \int_{A_{\mathbf{u}}} \frac{d^j}{dr^j} \bigl(\det_2(I_H + r\nabla\eta)\exp(-r\delta(\eta) - r^2/2)\bigr)\Big|_{r=0} da^{\partial A_{\mathbf{u}}}.$$

We note here that ρ must be within the radius of convergence of the Taylor series of Y_r^η, which in turn will ensure the convergence of the above series.

Finally, plugging the above expression in (5.1), we get

$$\begin{aligned}\mu(\mathrm{Tube}(A_{\mathbf{u}}, \rho)) &= \mu(A_{\mathbf{u}}) \\ &\quad + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \int_{A_{\mathbf{u}}} \frac{d^j}{dr^j} \bigl(\det_2(I_H + r\nabla\eta)\exp(-r\delta(\eta) - r^2/2)\bigr)\Big|_{r=0} da^{\partial A_{\mathbf{u}}} \\ &= \mu(A_{\mathbf{u}}) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \mathcal{M}_n^\mu(A_{\mathbf{u}}),\end{aligned} \tag{5.3}$$

where $\mathcal{M}_n^\mu(A_{\mathbf{u}})$ are Gaussian Minkowski functionals of the infinite dimensional set $A_{\mathbf{u}}$, given by

$$\mathcal{M}_n^\mu(A_{\mathbf{u}}) = \int_{A_{\mathbf{u}}} \frac{d^n}{dr^n} \bigl(\det_2(I_H + r\nabla\eta)\exp(-r\delta(\eta) - r^2/2)\bigr)\Big|_{r=0} da^{\partial A_{\mathbf{u}}}, \tag{5.4}$$

which proves the theorem. □

6. Applications. In this section we shall invoke the existential results from the previous section to obtain a kinematic fundamental formula akin to the one obtained in Theorem 15.9.5 of [1], though, for a larger class of random fields.

Let us consider a real-valued random field f defined on a compact Riemannian manifold M equipped with a metric τ. Then the modulus of continuity Ξ of a function $F : M \to \mathbb{R}$ is defined as

$$\Xi_F(\eta) \stackrel{\Delta}{=} \sup_{\tau(x,y)\le\eta} |F(x) - F(y)|$$

for all $\eta > 0$. Continuing the setup introduced in the example stated in Section 1, we shall consider a specific class of random fields f which can be represented as

$$f(x) = \sum_{i=1}^{N} \int_0^1 V_i(B_i^x(s))\, dB_i^x(s), \tag{6.1}$$

where the integral is to be interpreted in the Itô sense, each $V_i : \mathbb{R} \to \mathbb{R}$ is a smooth function, and $B^x(\cdot) = (B_i^x(\cdot))_{i=1}^N$ is a $\mathbb{R}^N$-valued, zero-mean Gaussian process defined on $M \times [0, 1]$, whose covariance is given by

$$E(B_i^x(s)B_j^y(t)) = (s \wedge t)C_{ij}(x, y) = (s \wedge t)C(x, y), \tag{6.2}$$

where $C : M \times M \to \mathbb{R}$ is a smooth function, such that for each fixed $t \in [0, 1]$, the field $B^{\cdot}(t)$ is an isotropic Gaussian field over M (see Sections 5.7 and 5.8 of [1]).

The following are the basic assumptions on the functions V_i and the Gaussian process $B^x(t)$:

(A1) $V_i \in C^4$, $1 \leq i \leq N$, the class of all 4-continuously differentiable functions.

(A2) Writing $V^{(k)}$ as the kth derivative of V, let us define

$$C_i(k, s, x, y) \stackrel{\Delta}{=} \sup_{0 \leq \alpha \leq 1} V_i^{(k)}\big(\alpha B^x(s) + (1 - \alpha)B^y(s)\big)$$

for any $x, y \in M$ and $k = 0, 1, 2, 3, 4$. Then, for some $p \gg \dim(M)$ and for all $i = 1, \dots, N$,

$$\sup_{0 \leq s \leq 1} \|C_i(k, s, x, y)\|_p^p = c_{i,k}(x, y, p) < \infty.$$

Also, $\sup_{x \neq y} c_{i,k}(x, y, p) < \infty$, for all $k = 0, 1, 2, 3, 4$. Note that this is satisfied whenever the V_i's are C^4 with polynomial growth.

(A3) For each $r \geq 1$, there exists a constant m_r, such that

$$E|B^x(s) - B^y(s)|^r \leq m_r|x - y|^r \qquad \forall 0 \leq s \leq 1,$$

where m_r depends solely on r.

(A4) All the above assumptions also hold true with $B^x(s)$ replaced by $\nabla B^x(s)$ and $\nabla^2 B^x(s)$, respectively.

Let us define the *excursion set* A_u corresponding to $f : M \to \mathbb{R}$ as

$$A_u(f; M) \stackrel{\Delta}{=} \{x \in M : f(x) \geq u\}.$$

Also, note that writing $F(\omega) = \int_0^1 V(\omega_s)\, d\omega_s$, one can consider the random field defined above as $f(x) = F(B^x)$.

THEOREM 6.1. *Let M be a m-dimensional manifold and f be a random field defined on M, represented as in* (6.1), *and satisfying the conditions* (A1)–(A4). *Also, let $f(x)$ and $\nabla f(x)$ be nondegenerate in the sense of Malliavin, for some $x \in M$, and that the corresponding Wiener functional F satisfies the exponential moment condition specified in Theorem* 4.2. *Then writing $A_u(f; M)$ as the excursion set for the random field f, and $\mathcal{L}_i(\cdot)$ as the ith Lipschitz–Killing curvature under the Gaussian induced metric, we have*

$$E(\mathcal{L}_0(A_u(f; M))) = \sum_{j=0}^{m} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\mu}(F^{-1}[u, \infty)), \tag{6.3}$$

where $F^{-1}([u,\infty))$ *is a subset of the Wiener space* X *with* $\mathcal{M}_j^{\mu}$ *as its GMF, as defined in the previous section.*

We shall approximate the LHS of (6.3) such that it has the form of the RHS and that this approximation of the RHS indeed converges to the RHS of (6.3).

At this point, we note that all the results obtained below are for the case of $N=1$, whereas using similar methods, the same results are true for general N. We shall start proving Theorem 6.1 by first listing some regularity properties of field f defined in (6.1), in the form of the following theorem.

THEOREM 6.2. *Let the random field* f *be as defined in* (6.1), *such that it also satisfies* (A1)–(A4), *then*:

(a) $F \in D_3^{\infty-}(X;\mathbb{R})$, *and under the assumption of nondegeneracy of* F, *the density* p_F *of* F *is bounded*,

(b) f *is continuous*, *and that for any* $\varepsilon > 0$

$$P\big(\Xi_f(\eta) > \varepsilon\big) = o\big(\eta^{\dim(M)}\big) \qquad \text{as } \eta \downarrow 0.$$

Also, *the same is true for* ∇f *and* $\nabla^2 f$.

PROOF. Clearly, for the Wiener functional $F = \int_0^1 V(B(t))\,dB(t)$, we have $DF \in H$. Therefore, by definition, there exists a unique $(\widehat{DF}) \in L^2([0,1])$ such that $D_r F \triangleq (DF)(r) = \int_0^r (\widehat{D_s F})\,ds$. Using this notation, we shall have

$$(\widehat{D_r F}) = V(B(r)) + \int_0^1 V^{(1)}(B(t))1_{[0,t]}(r)\,dB(t),$$

$$\begin{aligned}(\widehat{D_s(\widehat{D_r F})}) &= V^{(1)}(B(r))1_{[0,r]}(s) + V^{(1)}(B(s))1_{[0,s]}(r)\\ &\quad + \int_0^1 V^{(2)}(B(t))1_{[0,t]}(r)1_{[0,t]}(s)\,dB(t).\end{aligned}$$

Clearly, due to the moment conditions imposed on V and its derivatives, we can conclude that $F \in D_3^{\infty-}(X;\mathbb{R})$, and the boundedness of the density p_F follows using Proposition 2.1.1 of [5].

Now to prove continuity of f and its derivatives, we shall use Kolmogorov's continuity criterion for processes defined on smooth Riemannian manifolds. Note that Kolmogorov's continuity criterion is usually stated for processes with Euclidean parameter space, but since continuity is a local phenomena, thus, it can easily be extended to processes defined on smooth locally Euclidean spaces. Therefore, we present the proof of continuity related results for the field $f \circ \phi^{-1}$ where ϕ is the local chart, but we shall suppress the chart map, and will write f for both the field f and its counterpart $f \circ \phi^{-1}$.

In order to use Kolmogorov's continuity theorem, we must obtain L^p estimates for $(f(x) - f(y))$. Writing V^* as any antiderivative of V, we have

$$V^*(B^x(1)) = V^*(B^x(0)) + \int_0^1 V(B^x(s))\,dB^x(s) + \int_0^1 V^{(1)}(B^x(s))\,ds.$$

Thus, for $p \geq 1$, there exist $m_{1,p}$ and $m_{2,p}$ such that

$$\begin{aligned}
\|f(x) - f(y)\|_p^p &= E|f(x) - f(y)|^p \\
&\leq m_{1,p} E|V^*(B^x(1)) - V^*(B^y(1))|^p \\
&\quad + m_{2,p} E\left|\int_0^1 \bigl(V^{(1)}(B^x(s)) - V^{(1)}(B^y(s))\bigr)\,ds\right|^p \\
&\leq m_{1,p} E\left|\sup_\alpha V[\alpha B^x(1) + (1-\alpha)B^x(1)] \times \bigl(B^x(1) - B^y(1)\bigr)\right|^p \\
&\quad + m_{2,p} \int_0^1 E\bigl|V^{(1)}(B^x(s)) - V^{(1)}(B^y(s))\bigr|^p\,ds \\
&\leq m_{1,p} E\bigl|C(0,1,x,y)\bigl(B^x(1) - B^y(1)\bigr)\bigr|^p \\
&\quad + m_{2,p} E\bigl\|C(2,\cdot,x,y)\bigl(B^x(\cdot) - B^y(\cdot)\bigr)\bigr\|_{L^p[0,1]}^p.
\end{aligned}$$

Now choosing $p_1, p_2 > 0$ such that $p_1^{-1} + p_2^{-1} = p^{-1}$, we get

$$\begin{aligned}
\|f(x) - f(y)\|_p^p &\leq m_{1,p}\bigl(\|C(0,1,x,y)\|_{p_1}\|B^x(1) - B^y(1)\|_{p_2}\bigr)^p \\
&\quad + m_{2,p}\int_0^1 \bigl(\|C(2,s,x,y)\|_{p_1}\|B^x(s) - B^y(s)\|_{p_2}\bigr)^p\,ds \\
&\leq m_{1,p} c_0(x,y,p_1)^{p/p_1} m_{p_2}^{p/p_2}|x-y|^p \\
&\quad + m_{2,p} c_2(x,y,p_1)^{p/p_1} m_{p_2}^{p/p_2}|x-y|^p.
\end{aligned}$$

Next, fixing

$$M(p,p_1,p_2) = \sup_{x\neq y}\bigl(m_{1,p}c_0(x,y,p_1)^{p/p_1} m_{p_2}^{p/p_2} + m_{2,p}c_2(x,y,p_1)^{p/p_1} m_{p_2}^{p/p_2}\bigr),$$

we have

$$\|f(x) - f(y)\| \leq M(p,p_1,p_2)|x-y|^p. \tag{6.4}$$

For large enough p we can use Theorem 1.4.1 in [3] to deduce that there exists $\tilde{f}$, which is the continuous modification of f. Abusing the notation, we shall write f for $\tilde{f}$. Also, using the same result, we can infer that for any $\varepsilon > 0$, the modulus of continuity of f satisfies

$$P\bigl(\Xi_f(\eta) > \varepsilon\bigr) = o\bigl(\eta^{\dim(M)}\bigr) \qquad \text{as } \eta \downarrow 0.$$

Note that we needed supremum of $c_2(x,y)$ to be bounded to prove the continuity of f and to control its modulus of continuity. We can further conclude that the conditions stated in (A1)–(A4) suffice to obtain similar results for the modulus of continuity of ∇f and $\nabla^2 f$. $\square$

Recall from [1] that $\mathcal{L}_0$, also known as the Euler–Poincaré characteristic, of the excursion set $A_u(f; M)$ can be expressed as

$$\begin{aligned}\mathcal{L}_0(A_u(f; M)) &= \sum_{k=0}^{m} (-1)^k \#\{x \in M : f(x) \geq u, \nabla f(x) = 0, \text{index}(\nabla^2 f) = k\} \\ &= \sum_{k=0}^{m} (-1)^k \mu_k.\end{aligned}$$

Now using the *expectation metatheorem* (Theorem 11.2.1 of [1]), and replacing G and H by ∇f and $(\nabla^2 f, f)$, respectively, and B by $D_k \times [u, \infty)$, where D_k is the space of $m \times m$ matrices with index k, we can obtain a formula for the expected value of μ_k as defined above. However, in order to use this result for our purpose, we must also check the conditions involving conditional densities, for which we refer to Theorem 4.1 of [6]. Thus, using these results, we can write

$$\begin{aligned}&E(\mathcal{L}_0(A_u(f; M))) \\ &\quad = \int_M E\bigl(\det(-\nabla^2 f(x)) 1_{[u,\infty)}(f(x)) | \nabla f(x) = 0\bigr) p_{\nabla f(x)}(0)\, dx.\end{aligned} \tag{6.5}$$

Next, in order to construct an approximating sequence to the LHS of (6.3) and appeal to the results in [1], we shall use a cylindrical approximation of $f(x)$. Let $\{(i/n, (i+1)/n]\}_{i=0}^{n-1}$ be a partition of (0, 1], then define

$$f_n(x) = \sum_{i=0}^{n-1} V\bigl(B^x(i/n)\bigr)\bigl(B^x\bigl((i+1)/n\bigr) - B^x(i/n)\bigr).$$

Standard results from stochastic analysis ensure the convergence of $f_n(x)$ to $f(x)$. Moreover, note that $(B^x((i+1)/n) - B^x(i/n))_{i=0}^{n-1}$ forms an i.i.d. $0 \leq i \leq (n-1)$. Therefore, we can write

$$f_n(x) = F_n\bigl(y_1^{(n)}(x), \dots, y_n^{(n)}(x)\bigr),$$

where $y_{i+1}^{(n)}(x)$ are i.i.d. with the same distribution as $\sqrt{n}(B^x((i+1)/n) - B^x(i/n))$ and F_n is the appropriately defined real-valued function. Under the conditions imposed on f for the expectation metatheorem to be true, f_n also becomes a valid candidate to apply the metatheorem, thereby giving us

$$\begin{aligned}&E(\mathcal{L}_0(A_u(f_n; M))) \\ &\quad = \sum_{k=0}^{N} \int_M E\bigl(\det(-\nabla^2 f_n(x)) 1_{[u,\infty)}(f(x)) | \nabla f_n(x) = 0\bigr) p_{\nabla f_n(x)}(0)\, dx.\end{aligned} \tag{6.6}$$

Using Theorem 15.9.5 of [1] for the random field f_n, we shall have

$$E(\mathcal{L}_0(A_u(f_n;M))) = \sum_{j=0}^{m} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\mu}(F_n^{-1}[u,\infty)), \tag{6.7}$$

where $F_n^{-1}[u,\infty)$ is a subset of $\mathbb{R}^n$.

THEOREM 6.3. *Let $\{G_n\}_{n\geq 1}$ be a sequence of real-valued Wiener functionals, such that G_n belongs to the nth Wiener chaos, and $G_n \to G$ in $D_3^{\infty-}$, for some $G \in D_3^{\infty-}$. Also, let that each G_n and G satisfy all the assumptions of Theorem 4.2, then $\mathcal{M}_j^{\gamma_k}(G_n^{-1}[u,\infty)) \to \mathcal{M}_j^{\mu}(G^{-1}[u,\infty))$, as $n \to \infty$.*

PROOF. Using the definition of GMFs in Theorem 4.2 and the convergence of the densities p_{G_n} to p_G, it suffices to prove that

$$\begin{aligned} &E^{G_n=u}\big([\det(\sigma_{G_n})]^{1/2} \det{}_2(I_H + rD\eta_n) \exp\big(-r\delta(\eta_n) - \tfrac{1}{2}r^2\big)\big) \\ &\quad \to E^{G=u}\big([\det(\sigma_G)]^{1/2} \det{}_2(I_H + rD\eta) \exp\big(-r\delta(\eta) - \tfrac{1}{2}r^2\big)\big), \end{aligned}$$

where $\eta = DG/\|DG\|_H$ and $\eta_n = DG_n/\|DG_n\|$. Writing

$$A_n = \big([\det(\sigma_{G_n})]^{1/2} \det{}_2(I_H + rD\eta_n) \exp\big(-r\delta(\eta_n) - \tfrac{1}{2}r^2\big)\big),$$

and similarly defining A, we get

$$\begin{aligned} &|E^{G_n=u}A_n - E^{G=u}A| \\ &\quad = |E(A_n\delta_u \circ G_n) - E(A\delta_u \circ G)| \\ &\quad \leq |E(A_n\delta_u(G_n)) - E(A_n\delta_u(G))| + |E(A_n\delta_u(G)) - E(A\delta_u(G))| \\ &\quad \leq \|A_n\|_{D_\alpha^{p/(p-1)}} \|\delta_u(G_n) - \delta_u(G)\|_{D_{-\alpha}^p} + \|A_n - A\|_{D_\alpha^{p/(p-1)}} \|\delta_u(G)\|_{D_{-\alpha}^p}, \end{aligned} \tag{6.8}$$

where we recall Theorem 4.1 for definitions of p and α. We also note that, since G_n and G are elements of $D_3^{\infty-}$ and they are nondegenerate, existence of such p and α is ensured. Moreover, since $G_n \to G$ in $D_3^{\infty-}$, it's easy to see that $\sup_n \|A_n\|_{D_\alpha^{p/(p-1)}} < \infty$ and $\|A_n - A\|_{D_\alpha^{p/(p-1)}} \to 0$. Also, $\|\delta_u(G_n) - \delta_u(G)\|_{D_{-\alpha}^p} \to 0$, which proves the result. $\square$

PROOF OF THEOREM 6.1 (Continued). We extend F_n from $\mathbb{R}^n$ to $\mathbb{R}^\infty$ or, equivalently, to X, by suppressing all the indices after the first n, that is, considering F_n as cylindrical Wiener functionals. Then, by using the invariance property of GMFs, the $\mathcal{M}_j^{\mu}$'s of the extended F_n remain the same as that of F_n when restricted to $\mathbb{R}^n$. Together with this, using the fact that F_n converges to F in $D_3^{\infty-}$, we clearly have

$$\lim_{n\to\infty} \mathcal{M}_j^{\mu}(F_n^{-1}[u,\infty)) = \mathcal{M}_j^{\mu}(F^{-1}[u,\infty)). \tag{6.9}$$

Therefore, using (6.7) and (6.9), we shall have

$$\lim_{n\to\infty} E(\mathcal{L}_0(A_u(f_n;M))) = \sum_{j=0}^{N} c_j \mathcal{L}_j(M)\mathcal{M}_j^{\mu}(F^{-1}[u,\infty)). \tag{6.10}$$

Now, it suffices to prove that the right-hand side of (6.6) converges to the right-hand side of (6.5). Clearly,

$$\lim_{n\to\infty} p_{\nabla f_n}(y) = p_{\nabla f}(y), \tag{6.11}$$

which follows from the fact that ∇f_n converges to ∇f in a much stronger sense, as is clear from the assumption $f_n \to f$ in $D^{\infty}_{3+\delta}$. Next, we need to prove

$$\begin{aligned} &\big|E\big(|\det \nabla^2 f_n(x)| 1_{D_k}(\nabla^2 f_n(x)) 1_{[u,\infty)}(f_n(x)) | \nabla f_n(x) = 0\big) \\ &\quad - E\big(|\det \nabla^2 f(x)| 1_{D_k}(\nabla^2 f(x)) 1_{[u,\infty)}(f(x)) | \nabla f(x) = 0\big)\big| \to 0, \end{aligned} \tag{6.12}$$

which is similar to the proof of Theorem 6.3. Using precisely the same techniques, writing $B_n(x) = (|\det \nabla^2 f_n(x)| 1_{D_k}(\nabla^2 f_n(x)) 1_{[u,\infty)}(f_n(x)))$ and defining $B(x)$ in a similar fashion, we have

$$\begin{aligned} &\big|E^{\nabla f_n(x)=0} B_n(x) - E^{\nabla f(x)=0} B(x)\big| \\ &\quad \le \big|E^{\nabla f_n(x)=0} B_n(x) - E^{\nabla f(x)=0} B_n(x)\big| \\ &\qquad + \big|E^{\nabla f(x)=0} B_n(x) - E^{\nabla f(x)=0} B(x)\big| \\ &\quad = |E(B_n(x)\delta_0(\nabla f_n(x))) - E(B_n(x)\delta_0(\nabla f(x)))| \\ &\qquad + |E(B_n(x)\delta_0(\nabla f(x))) - E(B(x)\delta_0(\nabla f(x)))| \\ &\quad \le \|B_n(x)\|_{L^{p/(p-1)}} \|\delta_0(\nabla f_n(x)) - \delta_0(\nabla f(x))\|_{L^p} \\ &\qquad + \|B_n(x) - B(x)\|_{L^{p/(p-1)}} \|\delta_0(\nabla f(x))\|_{L^p}, \end{aligned}$$

which, under the assumptions of $f_n(x) \to f(x)$ in $D_3^{\infty-}$ and nondegeneracy of $\nabla f(x)$, converges to zero as $n \to \infty$, thus proving that the integrand of (6.6) converges to that of (6.5) for each $x \in M$. Then, in order to prove that the integral involved in equation (6.6) converges to the integral in (6.5), note that the random fields f_n and f defined on the manifold M are chosen to be sufficiently smooth so that we can use an uniform integrability argument to conclude that the right-hand side of (6.6) converges to the right-hand side of (6.5). Therefore, we shall have

$$\begin{aligned} E(\mathcal{L}_0(A_u(f;M))) &= \lim_{n\to\infty} E(\mathcal{L}_0(A_u(f_n;M))) \\ &= \lim_{n\to\infty} \sum_{j=0}^{m} c_j \mathcal{L}_j(M)\mathcal{M}_j^{\mu}(F_n^{-1}[u,\infty)) \\ &= \sum_{j=0}^{m} c_j \mathcal{L}_j(M)\mathcal{M}_j^{\mu}(F^{-1}[u,\infty)), \end{aligned}$$

where in going from the first line to the second, we have used the finite dimensional results set forth in [1], and in going from the second to the third line, we have used Theorem 6.3. □

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