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FUNCTIONAL ITÔ CALCULUS AND STOCHASTIC INTEGRAL REPRESENTATION OF MARTINGALES

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We develop a nonanticipative calculus for functionals of a continuous semimartingale, using an extension of the Itô formula to path-dependent functionals which possess certain directional derivatives. The construction is based on a pathwise derivative, introduced by Dupire, for functionals on the space of right-continuous functions with left limits. We show that this functional derivative admits a suitable extension to the space of square-integrable martingales. This extension defines a weak derivative which is shown to be the inverse of the Itô integral and which may be viewed as a nonanticipative "lifting" of the Malliavin derivative.

These results lead to a constructive martingale representation formula for Itô processes. By contrast with the Clark–Haussmann–Ocone formula, this representation only involves nonanticipative quantities which may be computed pathwise.

1. Introduction. In the analysis of phenomena with stochastic dynamics, Itô's stochastic calculus [8, 15, 16, 19, 23, 28, 29] has proven to be a powerful and useful tool. A central ingredient of this calculus is the *Itô formula* [15, 16, 23], a change of variable formula for functions $f(X_t)$ of a *semimartingale* X which allows one to represent such quantities in terms of a stochastic integral. Given that in many applications such as statistics of processes, physics or mathematical finance, one is led to consider path-dependent functionals of a semimartingale X and its quadratic variation process $[X]$ such as

$$\int_0^t g(t, X_t)\, d[X](t), \qquad G(t, X_t, [X]_t), \quad \text{or} \qquad E[G(T, X(T), [X](T))|\mathcal{F}_t] \tag{1}$$

(where $X(t)$ denotes the value at time t and $X_t = (X(u), u \in [0, t])$ the path up to time t), there has been a sustained interest in extending the framework of stochastic calculus to such path-dependent functionals.

In this context, the Malliavin calculus [3, 22, 24, 25, 30–32] has proven to be a powerful tool for investigating various properties of Brownian functionals.

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Since the construction of Malliavin derivative does not refer to an underlying filtration $\mathcal{F}_t$, it naturally leads to representations of functionals in terms of *anticipative* processes [4, 14, 25]. However, in most applications it is more natural to consider nonanticipative versions of such representations.

In a recent insightful work, Dupire [9] has proposed a method to extend the Itô formula to a functional setting using a pathwise functional derivative which quantifies the sensitivity of a functional $F_t : D([0,t],\mathbb{R}) \to \mathbb{R}$ to a variation in the endpoint of a path $\omega \in D([0,t],\mathbb{R})$.

$$\nabla_\omega F_t(\omega) = \lim_{\varepsilon \to 0} \frac{F_t(\omega + \varepsilon 1_t) - F_t(\omega)}{\varepsilon}.$$

Building on this insight, we develop hereafter a nonanticipative calculus [5] for a class of processes—including the above examples—which may be represented as

$$Y(t) = F_t\big(\{X(u), 0 \le u \le t\}, \{A(u), 0 \le u \le t\}\big) = F_t(X_t, A_t), \tag{2}$$

where A is the local quadratic variation defined by $[X](t) = \int_0^t A(u)\,du$, and the functional

$$F_t : D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+) \to \mathbb{R}$$

represents the dependence of Y on the path $X_t = \{X(u), 0 \le u \le t\}$ of X and its quadratic variation.

Our first result (Theorem 4.1) is a change of variable formula for path-dependent functionals of the form (2). Introducing A_t as an additional variable allows us to control the dependence of Y with respect to the "quadratic variation" $[X]$ by requiring smoothness properties of F_t with respect to the variable A_t in the supremum norm, without resorting to p-variation norms as in "rough path" theory [20]. This allows our result to cover a wide range of functionals, including the examples in (1).

We then extend this notion of functional derivative to *processes*: we show that for Y of the form (2) where F satisfies some regularity conditions, the process $\nabla_X Y = \nabla_\omega F(X_t, A_t)$ may be defined intrinsically, independently of the choice of F in (2). The operator ∇_X is shown to admit an extension to the space of square-integrable martingales, which is the inverse of the Itô integral with respect to X: for $\phi \in \mathcal{L}^2(X)$, $\nabla_X(\int \phi \cdot dX) = \phi$ (Theorem 5.8). In particular, we obtain a constructive version of the martingale representation theorem (Theorem 5.9), which states that for any square-integrable $\mathcal{F}_t^X$-martingale Y,

$$Y(T) = Y(0) + \int_0^T \nabla_X Y \cdot dX, \qquad \mathbb{P}\text{-a.s.}$$

This formula can be seen as a nonanticipative counterpart of the Clark–Haussmann–Ocone formula [4, 13, 14, 18, 25]. The integrand $\nabla_X Y$ is an adapted process

which may be computed pathwise, so this formula is more amenable to numerical computations than those based on Malliavin calculus.

Finally, we show that this functional derivative ∇_X may be viewed as a nonanticipative "lifting" of the Malliavin derivative (Theorem 6.1): for square-integrable martingales Y whose terminal values is differentiable in the sense of Malliavin $Y(T) \in \mathbf{D}^{1,2}$, we show that $\nabla_X Y(t) = E[\mathbb{D}_t H | \mathcal{F}_t]$.

These results provide a rigorous mathematical framework for developing and extending the ideas proposed by Dupire [9] for a large class of functionals. In particular, unlike the results derived from the pathwise approach presented in [6, 9], Theorems 5.8 and 5.9 do not require any pathwise regularity of the functionals and hold for nonanticipative square-integrable processes, including stochastic integrals and functionals which may depend on the quadratic variation of the process.

2. Functional representation of nonanticipative processes. Let $X : [0, T] \times \Omega \mapsto \mathbb{R}^d$ be a continuous, $\mathbb{R}^d$-valued semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ assumed to satisfy the usual hypotheses [8]. Denote by $\mathcal{P}$ (resp., $\mathcal{O}$) the associated *predictable* (resp., optional) sigma-algebra on $[0, T]$. $\mathcal{F}_t^X$ denotes the ($\mathbb{P}$-completed) natural filtration of X. The paths of X then lie in $C_0([0, T], \mathbb{R}^d)$, which we will view as a subspace of $D([0, T], \mathbb{R}^d)$ the space of cadlag functions with values in $\mathbb{R}^d$. We denote by $[X] = ([X^i, X^j], i, j = 1, \dots, d)$ the quadratic (co-)variation process associated to X, taking values in the set S_d^+ of positive $d \times d$ matrices. We assume that

$$[X](t) = \int_0^t A(s)\, ds \tag{3}$$

for some cadlag process A with values in S_d^+. Note that A need not be a semimartingale. The paths of A lie in $\mathcal{S}_t = D([0, t], S_d^+)$, the space of cadlag functions with values S_d^+.

2.1. *Horizontal extension and vertical perturbation of a path.* Consider a path $x \in D([0, T]), \mathbb{R}^d)$ and denote by $x_t = (x(u), 0 \le u \le t) \in D([0, t], \mathbb{R}^d)$ its restriction to $[0, t]$ for $t < T$. For a process X we shall similarly denote $X(t)$ its value at t and $X_t = (X(u), 0 \le u \le t)$ its path on $[0, t]$.

For $h \ge 0$, we define the *horizontal* extension $x_{t,h} \in D([0, t + h], \mathbb{R}^d)$ of x_t to $[0, t + h]$ as

$$x_{t,h}(u) = x(u), \qquad u \in [0, t]; \qquad x_{t,h}(u) = x(t), \qquad u \in]t, t + h]. \tag{4}$$

For $h \in \mathbb{R}^d$, we define the *vertical* perturbation x_t^h of x_t as the cadlag path obtained by shifting the endpoint by h.

$$x_t^h(u) = x_t(u), \qquad u \in [0, t[, \qquad x_t^h(t) = x(t) + h, \tag{5}$$

or, in other words, $x_t^h(u) = x_t(u) + h 1_{t=u}$.

2.2. *Adapted processes as nonanticipative functionals.* A process $Y : [0, T] \times \Omega \mapsto \mathbb{R}^d$ adapted to $\mathcal{F}_t^X$ may be represented as

$$Y(t) = F_t\big(\{X(u), 0 \le u \le t\}, \{A(u), 0 \le u \le t\}\big) = F_t(X_t, A_t), \tag{6}$$

where $F = (F_t)_{t\in[0,T]}$ is a family of functionals

$$F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$$

representing the dependence of $Y(t)$ on the underlying path of X and its quadratic variation.

Since Y is nonanticipative, $Y(t, \omega)$ only depends on the restriction ω_t of ω on $[0, t]$. This motivates the following definition:

DEFINITION 2.1 (Nonanticipative functional). A nonanticipative functional is a family of functionals $F = (F_t)_{t\in[0,T]}$ where

$$F_t : D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+) \mapsto \mathbb{R},$$
$$(x, v) \to F_t(x, v)$$

is measurable with respect to $\mathcal{B}_t$, the canonical filtration on $D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+)$.

We can also view $F = (F_t)_{t\in[0,T]}$ as a map defined on the space Υ of *stopped paths*

$$\Upsilon = \{(t, \omega_{t,T-t}), (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d \times S_d^+)\}. \tag{7}$$

Whenever the context is clear, we will denote a generic element $(t, \omega) \in \Upsilon$ simply by its second component, the path ω stopped at t. Υ can also be identified with the "vector bundle"

$$\Lambda = \bigcup_{t\in[0,T]} D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+). \tag{8}$$

A natural distance on the space Υ of stopped paths is given by

$$d_\infty((t, \omega), (t', \omega')) = |t - t'| + \sup_{u\in[0,T]} |\omega_{t,T-t}(u) - \omega'_{t',T-t'}(u)|. \tag{9}$$

(Υ, d_∞) is then a metric space, a closed subspace of $([0, T] \times D([0, T], \mathbb{R}^d \times S_d^+), \|\cdot\|_\infty)$ for the product topology.

Introducing the process A as an additional variable may seem redundant at this stage: indeed $A(t)$ is itself $\mathcal{F}_t$-measurable, that is, a functional of X_t. However, it is not a *continuous* functional on (Υ, d_∞). Introducing A_t as a second argument in the functional will allow us to control the regularity of Y with respect to $[X]_t = \int_0^t A(u)\,du$ simply by requiring continuity of F_t in supremum or L^p norms with respect to the "lifted process" (X, A); see Section 2.3. This idea is analogous in

some ways to the approach of rough path theory [20], although here we do not resort to p-variation norms.

If Y is a $\mathcal{B}_t$-predictable process, then [8, Volume I, paragraph 97]

$$\forall t \in [0, T], \qquad Y(t, \omega) = Y(t, \omega_{t-}),$$

where ω_{t-} denotes the path defined on $[0, t]$ by

$$\omega_{t-}(u) = \omega(u), \qquad u \in [0, t[, \qquad \omega_{t-}(t) = \omega(t-).$$

Note that ω_{t-} is cadlag and should *not* be confused with the caglad path $u \mapsto \omega(u-)$.

The functionals discussed in the introduction depend on the process A via $[X] = \int_0^{\cdot} A(t)\,dt$. In particular, they satisfy the condition $F_t(X_t, A_t) = F_t(X_t, A_{t-})$. Accordingly, we will assume throughout the paper that all functionals $F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ considered have "predictable" dependence with respect to the second argument,

$$\forall t \in [0, T], \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \qquad F_t(x_t, v_t) = F_t(x_t, v_{t-}). \tag{10}$$

2.3. *Continuity for nonanticipative functionals*. We now define a notion of (left) continuity for nonanticipative functionals.

DEFINITION 2.2 (Continuity at fixed times). A functional F defined on Υ is said to be continuous at fixed times for the d_∞ metric if and only if

$$\begin{aligned} &\forall t \in [0, T), \forall \varepsilon > 0, \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \\ &\quad \exists \eta > 0, (x', v') \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \\ &\qquad d_\infty((x, v), (x', v')) < \eta \quad \Rightarrow \quad |F_t(x, v) - F_t(x', v')| < \varepsilon. \end{aligned} \tag{11}$$

We now define a notion of joint continuity with respect to time and the underlying path:

DEFINITION 2.3 (Continuous functionals). A nonanticipative functional $F = (F_t)_{t \in [0,T)}$ is said to be continuous at $(x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t$ if

$$\begin{aligned} &\forall \varepsilon > 0, \exists \eta > 0, \forall (x', v') \in \Upsilon, \\ &\qquad d_\infty((x, v), (x', v')) < \eta \quad \Rightarrow \quad |F_t(x, v) - F_{t'}(x', v')| < \varepsilon. \end{aligned} \tag{12}$$

We denote by $\mathbb{C}^{0,0}([0, T))$ the set of continuous nonanticipative functionals on Υ.

DEFINITION 2.4 (Left-continuous functionals). A nonanticipative functional $F = (F_t, t \in [0, T))$ is said to be left-continuous if for each $t \in [0, T)$, $F_t : D([0, t],$

$\mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ in the sup norm and

$$\begin{aligned} &\forall \varepsilon > 0, \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \\ &\exists \eta > 0, \forall h \in [0, t], \forall (x', v') \in D([0, t-h], \mathbb{R}^d) \times \mathcal{S}_{t-h}, \\ &\quad d_\infty((x, v), (x', v')) < \eta \quad \Rightarrow \quad |F_t(x, v) - F_{t-h}(x', v')| < \varepsilon. \end{aligned} \tag{13}$$

We denote by $\mathbb{C}_l^{0,0}([0, T))$ the set of left-continuous functionals.

We define analogously the class of right-continuous functionals $\mathbb{C}_r^{0,0}([0, T))$.

We call a functional "boundedness preserving" if it is bounded on each bounded set of paths:

DEFINITION 2.5 (Boundedness-preserving functionals). Define $\mathbb{B}([0, T))$ as the set of nonanticipative functionals F such that for every compact subset K of $\mathbb{R}^d$, every $R > 0$ and $t_0 < T$,

$$\begin{aligned} &\exists C_{K,R,t_0} > 0, \forall t \le t_0, \forall (x, v) \in D([0, t], K) \times \mathcal{S}_t, \\ &\sup_{s \in [0,t]} |v(s)| < R \quad \Rightarrow \quad |F_t(x, v)| < C_{K,R,t_0}. \end{aligned} \tag{14}$$

2.4. *Measurability properties*. Composing a nonanticipative functional F with the process (X, A) yields an $\mathcal{F}_t$-adapted process $Y(t) = F_t(X_t, A_t)$. The results below link the measurability and pathwise regularity of Y to the regularity of the functional F.

LEMMA 2.6 (Pathwise regularity). *If $F \in \mathbb{C}_l^{0,0}$, then for any $(x, v) \in D([0, T], \mathbb{R}^d) \times \mathcal{S}_T$, the path $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.*

PROOF. Let $F \in \mathbb{C}_l^{0,0}$ and $t \in [0, T)$. For $h > 0$ sufficiently small,

$$\begin{aligned} d_\infty((x_{t-h}, v_{t-h}), (x_{t-}, v_{t-})) = &\sup_{u \in (t-h,t)} |x(u) - x(t-h)| \\ &+ \sup_{u \in (t-h,t)} |v(u) - v(t-h)| + h. \end{aligned} \tag{15}$$

Since x and v are cadlag, this quantity converges to 0 as $h \to 0+$, so

$$F_{t-h}(x_{t-h}, v_{t-h}) - F_t(x_{t-}, v_{t-}) \overset{h \to 0^+}{\to} 0,$$

so $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous. $\square$

THEOREM 2.7. (i) *If F is continuous at fixed times, then the process Y defined by $Y((x, v), t) = F_t(x_t, v_t)$ is adapted.*

(ii) *If $F \in \mathbb{C}_l^{0,0}([0, T))$, then the process $Z(t) = F_t(X_t, A_t)$ is optional.*

(iii) *If* $F \in \mathbb{C}_l^{0,0}([0,T))$, *and if either* A *is continuous or* F *verifies* (10), *then* Z *is a predictable process.*

In particular, any $F \in \mathbb{C}_l^{0,0}$ is a nonanticipative functional in the sense of Definition 2.1. We propose an easy-to-read proof of points (i) and (iii) in the case where A is continuous. The (more technical) proof for the cadlag case is given in the Appendix.

CONTINUOUS CASE. Assume that F is continuous at fixed times and that the paths of (X, A) are almost-surely continuous. Let us prove that Y is $\mathcal{F}_t$-adapted: $X(t)$ is $\mathcal{F}_t$-measurable. Introduce the partition $t_n^i = \frac{iT}{2^n}$, $i = 0, \dots, 2^n$ of $[0, T]$, as well as the following piecewise-constant approximations of X and A:

$$\begin{aligned} X^n(t) &= \sum_{k=0}^{2^n} X(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + X_T 1_{\{T\}}(t), \\ A^n(t) &= \sum_{k=0}^{2^n} A(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + A_T 1_{\{T\}}(t). \end{aligned} \tag{16}$$

The random variable $Y^n(t) = F_t(X_t^n, A_t^n)$ is a continuous function of the random variables $\{X(t_k^n), A(t_k^n), t_k^n \le t\}$ and hence is $\mathcal{F}_t$-measurable. The representation above shows in fact that $Y^n(t)$ is $\mathcal{F}_t$-measurable. X_t^n and A_t^n converge respectively to X_t and A_t almost-surely so $Y^n(t) \to^{n\to\infty} Y(t)$ a.s., and hence $Y(t)$ is $\mathcal{F}_t$-measurable.

(i) implies point (iii) since the path of Z are left-continuous by Lemma 2.6.

3. Pathwise derivatives of nonanticipative functionals.

3.1. *Horizontal and vertical derivatives*. We now define pathwise derivatives for a nonanticipative functional, following Dupire [9].

DEFINITION 3.1 (Horizontal derivative). The *horizontal derivative* at $(x, v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t$ of nonanticipative functional $F = (F_t)_{t\in[0,T)}$ is defined as

$$\mathcal{D}_t F(x, v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h} \tag{17}$$

if the corresponding limit exists. If (17) is defined for all $(x, v) \in \Upsilon$, the map

$$\begin{aligned} \mathcal{D}_t F : D([0,t], \mathbb{R}^d) \times \mathcal{S}_t &\mapsto \mathbb{R}^d, \\ (x, v) &\to \mathcal{D}_t F(x, v) \end{aligned} \tag{18}$$

defines a nonanticipative functional $\mathcal{D}F = (\mathcal{D}_t F)_{t\in[0,T]}$, the *horizontal derivative* of F.

Note that our definition (17) is different from the one in [9] where the case $F(x, v) = G(x)$ is considered.

Dupire [9] also introduced a pathwise spatial derivative for such functionals, which we now introduce. Denote $(e_i, i = 1, \dots, d)$ the canonical basis in $\mathbb{R}^d$.

DEFINITION 3.2. A nonanticipative functional $F = (F_t)_{t\in[0,T)}$ is said to be *vertically differentiable* at $(x, v) \in D([0, t]), \mathbb{R}^d) \times D([0, t], S_d^+)$ if

$$\begin{aligned} &\mathbb{R}^d \mapsto \mathbb{R}, \\ &e \to F_t(x_t^e, v_t) \end{aligned}$$

is differentiable at 0. Its gradient at 0,

$$\begin{aligned} \nabla_x F_t(x, v) &= \big(\partial_i F_t(x, v), i = 1, \dots, d\big) \\ \text{where } \partial_i F_t(x, v) &= \lim_{h\to 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h} \end{aligned} \tag{19}$$

is called the *vertical derivative* of F_t at (x, v). If (19) is defined for all $(x, v) \in \Upsilon$, the maps

$$\begin{aligned} &\nabla_x F : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \mapsto \mathbb{R}^d, \\ &(x, v) \to \nabla_x F_t(x, v) \end{aligned} \tag{20}$$

define a nonanticipative functional $\nabla_x F = (\nabla_x F_t)_{t\in[0,T]}$, the *vertical derivative* of F. F is then said to be *vertically differentiable* on Υ.

REMARK 3.3. $\partial_i F_t(x, v)$ is simply the directional derivative of F_t in the direction $(1_{\{t\}}e_i, 0)$. Note that this involves evaluating F at cadlag perturbations of the path x, even if x is continuous.

REMARK 3.4. If $F_t(x, v) = f(t, x(t))$ with $f \in C^{1,1}([0, T) \times \mathbb{R}^d)$, then we retrieve the usual partial derivatives

$$\mathcal{D}_t F(x, v) = \partial_t f(t, X(t)), \qquad \nabla_x F_t(X_t, A_t) = \nabla_x f(t, X(t)).$$

REMARK 3.5. Bismut [3] considered directional derivatives of functionals on $D([0, T], \mathbb{R}^d)$ in the direction of purely discontinuous (e.g., piecewise constant) functions with finite variation, which is similar to Definition 3.2. This notion, used in [3] to derive an integration by parts formula for pure-jump processes, is natural in the context of discontinuous semimartingales. We will show that the directional derivative (19) also intervenes naturally when the underlying process X is *continuous*, which is less obvious.

DEFINITION 3.6 (Regular functionals). Define $\mathbb{C}^{1,k}([0, T))$ as the set of functionals $F \in \mathbb{C}_l^{0,0}$ which are:

- horizontally differentiable with $\mathcal{D}_t F$ continuous at fixed times;
- k times vertically differentiable with $\nabla_x^j F \in \mathbb{C}_l^{0,0}([0,T))$ for $j = 1, \dots, k$.

Define $\mathbb{C}_b^{1,k}([0,T))$ as the set of functionals $F \in \mathbb{C}^{1,2}$ such that $\mathcal{D}F, \nabla_x F, \dots, \nabla_x^k F \in \mathbb{B}([0,T))$.

We denote $\mathbb{C}^{1,\infty}([0,T)) = \bigcap_{k\geq 1} \mathbb{C}^{1,k}([0,T)$.

Note that this notion of regularity only involves directional derivatives with respect to *local* perturbations of paths, so $\nabla_x F$ and $\mathcal{D}_t F$ seems to contain *less* information on the behavior of F than, say, the Fréchet derivative which considers perturbations in all directions in $C_0([0,T], \mathbb{R}^d)$ or the Malliavin derivative [21, 22] which examines perturbations in the direction of all absolutely continuous functions. Nevertheless we will show in Section 4 that knowledge of $\mathcal{D}F, \nabla_x F, \nabla_x^2 F$ along the paths of X is sufficient to reconstitute the path of $Y(t) = F_t(X_t, A_t)$.

EXAMPLE 1 (Smooth functions). In the case where F reduces to a smooth *function* of $X(t)$,

$$F_t(x_t, v_t) = f(t, x(t)), \tag{21}$$

where $f \in C^{1,k}([0,T] \times \mathbb{R}^d)$, the pathwise derivatives reduce to the usual ones

$$\mathcal{D}_t F(x_t, v_t) = \partial_t f(t, x(t)), \qquad \nabla_x^j F_t(x_t, v_t) = \partial_x^j f(t, x(t)). \tag{22}$$

In fact to have $F \in \mathbb{C}^{1,k}$ we just need f to be right-differentiable in the time variable, with right-derivative $\partial_t f(t, \cdot)$ which is continuous in the space variable and f, ∇f and $\nabla^2 f$ to be jointly left-continuous in t and continuous in the space variable.

EXAMPLE 2 (Cylindrical functionals). Let $g \in C^0(\mathbb{R}^d, \mathbb{R}), h \in C^k(\mathbb{R}^d, \mathbb{R})$ with $h(0) = 0$. Then

$$F_t(\omega) = h\big(\omega(t) - \omega(t_n-)\big) 1_{t\geq t_n} g(\omega(t_1-), \omega(t_2-), \dots, \omega(t_n-))$$

is in $\mathbb{C}_b^{1,k}$ with $\mathcal{D}_t F(\omega) = 0$ and

$$\forall j = 1, \dots, k,$$
$$\nabla_\omega^j F_t(\omega) = h^{(j)}\big(\omega(t) - \omega(t_n-)\big) 1_{t\geq t_n} g(\omega(t_1-), \omega(t_2-) \dots, \omega(t_n-)).$$

EXAMPLE 3 (Integrals with respect to quadratic variation). A process $Y(t) = \int_0^t g(X(u))\, d[X](u)$ where $g \in C^0(\mathbb{R}^d)$ may be represented by the functional

$$F_t(x_t, v_t) = \int_0^t g(x(u)) v(u)\, du. \tag{23}$$

It is readily observed that $F \in \mathbb{C}_b^{1,\infty}$, with

$$\mathcal{D}_t F(x_t, v_t) = g(x(t))v(t), \qquad \nabla_x^j F_t(x_t, v_t) = 0. \tag{24}$$

EXAMPLE 4. The martingale $Y(t) = X(t)^2 - [X](t)$ is represented by the functional

$$F_t(x_t, v_t) = x(t)^2 - \int_0^t v(u)\,du. \tag{25}$$

Then $F \in \mathbb{C}_b^{1,\infty}$ with

$$\begin{aligned} &\mathcal{D}_t F(x, v) = -v(t), \qquad \nabla_x F_t(x_t, v_t) = 2x(t), \\ &\nabla_x^2 F_t(x_t, v_t) = 2, \qquad \nabla_x^j F_t(x_t, v_t) = 0, \qquad j \geq 3. \end{aligned} \tag{26}$$

EXAMPLE 5. $Y = \exp(X - [X]/2)$ may be represented as $Y(t) = F(X_t)$

$$F_t(x_t, v_t) = e^{x(t) - 1/2 \int_0^t v(u)\,du}. \tag{27}$$

Elementary computations show that $F \in \mathbb{C}_b^{1,\infty}$ with

$$\mathcal{D}_t F(x, v) = -\frac{1}{2} v(t) F_t(x, v), \qquad \nabla_x^j F_t(x_t, v_t) = F_t(x_t, v_t). \tag{28}$$

Note that, although A_t may be expressed as a functional of X_t, this functional is not continuous and without introducing the second variable $v \in \mathcal{S}_t$, it is not possible to represent Examples 3, 4 and 5 as a left-continuous functional of x alone.

3.2. *Obstructions to regularity.* It is instructive to observe what prevents a functional from being regular in the sense of Definition 3.6. The examples below illustrate the fundamental obstructions to regularity:

EXAMPLE 6 (Delayed functionals). Let $\varepsilon > 0$. $F_t(x_t, v_t) = x(t - \varepsilon)$ defines a $\mathbb{C}_b^{0,\infty}$ functional. All vertical derivatives are 0. However, F fails to be horizontally differentiable.

EXAMPLE 7 (Jump of x at the current time). $F_t(x_t, v_t) = x(t) - x(t-)$ defines a functional which is infinitely differentiable and has regular pathwise derivatives

$$\mathcal{D}_t F(x_t, v_t) = 0, \qquad \nabla_x F_t(x_t, v_t) = 1. \tag{29}$$

However, the functional itself fails to be $\mathbb{C}_l^{0,0}$.

EXAMPLE 8 (Jump of x at a fixed time). $F_t(x_t, v_t) = 1_{t \geq t_0}(x(t_0) - x(t_0-))$ defines a functional in $\mathbb{C}_l^{0,0}$ which admits horizontal and vertical derivatives at any order at each point (x, v). However, $\nabla_x F_t(x_t, v_t) = 1_{t=t_0}$ fails to be either right- or left-continuous, so F is not $\mathbb{C}^{0,1}$ in the sense of Definition 3.2.

EXAMPLE 9 (Maximum). $F_t(x_t, v_t) = \sup_{s \le t} x(s)$ is $\mathbb{C}_l^{0,0}$ but fails to be vertically differentiable on the set

$$\Big\{(x_t, v_t) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, x(t) = \sup_{s \le t} x(s)\Big\}.$$

4. Functional Itô calculus.

4.1. *Functional Itô formula.* We are now ready to prove our first main result, which is a change of variable formula for nonanticipative functionals of a semimartingale [5, 9]:

THEOREM 4.1. *For any nonanticipative functional $F \in \mathbb{C}_b^{1,2}$ verifying* (10) *and any $t \in [0, T)$,*

$$\begin{aligned} F_t(X_t, A_t) - F_0(X_0, A_0) \\ = \int_0^t \mathcal{D}_u F(X_u, A_u)\, du + \int_0^t \nabla_x F_u(X_u, A_u) \cdot dX(u) \\ + \int_0^t \frac{1}{2} \operatorname{tr}(\nabla_x^2 F_u(X_u, A_u)\, d[X](u)) \qquad a.s. \end{aligned} \tag{30}$$

In particular, for any $F \in \mathbb{C}_b^{1,2}$, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

Theorem 4.1 shows that, for a regular functional $F \in \mathbb{C}^{1,2}([0,T))$, the process $Y = F(X, A)$ may be reconstructed from the second-order jet $(\mathcal{D}F, \nabla_x F, \nabla_x^2 F)$ of F along the paths of X.

PROOF. Let us first assume that X does not exit a compact set K and that $\|A\|_\infty \le R$ for some $R > 0$. Let us introduce a sequence of random partitions $(\tau_k^n, k = 0, \dots, k(n))$ of $[0, t]$, by adding the jump times of A to the dyadic partition $(t_i^n = \frac{it}{2^n}, i = 0, \dots, 2^n)$,

$$\tau_0^n = 0, \qquad \tau_k^n = \inf\Big\{s > \tau_{k-1}^n | 2^n s \in \mathbb{N} \text{ or } |A(s) - A(s-)| > \frac{1}{n}\Big\} \wedge t. \tag{31}$$

The following arguments apply pathwise. Lemma A.3 ensures that

$$\eta_n = \sup\Big\{|A(u) - A(\tau_i^n)| + |X(u) - X(\tau_i^n)| + \frac{t}{2^n}, i \le 2^n, u \in [\tau_i^n, \tau_{i+1}^n)\Big\} \underset{n\to\infty}{\to} 0.$$

Denote ${}_nX = \sum_{i=0}^\infty X(\tau_{i+1}^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + X(t) 1_{\{t\}}$ which is a cadlag piecewise constant approximation of X_t, and ${}_nA = \sum_{i=0}^\infty A(\tau_i^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + A(t) 1_{\{t\}}$ which is an adapted cadlag piecewise constant approximation of A_t. Denote $h_i^n = \tau_{i+1}^n - \tau_i^n$.

Start with the decomposition

$$
\begin{aligned}
&F_{\tau_{i+1}^n}({}_nX_{\tau_{i+1}^n-},{}_nA_{\tau_{i+1}^n-}) - F_{\tau_i^n}({}_nX_{\tau_i^n-},{}_nA_{\tau_i^n-}) \\
&\quad = F_{\tau_{i+1}^n}({}_nX_{\tau_{i+1}^n-},{}_nA_{\tau_i^n,h_i^n}) - F_{\tau_i^n}({}_nX_{\tau_i^n},{}_nA_{\tau_i^n}) \\
&\qquad + F_{\tau_i^n}({}_nX_{\tau_i^n},{}_nA_{\tau_i^n-}) - F_{\tau_i^n}({}_nX_{\tau_i^n-},{}_nA_{\tau_i^n-}),
\end{aligned} \tag{32}
$$

where we have used the fact that F has predictable dependence in the second variable to have $F_{\tau_i^n}({}_nX_{\tau_i^n},{}_nA_{\tau_i^n}) = F_{\tau_i^n}({}_nX_{\tau_i^n},{}_nA_{\tau_i^n-})$. The first term in (32) can be written $\psi(h_i^n) - \psi(0)$ where

$$
\psi(u) = F_{\tau_i^n+u}({}_nX_{\tau_i^n,u},{}_nA_{\tau_i^n,u}). \tag{33}
$$

Since $F \in \mathbb{C}^{1,2}([0,T])$, ψ is right-differentiable and left-continuous by Lemma 2.6, so:

$$
\begin{aligned}
&F_{\tau_{i+1}^n}({}_nX_{\tau_i^n,h_i^n},{}_nA_{\tau_i^n,h_i^n}) - F_{\tau_i^n}({}_nX_{\tau_i^n},{}_nA_{\tau_i^n}) \\
&\quad = \int_0^{\tau_{i+1}^n-\tau_i^n} \mathcal{D}_{\tau_i^n+u}F({}_nX_{\tau_i^n,u},{}_nA_{\tau_i^n,u})\,du.
\end{aligned} \tag{34}
$$

The second term in (32) can be written $\phi(X(\tau_{i+1}^n) - X(\tau_i^n)) - \phi(0)$ where $\phi(u) = F_{\tau_i^n}({}_nX^u_{\tau_i^n-},{}_nA_{\tau_i^n})$. Since $F \in \mathbb{C}_b^{1,2}$, ϕ is a C^2 function and $\phi'(u) = \nabla_x F_{\tau_i^n}({}_nX^u_{\tau_i^n-},{}_nA_{\tau_i^n,h_i})$, $\phi''(u) = \nabla_x^2 F_{\tau_i^n}({}_nX^u_{\tau_i^n-},{}_nA_{\tau_i^n,h_i})$. Applying the Itô formula to ϕ between 0 and $\tau_{i+1}^n - \tau_i^n$ and the $(\mathcal{F}_{\tau_i+s})_{s\geq 0}$ continuous semimartingale $(X(\tau_i^n + s))_{s\geq 0}$, yields:

$$
\begin{aligned}
&\phi\big(X(\tau_{i+1}^n) - X(\tau_i^n)\big) - \phi(0) \\
&\quad = \int_{\tau_i^n}^{\tau_{i+1}^n} \nabla_x F_{\tau_i^n}\big({}_nX^{X(s)-X(\tau_i^n)}_{\tau_i^n-},{}_nA_{\tau_i^n}\big)\,dX(s) \\
&\qquad + \frac{1}{2}\int_{\tau_i^n}^{\tau_{i+1}^n} \operatorname{tr}\big[{}^t\nabla_x^2 F_{\tau_i^n}\big({}_nX^{X(s)-X(\tau_i^n)}_{\tau_i^n-},{}_nA_{\tau_i^n}\big)\,d[X](s)\big].
\end{aligned} \tag{35}
$$

Summing over $i \geq 0$ and denoting $i(s)$ the index such that $s \in [\tau_{i(s)}^n, \tau_{i(s)+1}^n)$, we have shown:

$$
\begin{aligned}
&F_t({}_nX_t,{}_nA_t) - F_0(X_0, A_0) \\
&\quad = \int_0^t \mathcal{D}_s F\big({}_nX_{\tau_{i(s)}^n,s-\tau_{i(s)}^n},{}_nA_{\tau_{i(s)}^n,s-\tau_{i(s)}^n}\big)\,ds \\
&\qquad + \int_0^t \nabla_x F_{\tau_{i(s)+1}^n}\big({}_nX^{X(s)-X(\tau_{i(s)}^n)}_{\tau_{i(s)}^n-},{}_nA_{\tau_{i(s)}^n,h_{i(s)}}\big)\,dX(s) \\
&\qquad + \frac{1}{2}\int_0^t \operatorname{tr}\big[\nabla_x^2 F_{\tau_{i(s)}^n}\big({}_nX^{X(s)-X(\tau_{i(s)}^n)}_{\tau_{i(s)}^n-},{}_nA_{\tau_{i(s)}^n}\big)\cdot d[X](s)\big]
\end{aligned} \tag{36}
$$

$F_t({}_nX_t, {}_nA_t)$ converges to $F_t(X_t, A_t)$ almost surely. Since all approximations of (X, A) appearing in the various integrals have a d_∞-distance from (X_s, A_s) less than $\eta_n \to 0$, the continuity at fixed times of $\mathcal{D}F$ and left-continuity $\nabla_x F$, $\nabla_x^2 F$ imply that the integrands appearing in the above integrals converge respectively to $\mathcal{D}_s F(X_s, A_s)$, $\nabla_x F_s(X_s, A_s)$, $\nabla_x^2 F_s(X_s, A_s)$ as $n \to \infty$. Since the derivatives are in $\mathbb{B}$ the integrands in the various above integrals are bounded by a constant dependent only on F, K and R and t does not depend on s nor on ω. The dominated convergence and the dominated convergence theorem for the stochastic integrals [28, Chapter IV, Theorem 32] then ensure that the Lebesgue–Stieltjes integrals converge almost surely, and the stochastic integral in probability, to the terms appearing in (30) as $n \to \infty$.

Consider now the general case where X and A may be unbounded. Let K_n be an increasing sequence of compact sets with $\bigcup_{n\geq 0} K_n = \mathbb{R}^d$ and define the optional stopping times

$$\tau_n = \inf\{s < t | X_s \notin K^n \text{ or } |A_s| > n\} \wedge t.$$

Applying the previous result to the stopped process $(X_{t\wedge\tau_n}, A_{t\wedge\tau_n})$ and noting that, by (10), $F_t(X_t, A_t) = F_t(X_t, A_{t-})$ leads to

$$\begin{aligned} &F_t(X_{t\wedge\tau_n}, A_{t\wedge\tau_n}) - F_0(Z_0, A_0) \\ &\quad = \int_0^{t\wedge\tau_n} \mathcal{D}_u F_u(X_u, A_u)\, du + \frac{1}{2}\int_0^{t\wedge\tau_n} \operatorname{tr}({}^t\nabla_x^2 F_u(X_u, A_u)\, d[X](u)) \\ &\qquad + \int_0^{t\wedge\tau_n} \nabla_x F_u(X_u, A_u)\cdot dX + \int_{t\wedge\tau^n}^t D_u F(X_{u\wedge\tau_n}, A_{u\wedge\tau_n})\, du. \end{aligned}$$

The terms in the first line converges almost surely to the integral up to time t since $t \wedge \tau_n = t$ almost surely for n sufficiently large. For the same reason the last term converges almost surely to 0. □

REMARK 4.2. The above proof is probabilistic and makes use of the (classical) Itô formula [15]. In the companion paper [6] we give a nonprobabilistic proof of Theorem 4.1, using the analytical approach of Föllmer [12], which allows X to have discontinuous (cadlag) trajectories.

EXAMPLE 10. If $F_t(x_t, v_t) = f(t, x(t))$ where $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$, (30) reduces to the standard Itô formula.

EXAMPLE 11. For the functional in Example 5 $F_t(x_t, v_t) = e^{x(t) - 1/2\int_0^t v(u)\, du}$, the formula (30) yields the well-known integral representation

$$\exp\left(X(t) - \frac{1}{2}[X](t)\right) = \int_0^t e^{X(u) - 1/2[X](u)}\, dX(u). \tag{37}$$

An immediate corollary of Theorem 4.1 is that if X is a local martingale, any $\mathbb{C}_b^{1,2}$ functional of X which has finite variation is equal to the integral of its horizontal derivative:

COROLLARY 4.3. *If X is a local martingale and $F \in \mathbb{C}_b^{1,2}$, the process $Y(t) = F_t(X_t, A_t)$ has finite variation if only if $\nabla_x F_t(X_t, A_t) = 0$ $d[X] \times d\mathbb{P}$-almost everywhere.*

PROOF. $Y(t)$ is a continuous semimartingale by Theorem 4.1, with semimartingale decomposition given by (30). If Y has finite variation, then by formula (30), its continuous martingale component should be zero, that is, $\int_0^t \nabla_x F_t(X_t, A_t) \cdot dX(t) = 0$ a.s. Computing its quadratic variation, we obtain

$$\int_0^T \operatorname{tr}({}^t\nabla_x F_t(X_t, A_t) \cdot \nabla_x F_t(X_t, A_t) \cdot d[X]) = 0$$

which implies in particular that $\|\partial_t F_t(X_t, A_t)\|^2 = 0$ $d[X^i] \times d\mathbb{P}$-almost everywhere for $i = 1, \dots, d$. Thus, $\nabla_x F_t(X_t, A_t) = 0$ for $(t, \omega) \notin A \subset [0, T] \times \Omega$ where $\int_A d[X^i] \times d\mathbb{P} = 0$ for $i = 1, \dots, d$. $\square$

4.2. *Vertical derivative of an adapted process.* For a ($\mathcal{F}_t$-adapted) process Y, the the functional representation (42) is not unique, and the vertical $\nabla_x F$ depends on the choice of representation F. However, Theorem 4.1 implies that the *process* $\nabla_x F_t(X_t, A_t)$ has an intrinsic character, that is, independent of the chosen representation:

COROLLARY 4.4. *Let $F^1, F^2 \in \mathbb{C}_b^{1,2}([0, T))$, such that*

$$\forall t \in [0, T), \qquad F_t^1(X_t, A_t) = F_t^2(X_t, A_t) \qquad \mathbb{P}\text{-}a.s. \tag{38}$$

Then, outside an evanescent set,

$$\begin{aligned} &{}^t[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)]A(t-)[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)] \\ &\quad = 0. \end{aligned} \tag{39}$$

PROOF. Let $X(t) = B(t) + M(t)$, where B is a continuous process with finite variation and M is a continuous local martingale. There exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$, and for $\omega \in \Omega$ the path of $t \mapsto X(t, \omega)$ is continuous and $t \mapsto A(t, \omega)$ is cadlag. Theorem 4.1 implies that the local martingale part of $0 = F^1(X_t, A_t) - F^2(X_t, A_t)$ can be written as

$$0 = \int_0^t [\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]\, dM(u). \tag{40}$$

Considering its quadratic variation, we have, on Ω_1,

$$0 = \int_0^t \frac{1}{2}{}^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)] \times A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]\,du. \tag{41}$$

By Lemma 2.6 ($\nabla_x F^1(X_t, A_t) = \nabla_x F^1(X_{t-}, A_{t-})$ since X is continuous and F verifies (10). So on Ω_1 the integrand in (41) is left-continuous; therefore (41) implies that for $t < T$ and $\omega \in \Omega_1$,

$$^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u) = 0. \qquad \square$$

In the case where for all $t < T$, $A(t-)$ is almost surely positive definite, Corollary 4.4 allows us to define intrinsically the pathwise derivative of a process Y which admits a functional representation $Y(t) = F_t(X_t, A_t)$:

DEFINITION 4.5 (Vertical derivative of a process). Define $\mathcal{C}_b^{1,2}(X)$ the set of $\mathcal{F}_t$-adapted processes Y which admit a functional representation in $\mathbb{C}_b^{1,2}$,

$$\mathcal{C}_b^{1,2}(X) = \{Y, \exists F \in \mathbb{C}_b^{1,2}\ Y(t) = F_t(X_t, A_t)\ \mathbb{P}\text{-a.s.}\}. \tag{42}$$

If $A(t)$ is nonsingular, that is, $\det(A(t)) \neq 0\ dt \times d\mathbb{P}$ almost-everywhere, then for any $Y \in \mathcal{C}_b^{1,2}(X)$, the predictable process

$$\nabla_X Y(t) = \nabla_x F_t(X_t, A_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$ in the representation (42). We will call $\nabla_X Y$ the *vertical derivative* of Y with respect to X.

In particular this construction applies to the case where X is a standard Brownian motion, where $A = I_d$, so we obtain the existence of a vertical derivative process for $\mathbb{C}_b^{1,2}$ Brownian functionals:

DEFINITION 4.6 (Vertical derivative of nonanticipative Brownian functionals). Let W be a standard d-dimensional Brownian motion. For any $Y \in \mathcal{C}_b^{1,2}(W)$ with representation $Y(t) = F_t(W_t, t)$, the predictable process

$$\nabla_W Y(t) = \nabla_x F_t(W_t, t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$.

5. Martingale representation formulas. Consider now the case where X is a Brownian martingale:

ASSUMPTION 5.1. $X(t) = X(0) + \int_0^t \sigma(u) \cdot dW(u)$ where σ is a process adapted to $\mathcal{F}_t^W$ verifying

$$\det(\sigma(t)) \neq 0, \qquad dt \times d\mathbb{P}\text{-a.e.} \tag{43}$$

The functional Itô formula (Theorem 4.1) then leads to an explicit martingale representation formula for $\mathcal{F}_t$-martingales in $\mathcal{C}_b^{1,2}(X)$. This result may be seen as a nonanticipative counterpart of the Clark–Haussmann–Ocone formula [4, 14, 25] and generalizes other constructive martingale representation formulas previously obtained using Markovian functionals [7, 10, 11, 17, 26], Malliavin calculus [2, 14, 18, 24, 25] or other techniques [1, 27].

Consider an $\mathcal{F}_T$ measurable random variable H with $E|H| < \infty$, and consider the martingale $Y(t) = E[H|\mathcal{F}_t]$.

5.1. *A martingale representation formula.* If Y admits a representation $Y(t) = F_t(X_t, A_t)$ where $F \in \mathbb{C}_b^{1,2}$, we obtain the following stochastic integral representation for Y in terms of its derivative $\nabla_X Y$ with respect to X:

THEOREM 5.2. *If $Y(t) = F_t(X_t, A_t)$ for some functional $F \in \mathbb{C}_b^{1,2}$, then*

$$Y(T) = Y(0) + \int_0^T \nabla_x F_t(X_t, A_t)\, dX(t) = Y(0) + \int_0^T \nabla_X Y \cdot dX. \tag{44}$$

Note that regularity assumptions are not on $H = Y(T)$, but on the martingale $Y(t) = E[H|\mathcal{F}_t]$, $t < T$, which is typically more regular than H itself.

PROOF. Theorem 4.1 implies that for $t \in [0, T)$,

$$\begin{aligned} Y(t) = {} & \int_0^t \mathcal{D}_u F(X_u, A_u)\, du + \frac{1}{2} \int_0^t \operatorname{tr}[{}^t\nabla_x^2 F_u(X_u, A_u)\, d[X](u)] \\ & + \int_0^t \nabla_x F_u(X_u, A_u)\, dX(u). \end{aligned} \tag{45}$$

Given the regularity assumptions on F, the first term in this sum is a continuous process with finite variation, while the second is a continuous local martingale. However, Y is a martingale and its decomposition as sum of a finite variation process and a local martingale is unique [29]. Hence the first term is 0, and $Y(t) = \int_0^t F_u(X_u, A_u)\, dX_u$. Since $F \in \mathbb{C}_l^{0,0}([0, T])$ $Y(t)$ has limit $F_T(X_T, A_T)$ as $t \to T$, so the stochastic integral also converges. $\square$

EXAMPLE 12. If $e^{X(t)-1/2[X](t)}$ is a martingale, applying Theorem 5.2 to the functional $F_t(x_t, v_t) = e^{x(t)-\int_0^t v(u)\,du}$ yields the familiar formula

$$e^{X(t)-1/2[X](t)} = 1 + \int_0^t e^{X(s)-1/2[X](s)}\,dX(s). \tag{46}$$

5.2. *Extension to square-integrable functionals*. Let $\mathcal{L}^2(X)$ be the Hilbert space of progressively-measurable processes ϕ such that

$$\|\phi\|^2_{\mathcal{L}^2(X)} = E\left[\int_0^t \phi_s^2\,d[X](s)\right] < \infty \tag{47}$$

and $\mathcal{I}^2(X)$ be the space of square-integrable stochastic integrals with respect to X.

$$\mathcal{I}^2(X) = \left\{\int_0^{\cdot} \phi(t)\,dX(t), \phi \in \mathcal{L}^2(X)\right\} \tag{48}$$

endowed with the norm $\|Y\|_2^2 = E[Y(T)^2]$. The Itô integral $I_X : \phi \mapsto \int_0^{\cdot} \phi_s\,dX(s)$ is then a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$.

We will now show that the operator $\nabla_X : \mapsto \mathcal{L}^2(X)$ admits a suitable extension to $\mathcal{I}^2(X)$ which verifies

$$\forall \phi \in \mathcal{L}^2(X), \qquad \nabla_X\left(\int \phi \cdot dX\right) = \phi, \qquad dt \times d\mathbb{P}\text{-a.s.}; \tag{49}$$

that is, ∇_X is the inverse of the Itô stochastic integral with respect to X.

DEFINITION 5.3 (Space of test processes). The space of *test processes* $D(X)$ is defined as

$$D(X) = \mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X). \tag{50}$$

Theorem 5.2 allows us to define intrinsically the vertical derivative of a process in $D(X)$ as an element of $\mathcal{L}^2(X)$.

DEFINITION 5.4. Let $Y \in D(X)$ define the process $\nabla_X Y \in \mathcal{L}^2(X)$ as the equivalence class of $\nabla_x F_t(X_t, A_t)$, which does not depend on the choice of the representation functional $Y(t) = F_t(X_t, A_t)$.

PROPOSITION 5.5 (Integration by parts on $D(X)$). *Let $Y, Z \in D(X)$. Then*

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y(t) \nabla_X Z(t)\,d[X](t)\right]. \tag{51}$$

PROOF. Let $Y, Z \in D(X) \subset \mathcal{C}_b^{1,2}(X)$. Then Y, Z are martingales with $Y(0) = Z(0) = 0$ and $E[|Y(T)|^2] < \infty$, $E[|Z(T)|^2] < \infty$. Applying Theorem 5.2 to Y and Z, we obtain

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y \, dX \int_0^T \nabla_X Z \, dX\right].$$

Applying the Itô isometry formula yields the result. □

Using this result, we can extend the operator ∇_X to define a weak derivative on the space of (square-integrable) stochastic integrals, where $\nabla_X Y$ is characterized by (51) being satisfied against all test processes.

The following definition introduces the Hilbert space $\mathcal{W}^{1,2}(X)$ of martingales on which ∇_X acts as a weak derivative, characterized by integration-by-part formula (51). This definition may be also viewed as a nonanticipative counterpart of Wiener–Sobolev spaces in the Malliavin calculus [22, 30].

DEFINITION 5.6 (Martingale Sobolev space). The martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is defined as the closure in $\mathcal{I}^2(X)$ of $D(X)$.

The martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is in fact none other than $\mathcal{I}^2(X)$, the set of square-integrable stochastic integrals:

LEMMA 5.7. *$\{\nabla_X Y, Y \in D(X)\}$ is dense in $\mathcal{L}^2(X)$ and*

$$\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X).$$

PROOF. We first observe that the set U of "cylindrical" processes of the form

$$\phi_{n,f,(t_1,\dots,t_n)}(t) = f(X(t_1), \dots, X(t_n)) 1_{t>t_n},$$

where $n \geq 1$, $0 \leq t_1 < \cdots < t_n \leq T$ and $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is a total set in $\mathcal{L}^2(X)$, that is, the linear span of U is dense in $\mathcal{L}^2(X)$. For such an integrand $\phi_{n,f,(t1,\dots,t_n)}$, the stochastic integral with respect to X is given by the martingale

$$Y(t) = I_X\big(\phi_{n,f,(t_1,\dots,t_n)}\big)(t) = F_t(X_t, A_t),$$

where the functional F is defined on Υ as

$$F_t(x_t, v_t) = f(x(t_1-), \dots, x(t_n-))\big(x(t) - x(t_n)\big) 1_{t>t_n},$$

so that

$$\nabla_x F_t(x_t, v_t) = f(x_{t_1-}, \dots, x_{t_n-}) 1_{t>t_n}, \qquad \nabla_x^2 F_t(x_t, v_t) = 0,$$
$$\mathcal{D}_t F(x_t, v_t) = 0$$

which shows that $F \in \mathbb{C}_b^{1,2}$; see Example 2. Hence, $Y \in \mathcal{C}_b^{1,2}(X)$. Since f is bounded, Y is obviously square integrable, so $Y \in D(X)$. Hence $I_X(U) \subset D(X)$.

Since I_X is a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$, the density of U in $\mathcal{L}^2(X)$ entails the density of $I_X(U)$ in $\mathcal{I}^2(X)$, so $\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X)$. □

THEOREM 5.8 (Extension of ∇_X to $\mathcal{W}^{1,2}(X)$). *The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry*

$$\begin{aligned} &\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X), \\ &\int_0^{\cdot} \phi \cdot dX \mapsto \phi \end{aligned} \tag{52}$$

characterized by the following integration by parts formula: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that

$$\forall Z \in D(X), \qquad E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y(t) \nabla_X Z(t)\, d[X](t)\right]. \tag{53}$$

In particular, ∇_X is the adjoint of the Itô stochastic integral

$$\begin{aligned} &I_X : \mathcal{L}^2(X) \mapsto \mathcal{W}^{1,2}(X), \\ &\phi \mapsto \int_0^{\cdot} \phi \cdot dX \end{aligned} \tag{54}$$

in the following sense:

$$\begin{aligned} &\forall \phi \in \mathcal{L}^2(X), \forall Y \in \mathcal{W}^{1,2}(X), \\ &E\left[Y(T) \int_0^T \phi \cdot dX\right] = E\left[\int_0^T \nabla_X Y \phi\, d[X]\right]. \end{aligned} \tag{55}$$

PROOF. Any $Y \in \mathcal{W}^{1,2}(X)$ may be written as $Y(t) = \int_0^t \phi(s)\, dX(s)$ with $\phi \in \mathcal{L}^2(X)$, which is uniquely defined $d[X] \times d\mathbb{P}$ a.e. The Itô isometry formula then guarantees that (53) holds for ϕ. To show that (53) uniquely characterizes ϕ, consider $\psi \in \mathcal{L}^2(X)$ which also satisfies (53), then, denoting $I_X(\psi) = \int_0^{\cdot} \psi\, dX$ its stochastic integral with respect to X, (53) then implies that

$$\forall Z \in D(X), \qquad \langle I_X(\psi) - Y, Z\rangle_{\mathcal{W}^{1,2}(X)} = E\left[\left(Y(T) - \int_0^T \psi\, dX\right) Z(T)\right] = 0$$

which implies $I_X(\psi) = Y$ $d[X] \times d\mathbb{P}$ a.e., since by construction $D(X)$ is dense in $\mathcal{W}^{1,2}(X)$. Hence, $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$.

This construction shows that $\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X)$ is a bijective isometry which coincides with the adjoint of the Itô integral on $\mathcal{W}^{1,2}(X)$. $\square$

Thus, the Itô integral I_X with respect to X,

$$I_X : \mathcal{L}^2(X) \mapsto \mathcal{W}^{1,2}(X),$$

admits an inverse on $\mathcal{W}^{1,2}(X)$ which is an extension of the (pathwise) vertical derivative ∇_X operator introduced in Definition 3.2, and

$$\forall \phi \in \mathcal{L}^2(X), \qquad \nabla_X\left(\int_0^{\cdot} \phi\, dX\right) = \phi \tag{56}$$

holds in the sense of equality in $\mathcal{L}^2(X)$.

The above results now allow us to state a general version of the martingale representation formula, valid for all square-integrable martingales:

THEOREM 5.9 (Martingale representation formula: general case). *For any square-integrable $(\mathcal{F}_t^X)_{t\in[0,T]}$-martingale Y,*

$$Y(T) = Y(0) + \int_0^T \nabla_X Y \, dX, \qquad \mathbb{P}\text{-}a.s.$$

6. Relation with the Malliavin derivative. The above results hold in particular in the case where $X = W$ is a Brownian motion. In this case, the vertical derivative ∇_W may be related to the *Malliavin derivative* [2, 3, 22, 31] as follows.

Consider the canonical Wiener space $(\Omega_0 = C_0([0,T], \mathbb{R}^d), \|\cdot\|_\infty, \mathbb{P})$ endowed with its Borelian σ-algebra, the filtration of the canonical process. Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0,T]) = H(X_T)$ with $E[|H|^2] < \infty$. If H is differentiable in the Malliavin sense [2, 22, 24, 31], for example, $H \in \mathbf{D}^{1,2}$ with Malliavin derivative $\mathbb{D}_t H$, then the Clark–Haussmann–Ocone formula [24, 25] gives a stochastic integral representation of H in terms of the Malliavin derivative of H.

$$H = E[H] + \int_0^T {}^pE[\mathbb{D}_t H | \mathcal{F}_t] \, dW_t, \tag{57}$$

where ${}^pE[\mathbb{D}_t H|\mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. This yields a stochastic integral representation of the martingale $Y(t) = E[H|\mathcal{F}_t]$,

$$Y(t) = E[H|\mathcal{F}_t] = E[H] + \int_0^t {}^pE[\mathbb{D}_t H|\mathcal{F}_u] \, dW_u.$$

Related martingale representations have been obtained under a variety of conditions [2, 7, 11, 18, 24, 26].

Denote by:

- $L^2([0,T] \times \Omega)$ the set of (anticipative) processes ϕ on $[0,T]$ with $E\int_0^T \|\phi(t)\|^2 \, dt < \infty$;
- $\mathbb{D}$ the Malliavin derivative operator, which associates to a random variable $H \in \mathbf{D}^{1,2}(0,T)$ the (anticipative) process $(\mathbb{D}_t H)_{t\in[0,T]} \in L^2([0,T] \times \Omega)$.

THEOREM 6.1 (Lifting theorem). *The following diagram is commutative is the sense of $dt \times d\mathbb{P}$ equality*:

$$\begin{array}{ccc} \mathcal{I}^2(W) & \overset{\nabla_W}{\to} & \mathcal{L}^2(W) \\ \uparrow (E[\cdot|\mathcal{F}_t])_{t\in[0,T]} & & \uparrow (E[\cdot|\mathcal{F}_t])_{t\in[0,T]} \\ \mathbf{D}^{1,2} & \overset{\mathbb{D}}{\to} & L^2([0,T] \times \Omega). \end{array}$$

In other words, the conditional expectation operator intertwines ∇_W *with the Malliavin derivative*,

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \qquad \nabla_W(E[H|\mathcal{F}_t]) = E[\mathbb{D}_t H|\mathcal{F}_t]. \tag{58}$$

PROOF. The Clark–Haussmann–Ocone formula [25] gives

$$\forall H \in \mathbf{D}^{1,2}, \qquad H = E[H] + \int_0^T {}^pE[\mathbb{D}_t H|\mathcal{F}_t]\, dW_t, \tag{59}$$

where ${}^pE[\mathbb{D}_t H|\mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. On other hand, Theorem 5.2 gives

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \qquad H = E[H] + \int_0^T \nabla_W Y(t)\, dW(t), \tag{60}$$

where $Y(t) = E[H|\mathcal{F}_t]$. Hence ${}^pE[\mathbb{D}_t H|\mathcal{F}_t] = \nabla_W E[H|\mathcal{F}_t]$, $dt \times d\mathbb{P}$ almost everywhere. □

Thus, the conditional expectation operator (more precisely: the *predictable* projection on $\mathcal{F}_t$ [8, Volume I]) can be viewed as a morphism which "lifts" relations obtained in the framework of Malliavin calculus into relations between nonanticipative quantities, where the Malliavin derivative and the Skorokhod integral are replaced, respectively, by the vertical derivative ∇_W and the Itô stochastic integral.

From a computational viewpoint, unlike the Clark–Haussmann–Ocone representation which requires to simulate the *anticipative* process $\mathbb{D}_t H$ and compute conditional expectations, $\nabla_X Y$ only involves nonanticipative quantities which can be computed path by path. It is thus more amenable to numerical computations. This topic is further explored in a forthcoming work.

APPENDIX: PROOF OF THEOREM 2.7

In order to prove Theorem 2.7 in the general case where A is only required to be cadlag, we need the following three lemmas. The first lemma states a property analogous to "uniform continuity" for cadlag functions:

LEMMA A.1. *Let f be a cadlag function on* $[0, T]$ *and define* $\Delta f(t) = f(t) - f(t-)$. *Then*

$$\begin{aligned} &\forall \varepsilon > 0, \exists \eta(\varepsilon) > 0, \\ &\quad |x - y| \le \eta \quad \Rightarrow \quad |f(x) - f(y)| \le \varepsilon + \sup_{t \in (x,y]} \{|\Delta f(t)|\}. \end{aligned} \tag{61}$$

PROOF. If (61) does not hold, then there exists a sequence $(x_n, y_n)_{n \ge 1}$ such that $x_n \le y_n$, $y_n - x_n \to 0$, but $|f(x_n) - f(y_n)| > \varepsilon + \sup_{t \in [x_n, y_n]}\{|\Delta f(t)|\}$. We can extract a convergent subsequence $(x_{\psi(n)})$ such that $x_{\psi(n)} \to x$. Noting that

either an infinity of terms of the sequence are less than x or an infinity are more than x, we can extract *monotone* subsequences $(u_n, v_n)_{n\geq 1}$ which converge to x. If $(u_n), (v_n)$ both converge to x from above or from below, $|f(u_n) - f(v_n)| \to 0$ which yields a contradiction. If one converges from above and the other from below, $\sup_{t\in[u_n,v_n]}\{|\Delta f(t)|\} \geq |\Delta f(x)|$, but $|f(u_n) - f(v_n)| \to |\Delta f(x)|$, which results in a contradiction as well. Therefore (61) must hold. □

LEMMA A.2. *If $\alpha \in \mathbb{R}$ and V is an adapted cadlag process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and σ is a optional time, then*

$$\tau = \inf\{t > \sigma, |V(t) - V(t-)| > \alpha\} \tag{62}$$

is a stopping time.

PROOF. We can write that

$$\{\tau \leq t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t)} (\{\sigma \leq t - q\} \cap \Big\{\sup_{t\in(t-q,t]} |V(u) - V(u-)| > \alpha\Big\} \tag{63}$$

and, using Lemma A.1,

$$\begin{aligned} &\Big\{\sup_{u\in(t-q,t]} |V(u) - V(u-)| > \alpha\Big\} \\ &\quad = \bigcup_{n_0>1} \bigcap_{n>n_0} \bigcup_{m\geq 1} \Big\{\sup_{1\leq i\leq 2^n} \Big|V\Big(t - q\frac{i-1}{2^n}\Big) - V\Big(t - q\frac{i}{2^n}\Big)\Big| > \alpha + \frac{1}{m}\Big\}. \end{aligned} \tag{64}$$

□

LEMMA A.3 (Uniform approximation of cadlag functions by step functions). *Let $f \in D([0,T], \mathbb{R}^d)$ and $\pi^n = (t_i^n)_{n\geq 1, i=0,\dots,k_n}$ a sequence of partitions $(0 = t_0^n < t_1 < \cdots < t_{k_n}^n = T)$ of $[0,T]$ such that*

$$\sup_{0\leq i\leq k_n-1} |t_{i+1}^n - t_i^n| \overset{n\to\infty}{\to} 0, \qquad \sup_{u\in[0,T]\setminus\pi^n} |\Delta f(u)| \overset{n\to\infty}{\to} 0$$

then

$$\sup_{u\in[0,T]} \Big| f(u) - \sum_{i=0}^{k_n-1} f(t_i^n) 1_{[t_i^n, t_{i+1}^n)}(u) + f(t_{k_n}^n) 1_{\{t_{k_n}^n\}}(u) \Big| \overset{n\to\infty}{\to} 0. \tag{65}$$

PROOF. Denote $h^n = f - \sum_{i=0}^{k_n-1} f(t_i^n) 1_{[t_i^n, t_{i+1}^n)} + f(t_{k_n}^n) 1_{\{t_{k_n}^n\}}$. Since $f - h^n$ is piecewise constant on π^n and $h^n(t_i^n) = 0$ by definition,

$$\sup_{t\in[0,T]} |h^n(t)| = \sup_{i=0,\dots,k_n-1} \sup_{[t_i^n, t_i^{n+1})} |h^n(t)| = \sup_{t_i^n < t < t_i^{n+1}} |f(t) - f(t_i^n)|.$$

Let $\varepsilon > 0$. For $n \geq N$ sufficiently large, $\sup_{u\in[0,T]\setminus\pi^n} |\Delta f(u)| \leq \varepsilon/2$ and $\sup_i |t_{i+1}^n - t_i^n| \leq \eta(\varepsilon/2)$ using the notation of Lemma A.1. Then, applying Lemma A.1 to f we obtain, for $n \geq N$,

$$\sup_{t\in[t_i^n, t_i^{n+1})} |f(t) - f(t_i^n)| \leq \frac{\varepsilon}{2} + \sup_{t_i^n < t < t_i^{n+1}} |\Delta f(u)| \leq \varepsilon. \qquad \square$$

We can now prove Theorem 2.7 in the case where A is a cadlag adapted process.

PROOF OF THEOREM 2.7. Let us first show that $F_t(X_t, A_t)$ is adapted. Define

$$\tau_0^N = 0, \qquad \tau_k^N = \inf\left\{t > \tau_{k-1}^N | 2^N t \in \mathbb{N} \text{ or } |A(t) - A(t-)| > \frac{1}{N}\right\} \wedge t. \tag{66}$$

From lemma A.2, τ_k^N are stopping times. Define the following piecewise constant approximations of X_t and A_t along the partition $(\tau_k^N, k \geq 0)$:

$$\begin{aligned} X^N(s) &= \sum_{k\geq 0} X_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N[}(s) + X(t) 1_{\{t\}}(s), \\ A^N(s) &= \sum_{k=0} A_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(t) + A(t) 1_{\{t\}}(s), \end{aligned} \tag{67}$$

as well as their truncations of rank K

$$ {}_K X^N(s) = \sum_{k=0}^K X_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(s), \qquad {}_K A^N(t) = \sum_{k=0}^K A_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(t). \tag{68}$$

Since $({}_K X_t^N, {}_K A_t^N)$ coincides with (X_t^N, A_t^N) for K sufficiently large,

$$F_t(X_t^N, A_t^N) = \lim_{K\to\infty} F_t({}_K X_t^N, {}_K A_t^N). \tag{69}$$

The approximations $F_t^n({}_K X_t^N, {}_K A_t^N)$ are $\mathcal{F}_t$-measurable as they are continuous functions of the random variables

$$\{(X(\tau_k^N) 1_{\tau_k^N \leq t}, A(\tau_k^N) 1_{\tau_k^N \leq t}), k \leq K\},$$

so their limit $F_t(X_t^N, A_t^N)$ is also $\mathcal{F}_t$-measurable. Thanks to Lemma A.3, X_t^N and A_t^N converge uniformly to X_t and A_t, and hence $F_t(X_t^N, A_t^N)$ converges to $F_t(X_t, A_t)$ since $F_t : (D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \|\cdot\|_\infty) \to \mathbb{R}$ is continuous.

To show the optionality of Z in point (ii), we will show that Z it as limit of right-continuous adapted processes. For $t \in [0,T]$, define $i^n(t)$ to be the integer such that $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$. Define the process $Z_t^n = F_{(i^n(t))T/n}(X_{(i^n(t))T/n}, A_{(i^n(t))T/n})$, which is piecewise-constant and has right-continuous trajectories, and is also adapted by the first part of the theorem. Since $F \in \mathbb{C}_l^{0,0}$, $Z^n(t) \to Z(t)$ almost surely, which proves that Z is optional. Point (iii) follows from (i) and Lemma 2.6, since in both cases $F_t(X_t, A_t) = F_t(X_{t-}, A_{t-})$, and hence Z has left-continuous trajectories. $\square$

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