THE SOLUTION OF THE PERTURBED TANAKA-EQUATION IS PATHWISE UNIQUE[1]

By Vilmos Prokaj

Eötvös Loránd University

The Tanaka equation $dX_t = \text{sign}(X_t)\, dB_t$ is an example of a stochastic differential equation (SDE) without strong solution. Hence pathwise uniqueness does not hold for this equation. In this note we prove that if we modify the right-hand side of the equation, roughly speaking, with a strong enough additive noise, independent of the Brownian motion B, then the solution of the obtained equation is pathwise unique.

1. Introduction. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space and $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$. In the simplest form we are interested in the uniqueness of the solution for the following equation:

$$\text{(1)} \qquad dX_t = \text{sign}(X_t)\, dB_t^{(1)} + \lambda\, dB_t^{(2)},$$

where $\lambda \in \mathbb{R}$ is a constant, and sign denotes the signum function taking -1 at zero, that is, $\text{sign}(x) = \mathbb{1}_{(x>0)} - \mathbb{1}_{(x\leq 0)}$. We call (1) the perturbed Tanaka equation, and the statement in title reads as follows:

THEOREM 1. *For $\lambda \neq 0$ the solution of* (1) *is pathwise unique.*

Actually we prove a more general statement than Theorem 1. For the sake of fluent composition, we use the term *strongly orthogonal* for continuous local martingales whose product is a local martingale, that is, for M, N if $\langle M, N \rangle = 0$. We say that *N dominates M* if for some constant $c > 0$ we have $d\langle M \rangle \leq c\, d\langle N \rangle$. In other words there is a process Q (it can be chosen to be predictable) such that $\langle M \rangle_t = \int_0^t Q_s\, d\langle N \rangle_s$ for all $t \geq 0$ and $\mathbf{P}(\forall s \geq 0, 0 \leq Q_s \leq c) = 1$. A localized version of this notion, namely *N locally dominates M*, holds if this Q is locally bounded.

THEOREM 2. *Let M, N be continuous local martingales in $(\mathcal{F}_t)_{t\geq 0}$. Assume that M and N are strongly orthogonal and N dominates M. Then, the solution of*

the equation

(2) $$dX_t = \text{sign}(X_t)\,dM_t + dN_t$$

is pathwise unique.

The interest in the uniqueness of the solution of this type of equation stems from the search for the strong solution of the drift hiding problem. Weak solution was given in [7], and the results of this paper make it possible to modify the construction to obtain a strong solution. It is presented in the forthcoming paper [8]. It uses Theorem 2 as a main new ingredient. Besides this particular application, we think that this problem is also interesting in its own right.

By standard localization argument, we obtain the following:

COROLLARY 3. *Let M, N be continuous local martingales in $(\mathcal{F}_t)_{t\geq 0}$. Assume that M and N are strongly orthogonal, and N locally dominates M. Then, the solution of* (2) *is pathwise unique.*

Another trivial extension is obtained by a measure change argument.

COROLLARY 4. *Let M, N be continuous semimartingales in $(\mathcal{F}_t)_{t\geq 0}$. Assume that for each $T \geq 0$ there is an equivalent probability measure $\mathbf{Q}$ on $\mathcal{F}_T$ such that $(M_t)_{t\in[0,T]}$ and $(N_t)_{t\in[0,T]}$ are strongly orthogonal local martingales under $\mathbf{Q}$, and N locally dominates M. Then, the solution of* (2) *is pathwise unique.*

For the proof of pathwise uniqueness, one usually considers $X - X'$ where X, X' are two processes satisfying the equation with the same driving semimartingale and starting from the same initial value. Here it is not enough; we also have to deal with $X + X'$. The next theorem essentially states the uniqueness in terms of $U = (X - X')/2$ and $V = (X + X')/2$.

THEOREM 5. *Assume that U, V are continuous, strongly orthogonal local martingales such that*

(3) $$dU_t = \mathbb{1}_{(|V_t|<|U_t|)}\,dU_t, \qquad U_0 = V_0 = 0.$$

If V dominates U, then U is trivial, that is, identically zero.

Without domination the statement is not true in general. In Section 3 below, we construct a pair (U, V) satisfying (3) such that U is nontrivial. By Remark 6 below, this example also shows that strong orthogonality together with the almost sure absolute continuity of $\langle M \rangle$ with respect to $\langle N \rangle$ is not enough in Theorem 2 and Corollary 3. Hence the assumption that N dominates M is essential. Moreover, it is possible to construct an example in which M is a Brownian motion, and the perturbation N is such that its quadratic variation is equivalent with the

Lebesgue measure almost surely, and still the pathwise uniqueness does not hold for (2). Even if the perturbation N is a Brownian motion, one can construct a local martingale M strongly orthogonal to N such that the solution of (2) is not pathwise unique. These claims are formulated as Theorem 22, 23 and 24 in Section 3.

We close the introduction with a remark on Theorem 1. After rearranging and conditioning on $B^{(2)}$, Theorem 1 says that for almost all sample path w's of a Brownian motion, the solution of the next equation is pathwise unique, hence strong:

$$(4) \qquad\qquad dX_t = \text{sign}(X_t + w_t)\, dB_t, \qquad X_0 = 0.$$

Denote by $H \subset C[0, \infty)$, the set of those deterministic functions w for which the solution of (4) is pathwise unique. Then H is not empty, and the above reasoning gives that it has full measure with respect to the Wiener measure on the path space. On the other hand, to construct one such example not using randomness seems to be difficult. One possible reason for it is that H might be small in the sense of category. So the natural question arises, for which we do not know the answer: is the set H meager, that is, of the first Baire category?

2. Proofs.
We prove Theorem 5 below, but first we show how to deduce Theorem 2 from Theorem 5.

PROOF OF THEOREM 2 USING THEOREM 5.   We have to show that if X and X' are two solutions of (2), such that $X_0 = X'_0$, then $X = X'$. We can assume that $X_0 = X'_0 = 0$, since up to the stopping time $\tau = \inf\{t > 0 : X_t = 0\}$, the solution is given by $X_0 + \text{sign}(X_0)M_t + N_t$.

So, we can assume that $X_0 = X'_0 = 0$. As indicated in the remark before Theorem 5, put $U_t = (X_t - X'_t)/2$. Then

$$(5) \qquad \begin{aligned} U_t &= \frac{1}{2}\int_0^t \text{sign}(X_s) - \text{sign}(X'_s)\, dM_s \\ &= \int_0^t \mathbb{1}_{(X_s X'_s < 0)}\, \text{sign}(X_s)\, dM_s = \int_0^t \mathbb{1}_{(X_s X'_s < 0)}\, dU_s. \end{aligned}$$

We obtain (5), by observing that

$$\text{sign}(x) - \text{sign}(x') = \begin{cases} 2\,\text{sign}(x), & \text{if } xx' < 0, \\ \text{sign}(x) + 1, & \text{if } x' = 0, \\ -1 - \text{sign}(x'), & \text{if } x = 0, \\ 0, & \text{if } xx' > 0 \end{cases}$$

and

$$(6) \quad \int_0^t \mathbb{1}_{(X_s=0)}\big(1 + \text{sign}(X'_s)\big)\, dM_s = \int_0^t \mathbb{1}_{(X'_s=0)}\big(1 + \text{sign}(X_s)\big)\, dM_s = 0.$$

To show (6) put $\xi_t = \int_0^t \mathbb{1}_{(X_s'=0)}(1+\mathrm{sign}(X_s))\,dM_s$, and use $\mathbf{E}(\xi_t^2) \leq \mathbf{E}(\langle\xi\rangle_t)$ combined with

$$\langle\xi\rangle_t = \int_0^t \mathbb{1}_{(X_s'=0)}(1+\mathrm{sign}(X_s))^2\,d\langle M\rangle_s \leq 4\int_0^t \mathbb{1}_{(X_s'=0)}\,d\langle X'\rangle_s = 0.$$

The latter is an easy consequence of the occupation time formula. The other part of (6) follows similarly, by changing the role of X and X'.

We can observe that $X_t X_t' < 0$ if and only if $|X_t - X_t'| > |X_t + X_t'|$, that is, $|U_t| > |V_t|$, where $V = (X + X')/2$. Hence equation (5) is just another form of (3). By definition,

$$\langle U\rangle_t = \int_0^t \mathbb{1}_{(X_s X_s'<0)}\,d\langle M\rangle_s, \qquad \langle V\rangle_t = \int_0^t \mathbb{1}_{(X_s X_s'\geq0)}\,d\langle M\rangle_s + \langle N\rangle_t.$$

So $\langle U, V\rangle = 0$, that is, U and V are strongly orthogonal, and V dominates U. By Theorem 5, $2U = X - X'$ is identically zero, hence $X = X'$. $\square$

REMARK 6. Observe that any nontrivial example to (3) can produce an example showing that the solution of the corresponding perturbed Tanaka equation is not pathwise unique. Indeed take strongly orthogonal U, V such that (3) holds and U is not identically zero. Define

$$X = V + U, \qquad X' = V - U,$$

$$Y_t = \int_0^t \mathbb{1}_{(|V_s|\geq|U_s|)}\,dV_s, \qquad W_t = \int_0^t \mathbb{1}_{(|V_s|<|U_s|)}\,dV_s.$$

By enlarging the probability space, one can assume that $Y_t = \xi_t + \xi_t'$, where ξ and ξ' are strongly orthogonal continuous local martingales and $\langle\xi\rangle = \langle\xi'\rangle$. To see this take the DDS Brownian motion B of Y and a Brownian motion B' independent from the original $\mathcal{F}_\infty$, and write $\xi_t = \frac{1}{2}(B+B')_{\langle Y\rangle_t}$, $\xi_t' = \frac{1}{2}(B-B')_{\langle Y\rangle_t}$. With this choice U, W, ξ, ξ' are pairwise strongly orthogonal.

Finally let

$$N = W + \xi' \quad \text{and} \quad M_t = \int_0^t \mathrm{sign}(X_s)(dU_s + d\xi_s).$$

The point here is that by (3),

$$dU_t = \mathbb{1}_{(X_t X_t'<0)}\,\mathrm{sign}(X_t)\,dM_t \quad \text{and} \quad d\xi_t = \mathbb{1}_{(X_t X_t'\geq0)}\,\mathrm{sign}(X_t)\,dM_t,$$

since $XX' < 0$ exactly when $|U| > |V|$. Hence

$$dX_t = dW_t + d\xi_t' + d\xi_t + dU_t = dN_t + \mathrm{sign}(X_t)\,dM_t.$$

Note also that the calculation leading to (6), and finally (5), applies with the current definition of M, X and X', since both X and X' dominate M. Hence

$$\big(\mathrm{sign}(X_t) - \mathrm{sign}(X_t')\big)\,dM_t = \mathbb{1}_{(X_t X_t'<0)}\,\mathrm{sign}(X_t)\,dM_t = 2\,dU_t,$$

and

$$dX'_t = dX_t - 2\,dU_t = dN_t - \text{sign}(X'_t)\,dM_t.$$

That is, both X and X' solves (2). Moreover, N dominates M exactly when V dominates U, since

(7) $$\langle N \rangle = \langle V \rangle, \qquad \langle M \rangle = \langle U \rangle + \langle Y \rangle,$$

and V dominates Y by definition.

2.1. *Outline of the proof of Theorem* 5. In the previous remark we already defined Y, W as

(8) $$Y_t = \int_0^t \mathbb{1}_{(|V_s| \geq |U_s|)}\,dV_s, \qquad W_t = \int_0^t \mathbb{1}_{(|V_s| < |U_s|)}\,dV_s.$$

Assume that (3) holds. Then the key feature of Y and (U, W) is that they cannot change "simultaneously." One of the simplest examples for two continuous martingales without simultaneous moving is used in one of the proofs of the arcsine law; see, for example, Theorem 2.7 of Chapter VI on page 242 of [9]. In this proof one splits the Brownian motion B with the formula

$$B_t^+ = \int_0^t \mathbb{1}_{(B_s > 0)}\,dB_s, \qquad B_t^- = \int_0^t -\mathbb{1}_{(B_s < 0)}\,dB_s,$$

and exploits the fact that the two processes B^+ and B^- are linked to each other through the local time of B at level zero, that is,

$$\inf_{s \leq t} B_s^+ = \inf_{s \leq t} B_s^- = -\frac{1}{2}L_t^0(B) \qquad \text{for all } t \geq 0.$$

It means that the excursions of B^+ and B^- from their running minimum are interlaced. Heuristically, after each excursion of B^+ the value of the running minimum process decreases with an infinitesimal value. Before these infinitesimal decrements sum up to a visible change, B^- performs some excursions as well, so the running minimum processes remain synchronized.

Now suppose, contrary to Theorem 5, that we have a nontrivial pair (U, V) of strongly orthogonal, continuous local martingales satisfying (3). Then, similarly as in the above example, Y and (U, W) are "linked" to each other, although the situation is somewhat more complex. To describe this link take the random sets

$$A^+ = \{t : |V_t| > |U_t|\}, \qquad A^- = \{t : |V_t| < |U_t|\}.$$

Say, (σ, τ) is a connected component of A^+. Then (U, W) is constant on (σ, τ) while the process Y takes a move. Then Y stays on one side of Y_σ, and at the end of the interval, that is, at τ, it returns to the starting value of the excursion, that is, $Y_\tau = Y_\sigma$.

The other case is when (σ, τ) is a component of A^-. Then Y is constant, and (U, W) makes a move. Since for $t \in (\sigma, \tau)$ we have $|Y_t + W_t| < |U_t|$, the two-dimensional process (U, W) moves in the interior of a "double cone" until it reaches the boundary. To be precise this double cone is $C(-Y_\sigma)$, where

$$C(y) = \{(u, w) \in \mathbb{R}^2 : y - |u| \leq w \leq y + |u|\}.$$

The best way to think of the above is that the two-dimensional process (U, W) moves in the plane under the constraint that it can not leave the (moving) double cone $C(-L_t)$, where $L_t = Y_{\sigma(t)}$ the value of Y at the last time epoch when $|Y + W| = |U|$. When (U, W) hits the boundary of $C(-L_t)$, it has to wait until the change in L_t enables it to move.

Recall that this is similar to the way B is obtained from B^+ and B^-. In the case of B, the constraint is that B^+ must be in the moving half line $\{x \in \mathbb{R} : x \geq \inf_{s \leq t} B_s^-\}$. Since there is a one-sided condition, both processes have only excursions from the running minimum.

By similar reasoning, when (U, W) hits the polyline $\{(u, w) \in \mathbb{R}^2 : w = -L_t + |u|\}$, then $-L$ is locally increasing, as (U, W) pushes the double cone $C(-L_t)$ upward on the plane. Actually, L locally follows the running minimum of Y, and as in the case of $B^\pm$, the changes in L can be described as the changes of a local time process; see Lemma 7 below.

The other case, that is, when (U, W) hits the polyline $\{(u, w) \in \mathbb{R}^2 : w = -L_t - |u|\}$ differs only in the direction of changes. In this regime, (U, W) tries to push downward the cone on the plane, and therefore $-L$ is decreasing. Then Y performs excursions below the actual value of L, and L locally follows the running maximum of Y.

The above reasoning is made precise in Lemma 7 and yields that L is a linear combination of local time processes, whence it has a continuous sample path with a locally bounded variation.

The end of our argument is that immediately after the moment that U leaves the origin, the total variation of L becomes infinite. Since L has locally bounded variation, this clearly implies that U is identically zero and, in other words, is trivial.

To do this last step, we only use that under the assumptions of Theorem 5 the local martingales U, W are strongly orthogonal, W dominates U and (U_t, W_t) remains in the double cone $C(-L_t)$ for all t, that is, $W - |U| \leq -L \leq W + |U|$. To fix ideas let us discuss here the simplest case; that is, assume that (U, W) is a two-dimensional Brownian motion, and L is continuous process such that $W - |U| \leq -L \leq W + |U|$. Denote by $\mathcal{V}_t$ the total variation of L on $[0, t]$. Next we give the reason why $\mathcal{V}_t$ becomes infinite immediately after starting.

During each excursion of $|U|$ away from zero, the process $\mathcal{V}$ increases. Take one such excursion which is performed on the time interval $I = [s, t]$. Then $-L_s = W_s$ and $-L_t = W_t$ since $U_s = U_t = 0$. The increment of $\mathcal{V}$ on I can be estimated as

$\mathcal{V}_t - \mathcal{V}_s \geq |L_t - L_s| = |W_t - W_s|$. Here $(W_t - W_s)/\sqrt{t-s}$ is a standard normal variable, by the independence of U and W. Moreover, if we take the usual measurable enumeration of the excursions, then the corresponding normal variables are independent of each other and also of U. Hence we have a lower bound for $\mathcal{V}_t$ in the form

$$\tag{9} \sum_n \sqrt{|I_n|}|\eta_n|,$$

where $\{I_n : n \geq 0\}$ is the enumeration of excursion intervals ending before t, and the variables $|\eta_n|$ are i.i.d., with positive expectation, independent of the sequence $|I_n|$. By a characterization of Brownian local time we have $\sum_n \sqrt{|I_n|} = \infty$ a.s., and this implies immediately that (9) is also almost surely infinite. This shows that $\mathcal{V}_t = \infty$ for $t > 0$.

With some modification the above reasoning also applies to U, W and L in the general case.

2.2. *Details of the proof of Theorem* 5.   Throughout this section, for $t \geq 0$ put

$$\sigma(t) = \sup\{s \in [0, t] : |U_s| = |V_s|\}.$$

$\sigma(t)$ is the last point before t where $|V| = |U|$ holds. The process σ is increasing, right continuous and adapted. It starts at zero, since by assumption $U_0 = V_0 = 0$.

Next, Y, W are defined by formula (8) and L by

$$\tag{10} L_t = Y_{\sigma(t)}.$$

The reasoning outlined in the preceding section is accomplished by proving two lemmas below. Lemma 7 gives that L has continuous sample path with locally bounded variation. Lemma 9 applies to L by Proposition 8 and formalizes the argument at the end of the heuristic argument. It shows that the assumption that U is not identically zero would lead to a contradiction proving Theorem 5 completely. The proof of Lemma 9 uses two more proposition and a slight addition to Knight's theorem; see Lemma 12.

LEMMA 7.   *Let U, V be continuous semimartingales satisfying* (3) *and L as above. Then L is a linear combination of local time processes, and hence it is of bounded variation on compact intervals. To be precise*,

$$2L_t = L_t^0(|U| + V) - L_t^0(|U| - V),$$

where $L^x(X)$ denotes the local time process of X at level x.

PROOF.   Put $\xi = \mathrm{med}(V + U, V - U, 0)$, where med denotes the median of its three argument. Then ξ_t follows the trajectory of $V + U$ if it is in the middle, that is, when $UV < 0$ and $|V| > |U|$. It follows the changes of $V - U$ when $UV > 0$ and $|V| > |U|$ and stays at zero when $|V| < |U|$. When ξ switches between the

above regimes, the corresponding local time process increases. So apart form the local time changes, ξ_t follows the changes in Y, since the other two processes W, U are locally constant on $\{t : |V_t| > |U_t|\}$.

We obtained that $\xi_t = Y_t - L'_t$, where L'_t is from the local time components. Now if $\sigma(t) = t$, that is, $|V_t| = |U_t|$ then we have $\xi_t = 0$. Hence $L_t = Y_{\sigma(t)} = L'_{\sigma(t)}$. This gives that L is of locally bounded variation.

To carry out this program observe that

$$\xi_t = \text{med}(V + U, V - U, 0) = (V_t + |U_t|) \wedge 0 + (V_t - |U_t|) \vee 0$$

(11)

$$= (|U_t| + V_t) \wedge 0 - (|U_t| - V_t) \wedge 0.$$

For the first term, the Tanaka formula gives that

$$d(|U|_t + V_t) \wedge 0 = \mathbb{1}_{(V_t \leq -|U_t|)} \, d(|U_t| + V_t) - \tfrac{1}{2} \, dL^0_t(|U| + V).$$

Note that since U, V satisfies (3), and the support of $dL^0_t(U)$ is the null set of U, the right-hand side simplifies to

$$d(|U|_t + V_t) \wedge 0 = \mathbb{1}_{(V_t \leq -|U_t|)} \, dV_t + \mathbb{1}_{(V_t \leq 0)} \, dL^0_t(U) - \tfrac{1}{2} \, dL^0_t(|U| + V).$$

Similar calculation for the second term in (11) yields

$$d(|U_t| - V_t) \wedge 0 = -\mathbb{1}_{(V_t \geq |U_t|)} \, dV_t + \mathbb{1}_{(V_t \geq 0)} \, dL^0_t(U) - \tfrac{1}{2} \, dL^0_t(|U| - V).$$

Hence

$$d\xi_t = \mathbb{1}_{(|V_t| \geq |U_t|)} \, dV_t - \text{sign}_0(V_t) \, dL^0_t(U) + \tfrac{1}{2} \, dL^0_t(|U| - V) - \tfrac{1}{2} \, dL^0_t(|U| + V),$$

where $\text{sign}_0 = \mathbb{1}_{(x>0)} - \mathbb{1}_{(x<0)}$.

The first term on the right is simply dY_t by definition. The support of $dL^0(U)$ is a subset of $\{t \geq 0 : V_t = U_t = 0\}$, since on the components of its complement either U is nonzero or U is locally constant. Hence the second term on the right is zero.

After these simplifications, using that $\xi_0 = 0$, we obtain

$$\xi_t = Y_t - L'_t,$$

$$L'_t = \tfrac{1}{2} L^0_t(|U| + V) - \tfrac{1}{2} L^0_t(|U| - V).$$

To finish the proof use that $\xi_{\sigma(t)} = 0$ for all $t \geq 0$; that is, $L_t = L'_{\sigma(t)}$ and that $(\sigma(t), t)$ is disjoint from the support of all the involved local time processes, and hence $L'_t = L'_{\sigma(t)}$. $\quad\square$

Note that the formula, obtained for ξ, is the special case of the general formula for ranked semimartingales proved recently in [1].

PROPOSITION 8. *Let the continuous semimartingales U, V satisfy* (3) *and L, W defined by* (8) *as above. Then $|L_t + W_t| \leq |U_t|$ for all $t \geq 0$.*

PROOF. By definition at $s = \sigma(t)$ we have $|L_s + W_s| = |U_s|$. It is enough to consider the case when $s < t$, since otherwise we are done. On the interval $(s, t]$ either $|V| > |U|$ or $|V| < |U|$. In the first case, W, U and L are constant on $[s, t]$, and we get the statement with equality. In the second case, Y is constant on $[s, t]$; hence $L_t = Y_t$, and the statement follows, since then $|L_t + W_t| = |Y_t + W_t| = |V_t| < |U_t|$. □

LEMMA 9. *Let U and W be strongly orthogonal continuous local martingales starting from zero. Assume that W dominates U, and for the continuous process L, we have $|L_t + W_t| \leq |U_t|$ for $t \geq 0$.*

Then, the total variation process $(\mathcal{V}_t)_{t \geq 0}$ of L satisfies

$$(12) \qquad \mathcal{V}_t = \begin{cases} 0, & \text{if } U_s = 0 \text{ for } s \leq t, \\ \infty, & \text{otherwise.} \end{cases}$$

That is, immediately after U leaves the origin, $\mathcal{V}$ becomes infinite.

REMARK. By enlarging the probability space if necessary, we may assume that both U and W are divergent martingales. Indeed, enlarge a probability space with a two-dimensional Brownian motion $B = (B^{(1)}, B^{(2)})$, independent of $\mathcal{F}_\infty$. Fix a $T > 0$, and define $\bar{U}$, $\bar{W}$ and a new filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ with the formulas

$$\bar{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{F}_t^B,$$
$$\bar{U}_t = U_{t \wedge T} + B_t^{(1)} - B_{t \wedge T}^{(1)},$$
$$\bar{W}_t = W_{t \wedge T} + B_t^{(2)} - B_{t \wedge T}^{(2)}.$$

Now we can define $\bar{L}$ to satisfy the assumption of Lemma 9 in many ways. One possibility is to define τ be the first time after T when $|L + \bar{W}|$ meets $|\bar{U}|$. Up to τ the process $\bar{L}$ is the same as the stopped process $L_{t \wedge T}$. After τ, the process $\bar{L}$ follows the changes of either $-\bar{W} - |\bar{U}|$ or $-\bar{W} + |\bar{U}|$ according to which hits before the level L_T. Formally one could define $\bar{L}$ as

$$\xi_t^\pm = -\bar{W}_t \pm |\bar{U}_t|,$$
$$\tau^\pm = \inf\{t \geq T : L_T = \xi_t^\pm\},$$
$$\bar{L}_t = L_{t \wedge T} + \mathbb{1}_{(\tau^+ < \tau^-)}(\xi_t^+ - \xi_{t \wedge \tau^+}^+) + \mathbb{1}_{(\tau^+ \geq \tau^-)}(\xi_t^- - \xi_{t \wedge \tau^-}^-).$$

Using the independence of B and $\mathcal{F}_\infty$, it follows that $\bar{U}$ and $\bar{W}$ are orthogonal continuous local martingales in $(\bar{\mathcal{F}}_t)_{t \geq 0}$. By construction $\bar{U}$, $\bar{W}$ are divergent, $\bar{W}$ dominates $\bar{U}$, the process $\bar{L}$ has continuous sample paths and $|\bar{L}_t + \bar{W}_t| \leq |\bar{U}_t|$ almost surely for all t.

Now, if the statement of Lemma 9 holds for the triple $(\bar{L}, \bar{U}, \bar{W})$, then it also holds for (L, U, W), provided that t in (12) is smaller than T. Since $T > 0$ was arbitrary, the Lemma follows from the special case when U and W are divergent.

NOTATION. To shorten formulas, we use $\Delta_I X$ for the change of the process X on the interval I.

PROOF OF LEMMA 9. According to the previous remark, we may and do assume that both U and W are divergent. For $\varepsilon > 0$, let $\tau(\varepsilon) = \tau(U, \varepsilon) = \inf\{t > 0 : L_t^0(U) > \varepsilon\}$. Since U is a divergent local martingale, $\tau(\varepsilon)$ is finite almost surely. Clearly it is enough to show that $\mathcal{V}_{\tau(\varepsilon)} = \infty$ for any fixed $\varepsilon > 0$. In the first part of the proof, we fix a "typical" $\omega \in \Omega$, but in the notation it is suppressed.

Let $\mathfrak{z} = \{t : U_t = 0\}$ denote the null set of U. Also, let $\mathcal{C}$ denote the collection of connected components of $\{t : U_t \neq 0\}$ and $\mathcal{C}(\varepsilon) = \{I \in \mathcal{C} : I \subset [0, \tau(\varepsilon)]\}$. Since U is divergent, for a typical ω, that is, with probability one, $\mathcal{C}$ and $\mathcal{C}(\varepsilon)$ has infinitely many elements.

Next, since W dominates U, there is a $c > 0$ such that $d\langle U \rangle_t \leq cd\langle W \rangle_t$, that is, the increase of $\langle W \rangle$ on any interval I is at least $\Delta_I \langle U \rangle / c$. Hence,

$$\gamma_{(a,b)} = \inf\left\{t \geq a : \langle W \rangle_t - \langle W \rangle_a = \frac{\langle U \rangle_b - \langle U \rangle_a}{2c}\right\}$$

defines a time-point in (a, b).

If $I = (a, b) \in \mathcal{C}$, then $U_a = 0$ and $L_a = -W_a$ by assumption. Also by our assumption, $|L_s + W_s| \leq |U_s|$ for $s = \gamma_{(a,b)}$, hence by the triangle inequality,

$$\Delta_I \mathcal{V} \geq |L_s - L_a| \geq |W_s - W_a| - |U_s| \geq |W_s - W_a| - \sup_{u \in (a,b)} |U_u|.$$

This gives

$$\mathcal{V}_{\tau(\varepsilon)} \geq \sum_{I \in \mathcal{C}(\varepsilon)} (\Delta_I \langle U \rangle)^{1/2}(|\xi_I| - \eta_I)^+,$$

where $(x)^+ = 0 \vee x$ is the positive part of x and for $I = [a, b]$

$$\xi_I = \frac{1}{(\Delta_I \langle U \rangle)^{1/2}}(W_{\gamma_I} - W_a), \qquad \eta_I = \frac{1}{(\Delta_I \langle U \rangle)^{1/2}} \sup_{s \in I} |U_s|.$$

We claim the following:

PROPOSITION 10. *There is a measurable enumeration of the random collection of intervals $\mathcal{C}(\varepsilon) = \{I_n : n \geq 1\}$ such that (ξ_{I_n}, η_{I_n}), $n \geq 1$ is an i.i.d. sequence independent of $\mathcal{A} = \sigma(\{\Delta_{I_n} \langle U \rangle : n \geq 1\})$. Moreover, $\mathbf{E}((|\xi_{I_n}| - \eta_{I_n})^+)$ is positive and finite.*

PROPOSITION 11.

$$\sum_{I \in \mathcal{C}(\varepsilon)} (\Delta_I \langle U \rangle)^{1/2} = \infty \qquad almost \ surely.$$

The end of the proof is then rather straightforward. For independent nonnegative random variables $X_1, X_2, \ldots,$ the sum $\sum_n X_n$ is finite if and only if $\sum_n \mathbf{E}(X_n \wedge 1) < \infty$; see Proposition 3.14 of [4]. When $\mathbf{E}(X_n) < \infty$ for all n and $X_n / \mathbf{E}(X_n)$ is an i.i.d. sequence the truncation can obviously be dropped. Thus, conditioning first on $\mathcal{A}$, we can apply this result to $X_n = (\Delta_{I_n} \langle U \rangle)^{1/2} (|\xi_{I_n}| - \eta_{I_n})^+$, by Proposition 10. Since by Proposition 11, $\sum_n \mathbf{E}(X_n | \mathcal{A}) = \infty$ almost surely, the lower bound for $\mathcal{V}_{\tau(\varepsilon)}$ is infinite almost surely. $\square$

Proposition 10 is probably the most delicate part of the proof. It is based on a slight extension to Knight's theorem, Lemma 12. For a divergent continuous local martingale M starting at zero, we say that β is the DDS Brownian motion of M if $\beta_t = M_{\rho(t)}$ where

$$\rho(t) = \inf\{s > 0 : \langle M \rangle_s > t\}.$$

Then β is a Brownian motion; see Chapter V in [9].

To prove Proposition 10 we use the next statement whose proof is deferred to the end of the section.

LEMMA 12. *Let M, N be divergent, continuous local martingales in the filtration $(\mathcal{F})_{t \geq 0}$. Assume that M and N are strongly orthogonal. Denote β the DDS Brownian–motion of N. Then M is a local martingale in the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$, where $\bar{\mathcal{F}}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\beta))$.*

PROOF OF PROPOSITION 10.   Denote by β the DDS Brownian motion of U, let $\bar{\mathcal{F}}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\beta))$. Then by Lemma 12 the process W is a local martingale in the larger filtration $\bar{\mathcal{F}}$ as well.

Let $\mathfrak{z}(\beta)$ be the null set of β, and denote by $\mathcal{C}(\beta)$ the connected components of the complement of $\mathfrak{z}(\beta)$ and $\mathcal{C}(\beta, \varepsilon) = \{I \in \mathcal{C} : I \subset [0, \tau(\beta, \varepsilon)]\}$ where $\tau(\beta, \varepsilon) = \inf\{t > 0 : L_t^0(\beta) > \varepsilon\}$. Besides, let $\sigma(\mathfrak{z}(\beta)) = \sigma\{C_{s,t} : 0 \leq s \leq t\}$ the smallest σ-algebra containing the events $C_{s,t} = \{[s, t] \cap \mathfrak{z}(\beta) = \varnothing\}$.

Then we define the enumeration of $\mathcal{C}(\varepsilon)$ based on the usual $\sigma(\beta)$ measurable enumeration $\{J_n : n \geq 1\}$ of $\mathcal{C}(\beta, \varepsilon)$. Indeed, $J_n = (a_n, b_n)$ with some $\sigma(\beta)$ measurable random time a_n, b_n; then let $I_n = (\rho(a_n), \rho(b_n))$, where $\rho(t) = \inf\{s > 0 : \langle U \rangle_s > t\}$. The point here is that the random times $\rho(a_n), \rho(b_n), \gamma_I$ are stopping times in the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$.

This implies that for any finite collection $F \subset \mathbb{N}$ the random variables $\{\xi_{I_n} : n \in F\}$ are independent also from each other and of $\bar{\mathcal{F}}_0$. To see this, we can define the simple $\bar{\mathcal{F}}$-predictable process

$$H_t = \sum_{n \in F} \frac{\alpha_n}{(\Delta_{I_n} \langle U \rangle)^{1/2}} \mathbb{1}_{(\rho(a_n) < t \leq \gamma_{I_n})}$$

with $\alpha_n \in \mathbb{R}$. Then $H \cdot W$ has uniformly bounded quadratic variation $\langle H \cdot W \rangle_\infty = \sum_{n \in F} \alpha_n^2 / 2c$, which is deterministic. Using that $\exp\{i H \cdot W + \frac{1}{2} \langle H \cdot W \rangle\}$ is a

bounded martingale, we get $\mathbf{E}(\exp\{i(H \cdot W)_\infty + \frac{1}{2}\langle H \cdot W\rangle_\infty\}|\bar{\mathcal{F}}_0) = 1$. This yields the joint conditional characteristic function of $\{\xi_{I_n} : n \in F\}$, given $\bar{\mathcal{F}}_0$

$$\mathbf{E}\left(\exp\left\{i\sum_{n \in F}\alpha_n\xi_{I_n}\right\}\Big|\bar{\mathcal{F}}_0\right) = \exp\left\{-\frac{1}{2}\sum_{n \in F}\frac{\alpha_n^2}{2c}\right\}.$$

That is, $\{\xi_{I_n} : n \geq 1\}$ is an i.i.d. sequence which is independent from $\bar{\mathcal{F}}_0$, and the common law is normal with expectation 0 and variance $1/2c$.

η_{I_n} is calculated from the normalized excursions of β on J_n; hence they form an i.i.d. sequence measurable with respect to $\sigma(\beta) \subset \bar{\mathcal{F}}_0$ and independent of $\sigma(\mathfrak{z}(\beta))$; see [3], Section 2.9.

Finally, $\Delta_{I_n}\langle U\rangle$ is the length of J_n, and hence it is $\sigma(\mathfrak{z}(\beta))$ measurable.

Putting these pieces together, we obtain that (ξ_{I_n}, η_{I_n}) is an i.i.d. sequence independent of $\sigma(\mathfrak{z}(\beta)) \supset \mathcal{A}$. The claim that $\mathbf{E}((|\xi_{I_n}| - \eta_{I_n})^+) > 0$ and finite is obvious from the joint law of (ξ_{I_n}, η_{I_n}). □

PROOF OF PROPOSITION 11.   With the notation introduced in the proof of Proposition 10, we can reformulate the statement. Using β the DDS Brownian motion of U, we have to show that for $\varepsilon > 0$,

$$\sum_{J \in \mathcal{C}(\beta,\varepsilon)} |J|^{1/2} = \infty \qquad \text{almost surely,}$$

which follows from a characterization of the local time.

Indeed, let n_k be the number of intervals in $\mathcal{C}(\beta, \varepsilon)$ longer than 2^{-k}. Then the limit $\lim_{k\to\infty} 2^{-k/2}n_k$ almost surely exists and is positive; it is $\sqrt{2/\pi}L^0_{\tau(\beta,\varepsilon)}(\beta)$; see, for example, [9], Proposition (2.9), Chapter XII. From this, the statement follows using elementary analysis, since

$$\sum_{J \in \mathcal{C}(\beta,\varepsilon)} |J|^{1/2} \geq \sum_{k=1}^{\infty} 2^{-k/2}(n_k - n_{k-1})$$

$$= -\frac{n_0}{\sqrt{2}} + (1 - 2^{-1/2})\sum_{k=1}^{\infty} 2^{-k/2}n_k = \infty. \qquad \square$$

REMARK 13.   With obvious modification the previous calculation also gives that for $\alpha > 1/2$, we have $\sum_{J \in \mathcal{C}(\beta,\varepsilon)} |J|^\alpha < \infty$ almost surely.

PROOF OF LEMMA 12. M is a divergent continuous local martingale, denote B its DDS Brownian motion. That is $B_t = M_{\rho(t)}$ with the time-change ρ associated with the quadratic variation of M. Then B is a Brownian motion in the time-changed filtration $(\mathcal{G}_t = \mathcal{F}_{\rho(t)})_{t \geq 0}$, and $(\langle M\rangle_t)_{t \geq 0}$ is continuous time-change in the filtration $(\mathcal{G}_t)_{t \geq 0}$.

We actually show that B is a martingale in the filtration $(\bar{\mathcal{G}}_t)_{t \geq 0}$, where $\bar{\mathcal{G}}_t = \mathcal{G}_t \vee \sigma(\beta)$; that is, for $0 \leq t \leq s$, we have $\mathbf{E}(B_s - B_t | \bar{\mathcal{G}}_t) = 0$.

To see this, fix $t \geq 0$, and observe first that the time-shifted processes $(M_{\rho(t)+u} - M_{\rho(t)})_{u \geq 0}$ and $(N_{\rho(t)+u} - N_{\rho(t)})_{u \geq 0}$ are divergent, continuous local martingales in the time-shifted filtration $(\mathcal{F}_{\rho(t)+u})_{u \geq 0}$. Their DDS Brownian motions are given by $(B_{t+s} - B_t)_{s \geq 0}$ and $(\beta_{\eta(t)+s} - \beta_{\eta(t)})_{s \geq 0}$, respectively, where $\eta(t) = \langle N \rangle_{\rho(t)}$.

By Knight's theorem (see Theorem 1.9 of Chapter 5 in [9]), the processes $(B_{t+s} - B_t)_{s \geq 0}$ and $(\beta_{\eta(t)+s} - \beta_{\eta(t)})_{s \geq 0}$ constitute a two-dimensional Brownian motion in its own filtration and, with a little extension of the original statement, independent of $\mathcal{G}_t = \mathcal{F}_{\rho(t)}$. The independence follows from considering the conditional law given $\mathcal{G}_t$.

Next, note that

$$\bar{\mathcal{G}}_t = \mathcal{G}_t \vee \sigma(\beta) = \mathcal{G}_t \vee \sigma\big(\{\beta_{\eta(t)+s} - \beta_{\eta(t)} : s \geq 0\}\big),$$

since $\eta(t) = \langle N \rangle_{\rho(t)}$ is $\mathcal{F}_{\rho(t)} = \mathcal{G}_t$ measurable.

Then, the three σ-algebras: $\mathcal{A}_1 = \sigma(\{\beta_{\eta(t)+s} - \beta_{\eta(t)} : s \geq 0\})$, $\mathcal{A}_2 = \sigma(\{B_{t+s} - B_t : s \geq 0\})$ and $\mathcal{G}_t$ are independent. For $s \geq 0$ this gives that $B_{t+s} - B_t$ is independent from $\mathcal{A}_1 \vee \mathcal{G}_t = \bar{\mathcal{G}}_t$, and $\mathbf{E}(B_{t+s} - B_t | \bar{\mathcal{G}}_t) = \mathbf{E}(B_{t+s} - B_t) = 0$, showing that B is not only a $\mathcal{G}$ Brownian motion, but also a $\bar{\mathcal{G}}$ Brownian motion.

Since M is obtained from B with a continuous $\bar{\mathcal{G}}$-time-change $(\langle M \rangle_t)_{t \geq 0}$, it is a local martingale in the filtration $(\bar{\mathcal{G}}_{\langle M \rangle_t})_{t \geq 0}$ and also in its right continuous hull. Now

$$\bar{\mathcal{G}}_{\langle M \rangle_t} \supset \mathcal{G}_{\langle M \rangle_t} \vee \sigma(\beta) \supset \mathcal{F}_t \vee \sigma(\beta)$$

finishes the proof. $\quad\square$

3. Examples, showing that domination is necessary. The aim of this section is to show that we cannot drop the domination condition in Theorems 2 and 5 completely. It is enough to give an example showing that without domination, Theorem 5 does not hold, since by Remark 6, it also provides an example for Theorem 2.

First we describe L in terms of U, W in a way which is invariant under time-change. This characterization is similar in spirit to the reflection lemma of Skorohod.

LEMMA 14. *Let $f, g, h : [0, \infty) \to \mathbb{R}$ be continuous functions satisfying the following properties*:

(i) $f(0) = h(0) = g(0)$;
(ii) $f \leq h \leq g$;
(iii) *h is locally nondecreasing on $\{g \neq h\}$ and locally nonincreasing on $\{h \neq f\}$. That is, for $s \leq t$ if $g \neq h$ on (s, t), then $h(s) \leq h(t)$, and if $h \neq f$ on (s, t), then $h(s) \geq h(t)$.*

Then

$$h(t) = F(d(t), t) = G(d(t), t),$$

where

$$F(s, t) = \max\{f(x) : x \in [s, t]\},$$
$$G(s, t) = \min\{g(x) : x \in [s, t]\},$$
$$d(t) = \sup\{s \le t : F(s, t) \ge G(s, t)\}.$$

In plain words, to calculate $h(t)$ go backward starting at t on the graph of f and g until there is common value in the range swept by these functions. The first such value is $h(t)$. We remark that with obvious modifications, Lemma 14 extends the explicit formula obtained in [5] for the two-sided reflection map on $D[0, \infty)$.

PROOF OF LEMMA 14.   Define t^g and t^f the last time before t, when $g = h$ or $f = h$, respectively; that is, $t^g = \max\{s \in [0, t] : g(s) = h(s)\}$ and $t^f = \max\{s \in [0, t] : f(s) = h(s)\}$. We can assume that $t^f \le t^g$; the other case is obtained by considering $-g \le -h \le -f$.

By our assumption (iii) the function h is nonincreasing on (t^f, t), and nondecreasing on (t^g, t). Since $t^f \le t^g \le t$, we have that $h(s) = h(t) = h(t^g) = g(t^g)$ for all $s \in [t^g, t]$ and also that $h(t^f) = f(t^f) \ge h(s) \ge h(t)$ for $s \in [t^f, t]$. Thus

$$h(t) = \min_{s \in [t^f, t]} h(s) \le G(t^f, t) \le g(t^g) = h(t),$$

that is, $h(t) = G(t^f, t) \le F(t^f, t)$. By definition, $d(t) \ge t^f$. On the other hand, $d(t) \le t^g$ follows from the fact that if $s \in (t^g, t)$, then $f(s) < h(t) < g(s)$.

Since $F(d(t), t) = G(d(t), t)$ by definition, we obtain that $h(t) = G(t^f, t) \le G(d(t), t) \le G(t^g, t) = h(t)$ and $h(t) = F(d(t), t) = G(d(t), t)$.   □

COROLLARY 15.   *Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous functions and assume that $f(0) = g(0)$ and $f \le g$. Then there is a unique continuous function denoted by $\bar{L}(f, g)$ such that (i), (ii) and (iii) holds for $f \le h = \bar{L}(f, g) \le g$.*

REMARK.   The function $(t, f, g) \mapsto \bar{L}_t(f, g)$ is clearly predictable; see [9], Chapter IX, for definition.

COROLLARY 16.   *Assume that U, V satisfies (3) and L, W are defined as above. Then $L_t = \bar{L}_t(-W - |U|, -W + |U|)$.*

PROOF.   By Proposition 8 $W - |U| \le -L \le W + |U|$ and by Lemma 7, L is continuous, nonincreasing on $|U| + L + W \ne 0$ and nondecreasing on $|U| - L - W \ne 0$.   □

Finally, we have the following result which will be proved below in Section 3.1.

LEMMA 17. *There is a two-dimensional local martingale $(\bar{U}, \bar{W})$ on some filtered probability space such that*:

(i) $\bar{U}$ *and* $\bar{W}$ *are strongly orthogonal*;

(ii) $d\langle\bar{U}\rangle \approx d\langle\bar{W}\rangle$ *almost surely, that is, the random measures induced by the changes of* $\langle\bar{U}\rangle$ *and* $\langle\bar{W}\rangle$ *are equivalent*;

(iii) $\bar{L} = \bar{L}(-\bar{W} - |\bar{U}|, -\bar{W} + |\bar{U}|)$ *has locally bounded variation*;

(iv) $\bar{U}$ *and* $\bar{L}$ *are divergent*.

Let $(\bar{U}, \bar{W})$ from Lemma 17 and $\bar{L}_t = \bar{L}_t(-\bar{W} - |\bar{U}|, -\bar{W} + |\bar{U}|)$. Then $|\bar{L}_t + \bar{W}_t| \le |\bar{U}_t|$ for $t \ge 0$. We can assume that $\bar{W}$ is a Brownian motion by applying an appropriate time-change; the proof is actually formulated in this way. So assume for the moment that $\langle\bar{W}\rangle_t = t$. Observe that by Lemma 9 the Brownian motion $\bar{W}$ cannot dominate $\bar{U}$ on any intervals of the form $[0, t]$, with $t > 0$. Since $\langle\bar{U}\rangle$ is equivalent with $\langle\bar{W}\rangle$, that is, with the Lebesgue measure, we can write it a- $\langle\bar{U}\rangle_t = \int_0^t Q_s \, ds$. The nondomination property means that $\operatorname{esssup}_{s \in [0,t]} Q_s = \infty$ almost surely for all $t > 0$.

PROPOSITION 18. $(\bar{U}, \bar{L} + \bar{W})$ *fulfills* (3), *that is*,

$$d\bar{U}_t = \mathbb{1}_{(|\bar{L}_t + \bar{W}_t| < |\bar{U}_t|)} \, d\bar{U}_t, \qquad \bar{U}_0 = \bar{W}_0 = \bar{L}_0 = 0.$$

Moreover decomposition (8) *gives back* $\bar{L}$ *and* $\bar{W}$, *that is*,

$$\bar{L}_t = \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| \ge |\bar{U}_s|)} \, d(\bar{L}_s + \bar{W}_s), \qquad \bar{W}_t = \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| < |\bar{U}_s|)} \, d(\bar{L}_s + \bar{W}_s),$$

and $\bar{L}_{\bar{\sigma}(t)} = \bar{L}_t$, *where* $\bar{\sigma}(t) = \sup\{s \le t : |\bar{U}_s| = |\bar{L}_s + \bar{W}_s|\}$.

PROOF. Since $|\bar{L} + \bar{W}| \le |\bar{U}|$, to show that $\bar{U}, \bar{L} + \bar{W}$ satisfies (3), we only need that

$$(13) \qquad\qquad \xi_t = \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| = |\bar{U}_s|)} \, d\bar{U}_s = 0.$$

This follows similarly as (6) above as $\mathbf{E}(\xi_t^2) \le \mathbf{E}(\langle\xi\rangle_t)$ and the latter can be estimated using the orthogonality of $\bar{U}$ and $\bar{W}$ by

$$(14) \qquad \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| = |\bar{U}_s|)} \, d\langle\bar{U}\rangle \le \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| = |\bar{U}_s|)} \, d\langle|\bar{L} + \bar{W}| - |\bar{U}|\rangle_s = 0$$

by the occupation time formula. The same applies if we integrate with respect to $\bar{W}$ in (13). Thus

$$\int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| \ge |\bar{U}_s|)} \, d(\bar{L}_s + \bar{W}_s) = \int_0^t \mathbb{1}_{(|\bar{L}_s + \bar{W}_s| = |\bar{U}_s|)} \, d\bar{L}_s = \bar{L}_t.$$

In the last step we used that $\bar{L}$ is locally constant on $\bar{U} \ne \bar{L} + \bar{W}$; cf. Property (iii) of Lemma 14. This proves the first part of the decomposition formula. The second part, that is, the formula for $\bar{W}$, obviously follows.

Finally, $(\bar\sigma(t), t) \subset \{s : |\bar U_s| \neq |\bar L_s + \bar W_s|\}$, hence $\bar L$ is constant on $[\bar\sigma(t), t]$ and $\bar L_{\sigma(t)} = \bar L_t$. $\quad\square$

Application of Lemma 7 proves the next representation of $\bar L$.

COROLLARY 19.

$$(15) \qquad 2\bar L_t = L_t^0(|\bar U| + (\bar L + \bar W)) - L_t^0(|\bar U| - (\bar L + \bar W)).$$

COROLLARY 20.

$$(16) \qquad \int \mathbb{1}_{(\bar U_t = 0)} \big(dL_t^0(|\bar U| + (\bar L + \bar W)) - dL_t^0(|\bar U| - (\bar L + \bar W))\big) = 0.$$

PROOF. We use that for a nonnegative continuous semimartingale X, we have

$$\frac{1}{2} L_t^0(X) = \int_0^t \mathbb{1}_{(X_s=0)} \, dX_s.$$

We apply it for $X = |\bar U| + (\bar L + \bar W)$. Using that $\mathbb{1}_{(\bar U = 0)} \mathbb{1}_{(X=0)} = \mathbb{1}_{(\bar U = 0)}$, we obtain that

$$\int_0^t \mathbb{1}_{(\bar U_s = 0)} \, dL_s^0(|\bar U| + (\bar L + \bar W))$$
$$= 2 \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, d\big(|\bar U|_s + (\bar L_s + \bar W_s)\big)$$
$$= L_t^0(|\bar U|) + 2 \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, d\bar L_s + 2 \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, dW_s.$$

Here the last term is zero. This can be seen by using isometry and the fact that $d\langle U \rangle \approx d\langle W \rangle$. For the second term use Corollary 19,

$$2 \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, d\bar L_s = \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, d\big(L_s^0(|\bar U| + (\bar L + \bar W)) - L_s^0(|\bar U| - (\bar L + \bar W))\big).$$

Rearranging gives that

$$(17) \qquad L_t^0(|\bar U|) = \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, dL_s^0(|\bar U| - (\bar L + \bar W)).$$

Making the same calculation for

$$\int_0^t \mathbb{1}_{(\bar U_s = 0)} \, dL_s^0(|\bar U| - (\bar L + \bar W)),$$

we obtain

$$(18) \qquad L_t^0(|\bar U|) = \int_0^t \mathbb{1}_{(\bar U_s = 0)} \, dL_s^0(|\bar U| + (\bar L + \bar W)).$$

(17) and (18) together prove the statement. $\quad\square$

Now, our example is obtained by interlacing the two-dimensional local martingale $(\bar{U}, \bar{W})$ from Lemma 17 with an independent Brownian motion $\bar{B}$. The linkage between the two processes is

$$\bar{\mathcal{V}}_t = \tfrac{1}{2}\left(L_t^0(|\bar{U}| + \bar{L} + \bar{W}) + L_t^0(|\bar{U}| - (\bar{L} + \bar{W}))\right)$$

on the one side, and

$$\bar{S}_t = \max_{s \le t} \bar{B}_s$$

on the other side. That is, the processes are time changed so that after the time change, $\bar{\mathcal{V}}$ and $\bar{S}$ coincide. To describe this, put

$$\alpha(t) = \inf\{u > 0 : \bar{\mathcal{V}}_u > \bar{S}_{t-u}\}, \qquad (\mathcal{V}, L, U, W)_t = (\bar{\mathcal{V}}, \bar{L}, \bar{U}, \bar{W})_{\alpha(t)},$$

$$\beta(t) = \inf\{u > 0 : \bar{S}_u > \bar{\mathcal{V}}_{t-u}\}, \qquad (B, S)_t = (\bar{B}, \bar{S})_{\beta(t)}.$$

PROPOSITION 21. *The following properties hold almost surely*:

(i) α, β *are nondecreasing, continuous and* $\alpha(t) + \beta(t) = t$ *for all* $t \ge 0$;
(ii) $\lim_{t \to \infty} \alpha(t) = \lim_{t \to \infty} \beta(t) = \infty$;
(iii) $S = \mathcal{V}$;
(iv) *for all* $t \ge 0$, *if* $B_t \ne S_t$ *then* $|L_t + W_t| = |U_t|$.

PROOF. The key property of $\bar{S}$ and $\bar{\mathcal{V}}$ is that they do not have a nondegenerate plateau (interval of constancy) at the same level. The sample path of $\bar{\mathcal{V}}$ is nondecreasing, and therefore $p(\bar{\mathcal{V}})$ the set of levels, at which $\bar{\mathcal{V}}$ spends positive amount of time, is at most countable. The same holds for $\bar{S}$. By the independence of the two processes, $p(\bar{\mathcal{V}})$ and $p(\bar{S})$ are disjoint almost surely.

By the continuity of $\bar{\mathcal{V}}$ and $\bar{S}$, we have

(19) $$\bar{\mathcal{V}}_{\alpha(t)} = \bar{S}_{t-\alpha(t)} \quad \text{and} \quad \bar{S}_{\beta(t)} = \bar{\mathcal{V}}_{t-\beta(t)}.$$

It follows that $\alpha(t) = t - \beta(t)$ almost surely for all t. To see this we can assume on the contrary that $\alpha(t) < t - \beta(t)$. Then

$$\bar{\mathcal{V}}_{\alpha(t)} = \bar{S}_{t-\alpha(t)} \ge \bar{S}_{\beta(t)} = \bar{\mathcal{V}}_{t-\beta(t)} \ge \bar{\mathcal{V}}_{\alpha(t)},$$

showing that $\bar{\mathcal{V}}$ and $\bar{S}$ have a nondegenerate plateau at the same level, which can happen only on a negligible exceptional event. Hence $\alpha(t) + \beta(t) = t$ for all $t \ge 0$ almost surely.

Since clearly, α, β are nondecreasing, the fact that $\alpha(t) + \beta(t) = t$ implies that they are continuous, even contractions, that is, $|\alpha(t) - \alpha(s)| \le |t - s|$ and similarly for β. This proves Property (i).

Property (ii) follows from the unboundedness of $\bar{\mathcal{V}}$ and $\bar{S}$, cf. (iv) of Lemma 17.

Property (iii) is an easy corollary of (19) and $\alpha(t) + \beta(t) = t$.

For Property (iv) note that if $B_t \ne S_t$ then $\bar{B}_{\beta(t)} \ne \bar{S}_{\beta(t)}$ and $\bar{S}$ has a nondegenerate plateau at the level S_t. But, then $\bar{\mathcal{V}}$ spends zero time at this level, that is, $\alpha(t)$

is a point of increase of $\bar{\mathcal{V}}$. Using (15) this implies that $|\bar{L} + \bar{W}| = |\bar{U}|$ holds at $\alpha(t)$, that is, $|L_t + W_t| = |U_t|$. $\square$

We obtained that $(\alpha(t))_{t \geq 0}$ and $(\beta(t))_{t \geq 0}$ are continuous time changes with respect to the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ and $(\bar{\mathcal{G}}_t)_{t \geq 0}$, respectively, where $\bar{\mathcal{F}}_t = \mathcal{F}_t^{\bar{U}, \bar{W}} \vee \sigma(\bar{B})$ and $\bar{\mathcal{G}}_t = \mathcal{F}_t^{\bar{B}} \vee \sigma(\bar{U}, \bar{W})$. Then $(U, W)_t = (\bar{U}, \bar{W})_{\alpha(t)}$ is a continuous local martingale in the time changed filtration $\bar{\mathcal{F}}_{\alpha(t)}$, and since it is clearly adapted to $\mathcal{F}_t = \bar{\mathcal{F}}_{\alpha(t)} \cap \bar{\mathcal{G}}_{\beta(t)}$ we get that (U, W) is a continuous local martingale in $(\mathcal{F}_t)_{t \geq 0}$. By similar reasoning, $B_t = \bar{B}_{\beta(t)}$ is also a continuous local martingale in $(\mathcal{F}_t)_{t \geq 0}$.

By the definition of $\bar{\mathcal{V}}$ and Corollaries 19 and 20, we have that

$$\bar{L}_t = \int_0^t -\operatorname{sign}_0(\bar{L}_s + \bar{W}_s) \, d\bar{\mathcal{V}}_s,$$

where $\operatorname{sign}_0(x) = \mathbb{1}_{(x>0)} - \mathbb{1}_{(x<0)}$. Then the same identity holds for the time changed processes, that is,

$$(20) \qquad L_t = \int_0^t -\operatorname{sign}_0(L_s + W_s) \, d\mathcal{V}_s.$$

The final step is to define

$$(21) \qquad Y_t = \int_0^t -\operatorname{sign}_0(L_s + W_s) \, dB_s \quad \text{and} \quad V = Y + W.$$

It is easy to check that U and V are strongly orthogonal, and U is divergent. Property (ii) of Lemma 17 is inherited by U, V, that is, $\langle U \rangle_t = \int_0^t Q_s \, d\langle V \rangle_s$ with some Q. To show that the pair U, V satisfies (3) we apply the balayage formula: for a predictable bounded process ξ and a continuous semimartingale Z, we have

$$\xi_{\gamma(t)} Z_t = \int_0^t \xi_{\gamma(u)} \, dZ_u,$$

where $\gamma(t) = \sup\{s \leq t : Z_s = 0\}$; see [6], Lemma 0.2, or [9], Chapter VI. We apply this for $Z = \mathcal{V} - B$ and $\xi_t = -\operatorname{sign}_0(L_t + W_t)$, that is, $\gamma(t) = \sup\{s \leq t : \mathcal{V}_s = B_s\}$. Observe that on the interval $[\gamma(t), t]$ the time change α is constant, hence $\xi_t = \xi_{\gamma(t)}$ for all $t \geq 0$. Then

$$L_t - Y_t = \int_0^t \xi_s \, d(\mathcal{V}_s - B_s) = \xi_t(\mathcal{V}_t - B_t) = \xi_t(S_t - B_t).$$

This formula shows that $L_t \neq Y_t$ implies that $S_t \neq B_t$ and hence $|L_t + W_t| = |U_t|$ by Property (iv) of Proposition 21. That is, if $|V_t| \geq |U_t|$ for some t then either $Y_t \neq L_t$ and then $|L_t + W_t| = |U_t|$, or $Y_t = L_t$ and we get that $|L_t + W_t| \geq |U_t|$. Since $|L + W| \leq |U|$ by the definition of L, we obtain in both cases that $|L_t + W_t| = |U_t|$. In formula,

$$\mathbb{1}_{(|V_t| \geq |U_t|)} \leq \mathbb{1}_{(|L_t + W_t| = |U_t|)} \quad \text{and} \quad \mathbb{1}_{(|V_t| < |U_t|)} \geq \mathbb{1}_{(|L_t + W_t| < |U_t|)}.$$

Finally, we can write the time-changed version of Proposition 18 [the time-change $(\alpha(t))_{t \geq 0}$ is continuous]

$$dU_t = \mathbb{1}_{(|L_t + W_t| < |U_t|)} \, dU_t = \mathbb{1}_{(|V_t| < |U_t|)} \, dU_t;$$

that is, (3) holds.

We can summarize this section in the next theorem.

THEOREM 22. *There is a pair (U, V) of strongly orthogonal continuous local martingales such that* (3) *holds, U, V are divergent, and $d\langle U \rangle$ is absolutely continuous with respect to $d\langle V \rangle$.*

For our final statement in this subsection, recall that by (7) when we reformulate the example in terms of M and N, we have

$$\langle M \rangle = \langle V \rangle = \langle W \rangle + \langle Y \rangle \quad \text{and} \quad \langle N \rangle = \langle U \rangle + \langle Y \rangle.$$

Now, since our construction yields an example in which $d\langle U \rangle$ and $d\langle W \rangle$ are equivalent and $\langle U \rangle$, $\langle V \rangle$ are divergent, the same properties hold for $\langle M \rangle$ and $\langle N \rangle$. Then, by time change we can transform (M, N) such that M becomes a Brownian motion and N a continuous local martingale in the time-changed filtration.

THEOREM 23. *There is a pair B, N of continuous strongly orthogonal local martingales such that B is a Brownian motion, $\langle N \rangle_t = \int_0^t Q_s \, ds$ with some strictly positive Q such that the solution of*

$$dX_t = \text{sign}(X_t) \, dB_t + dN_t$$

is not pathwise unique.

In other words, if the perturbation of the Tanaka equation is not strong enough, then pathwise uniqueness of the solution does not hold.

The other possibility is that we transform N into a Brownian motion. Then we obtain an example showing that in some cases even a Brownian motion is not strong enough as a perturbation.

THEOREM 24. *There is a pair M, B of continuous strongly orthogonal local martingales such that B is a Brownian motion, $\langle M \rangle_t = \int_0^t Q_s \, ds$ with some strictly positive Q such that the solution of*

$$dX_t = \text{sign}(X_t) \, dM_t + dB_t$$

is not pathwise unique.

3.1. *Proof of Lemma* 17. Lemma 17 states the existence of two-dimensional local martingale (U, W) with essentially the following property holding almost surely: one can draw the graph of a continuous function with locally bounded variation into the plane region

$$\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : -W_t - |U|_t \leq x \leq -W_t + |U_t|\}, \tag{22}$$

since this property together with Proposition 27 below ensures (iii) of Lemma 17.

To achieve this we start with two independent Brownian motions $\bar{U}$ and W. Then we apply a time change onto $\bar{U}$ to obtain $U_t = \bar{U}_{\eta(t)}$. This time change is in the form

$$\eta(t) = \inf\left\{s : \int_0^s |\bar{U}_u|^\kappa \, du > t\right\}, \tag{23}$$

with a suitably chosen $\kappa > 0$. This way of construction guarantees that (i), (ii) and even (iv) of Lemma 17 hold.

As a result of the time-change the Brownian motion $\bar{U}$ is accelerated when it is near the origin. It has three effects:

(1) The Hausdorff dimension of the zero level set $\mathfrak{z}(U)$ will decrease below $1/2$.

(2) Short excursions of $\bar{U}$ after the time change will be even shorter, and therefore the sum, which played a crucial role in the proof of Theorem 5, will be finite, that is,

$$\sum_{I \in \mathcal{C}(U, s)} |\Delta_I W| < \infty \qquad \text{almost surely for all } s \geq 0. \tag{24}$$

(3) To describe the third effect we denote by K the continuous process with $K_t = -W_t$ whenever $U_t = 0$ and linear in between.

Then, the random closed set $\{t \geq 0 : |K_t + W_t| \leq |U_t|\}$ contains in its interior $\mathfrak{z}(U)$, the zero level set of U, almost surely. Moreover, if I is a short excursion interval of U, then $|K + W| \leq |U|$ with high probability. Then by means of the Borel–Cantelli lemma it follows that $|K + W| \leq |U|$ on all, but finitely many excursion intervals ending before t, for any $t > 0$. That is, the number of exceptional excursion intervals is locally finite.

Properties (1) and (2) imply that the process K defined in (3) has locally bounded variation. Then property (3) implies it is possible to draw a graph of locally bounded variation into the plain region (22): one has to modify K on the finitely many exceptional excursion intervals. It is possible since $|U| \geq \varepsilon$ with some $\varepsilon > 0$ on the closed set $\overline{A_T}$, where $A_T = \{t \in [0, T] : |K_t + W_t| > |U_t|\}$.

So we only have to show that with suitable choice of $\kappa > 0$ properties (1), (2) and (3) are fulfilled.

Property (1) is a classical fact (see, e.g., [3], Section 6.7), where it was proved that $\dim \mathfrak{z}(U) = (2 + \kappa)^{-1}$.

The finiteness of (24) is a corollary of

$$\sum_{I \in \mathcal{C}(U,s)} |I|^{1/2} < \infty \qquad \text{for all } s > 0.$$

This latter follows from the rather crude estimation on the length of I. If the corresponding excursion interval of $\bar{U}$ is J, then

$$|I| \le |J| \sup_{s \in J} |\bar{U}_s|^{\kappa} = |J|^{1+\kappa/2} \sup_{s \in J} \left(\frac{|\bar{U}_s|}{|J|^{1/2}} \right)^{\kappa}.$$

Here $\sup_{s \in J} (|\bar{U}_s|/|J|^{1/2})^{\kappa}$ where J run through $\mathcal{C}(\bar{U}, s)$ is an i.i.d. sequence with finite expectation, and hence it is enough to show that

$$\sum_{J \in \mathcal{C}(U,s)} |J|^{1/2+\kappa/4} < \infty.$$

This follows from a trivial modification of Proposition 11, as already mentioned in Remark 13.

It remains to show property (3). In this step the crucial issue is the estimation of the probability

(25) $$\mathbf{P}(\exists t \in I_n, |K_t + W_t| > |U_t|),$$

where $(I_n)_{n \ge 1}$ is the usual $\sigma(U)$ measurable enumeration of the excursions of U.

Let us fix n and drop the index from the notation. By the definition of K the process $K + W$ is a Brownian bridge on the interval I and is independent of U. Let us map $[0, 1]$ onto $I = (a, b)$ linearly by $\varphi(t) = t(b - a) + a$ and scale both $K + W$ and U with $|I|^{-1/2}$. This way we obtain

$$B_t = |I|^{-1/2} \big(K_{\varphi(t)} + W_{\varphi(t)} \big),$$

$$E_t = |I|^{-1/2} |U_{\varphi(t)}|.$$

Then B is a standard Brownian bridge, and E is a distorted Brownian excursion. Now the question is the probability

$$\mathbf{P}(\exists t \in [0, 1], B_t > E_t),$$

since by symmetry the twice of this probability gives an upper bound for (25). We can describe the graph of the distorted excursion $(E_t)_{t \in [0,1]}$ in terms of a standard Brownian excursion $(\bar{E}_t)_{t \in [0,1]}$ and the length $|J|$ of the excursion interval of $\bar{U}$ which is transformed after the time change into I. Indeed, the excursion of $\bar{U}$ is obtained by scaling form $\bar{E}$; that is, its graph can be described as

$$\{(\bar{a} + |J|t, |J|^{1/2} \bar{E}_t) : t \in [0, 1]\},$$

where $\bar{a} = \inf J$. To describe the effect of the time-change on the graph introduce the process

$$r(t) = \int_0^t |\bar{E}_s|^{\kappa} ds, \qquad t \in [0, 1].$$

Then $|I| = |J|^{1+\kappa/2} r(1)$, and we can parametrize the graph of E as

$$\left\{ \left(\frac{r(t)}{r(1)}, |J|^{-\kappa/4} \frac{\bar{E}_t}{r(1)^{1/2}} \right) : t \in [0,1] \right\}.$$

Next we define independent variables

$$\xi = \sup_{t \in (0,1)} \frac{B_t}{(t(1-t))^{1/4}},$$

$$\zeta = \sup_{t \in (0,1)} \frac{(r(t)(r(1) - r(t)))^{1/4}}{\bar{E}_t} = |J|^{-\kappa/4} \sup_{t \in (0,1)} \frac{(t(1-t))^{1/4}}{E_t}.$$

The point here is that if $B_{t_0} > E_{t_0}$ for some $t_0 \in [0,1]$, then $\xi \zeta |J|^{\kappa/4} > 1$. Whence, by the independence of ζ, ξ and $|J|$, we have the next estimate for the conditional probability,

$$(26) \qquad \mathbf{P}(\exists t \in [0,1], B_t > E_t \,|\, |J|) \leq \mathbf{P}(\xi \zeta > x)|_{x=|J|^{-\kappa/4}}.$$

Hence we are interested in the tail of ξ and ζ. Although it would be nice to find some explicit formulas, a rather coarse estimate is sufficient for our purposes. We use that if B is a Brownian bridge, then $W_t = (1+t) B_{t/(1+t)}$ is a Brownian motion, and

$$\mathbf{P}(\xi > x) \leq 2\mathbf{P}\left(\sup_{t \in (1/2,1)} \frac{B_t}{(t(1-t))^{1/4}} > x \right)$$

$$\leq 2\mathbf{P}\left(\exists t \geq 0, W_t > \frac{1}{2} x (1+t)^{3/4} \right).$$

The next lemma shows that the tail of ξ is really thin.

LEMMA 25. *Let W be a Brownian motion. Then for $\beta > 1/2$,*

$$\mathbf{P}\big(\exists t \geq 0, W_t > x(1+t)^\beta\big) \leq \frac{e^{-cx^2}}{1 - e^{-cx^2}},$$

where $c > 0$ depends only on β. For $\beta \in (1/2, 1)$ with $c(\beta) = 2\beta(1 - \beta) \times (1/2)^{1/(2\beta-1)}$ the estimate holds.

PROOF. It is enough to prove for $\beta \in (1/2, 1)$. Take an increasing sequence $(t_n)_{n \geq 0}$ such that $t_0 = 0$ and $\lim_{n \to \infty} t_n = \infty$. Let e_k denote the secant line through t_k, t_{k+1}, that is,

$$e_k(t) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k}(t - t_k) + f(t_k) = a_k t + b_k,$$

where $f(t) = (1+t)^\beta$. Since $e_k(t) \leq f(t)$ for $t \in [t_k, t_{k+1}]$ we have that

$$\mathbf{P}\big(\exists t \geq 0, W_t \geq x f(t)\big) \leq \sum_{k=0}^\infty \mathbf{P}\big(\exists t \geq 0, W_t \geq x e_k(t)\big) = \sum_{k=0}^\infty e^{-2x^2 a_k b_k}.$$

In the last step we have used that for the Brownian motion W, and $x, y > 0$, we have $\mathbf{P}(\exists t \geq 0, W_t \geq x + yt) = e^{-2xy}$; see, for example, (1) on page 251 of [2].

To finish the proof we need to esimate $a_k b_k$ from below, where

$$a_k = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} \geq f'(t_{k+1}) = \beta(1 + t_{k+1})^{\beta-1},$$

$$b_k = f(t_k) - t_k \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} \geq t_k\left(\frac{f(t_k)}{t_k} - f'(t_k)\right)$$

$$= t_k(1 + t_k)^{\beta-1}\left(1 + \frac{1}{t_k} - \beta\right) \geq (1 - \beta)(1 + t_k)^\beta.$$

Hence

$$a_k b_k \geq \beta(1 - \beta)(1 + t_k)^\beta(1 + t_{k+1})^{\beta-1} \geq \beta(1 - \beta)\frac{(1 + t_k)^{2\beta}}{1 + t_{k+1}}.$$

Taking $t_k = (k + 1)^{1/(2\beta-1)} - 1$, we get that $a_k b_k \geq (k + 1)\beta(1 - \beta)(1/2)^{1/(2\beta-1)}$ and

$$\mathbf{P}(\exists t \geq 0, W_t \geq xf(t)) \leq \sum_{k=0}^\infty e^{-2x^2 a_k b_k} \leq \frac{e^{-cx^2}}{1 - e^{-cx^2}}$$

with $c(\beta) = 2\beta(1 - \beta)(1/2)^{1/(2\beta-1)}$. $\square$

COROLLARY 26. *There are $c_1, c_2 > 0$ such that*

$$\mathbf{P}(\xi > x) \leq c_1 e^{-c_2 x^2} \quad and \quad \mathbf{P}(\bar{\xi} > x) \leq c_1 e^{-c_2 x^2},$$

where $\bar{\xi} = \sup_{t\in(0,1)}(t(1 - t))^{-1/4}\bar{E}_t$.

The estimation for the standard Brownian excursion $\bar{E}$ follows from the description of $\bar{E}$ as a three-dimensional Bessel bridge; that is, $\rho_t = (1 + t)\bar{E}_{t/(1+t)}$ is a three-dimensional Bessel process starting from zero; see [9], XII, Theorem 4.2. Then, it follows that $\bar{E}^2 \stackrel{d}{=} B^2(1) + B^2(2) + B^2(3)$ where $B(1), B(2), B(3)$ are three independent Brownian bridges. This explains the second part of the corollary.

We will also use the well-known fact about the three-dimensional Bessel process ρ, that

$$J = \frac{\inf\{\rho_t : t \geq 1\}}{\rho_1}$$

is independent of $\sigma(\{\rho_s : s \leq 1\})$ and uniformly distributed on $[0, 1]$. Formulating this with $\bar{E}$ and $\bar{E}_{1-t}$ we obtain that

$$J_1 = \frac{1}{2\bar{E}_{1/2}} \cdot \min_{t\in[1/2,1)} \frac{\bar{E}_t}{1 - t} \quad and \quad J_2 = \frac{1}{2\bar{E}_{1/2}} \cdot \min_{t\in(0,1/2]} \frac{\bar{E}_t}{t}$$

are uniformly distributed on [0, 1], and J_1, J_2, $\bar{E}_{1/2}$ are independent.

Using these tools we want to estimate

$$\mathbf{P}(\zeta > x) = \mathbf{P}\left(\sup_{t \in (0,1)} \frac{(r(t)(r(1) - r(t)))^{1/4}}{\bar{E}_t} > x\right).$$

With the notation of the previous corollary,

$$r(t)\big(r(1) - r(t)\big) \leq \bar{\xi}^{2\kappa}\big(t \wedge (1 - t)\big)^{1+\kappa/4}.$$

For the denominator we have the following lower bound:

$$\bar{E}_t \geq \left(t \cdot \min_{t \in (0,1/2]} \frac{\bar{E}_t}{t}\right) \wedge \left((1 - t) \cdot \min_{t \in [1/2,1)} \frac{\bar{E}_t}{1 - t}\right)$$

$$\geq 2\big(t \wedge (1 - t)\big)\bar{E}_{1/2}(J_1 \wedge J_2).$$

Thus for $\kappa \geq 12$,

$$\zeta \leq \frac{\bar{\xi}^{\kappa/2}}{2\bar{E}_{1/2}(J_1 \wedge J_2)}.$$

The tail of $\bar{\xi}$ and ξ goes to zero exponentially fast, while on the other hand the tail of $(\bar{E}_{1/2}(J_1 \wedge J_2))^{-1}$ is polynomial, more precisely,

$$\mathbf{P}\left(\frac{1}{\bar{E}_{1/2}(J_1 \wedge J_2)} > x\right)$$

$$\leq \mathbf{P}(\bar{E}_{1/2} < x^{-1/2}) + \mathbf{P}(J_1 \wedge J_2 < x^{-1/2}) \leq c_3 x^{-1/2}$$

with some positive c_3. So we obtain that

(27) $$\mathbf{P}(\xi\zeta > x) \leq c(\varepsilon)x^{-1/2+\varepsilon},$$

where $\varepsilon > 0$ arbitrary small, and $c(\varepsilon)$ is a positive constant depending on ε.

Combining (27) with (26) and taking into account Remark 13, we get

$$\mathbf{P}\left(\left\{I \in \mathcal{C}(U, s) : \sup_{t \in I} |K_t + W_t| - |U_t| > 0\right\} \text{ is finite}\right) = 1 \qquad \text{for all } s > 0,$$

that is, the number of excursion intervals of U on which $|K + W| \leq |U|$ does not hold, is locally finite almost surely, provided that $\kappa \geq 12$. This proves property (3) completely.

The next proposition showing the extremal property of L finishes the proof of Lemma 17.

PROPOSITION 27. *Assume that $f, g, h : [0, \infty) \to \mathbb{R}$ are continuous functions, satisfying $f \leq h \leq g$ and $f(0) = g(0)$. Then, for any $t \geq 0$ the total variation of $L = \bar{L}(f, g)$ on $[0, t]$ is not greater than that of h.*

PROOF.   Take $t \geq 0$ and a subdivision $t_0 = 0 < t_1 < \cdots < t_n = t$. It is enough to show that there is a subdivision $s_0 = 0 < s_1 < \cdots < s_m = t$ such that

$$\sum_{j=1}^{n} |L(t_j) - L(t_{j-1})| \leq \sum_{j=1}^{m} |h(s_j) - h(s_{j-1})|.$$

We may and do assume that the sign of the increments $L(t_j) - L(t_{j-1})$ is alternating on the left. We can simply leave out those t_j at which the sign of the increments does not alternate without affecting the left-hand side.

The case $n = 1$ and $L(t) = L(0) = 0$ is trivial. In all other cases the increments $L(t_j) - L(t_{j-1})$, $j = 1, \ldots, n$ are nonzero.

If $L(t_j) - L(t_{j-1}) > 0$, then there is $s_j \in [t_{j-1}, t_j]$ such that $L(t_j) = f(s_j) \leq h(s_j)$; similarly if $L(t_j) - L(t_{j-1}) < 0$, then there is $s_j \in [t_{j-1}, t_j]$ such that $L(t_j) = g(s_j) \geq h(s_j)$. Defining $s_0 = 0$ and $s_{n+1} = t$ we get $|L(t_j) - L(t_{j-1})| \leq |h(s_j) - h(s_{j-1})|$ for $j = 1, \ldots, n$ and the statement follows.   □

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DEPARTMENT OF PROBABILITY THEORY
   AND STATISTICS
EÖTVÖS LORÁND UNIVERSITY
1117 BUDAPEST, PÁZMÁNY P. SÉTÁNY 1/C
HUNGARY
E-MAIL: prokaj@cs.elte.hu