

ROSENTHAL-TYPE INEQUALITIES FOR THE MAXIMUM OF PARTIAL SUMS OF STATIONARY PROCESSES AND EXAMPLES

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The aim of this paper is to propose new Rosenthal-type inequalities for moments of order higher than 2 of the maximum of partial sums of stationary sequences including martingales and their generalizations. As in the recent results by Peligrad et al. [*Proc. Amer. Math. Soc.* **135** (2007) 541–550] and Rio [*J. Theoret. Probab.* **22** (2009) 146–163], the estimates of the moments are expressed in terms of the norms of projections of partial sums. The proofs of the results are essentially based on a new maximal inequality generalizing the Doob maximal inequality for martingales and dyadic induction. Various applications are also provided.

1. Introduction. For independent random variables, the Rosenthal inequalities relate moments of order higher than 2 of partial sums of random variables to the variance of partial sums. One variant of this inequality is the following [see Rosenthal (1970), page 279]: let $(X_k)_k$ be independent and centered real valued random variables with finite moments of order p , $p \geq 2$. Then for every positive integer n ,

$$(1) \quad \mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll \sum_{k=1}^n \mathbf{E}(|X_k|^p) + \left(\sum_{k=1}^n \mathbf{E}(X_k^2)\right)^{p/2},$$

where $S_j = \sum_{k=1}^j X_k$. Unless otherwise specified, throughout the paper the notation $a_n \ll b_n$ means that there exists a numerical constant C_p depending only on p (and not on the underlying random variables and neither on n) such that $a_n \leq C_p b_n$, for all positive integers n .

Besides being useful to compare the norms \mathbf{L}^p and \mathbf{L}^2 of partial sums, these inequalities are important tools for obtaining a variety of results, including tightness of the empirical process [see the proof of Theorem 22.1 in Billingsley (1968)], convergence rates with respect to the strong law of large numbers [see, e.g., Wittmann (1985)] or almost sure invariance principles; see Wu (2007) and Gouëzel (2010) for recent results. Since the 1970s, there has been a great amount of work which

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has extended inequality (1) to dependent sequences. See, for instance, among many others: Peligrad (1985) and Shao (1995) for the case of ρ -mixing sequences; Shao (1988), Peligrad (1989) and Utev (1991) for the case of ϕ -mixing sequences; Peligrad and Gut (1999) and Utev and Peligrad (2003) for interlaced mixing; Theorem 2.2 in Viennet (1997) for β -mixing processes; Theorem 6.3 in Rio (2000) for the strongly mixing case; Dedecker (2001) and Rio (2009) for projective criteria.

The main goal of this paper is to generalize the Rosenthal inequality from sequences of independent variables to stationary dependent sequences including martingales, allowing then to consider examples that are not necessarily dependent in the sense of the dependence structures mentioned above.

In order to present our results, let us first introduce some notations and definitions used throughout the paper.

NOTATION 1. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let $T : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability \mathbf{P} . Let \mathcal{F}_0 be a σ -algebra of \mathcal{A} satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$. We then define the nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$ and the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ by $X_i = X_0 \circ T^i$, where X_0 is a real-valued random variable. The sequence will be called adapted to the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ if X_0 is \mathcal{F}_0 -measurable. The following notations will also be used: $\mathbf{E}_k(X) = \mathbf{E}(X|\mathcal{F}_k)$ and the norm in \mathbf{L}^p of X is denoted by $\|X\|_p$. Let $S_n = \sum_{j=1}^n X_j$.

In the rest of this section the sequence $(X_i)_{i \in \mathbb{Z}}$ is assumed to be stationary and adapted to $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ and the variables are in \mathbf{L}^p .

If $(X_k)_k$ are stationary martingale differences, the martingale form of inequality (1) is

$$(2) \quad \left\| \max_{1 \leq j \leq n} |S_j| \right\|_p \ll n^{1/p} \|X_1\|_p + \left\| \sum_{k=1}^n \mathbf{E}_{k-1}(X_k^2) \right\|_{p/2}^{1/2} \quad \text{for any } p \geq 2;$$

see Burkholder (1973). One of our goals is to replace the last term in this inequality with a new one containing terms of the form $\|\mathbf{E}_0(S_n^2)\|_{p/2}$. The reason for introducing this term comes from the fact that for many stationary sequences $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ is closer to the variance of partial sums. In addition, we are interested in pointing out a Rosenthal-type inequality for a larger class of stationary adapted sequences that includes the martingale differences as a special case.

Two recent results by Peligrad and Utev (2005) and Wu and Zhao (2008) show that

$$\left\| \max_{1 \leq j \leq n} |S_j| \right\|_p \ll n^{1/p} \left(\|X_1\|_p + \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \right) \quad \text{for any } 1 \leq p \leq 2.$$

To find a suitable extension of this inequality for $p > 2$, the first step in our approach is to establish the following maximal inequality that has interest in itself:

$$(3) \quad \left\| \max_{1 \leq j \leq n} |S_j| \right\|_p \ll n^{1/p} \left(\max_{1 \leq j \leq n} \|S_j\|_p / n^{1/p} + \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \right)$$

for any $p > 1$.

This inequality can be viewed as generalization of the well-known Doob maximal inequality for martingales. For a more precise version than (3), with constants specified, see our inequality (7).

Then we combine inequality (3) with several inequalities for $\|S_n\|_p$ that will further be established in this paper.

As we shall see in Section 3.1, by a direct approach using dyadic induction combined with maximal inequality (3), we shall prove that, for any $p > 2$,

$$(4) \quad \left\| \max_{1 \leq j \leq n} |S_j| \right\|_p \ll n^{1/p} \left(\|X_1\|_p + \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p + \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{1/(2\delta)} \right),$$

where $\delta = \min(1, 1/(p - 2))$. For $2 < p \leq 3$ our inequality provides a maximal form for Theorem 3.1 in Rio (2009). When $p \geq 4$, we shall see that the last term in the right-hand side dominates the second term, so that the second term can be omitted in this case. Inequality (4) shows that in order to relate $\|\max_{1 \leq j \leq n} |S_j|\|_p$ to the vector $(\|S_j\|_2)_{1 \leq j \leq n}$ we have to control $\sum_{k=1}^n k^{-(p+1)/p} \|\mathbf{E}_0(S_k)\|_p$ and $\sum_{k=1}^n k^{-(p+2\delta)/p} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^\delta$.

In Section 3.3.1, we study the case of stationary martingale difference sequences showing that for all even powers $p \geq 4$, inequality (4) holds with $\delta = 2/(p - 2)$. This result is possible for stationary martingale differences with the help of a special symmetrization for martingales initiated by Kwapien and Woyczyński (1991). In addition, by using martingale approximation techniques, we obtain, for any even integer p , another form of Rosenthal-type inequality, other than (4), for stationary adapted processes (see Section 3.3.2), that gives, for instance, better results for functionals of linear processes with independent innovations.

We also investigate the situation when the conditional expectation with respect to both the past and the future of the process is used. For instance, when $p \geq 4$ is an even integer, and the process is reversible, then inequality (4) holds (see Theorem 9 and Corollary 29) with $\delta = 1$.

In Section 3.2 we show that our inequalities imply the Burkholder-type inequality as stated in Theorem 1 of Peligrad, Utev and Wu (2007). For the sake of applications in Section 3.4 we express the terms that appear in our Rosenthal inequalities in terms of individual summands.

Our paper is organized as follows. In Section 2, we prove a new maximal inequality allowing us to relate the moments of the maximum of partial sums of an adapted sequence, that is not necessarily stationary, to the moments of its partial sums. Maximal inequality (3), combined with moment estimates, allows us to obtain the Rosenthal-type inequalities stated in Theorems 6 and 9 of Section 3.1. Section 3.3 is devoted to Rosenthal-type inequalities for even powers for the special case of stationary martingale differences and to an application to stationary processes via a martingale approximation technique. In Section 4, we give other applications of the maximal inequalities stated in Section 2 and provide examples for which we compute the quantities involved in the Rosenthal-type inequalities of Section 3. One of the applications presented in this section is a Bernstein inequality for the maximum of partial sums for strongly mixing sequences that extends the inequality in Merlevède, Peligrad and Rio (2009). The applications are given to ARCH models, to functions of linear processes and reversible Markov chains. In Section 5, we apply the inequality (4) to estimate the random term of the \mathbf{L}^p -integrated risk of kernel estimators of the unknown marginal density of a stationary sequence that is assumed to be β -mixing in the weak sense (see Definition 32). Some technical results are postponed to the Appendix.

2. Maximal inequalities for adapted sequences. The next proposition is a generalization of the well-known Doob maximal inequality for martingales to adapted sequences. It states that the moment of order p of the maximum of the partial sums of an adapted process can be compared to the corresponding moment of the partial sum plus a correction term which is zero for martingale differences sequences. The proof is based on convexity and chaining arguments.

PROPOSITION 2. *Let $p > 1$ and $q = p/(p - 1)$. Let Y_i , $1 \leq i \leq 2^r$, be real random variables in \mathbf{L}^p , where r is a positive integer. Assume that the random variables are adapted to an increasing filtration $(\mathcal{F}_i)_i$. Let $S_r = Y_1 + \dots + Y_r$. Then the following inequality holds:*

$$(5) \quad \left\| \max_{1 \leq i \leq 2^r} |S_i| \right\|_p \leq q \|S_{2^r}\|_p + q \sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})\|_p^p \right)^{1/p}.$$

COROLLARY 3. *In the stationary case, we get that for any integer $r \geq 1$,*

$$\left\| \max_{1 \leq i \leq 2^r} |S_i| \right\|_p \leq q \|S_{2^r}\|_p + q 2^{r/p} \sum_{l=0}^{r-1} 2^{-l/p} \|\mathbf{E}(S_{2^l} | \mathcal{F}_0)\|_p.$$

REMARK 4. The inequality in Corollary 3 easily implies that

$$(6) \quad \left\| \max_{1 \leq i \leq n} |S_i| \right\|_p \leq 2q \max_{1 \leq m \leq n} \|S_m\|_p + (2^{1/p} q) n^{1/p} \sum_{l=0}^{r-1} 2^{-l/p} \|\mathbf{E}(S_{2^l} | \mathcal{F}_0)\|_p$$

for any integer $n \in [2^{r-1}, 2^r[$, where r is a positive integer. Moreover, due to the subadditivity of the sequence $(\|\mathbf{E}(S_n|\mathcal{F}_0)\|_p)_{n \geq 1}$, according to Lemma 37, we also have that for any positive integer n ,

$$(7) \quad \left\| \max_{1 \leq i \leq n} |S_i| \right\|_p \leq 2q \max_{1 \leq m \leq n} \|S_m\|_p + \left(q \frac{2^{2+2/p}}{2^{1+1/p} - 1} \right) n^{1/p} \sum_{j=1}^n j^{-1-1/p} \|\mathbf{E}(S_j|\mathcal{F}_0)\|_p.$$

Inequalities (6) and (7) are true, even if the variables are not centered.

For several applications involving exponential bounds we point out the following proposition.

PROPOSITION 5. *Let $p > 1$ and $q = p/(p - 1)$. Let $(Y_i)_{i \geq 1}$ be real random variables in \mathbf{L}^p . Assume that the random variables are adapted to an increasing filtration $(\mathcal{F}_i)_i$. Let $S_n = Y_1 + \dots + Y_n$. Let φ be a nondecreasing, nonnegative, convex and even function. Then for any positive real x , and any positive integer r , the following inequality holds:*

$$(8) \quad \mathbf{P}\left(\max_{1 \leq i \leq 2^r} |S_i| \geq 2x \right) \leq \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r})) + q^p x^{-p} \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{(k+1)2^l} - S_{k2^l}|\mathcal{F}_{k2^l})\|_p^p \right)^{1/p} \right)^p.$$

Assume, in addition, that there exists a positive real M such that $\sup_i \|Y_i\|_\infty \leq M$. Then for any positive real x , and any positive integer r , the following inequality holds:

$$(9) \quad \mathbf{P}\left(\max_{1 \leq i \leq 2^r} |S_i| \geq 4x \right) \leq \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r})) + q^p x^{-p} \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})\|_p^p \right)^{1/p} \right)^p$$

with $v = [x/M]$ (where $[\cdot]$ denotes the integer part).

PROOFS OF PROPOSITIONS 2 AND 5. Denote $S_{2^r}^* = \max_{1 \leq i \leq 2^r} |S_i|$. For any $m \in [0, 2^r - 1]$, we have that

$$S_{2^r-m} = \mathbf{E}(S_{2^r}|\mathcal{F}_{2^r-m}) - \mathbf{E}(S_{2^r} - S_{2^r-m}|\mathcal{F}_{2^r-m}).$$

So

$$(10) \quad S_{2^r}^* \leq \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| + \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})|.$$

Since $(\mathbf{E}(S_{2^r} | \mathcal{F}_k))_{k \geq 1}$ is a martingale, we shall use Doob's maximal inequality to deal with the first term in the right-hand side of (10). Hence, since φ is a nondecreasing, nonnegative, convex and even function, we get that

$$(11) \quad \mathbf{P}\left(\max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| \geq x\right) \leq \frac{1}{\varphi(x)} \mathbf{E}(\varphi(S_{2^r}))$$

and also that

$$(12) \quad \left\| \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| \right\|_p \leq q \|S_{2^r}\|_p.$$

Write now m in basis 2 as follows:

$$m = \sum_{i=0}^{r-1} b_i(m) 2^i \quad \text{where } b_i(m) = 0 \text{ or } b_i(m) = 1.$$

Set $m_l = \sum_{i=l}^{r-1} b_i(m) 2^i$ and notice that for any $p \geq 1$, we have

$$|\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})|^p \leq \left(\sum_{l=0}^{r-1} |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})| \right)^p.$$

Hence setting

$$\alpha_l = \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})\|_p^p \right)^{1/p} \quad \text{and} \quad \lambda_l = \frac{\alpha_l}{\sum_{l=0}^{r-1} \alpha_l},$$

we get by convexity

$$\begin{aligned} & |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})|^p \\ & \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})|^p \\ & \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |\mathbf{E}(|\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})| | \mathcal{F}_{2^r - m_l})|^p. \end{aligned}$$

Now $m_l \neq m_{l+1}$, only if $b_l(m) = 1$, and in that case $m_l = k_m 2^l$ with k_m odd. It follows that

$$\begin{aligned} & |\mathbf{E}(S_{2^r - m_{l+1}} - S_{2^r - m_l} | \mathcal{F}_{2^r - m_l})| \\ & \leq \max_{1 \leq k \leq 2^{r-l}, k \text{ odd}} |\mathbf{E}(S_{2^r - (k-1)2^l} - S_{2^r - k2^l} | \mathcal{F}_{2^r - k2^l})| \\ & := A_{r,l}. \end{aligned}$$

Hence, using the fact that if $|X| \leq |Y|$, then $\mathbf{E}(|X||\mathcal{F}) \leq \mathbf{E}(|Y||\mathcal{F})$, we get that

$$\left\| \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})| \right\|_p^p \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} \mathbf{E} \left(\max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(A_{r,l} | \mathcal{F}_{2^r - m})| \right)^p.$$

Notice that $(\mathbf{E}(A_{r,l} | \mathcal{F}_k))_{k \geq 1}$ is a martingale and by Doob’s maximal inequality, we obtain

$$\mathbf{E} \left(\max_{0 \leq m \leq 2^r - 1} \mathbf{E}(A_{r,l} | \mathcal{F}_{2^r - m}) \right)^p \leq q^p \|A_{r,l}\|_p^p \leq q^p \alpha_l^p.$$

Using the definition of λ_l , it follows that

$$\begin{aligned} & \left\| \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} - S_{2^r - m} | \mathcal{F}_{2^r - m})| \right\|_p^p \\ (13) \quad & \leq q^p \left(\sum_{l=0}^{r-1} \alpha_l \right)^p \\ & \leq q^p \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{(k+1)2^l} - S_{k2^l} | \mathcal{F}_{k2^l})\|_p^p \right)^{1/p} \right)^p. \end{aligned}$$

Starting from (10) and using (11) and (13) combined with Markov’s inequality, inequality (8) of Proposition 5 follows. To end the proof of Proposition 2, we start from (10) and consider bounds (12) and (13).

We turn now to the proof of inequality (9). We start from (10) and we write that

$$\begin{aligned} S_{2^r}^* & \leq \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| \\ & \quad + \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r + v} - S_{2^r + v - m} | \mathcal{F}_{2^r - m})| \\ & \quad + \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r + v} - S_{2^r} | \mathcal{F}_{2^r - m})| \\ & \quad + \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r + v - m} - S_{2^r - m} | \mathcal{F}_{2^r - m})|. \end{aligned}$$

By the fact that the variables are uniformly bounded by M , we then derive that

$$S_{2^r}^* \leq \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| + \max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r + v} - S_{2^r + v - m} | \mathcal{F}_{2^r - m})| + 2vM.$$

Since $vM \leq x$, it follows that

$$\begin{aligned} (14) \quad \mathbf{P}(S_{2^r}^* \geq 4x) & \leq \mathbf{P} \left(\max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r} | \mathcal{F}_{2^r - m})| \geq x \right) \\ & \quad + \mathbf{P} \left(\max_{0 \leq m \leq 2^r - 1} |\mathbf{E}(S_{2^r + v} - S_{2^r + v - m} | \mathcal{F}_{2^r - m})| \geq x \right). \end{aligned}$$

Using chaining arguments, convexity and Doob’s maximal inequality, as above, we infer that for any $p > 1$,

$$(15) \quad \begin{aligned} & \left\| \max_{0 \leq m \leq 2^r-1} |\mathbf{E}(S_{2^r+v} - S_{2^r+v-m} | \mathcal{F}_{2^r-m})| \right\|_p^p \\ & \leq q^p \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l} | \mathcal{F}_{k2^l})\|_p^p \right)^{1/p} \right)^p. \end{aligned}$$

Starting from (14) and using (11) and (15) combined with Markov’s inequality, inequality (9) of Proposition 5 follows. \square

3. Moment inequalities for the maximum of partial sums under projective conditions.

3.1. *Rosenthal-type inequalities for stationary processes.* Using a direct approach that combines maximal inequality (7) and Lemma 36, we obtain the following Rosenthal inequality for the maximum of the partial sums of a stationary process for all powers $p > 2$.

THEOREM 6. *Let $p > 2$ be a real number, and let $(X_i)_{i \in \mathbb{Z}}$ be an adapted stationary sequence in the sense of Notation 1. Then, for any positive integer n , the following inequality holds:*

$$\begin{aligned} \mathbf{E} \left(\max_{1 \leq j \leq n} |S_j|^p \right) & \ll n \mathbf{E}(|X_1|)^p + cn \left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \right)^p \\ & \quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{p/(2\delta)}, \end{aligned}$$

where $\delta = \min(1, 1/(p - 2))$ and $c = 1$. When $p \geq 4$ we can take $c = 0$ by enlarging the constant involved.

COMMENT 7. (1) Notice that for $2 < p \leq 3$ the inequality holds with $\delta = 1$, and therefore it provides a maximal form for Theorem 3.1 in Rio (2009).

(2) It is interesting to indicate the monotonicity of the right-hand side of the inequality in δ . To be more precise, for any $0 < \delta \leq \gamma \leq 1$, the following inequality holds:

$$\begin{aligned} & \left(\sum_{k=1}^n k^{-1-2\gamma/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\gamma \right)^{1/\gamma} \\ & \leq 2^{(1+\gamma)(\gamma-\delta)/(\delta\gamma)} \left(\sum_{k=1}^n k^{-1-2\delta/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{1/\delta}. \end{aligned}$$

To see this, we notice the subadditivity property $\|\mathbf{E}_0(S_{i+j}^2)\|_{p/2} \leq 2\|\mathbf{E}_0(S_i^2)\|_{p/2} + 2\|\mathbf{E}_0(S_j^2)\|_{p/2}$ and apply then item (3) of Lemma 37 with $C = 2$.

(3) On the other hand, for any $0 < \delta < 1$ and any $\gamma > 1/\delta - 1$, by Hölder’s inequality, there exists a positive constant C depending on p, γ and δ , such that

$$(16) \quad \left(\sum_{k=1}^n k^{-1-2\delta/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{p/(2\delta)} \leq C \left(\sum_{k=1}^n k^{-1-2/p} (\log k)^\gamma \|\mathbf{E}_0(S_k^2)\|_{p/2} \right)^{p/2}.$$

(4) As a matter of fact we shall prove first the inequality from Theorem 6 in a slightly different form which is equivalent up to multiplicative constants: for any positive integer r and any integer n such that $2^{r-1} \leq n < 2^r$ (δ and c as above)

$$(17) \quad \mathbf{E} \left(\max_{1 \leq j \leq n} |S_j|^p \right) \ll n \mathbf{E}(|X_1|)^p + cn \left(\sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \right)^p + n \left(\sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \right)^{p/(2\delta)}.$$

(5) In the traditional Rosenthal inequality for martingales, the lower bound and the upper one have the same order of magnitude [we refer to the paper by Hitzenko (1990) for results on the best constants]. In general the upper bound obtained in Theorem 6 cannot be a lower bound for $\mathbf{E}(|S_n|^p)$ (up to some constants) without any additional assumptions. Indeed, if $X_k = Z_0 \circ T^{k-1} - Z_0 \circ T^k$ where Z_0 is \mathcal{F}_0 -measurable and in \mathbf{L}^p , then $\mathbf{E}(|S_n|^p) \leq 2^p \|Z_0\|_p^p$, whereas the upper bound given in Theorem 6 is of order n . However, we notice that Theorem 6 is sharp in some sense. Indeed, a simple lower bound for $\|S_n\|_p$ can be obtained via the following inequality: $\|\mathbf{E}_0(S_n^2)\|_{p/2} \leq \|S_n\|_p^2$. Hence for $p > 2$, if $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ is of order n^γ with $\gamma > 2/p$, then the lower bound, $\|\mathbf{E}_0(S_n^2)\|_{p/2}^{p/2}$ and the term in the right-hand side of the inequality of Theorem 6 have the same order of magnitude [to see this, we also use the fact that $\|\mathbf{E}_0(S_n)\|_p^2 \leq \|\mathbf{E}_0(S_n^2)\|_{p/2}$].

With applications to Markov processes in mind, by conditioning with respect to both the future and the past of the process, our next result gives an alternative inequality to the one given in Theorem 6 when p is an even integer. For this case, the power δ appearing in Theorem 6 is always equal to one. Before stating the result, we first introduce the following notation to define the additional nonincreasing filtration that we consider.

NOTATION 8. Let $\bar{\mathcal{F}}_0$ be a σ -algebra of \mathcal{A} satisfying $T^{-1}(\bar{\mathcal{F}}_0) \subseteq \bar{\mathcal{F}}_0$. We then define the nonincreasing filtration $(\bar{\mathcal{F}}_i)_{i \in \mathbb{Z}}$ by $\bar{\mathcal{F}}_i = T^{-i}(\bar{\mathcal{F}}_0)$. In what follows, we use the notation $\bar{\mathbf{E}}_k(Y) = \mathbf{E}(Y|\bar{\mathcal{F}}_k)$.

THEOREM 9. *Let $p \geq 4$ be an even integer, and let X_0 be a real valued random variable such that $\|X_0\|_p < \infty$ and measurable with respect to \mathcal{F}_0 and to $\bar{\mathcal{F}}_0$. We construct the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ as in Notation 1. Then for any integer n ,*

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll n\mathbf{E}(|X_1|^p) + n\left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} (\|\mathbf{E}_0(S_k)\|_p + \|\bar{\mathbf{E}}_{k+1}(S_k)\|_p)\right)^p + n\left(\sum_{k=1}^n \frac{1}{k^{1+2/p}} (\|\mathbf{E}_0(S_k^2)\|_{p/2} + \|\bar{\mathbf{E}}_{k+1}(S_k^2)\|_{p/2})\right)^{p/2}.$$

As a corollary to the proof of Theorem 9, we obtain:

THEOREM 10. *Let $p \geq 4$ be a real number, and let X_0 be a real-valued random variable such that $\|X_0\|_p < \infty$ and measurable with respect to \mathcal{F}_0 and to $\bar{\mathcal{F}}_0$. We construct the stationary sequence $(X_i)_{i \in \mathbb{Z}}$ as in Notation 1. Then for any integer n ,*

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll n\mathbf{E}(|X_1|^p) + n\left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} (\|\mathbf{E}_0(S_k^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{k+1}(S_k^2)\|_{p/2}^{1/2})\right)^p.$$

This theorem is also valid for $2 < p < 4$. In this range, however, according to item (2) of Comment 7, Theorem 6 gives better bounds.

PROOF OF THEOREM 6. The proof of this theorem is based on dyadic induction and involves several steps. With the notation $a_n = \|S_n\|_p$ we shall establish a recurrence formula: for any positive integer r ,

$$(18) \quad a_{2n}^p \leq 2a_n^p + 2c_1 a_n^{p-1} \|\mathbf{E}_0(S_n)\|_p + 2c_2 a_n^{p-2\delta} \|\mathbf{E}_0(S_n^2)\|_{p/2}^\delta,$$

where c_1 and c_2 are positive constants depending only on p . Before proving it, let us show that (18) implies our result.

LEMMA 11. *Assume that for some $0 < \delta \leq 1$ the recurrence formula (18) holds. Then inequalities (17) and (4) hold with the same δ .*

Let us prove the lemma. From inequality (18), by recurrence on the first term, we obtain for any positive integer r ,

$$(19) \quad a_{2^r}^p \leq 2^r \left(a_{2^0}^p + c_1 \sum_{k=0}^{r-1} 2^{-k} a_{2^k}^{p-1} \|\mathbf{E}_0(S_{2^k})\|_p + c_2 \sum_{k=0}^{r-1} 2^{-k} a_{2^k}^{p-2\delta} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \right).$$

We shall establish first inequality (17). Due to maximal inequality (6), it suffices to prove that the inequality is satisfied for $\max_{1 \leq j \leq n} \mathbf{E}(|S_j|^p)$ instead of $\mathbf{E}(\max_{1 \leq j \leq n} |S_j|^p)$.

The proof is divided in several steps. The goal is to establish that for any positive integer r and any integer n such that $2^{r-1} \leq n < 2^r$,

$$(20) \quad \begin{aligned} \max_{1 \leq j \leq n} \mathbf{E}(|S_j|^p) &\ll n \mathbf{E}(|X_1|)^p + cn \left(\sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \right)^p \\ &\quad + n \left(\sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \right)^{p/2\delta}. \end{aligned}$$

With the notation $B_r = \max_{0 \leq k \leq r} (a_{2^k}^p / 2^k)$, starting from (19), we get

$$\begin{aligned} B_r &\leq a_{2^0}^p + c_1 B_r^{1-1/p} \sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \\ &\quad + c_2 B_r^{1-2\delta/p} \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta. \end{aligned}$$

Therefore, taking into account that either $B_r \leq 3a_{2^0}^p$ or

$$B_r^{1/p} \leq 3c_1 \sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \quad \text{or} \quad B_r^{2\delta/p} \leq 3c_2 \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta,$$

we derive that

$$(21) \quad \begin{aligned} a_{2^r}^p &\leq 2^r \left(3a_{2^0}^p + \left(3c_1 \sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \right)^p \right. \\ &\quad \left. + \left(3c_2 \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \right)^{p/2\delta} \right). \end{aligned}$$

Let now $2^{r-1} \leq n < 2^r$, and write its binary expansion,

$$(22) \quad n = \sum_{k=0}^{r-1} 2^k b_k \quad \text{where } b_{r-1} = 1 \text{ and } b_k \in \{0, 1\} \quad \text{for } k = 0, \dots, r-2.$$

Notice that

$$\begin{aligned} S_n &= \sum_{k=0}^{r-1} b_k T_{2^k} \quad \text{where } T_{2^k} = \sum_{i=n_{k-1}+1}^{n_k} X_i, \\ n_k &= \sum_{j=0}^k b_j 2^j \quad \text{and } n_{-1} = 0. \end{aligned}$$

Hence, by stationarity,

$$\|S_n\|_p \leq \sum_{k=0}^{r-1} b_k \|T_{2^k}\|_p \leq \sum_{k=0}^{r-1} b_k \|S_{2^k}\|_p.$$

Then, by using (21) and the fact that $\sum_{k=0}^{r-1} b_k 2^{k/p} \leq 2^{r/p}/(1 - 2^{-1/p})$, we derive inequality (20) for $\mathbf{E}(|S_n|^p)$ and also for $\max_{1 \leq j \leq n} \mathbf{E}(|S_j|^p)$. Inequality (17) follows now by maximal inequality (6).

We indicate now how to derive from (17) the inequality stated in Theorem 6. Notice that, by stationarity, for any integers i and j ,

$$\|\mathbf{E}_0(S_{i+j})\|_p \leq \|\mathbf{E}_0(S_i)\|_p + \|\mathbf{E}_0(S_j)\|_p,$$

and also that for any $0 < \delta \leq 1$,

$$\|\mathbf{E}_0(S_{i+j}^2)\|_{p/2}^\delta \leq 2^\delta \|\mathbf{E}_0(S_i^2)\|_{p/2}^\delta + 2^\delta \|\mathbf{E}_0(S_j^2)\|_{p/2}^\delta.$$

Using item (1) of Lemma 37, it follows that

$$\sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p \ll \sum_{k=1}^n k^{-1-1/p} \|\mathbf{E}_0(S_k)\|_p$$

and

$$(23) \quad \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \ll \sum_{k=1}^n k^{-1-2\delta/p} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta.$$

The results follows by the above considerations via inequality (17). \square

END OF THE PROOF OF THEOREM 6. It remains to establish the recurrence formula (18). We divide the proof into three cases according to the values of p . Denote $\bar{S}_n = X_{n+1} + \cdots + X_{2n}$.

The case $2 < p \leq 3$ was discussed in Rio (2009). We give here a shorter alternative proof. We apply inequality (85) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$. Then, by taking the expectation and using stationarity and properties of conditional expectation, we obtain

$$\mathbf{E}(|S_{2n}|)^p \leq 2\mathbf{E}(|S_n|)^p + p\mathbf{E}(|S_n|^{p-1} \text{sign}(S_n) \mathbf{E}_n(\bar{S}_n)) + C_p^2 \mathbf{E}(|S_n|^{p-2} \mathbf{E}_n(\bar{S}_n^2)),$$

where $C_p^2 = p(p-1)/2$. This inequality combined with Hölder's inequality gives

$$a_{2n}^p \leq 2a_n^p + pa_n^{p-1} \|\mathbf{E}_0(S_n)\|_p + C_p^2 a_n^{p(1-2/p)} \|\mathbf{E}_0(S_n^2)\|_{p/2},$$

and therefore (18) holds with $\delta = 1$, $c_1 = 2^{-1}p$ and $c_2 = 2^{-1}C_p^2$.

Assume now that $p \in]3, 4[$. Using inequality (86) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$, taking the expectation, and using Lemma 34, we get by stationarity that for any positive integer r ,

$$a_{2n}^p \leq 2a_n^p + pa_n^{p-1} \|\mathbf{E}_0(S_n)\|_p + C_p^2 a_n^{p(1-2/p)} \|\mathbf{E}_0(S_n^2)\|_{p/2} + 2p(p-2)^{-1} a_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}.$$

Since $\|\mathbf{E}_0(S_n^2)\|_{p/2} \leq a_n^{2(p-3)/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}$, it follows that

$$a_{2n}^p \leq 2a_n^p + pa_n^{p-1} \|\mathbf{E}_0(S_n)\|_p + 4pa_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}.$$

It follows that (18) holds with $\delta = 1/(p-2)$, $c_1 = 2^{-1}p$ and $c_2 = 2p$.

It remains to prove inequality (4) for $p \geq 4$. Using inequality (87) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$, and taking the expectation, we get by stationarity that

$$a_{2n}^p \leq 2a_n^p + 4^p \mathbf{E}(|S_n|^{p-1} |\bar{S}_n| + |\bar{S}_n|^{p-1} |S_n|).$$

Using Lemma 34 together with stationarity, it follows that

$$\mathbf{E}(|S_n| |\bar{S}_n|^{p-1}) \leq a_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}$$

and that

$$\mathbf{E}(|S_n|^{p-1} |\bar{S}_n|) \leq a_n^{p-1} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/2} \leq a_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)}.$$

From these estimates we deduce

$$a_{2n}^p \leq 2a_n^p + 2(4^p) a_n^{p-2/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/(p-2)},$$

and then (18) holds with $\delta = 1/(p-2)$, $c_1 = 0$ and $c_2 = 4^p$. Therefore, in this case (20) holds with $c = 0$. Then by maximal inequality (6), inequality (17) holds with $c = 1$. We show now that, in this case, the second term in the inequality can be bounded up to a multiplicative constant by the third term. By Jensen’s inequality and since in this case $\delta < 1/2$, we have

$$(24) \quad \begin{aligned} \sum_{k=0}^{r-1} 2^{-k/p} \|\mathbf{E}_0(S_{2^k})\|_p &\leq \left(\sum_{k=0}^{r-1} 2^{-(2k/p)(1/2)} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{1/2} \right)^2 \\ &\leq \left(\sum_{k=0}^{r-1} 2^{-(2k/p)\delta} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^\delta \right)^{1/\delta}. \end{aligned}$$

We then finish the proof by using (23). \square

PROOF OF THEOREM 9. Denote $\bar{S}_n = X_{n+1} + \dots + X_{2n}$. Starting from inequality (88) of Lemma 36 applied with $x = S_n$ and $y = \bar{S}_n$ and using the notation $a_n = \|S_n\|_p$, by stationarity, we get that

$$(25) \quad \begin{aligned} a_{2n}^p &\leq 2a_n^p + p(\mathbf{E}(S_n^{p-1} \bar{S}_n) + \mathbf{E}(S_n \bar{S}_n^{p-1})) \\ &\quad + 2^p (\mathbf{E}(S_n^{p-2} \bar{S}_n^2) + \mathbf{E}(\bar{S}_n^{p-2} S_n^2)). \end{aligned}$$

By using Hölder’s inequality and recurrence, we then derive that for any positive integer r ,

$$a_{2^r}^p \leq 2^r a_{2^0}^p + 2^{-1} p \sum_{k=0}^{r-1} 2^{r-k} a_{2^k}^{p-1} (\|\mathbf{E}_0(S_{2^k})\|_p + \|\bar{\mathbf{E}}_{2^{k+1}}(S_{2^k})\|_p) + 2^{p-1} \sum_{k=0}^{r-1} 2^{r-k} a_{2^k}^{p-2} (\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} + \|\bar{\mathbf{E}}_{2^{k+1}}(S_{2^k}^2)\|_{p/2}).$$

By using the arguments of the proof of Lemma 11, we get for $2^{r-1} \leq n < 2^r$,

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll n \mathbf{E}(|X_1|^p) + n \left(\sum_{k=1}^{r-1} 2^{-k/p} (\|\mathbf{E}_0(S_{2^k})\|_p + \|\bar{\mathbf{E}}_{k+1}(S_{2^k})\|_p)\right)^p + n \left(\sum_{k=1}^n 2^{-2k/p} (\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} + \|\bar{\mathbf{E}}_{k+1}(S_{2^k}^2)\|_{p/2})\right)^{p/2}.$$

Noticing, in addition, that, by stationarity, for any integer i and j ,

$$\begin{aligned} \|\mathbf{E}_0(S_{i+j})\|_p &\leq \|\mathbf{E}_0(S_i)\|_p + \|\mathbf{E}_0(S_j)\|_p, \\ (26) \quad \|\bar{\mathbf{E}}_{i+j+1}(S_{i+j})\|_p &\leq \|\bar{\mathbf{E}}_{i+1}(S_i)\|_p + \|\bar{\mathbf{E}}_{j+1}(S_j)\|_p, \\ \|\bar{\mathbf{E}}_{i+j+1}(S_{i+j}^2)\|_{p/2} &\leq 2\|\bar{\mathbf{E}}_{i+1}(S_i^2)\|_{p/2} + 2\|\bar{\mathbf{E}}_{j+1}(S_j^2)\|_{p/2} \end{aligned}$$

and that

$$(27) \quad \|\mathbf{E}_0(S_{i+j}^2)\|_{p/2} \leq 2\|\mathbf{E}_0(S_i^2)\|_{p/2} + 2\|\mathbf{E}_0(S_j^2)\|_{p/2}.$$

We obtain the desired result by using Lemma 37. \square

PROOF OF THEOREM 10. To prove this theorem we apply inequality (87) of Lemma 36 with $x = S_n$ and $y = \bar{S}_n$, where $\bar{S}_n = X_{n+1} + \dots + X_{2n}$. With the notation $a_n = \|S_n\|_p$, we then have by stationarity that

$$a_{2n}^p \leq 2a_n^p + 4^p (\mathbf{E}(|S_n|^{p-1}|\bar{S}_n|) + \mathbf{E}(|S_n|\bar{S}_n|^{p-1})).$$

By conditioning and then applying Jensen’s inequality followed by Hölder’s inequality we obtain

$$\begin{aligned} a_{2n}^p &\leq 2a_n^p + 4^p (\mathbf{E}(|S_n|^{p-1}\mathbf{E}_n^{1/2}(\bar{S}_n^2)) + \mathbf{E}(|\bar{S}_n|^{p-1}\bar{\mathbf{E}}_{n+1}^{1/2}(S_n^2))) \\ &\leq 2a_n^p + 4^p a_n^{p-1} (\|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{n+1}(S_n^2)\|_{p/2}^{1/2}). \end{aligned}$$

By recurrence, we then derive that for any positive integer r ,

$$a_{2^r}^p \leq 2^r \left(a_0^p + 2^{2p-1} \sum_{k=0}^{r-1} 2^{-k} a_{2^k}^{p-1} (\|\mathbf{E}_0(S_{2^k}^2)\|_{p/2}^{1/2} + \|\bar{\mathbf{E}}_{2^{k+1}}(S_{2^k}^2)\|_{p/2}^{1/2}) \right).$$

The proof is completed by the arguments developed in the proof of Lemma 11 and by using Lemma 37 via inequalities (26) and (27). \square

3.2. *Relation with the Burkholder-type inequality.* The next lemma shows how to compare $\|\mathbf{E}_0(S_n^2)\|_{p/2}$ with quantities involving only $\|\mathbf{E}_0(S_n)\|_p$.

LEMMA 12. *Let $p \geq 2$ be a real number, and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then, for any positive integer n ,*

$$(28) \quad \|\mathbf{E}_0(S_n^2)\|_{p/2} \ll n \|\mathbf{E}_0(X_1^2)\|_{p/2} + n \left(\sum_{j=1}^n \frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}} \right)^2.$$

As a consequence of the above lemma, we get that for any $0 < \delta \leq 1$ and any real $p > 2$,

$$n \left(\sum_{j=1}^n \frac{\|\mathbf{E}_0(S_j^2)\|_{p/2}^\delta}{j^{1+2\delta/p}} \right)^{p/(2\delta)} \ll n^{p/2} \|\mathbf{E}_0(X_1^2)\|_{p/2}^{p/2} + n^{p/2} \left(\sum_{j=1}^n \frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}} \right)^p.$$

Theorem 6 then implies the following Burkholder-type inequality that was established by Peligrad, Utev and Wu (2007), Theorem 1:

COROLLARY 13. *Let $p > 2$ be a real number, and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then, for any integer n ,*

$$\mathbf{E} \left(\max_{1 \leq j \leq n} |S_j|^p \right) \ll n^{p/2} \mathbf{E}(|X_1|^p) + n^{p/2} \left(\sum_{j=1}^n \frac{\|\mathbf{E}_0(S_j)\|_p}{j^{3/2}} \right)^p.$$

PROOF OF LEMMA 12. We shall first prove that for any positive integer k ,

$$(29) \quad \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} \leq 2^{k+1} \|\mathbf{E}_0(X_1^2)\|_{p/2} + 2^{k+2} \left(\sum_{j=0}^{k-1} \frac{\|\mathbf{E}_0(S_{2^j})\|_p}{2^{j/2}} \right)^2.$$

By using the notation $\bar{S}_n = X_{n+1} + \dots + X_n$ and the fact that $S_{2n}^2 = S_n^2 + \bar{S}_n^2 + 2S_n\bar{S}_n$, we get, by stationarity, that

$$\|\mathbf{E}_0(S_{2n}^2)\|_{p/2} \leq 2\|\mathbf{E}_0(S_n^2)\|_{p/2} + 2\|\mathbf{E}_0(S_n\mathbf{E}_n(\bar{S}_n))\|_{p/2}.$$

Now, the Cauchy–Schwarz inequality, applied twice, gives

$$\|\mathbf{E}_0(S_n\mathbf{E}_n(\bar{S}_n))\|_{p/2} \leq \|\mathbf{E}_0^{1/2}(S_n^2)\mathbf{E}_0^{1/2}(\mathbf{E}_n^2(\bar{S}_n))\|_{p/2} \leq \|\mathbf{E}_0(S_n^2)\|_{p/2}^{1/2} \|\mathbf{E}_0(S_n)\|_p.$$

Hence, setting $b_n = \|\mathbf{E}_0(S_n^2)\|_{p/2}$, it follows that

$$b_{2n} \leq 2b_n + 2b_n^{1/2} \|\mathbf{E}_0(S_n)\|_p.$$

By recurrence, this gives that

$$b_{2^k} \leq 2^k b_0 + \sum_{j=0}^{k-1} 2^{k-j} b_{2^j}^{1/2} \|\mathbf{E}_0(S_{2^j})\|_p.$$

With the notation $B_k = \max_{0 \leq j \leq k} 2^{-j} b_{2^j}$, we derive that

$$B_k \leq 2 \max \left(b_0, B_k^{1/2} \sum_{j=0}^{k-1} 2^{-j/2} \|\mathbf{E}_0(S_{2^j})\|_p \right)$$

implying that

$$2^{-k} b_{2^k} \leq B_k \leq 2b_0 + 2^2 \left(\sum_{j=0}^{k-1} 2^{-j/2} \|\mathbf{E}_0(S_{2^j})\|_p \right)^2.$$

This ends the proof of inequality (29).

We turn now to the proof of (28). Let r be the positive integer such that $2^{r-1} \leq n < 2^r$. Starting with the binary expansion (22), and using Minkowski’s inequality twice, first with respect to the conditional expectation, and second with respect to the norm in \mathbf{L}^p , we get by stationarity that

$$\|\mathbf{E}_0(S_n^2)\|_{p/2} \leq \left(\sum_{k=0}^{r-1} b_k \|\mathbf{E}_0(S_{2^k}^2)\|_p \right)^2 \leq \left(\sum_{k=0}^{r-1} \|\mathbf{E}_0(S_{2^k}^2)\|_{p/2} \right)^2.$$

Using then inequality (29), we derive that

$$(30) \quad \|\mathbf{E}_0(S_n^2)\|_{p/2} \ll n \|\mathbf{E}_0(X_1^2)\|_{p/2} + n \left(\sum_{j=0}^{r-1} \frac{\|\mathbf{E}_0(S_{2^j})\|_p}{2^{j/2}} \right)^2.$$

Since $(\|\mathbf{E}_0(S_n)\|_p)_{n \geq 1}$ is subadditive, using item (1) of Lemma 37, inequality (28) follows from (30). \square

3.3. Rosenthal inequalities for martingales and the case of even powers.

3.3.1. *The martingale case.* For any real $p > 2$, Theorem 6 applied to stationary martingale differences gives the following inequality:

$$\mathbf{E} \left(\max_{1 \leq j \leq n} |S_j|^p \right) \ll n \mathbf{E}(|X_1|^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^\delta \right)^{p/(2\delta)},$$

where $\delta = \min(1, 1/(p - 2))$.

Since for stationary martingale differences we have $\mathbf{E}(S_n^2) = n\mathbf{E}(X_1^2)$, we can express the inequality in the following form useful for applications:

$$(31) \quad \mathbf{E} \left(\max_{1 \leq j \leq n} |S_j|^p \right) \ll n^{p/2} (\mathbf{E}(X_1^2))^{p/2} + n \mathbf{E}(|X_1|^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^\delta \right)^{p/(2\delta)}.$$

As we shall see in the next result, for a stationary sequence $(d_i)_{i \in \mathbb{Z}}$ of martingale differences in \mathbf{L}^p with $p \geq 4$ an even integer, this inequality can be sharpened since it holds with $\delta = 2/(p - 2)$ (see Comment 7). As a consequence, we recover, in case $p = 4$, inequality (1.6) stated in Rio (2009) that was obtained for variables in \mathbf{L}^q with $q = p/2$, by using Burkholder’s classical inequality combined with Theorem 3 in Wu and Zhao (2008). Notice that inequality (1.6) stated in Rio (2009) cannot be generalized for $p > 4$ since Theorem 3 in Wu and Zhao (2008) is only valid for variables in \mathbf{L}^q with $1 < q \leq 2$.

THEOREM 14. *Let $p \geq 4$ be an even integer, and let d_0 be a real random variable in \mathbf{L}^p , measurable with respect to \mathcal{F}_0 and such that $\mathbf{E}(d_0 | \mathcal{F}_{-1}) = 0$. Let $d_i = d_0 \circ T^i$ and $S_n = \sum_{i=1}^n d_i$. Then for any integer n ,*

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll n \mathbf{E}(|d_1|^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\right)^{p(p-2)/4}.$$

The technique that makes this result possible is a special symmetrization for martingales initiated by Kwapien and Woyczyński (1991).

PROPOSITION 15. *Assume that $(e_k)_k$ are stationary martingale differences adapted to an increasing filtration $(\mathcal{F}_k)_k$ that are conditionally symmetric (the distribution of e_k given \mathcal{F}_{k-1} is equal to the distribution of $-e_k$ given \mathcal{F}_{k-1}). Assume, in addition, that the e_k ’s are conditionally independent given a sigma algebra \mathcal{G} , and such that the law of e_k given \mathcal{G} is the same as the law of e_k given \mathcal{F}_{k-1} . Let $S_n = \sum_{i=1}^n e_i$. Then for any even integer $p \geq 4$ and any integer $n \geq 1$,*

$$(32) \quad \mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll n \mathbf{E}(|e_1|^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\right)^{p(p-2)/4}.$$

PROOF. Due to Doob’s maximal inequality, we have that $\|\max_{1 \leq j \leq n} |S_j|\|_p \leq q \|S_n\|_p$ where $q = p(p - 1)^{-1}$. Then it suffices to show that inequality (32) holds for $\mathbf{E}(|S_n|^p)$. We shall base this proof again on dyadic induction. Denote $\bar{S}_n = e_{n+1} + \dots + e_{2n}$ and $a_n = \|S_n\|_p$.

We start from inequality (25). Since the sequence of martingale differences (e_k) is conditionally symmetric and conditionally independent given a master sigma algebra \mathcal{G} , we have $\mathbf{E}(S_n^{p-1} \bar{S}_n) + \mathbf{E}(S_n \bar{S}_n^{p-1}) = 0$ and therefore

$$a_{2n}^p \leq 2a_n^p + 2^p (\mathbf{E}(S_n^{p-2} \bar{S}_n^2) + \mathbf{E}(\bar{S}_n^{p-2} S_n^2)).$$

Using Lemma 34, we have that

$$\mathbf{E}(S_n^{p-2} \bar{S}_n^2) \leq a_n^{p-2} \|\mathbf{E}_0(S_n^2)\|_{p/2}$$

and

$$\mathbf{E}(S_n^2 \bar{S}_n^{p-2}) \leq a_n^{p-4/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)}.$$

Therefore, by combining all these bounds, we obtain for every even integer $p \geq 4$,

$$\begin{aligned} a_{2n}^p &\leq 2a_n^p + 2^p(a_n^{p-2} \|\mathbf{E}_0(S_n^2)\|_{p/2} + a_n^{p-4/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)}) \\ &\leq 2a_n^p + 2^{p+1} a_n^{p-4/(p-2)} \|\mathbf{E}_0(S_n^2)\|_{p/2}^{2/(p-2)}. \end{aligned}$$

We end the proof by using Lemma 11. \square

PROOF OF THEOREM 14. We consider our martingale differences sequence $(d_k)_k$, and we construct two decoupled tangent versions $(e_k)_k$ and $(\tilde{e}_k)_k$ to $(d_k)_k$ that are \mathcal{G} -conditionally independent between them [here $\mathcal{G} = \sigma(\{d_i\})$]. This means that these two sequences are martingale differences with the additional property that the conditional distribution of d_k given \mathcal{F}_{k-1} is equal to the distribution of e_k given \mathcal{F}_{k-1} and also to the distribution of \tilde{e}_k given \mathcal{F}_{k-1} ; see Definition 6.1.4 in de la Peña and Giné (1999) for the definition of a decoupled tangent sequence, and their Proposition 6.1.5. for the crucial fact that decoupled sequences always exist. We refer also to their Remark 6.1.6, for the construction of $(e_k)_k$ and $(\tilde{e}_k)_k$. Therefore, for any even integer p ,

$$\mathbf{E}\left(\sum_{i=1}^n d_i\right)^p = \mathbf{E}\left(\sum_{i=1}^n (d_i - \mathbf{E}(e_i|\mathcal{G}))\right)^p \leq \mathbf{E}\left(\sum_{i=1}^n (d_i - e_i)\right)^p.$$

Now we use Corollary 6.6.8. in de la Peña and Giné (1999); see also Zinn (1985). Since $(e_i - \tilde{e}_i)_i$ is a decoupled tangent sequence of $(d_i - e_i)_i$, it follows that

$$\mathbf{E}\left(\sum_{i=1}^n d_i\right)^p \ll \mathbf{E}\left(\sum_{i=1}^n (e_i - \tilde{e}_i)\right)^p.$$

Notice that the distribution of $e_i - \tilde{e}_i$ is conditionally symmetric given \mathcal{G} . Therefore, using Doob’s maximal inequality and applying Proposition 15, we obtain that for every even integer $p \geq 4$ and any integer n ,

$$\begin{aligned} &\mathbf{E}\left(\max_{1 \leq k \leq n} \left(\sum_{i=1}^k d_i\right)^p\right) \\ &\ll \mathbf{E}\left(\sum_{i=1}^n (e_i - \tilde{e}_i)\right)^p \\ (33) \quad &\ll n \mathbf{E}(|e_1 - \tilde{e}_1|^p) \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \left\| \mathbf{E}_0\left(\left(\sum_{i=1}^k (e_i - \tilde{e}_i)\right)^2\right) \right\|_{p/2}^{2/(p-2)}\right)^{p(p-2)/4}. \end{aligned}$$

Notice now that $\sum_{i=1}^k \mathbf{E}(e_i^2 | \mathcal{F}_{i-1}) = \sum_{i=1}^k \mathbf{E}(d_i^2 | \mathcal{F}_{i-1})$ since both quantities are obtained using only the conditional distributions of the d_i 's and e_i 's, respectively, and these two sequences are tangent. Tangency also implies that d_i and e_i have the same distributions. Hence $\|d_1\|_p = \|e_1\|_p$. For the same reasons, we also have $\sum_{i=1}^k \mathbf{E}(\tilde{e}_i^2 | \mathcal{F}_{i-1}) = \sum_{i=1}^k \mathbf{E}(d_i^2 | \mathcal{F}_{i-1})$ and $\|d_1\|_p = \|\tilde{e}_1\|_p$. Therefore, $\|e_1 - \tilde{e}_1\|_p \leq 2\|d_1\|_p$ and

$$\begin{aligned} \left\| \mathbf{E}_0 \left(\left(\sum_{i=1}^k (e_i - \tilde{e}_i) \right)^2 \right) \right\|_{p/2} &\leq 2 \left\| \mathbf{E}_0 \left(\sum_{i=1}^k \mathbf{E}(e_i^2 | \mathcal{F}_{i-1}) \right) \right\|_{p/2} \\ &\quad + 2 \left\| \mathbf{E}_0 \left(\sum_{i=1}^k \mathbf{E}(\tilde{e}_i^2 | \mathcal{F}_{i-1}) \right) \right\|_{p/2} \\ &= 4 \left\| \mathbf{E}_0 \left(\sum_{i=1}^k \mathbf{E}(d_i^2 | \mathcal{F}_{i-1}) \right) \right\|_{p/2} \\ &= 4 \left\| \sum_{i=1}^k \mathbf{E}_0(d_i^2) \right\|_{p/2}. \end{aligned}$$

Theorem 14 follows by introducing these bounds in inequality (33). \square

3.3.2. *Application to stationary processes via martingale approximation.* Theorem 14 together with the martingale approximation provide an alternative Rosenthal-type inequality involving the projection operator, very useful for analyzing linear processes.

The next lemma is a slight reformulation of the martingale approximation result that can be found in the paper by Wu and Woodroffe (2004), Theorem 1; see also Zhao and Woodroffe (2008) and Gordin and Peligrad (2011).

LEMMA 16. *Let $p \geq 1$, and let (X_n) be an adapted stationary sequence in the sense of Notation 1. Then there is a triangular array of row-wise stationary martingale differences satisfying*

$$(34) \quad D_0^n = \frac{1}{n} \sum_{i=1}^n (\mathbf{E}_1(S_i) - \mathbf{E}_0(S_i)); \quad D_k^n = D_0^n \circ T^k,$$

such that for any $1 \leq k \leq n$ we have

$$(35) \quad S_k = M_k^n + R_k^n \quad \text{where } M_k^n = \sum_{i=1}^k D_i^n$$

and

$$\max_{1 \leq k \leq n} \|R_k^n\|_p \leq 2\|X_0\|_p + \frac{3}{n} \sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p.$$

We state now the Rosenthal-type inequality that we shall establish with the help of the approximation result above.

THEOREM 17. *Let $p \geq 4$ be an even integer, and let $(X_i)_{i \in \mathbb{Z}}$ be as in Theorem 6. Then the following inequality is valid: for any integer n ,*

$$\begin{aligned} \mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) &\ll n \|X_0\|_p^p + n^{1-p} \left(\sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p\right)^p \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)}\right)^{p(p-2)/4}. \end{aligned}$$

REMARK 18. Theorems 6 and 17 are, in general, not comparable. Indeed, for $p \geq 4$, Theorem 6 applies with $\delta = 1/(p-2)$, so the last term of the inequality stated in Theorem 17 can be bounded by the last term in the inequality from Theorem 6 [see item (2) of Comment 7]. However, the second term in Theorem 17 gives additional contribution.

PROOF OF THEOREM 17. We bound first $\max_{1 \leq k \leq n} \mathbf{E}(|S_k|^p)$. By the martingale approximation of Lemma 16 combined with Theorem 14, we get that

$$\begin{aligned} \max_{1 \leq k \leq n} \mathbf{E}(|S_k|^p) &\ll \max_{1 \leq k \leq n} \mathbf{E}(|M_k^n|^p) + \max_{1 \leq k \leq n} \mathbf{E}(|R_k^n|^p) \\ &\ll n \mathbf{E}(|D_0^n|^p) + \|X_0\|_p^p + \left(\frac{1}{n} \sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p\right)^p \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0((M_k^n)^2)\|_{p/2}^{2/(p-2)}\right)^{p(p-2)/4}. \end{aligned}$$

It remains to analyze the first and the last terms. By (35) applied with $k = 1$, we notice that $D_1^n = X_1 - R_1^n$, and by the triangle inequality,

$$\|D_0^n\|_p^p \leq (\|X_0\|_p + \|R_1^n\|_p)^p \ll \|X_0\|_p^p + \frac{1}{n^p} \left(\sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p\right)^p.$$

For analyzing the last term, we use the fact that

$$\mathbf{E}_0((M_k^n)^2) \leq 2\mathbf{E}_0(S_k^2) + 2\mathbf{E}_0((R_k^n)^2).$$

By using Lemma 16 it follows that

$$\|\mathbf{E}_0((M_k^n)^2)\|_{p/2} \ll \|\mathbf{E}_0(S_k^2)\|_{p/2} + \|X_0\|_p^2 + n^{-2} \left(\sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p\right)^2$$

and overall

$$\begin{aligned} \max_{1 \leq k \leq n} \mathbf{E}(|S_k|^p) &\leq n \|X_0\|_p^p + n^{1-p} \left(\sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p \right)^p \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2)\|_{p/2}^{2/(p-2)} \right)^{p(p-2)/4}. \end{aligned}$$

We apply now inequality (7) that has the effect of an addition of a fourth term, namely $n(\sum_{k=1}^n k^{-1-1/p} \|\mathbf{E}_0(S_k)\|_p)^p$. However, we can express our inequality without including this term because it can be bounded, up to a multiplicative constant, by $n^{1-p}(\sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p)^p$. Indeed, notice that it is enough to show that for a certain universal constant C ,

$$\max_{1 \leq i \leq n} \|\mathbf{E}_0(S_i)\|_p \leq \frac{C}{n} \sum_{i=1}^n \|\mathbf{E}_0(S_i)\|_p.$$

To prove it, we first notice that

$$\max_{1 \leq i \leq n} \|\mathbf{E}_0(S_i)\|_p \leq \max_{1 \leq i \leq [n/2]} \|\mathbf{E}_0(S_i)\|_p + \max_{[n/2] < i \leq n} \|\mathbf{E}_0(S_i)\|_p$$

and that for any $i \in \{1, \dots, [n/2]\}$,

$$\|\mathbf{E}_0(S_i)\|_p \leq \|\mathbf{E}_0(S_{i+[n/2]})\|_p + \|\mathbf{E}_0(S_{i+[n/2]} - S_i)\|_p.$$

Therefore, by the properties of conditional expectation and stationarity, it follows that

$$\max_{1 \leq i \leq n} \|\mathbf{E}_0(S_i)\|_p \leq \|\mathbf{E}_0(S_{[n/2]})\|_p + 2 \max_{[n/2] < i \leq n} \|\mathbf{E}_0(S_i)\|_p.$$

To complete the proof, it remains to apply inequality (92) to the subadditive sequence $(\|\mathbf{E}_0(S_i)\|_p)_{i \geq 1}$. \square

3.4. *Rosenthal inequality in terms of individual summands.* For the sake of applications in this section we indicate how to estimate the terms that appear in our Rosenthal inequalities in terms of individual summands and formulate some specific inequalities. By subtracting $\mathbf{E}(S_k^2)$ and applying the triangle inequality we can reformulate all the inequalities in terms of the quantities $\mathbf{E}(S_k^2)$, $\|\mathbf{E}_0(S_k)\|_p$ and $\|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}$. The next lemma proposes a simple way to estimate these quantities in terms of coefficients in the spirit of [Gordin \(1969\)](#).

LEMMA 19. *Under the stationary setting assumptions in Notation 1, we have the following estimates:*

$$(36) \quad \mathbf{E}(S_k^2) \leq 2k \sum_{j=0}^{k-1} |\mathbf{E}(X_0 X_j)|,$$

$$(37) \quad \|\mathbf{E}_0(S_k)\|_p \leq \sum_{\ell=1}^n \|\mathbf{E}_0(X_\ell)\|_p$$

and

$$(38) \quad \begin{aligned} & \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2} \\ & \leq 2 \sum_{i=1}^k \sum_{j=0}^{k-i} \|\mathbf{E}_0(X_i X_{i+j}) - \mathbf{E}(X_i X_{i+j})\|_{p/2} \\ & \leq 2 \sum_{i=1}^k \sum_{j=0}^{k-i} \sup_{\ell \geq 0} \|\mathbf{E}_0(X_i X_{i+\ell}) - \mathbf{E}(X_i X_{i+\ell})\|_{p/2} \wedge (2\|X_0 \mathbf{E}_0(X_j)\|_{p/2}) \\ & \leq 4 \sum_{j=1}^k j \|X_0 \mathbf{E}_0(X_j)\|_{p/2} + 2 \sum_{i=1}^k i \sup_{j \geq i} \|\mathbf{E}_0(X_i X_j) - \mathbf{E}(X_i X_j)\|_{p/2}. \end{aligned}$$

Mixing coefficients are useful to continue the estimates from Lemma 19. We refer to the books by Bradley [(2007), Theorem 4.13 via Remark 4.7, item VI], Rio [(2000), Theorem 2.5 and Appendix, Section C] and Dedecker et al. [(2007), Remark 2.5 and Chapter 3] for various estimates of the coefficients involved in Lemma 19 and examples. We shall also provide applications and explicit computations of the quantities involved.

We formulate the following proposition:

PROPOSITION 20. *Let $p > 2$ be a real number, and let $(X_i)_{i \in \mathbf{Z}}$ be a stationary sequence of real-valued random variables in \mathbf{L}_p adapted to an increasing filtration (\mathcal{F}_i) . For any $j \geq 1$, let*

$$(39) \quad \lambda(j) = \max\left(\|X_0 \mathbf{E}_0(X_j)\|_{p/2}, \sup_{i \geq j} \|\mathbf{E}_0(X_i X_j) - \mathbf{E}(X_i X_j)\|_{p/2}\right).$$

Then for every positive integer n ,

$$\begin{aligned} \left\| \max_{1 \leq j \leq n} |S_j| \right\|_p & \ll n^{1/2} \left(\sum_{k=0}^{n-1} |\mathbf{E}(X_0 X_k)| \right)^{1/2} \\ & + n^{1/p} \|X_1\|_p + cn^{1/p} \sum_{k=1}^n \frac{1}{k^{1/p}} \|\mathbf{E}_0(X_k)\|_p \\ & + n^{1/p} \left(\sum_{k=1}^n \frac{1}{k^{(2/p)-1}} (\log k)^\gamma \lambda(k) \right)^{1/2}, \end{aligned}$$

where γ can be taken $\gamma = 0$ for $2 < p \leq 3$ and $\gamma > p - 3$ for $p > 3$; $c = 1$ for $2 < p < 4$ and $c = 0$ for $p \geq 4$. The constant that is implicitly involved in the notation \ll depends on p and γ , but it does not depend on n and on the X_i 's.

PROOF. The proof of this proposition is basically a combination of Theorem 6 and Lemma 19. By the triangle inequality,

$$\|\mathbf{E}_0(S_k^2)\|_{p/2} \leq \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2} + \mathbf{E}_0(S_k^2).$$

By (36), for any $p > 2$ and any $\delta > 0$, we easily obtain

$$\left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} (\mathbf{E}(S_k^2))^\delta\right)^{1/(2\delta)} \ll n^{1/2-1/p} \left(\sum_{j=0}^{n-1} |\mathbf{E}(X_0 X_j)|\right)^{1/2}.$$

Then we use inequality (37) and changing the order of summation

$$\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \ll \sum_{k=1}^n \frac{1}{k^{1/p}} \|\mathbf{E}_0(X_k)\|_p.$$

Now for the situation $0 < \delta < 1$, by Hölder’s inequality,

$$\begin{aligned} &\left(\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^\delta\right)^{p/(2\delta)} \\ &\ll \left(\sum_{k=1}^n \frac{(\log k)^\gamma}{k^{1+2/p}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}\right)^{p/2}, \end{aligned}$$

where $\gamma > 1/\delta - 1$. We continue the proof by using (38) and get

$$\sum_{k=1}^n \frac{(\log k)^\gamma}{k^{1+2/p}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2} \ll \sum_{k=1}^n \frac{(\log k)^\gamma}{k^{(2/p)-1}} \lambda(k).$$

Proposition 20 follows by using Theorem 6 combined with all the above upper bounds. \square

We give now a consequence of Theorem 6 that will be used in one of our applications. The proof is omitted since it is in the spirit of the proof of Proposition 20; namely, we use Lemma 19 combined with Hölder’s inequality and the fact that $\|\mathbf{E}_0(S_k)\|_p \leq \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^{1/2} + (\mathbf{E}(S_k^2))^{1/2}$.

PROPOSITION 21. *Let $p > 2$ be a real number, and let $(X_i)_{i \in \mathbf{Z}}$ be a stationary sequence of real-valued random variables in \mathbf{L}_p adapted to an increasing filtration (\mathcal{F}_i) . Let $(\lambda(j))_{j \geq 1}$ be defined by (39). For every positive integer n , the following inequality holds: for any $\varepsilon > 0$,*

$$\begin{aligned} \mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) &\ll n^{p/2} \left(\sum_{k=0}^{n-1} |\mathbf{E}(X_0 X_k)|\right)^{p/2} + n \mathbf{E}(|X_1|^p) \\ &\quad + n \sum_{k=1}^n k^{p-2+\varepsilon} \lambda^{p/2}(k). \end{aligned}$$

The constant that is implicitly involved in the notation \ll depends on p and ε , but it does not depend on n .

4. Applications and examples. As we have seen, Propositions 2 and 5 give a direct approach to compare the moments of order p of the maximum of the partial sums to the moments of order p of the partial sum. We start this section by presenting two additional applications of these propositions to the convergence of maximum of partial sums and to the maximal Bernstein inequality for dependent structures. In the last three examples, we apply our results on the Rosenthal-type inequalities to different classes of processes. In all the examples given in Sections 4.3, 4.4 and 4.5, we show that, for real numbers p larger than 2, the order of magnitude of $\|\max_{1 \leq k \leq n} |S_k|\|_p$ is essentially given by the order of magnitude of $\|S_n\|_2$ (or by a bound of it).

4.1. Convergence of the maximum of partial sums in \mathbf{L}^p .

COROLLARY 22. Let $p > 1$, and let $(X_i)_{i \in \mathbf{Z}}$ be a strictly stationary sequence of centered real-valued random variables in \mathbf{L}^p adapted to an increasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbf{Z}}$. Assume that

$$(40) \quad \lim_{n \rightarrow \infty} n^{-1/p} \|S_n\|_p = 0.$$

Assume, in addition, that

$$(41) \quad \sum_{n \geq 1} \frac{\|\mathbf{E}(S_n | \mathcal{F}_0)\|_p}{n^{1+1/p}} < \infty.$$

Then

$$(42) \quad \lim_{n \rightarrow \infty} n^{-1/p} \left\| \max_{1 \leq k \leq n} |S_k| \right\|_p = 0.$$

REMARK 23. This corollary is particularly useful for studying the asymptotic behavior of a partial sum via a martingale approximation. Assume there exists a strictly stationary sequence $(d_i)_{i \in \mathbf{Z}}$ of martingale differences with respect to $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ that are in \mathbf{L}^p , such that $\lim_{n \rightarrow \infty} n^{-1/p} \|S_n - \sum_{i=1}^n d_i\|_p = 0$. Then, if the condition (41) holds for the sequence $(X_i)_{i \in \mathbf{Z}}$, by a construction in Zhao and Woodroffe (2008) and by the uniqueness of the martingale approximation, the sequence $(X_i - d_i)_{i \in \mathbf{Z}}$ is still a strictly stationary sequence and by our theorem $\lim_{n \rightarrow \infty} n^{-1/p} \|\max_{1 \leq k \leq n} |S_k - \sum_{i=1}^k d_i|\|_p = 0$. As a matter of fact, for $p = 2$, our corollary leads to the functional form of the central limit theorem for $\{n^{-1/2} S_{[nt]}, t \in [0, 1]\}$; see also Theorem 1.1 in Peligrad and Utev (2005).

PROOF OF COROLLARY 22. Let m be an integer and $k = k_{n,m} = [n/m]$ (where $[x]$ denotes the integer part of x).

The initial step of the proof is to divide the variables in blocks of size m and to make the sums in each block. Let

$$X_{i,m} = \sum_{j=(i-1)m+1}^{im} X_j, \quad i \geq 1.$$

Notice first that

$$\begin{aligned} \left\| \sup_{t \in [0,1]} \left| \sum_{j=1}^{[nt]} X_j - \sum_{i=1}^{[kt]} X_{i,m} \right| \right\|_p &\leq \left\| \sup_{t \in [0,1]} \left| \sum_{i=[kt]m+1}^{[nt]} X_i \right| \right\|_p \\ &\leq m \left\| \max_{1 \leq i \leq n} X_i \right\|_p. \end{aligned}$$

Since for every $\varepsilon > 0$,

$$\mathbf{E} \left(\max_{1 \leq i \leq n} |X_i|^p \right) \leq \varepsilon^p + \sum_{i=1}^n \mathbf{E} (|X_i|^p \mathbf{1}_{\{|X_i| > \varepsilon\}}),$$

and since $\|X_i\|_p < \infty$ for all i , we derive that $\lim_{n \rightarrow \infty} \|\max_{1 \leq i \leq n} X_i\|_p / n^{1/p} = 0$. Hence, in order to prove (42), it remains to show that

$$(43) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-1/p} \left\| \sup_{t \in [0,1]} \left| \sum_{i=1}^{[kt]} X_{i,m} \right| \right\|_p = 0.$$

Applying Proposition 2 to the variables $(X_{i,m})_{1 \leq i \leq k}$ which are adapted with respect to \mathcal{F}_{im} , and taking into account Remark 4, we get that

$$\left\| \sup_{t \in [0,1]} \left| \sum_{i=1}^{[kt]} X_{i,m} \right| \right\|_p \ll \max_{1 \leq j \leq k} \left\| \sum_{\ell=1}^{jm} X_\ell \right\|_p + k^{1/p} \sum_{j=1}^k \frac{\|\mathbf{E}(S_{jm} | \mathcal{F}_0)\|_p}{j^{1+1/p}},$$

where for the last term we used the fact that for any positive integer u , $\|\mathbf{E}(\sum_{j=um2^\ell+1}^{(u+1)m2^\ell} X_j | \mathcal{F}_{um2^\ell})\|_p = \|\mathbf{E}(\sum_{j=1}^{m2^\ell} X_j | \mathcal{F}_0)\|_p$. Condition (40) implies that

$$\max_{1 \leq j \leq k} \left\| \sum_{\ell=1}^{jm} X_\ell \right\|_p = o((km)^{1/p}) = o(n^{1/p}).$$

Now, by the subadditivity of the sequence $(\|\mathbf{E}(S_n | \mathcal{F}_0)\|_p)_{n \geq 1}$ and applying Lemma 37 we have

$$(44) \quad n^{-1/p} k^{1/p} \sum_{j=1}^k \frac{\|\mathbf{E}(S_{jm} | \mathcal{F}_0)\|_p}{j^{1+1/p}} \ll \sum_{\ell=1}^m \frac{\|\mathbf{E}(S_\ell | \mathcal{F}_0)\|_p}{(\ell + m)^{1+1/p}} + \sum_{j \geq m} \frac{\|\mathbf{E}(S_j | \mathcal{F}_0)\|_p}{j^{1+1/p}}.$$

Hence, by (41) and by using the dominated convergence theorem for discrete measures applied to the first term in the right-hand side of (44), we get that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{-1/p} k^{1/p} \sum_{j=1}^k \frac{\|\mathbf{E}(S_{jm} | \mathcal{F}_0)\|_p}{j^{1+1/p}} = 0,$$

which ends the proof of the corollary. \square

4.2. *Maximal exponential inequalities for strong mixing sequences.* Let us first recall the definition of strongly mixing sequences, introduced by Rosenblatt (1956): for any two σ algebras \mathcal{A} and \mathcal{B} , we define the α -mixing coefficient by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)|.$$

Let $(X_k, k \geq 1)$ be a sequence of real-valued random variables defined on $(\Omega, \mathcal{A}, \mathbf{P})$. This sequence will be called strongly mixing if

$$(45) \quad \alpha(n) := \sup_{k \geq 1} \alpha(\mathcal{F}_k, \mathcal{G}_{k+n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{F}_j := \sigma(X_i, i \leq j)$ and $\mathcal{G}_j := \sigma(X_i, i \geq j)$ for $j \geq 1$.

In 2009, Merlevède, Peligrad and Rio proved (see their Theorem 2) that for a strongly mixing sequence of centered random variables satisfying $\sup_{i \geq 1} \|X_i\|_\infty \leq M$ and for a certain $c > 0$

$$(46) \quad \alpha(n) \leq \exp(-cn),$$

the following Bernstein-type inequality is valid: there is a constant C depending only on c such that for all $n \geq 2$,

$$(47) \quad \mathbf{P}(|S_n| \geq x) \leq \exp\left(-\frac{Cx^2}{v^2n + M^2 + xM(\log n)^2}\right),$$

where

$$(48) \quad v^2 = \sup_{i > 0} \left(\text{Var}(X_i) + 2 \sum_{j > i} |\text{Cov}(X_i, X_j)| \right).$$

Proving the maximal version of inequality (47) cannot be handled directly using either Theorem 2.2 in Móricz, Serfling and Stout (1982) or using Theorem 1 in Kevei and Mason (2011) since the right-hand side of (47) does not satisfy the assumptions of these papers. However, an application of Proposition 5 leads to the maximal version of Theorem 2 in Merlevède, Peligrad and Rio (2009).

COROLLARY 24. *Let $(X_j)_{j \geq 1}$ be a sequence of centered real-valued random variables. Suppose that there exists a positive M such that $\sup_{i \geq 1} \|X_i\|_\infty \leq M$ and that the strongly mixing coefficients $(\alpha(n))_{n \geq 1}$ of the sequence satisfy (46). Then there exist constants $C = C(c)$ and $K = K(M, c)$ such that for all integers $n \geq 2$ and all real $x > K \log n$,*

$$(49) \quad \mathbf{P}\left(\max_{1 \leq k \leq n} |S_k| \geq x\right) \leq \exp\left(-\frac{Cx^2}{v^2n + M^2 + xM(\log n)^2}\right).$$

PROOF. We first apply inequality (9) of Proposition 5 with $p = 2$ and $\varphi(x) = e^{t|x|}$ where t is a positive integer. According to Theorem 2 in Merlevède, Peligrad and Rio (2009), there exist positive constants C_1 and C_2 depending only on c such that for all $n \geq 2$ and any positive t such that $t < \frac{1}{C_1 M(\log n)^2}$, the following inequality holds:

$$\log \mathbf{E}(\exp(tS_n)) \leq \frac{C_2 t^2 (nv^2 + M^2)}{1 - C_1 t M(\log n)^2}.$$

Then an optimization on t gives that there is a constant C_3 depending only on c such that for all $n \geq 2$ and any positive real x ,

$$\begin{aligned} & \mathbf{P}\left(\max_{1 \leq k \leq n} |S_k| \geq 4x\right) \\ (50) \quad & \leq \exp\left(-\frac{C_3 x^2}{v^2 n + M^2 + x M(\log n)^2}\right) \\ & \quad + 4x^{-2} \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l} | \mathcal{F}_{k2^l})\|_2^2\right)^{1/2}\right)^2, \end{aligned}$$

where r is the positive integer satisfying $2^{r-1} < n \leq 2^r$ and $v = \lfloor x/M \rfloor$.

It remains to bound up the second term in the right-hand side of the above inequality. Notice first that for any centered variable Z such that $\|Z\|_\infty \leq B$, Ibragimov’s covariance inequality [see Theorem 1.11 in Bradley (2007)] gives

$$\|\mathbf{E}(Z|\mathcal{F})\|_2^2 = \text{Cov}(\mathbf{E}(Z|\mathcal{F}), Z) \leq 4B^2 \alpha(\mathcal{F}, \sigma(Z)).$$

Therefore, applying this last estimate with $Z = S_{v+(k+1)2^l} - S_{v+k2^l}$ and $\mathcal{F} = \mathcal{F}_{k2^l}$, we get that

$$\|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l} | \mathcal{F}_{k2^l})\|_2^2 \leq 4M^2 2^{2l} \alpha(v)$$

implying that

$$\left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l} | \mathcal{F}_{k2^l})\|_2^2\right)^{1/2}\right)^2 \leq 4M^2 2^{2r} (\sqrt{2} + 1)^2 \alpha(v).$$

Since $2^{2r} \leq 4n^2$ and $\lfloor x/M \rfloor \geq x/(2M)$, for $x \geq 2M$, by using (46), we get that, for any $x \geq 2M$,

$$\begin{aligned} (51) \quad & \left(\sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l}-1} \|\mathbf{E}(S_{v+(k+1)2^l} - S_{v+k2^l} | \mathcal{F}_{k2^l})\|_2^2\right)^{1/2}\right)^2 \\ & \leq 3(2^5)M^2 n^2 \exp(-cx/(2M)). \end{aligned}$$

Starting from (50) and using (51), we then derive that for any $x \geq 2M \max(1, 4c^{-1} \log n)$,

$$\mathbf{P}(S_{2r}^* \geq 4x) \leq \exp\left(-\frac{C_3 x^2}{v^2 n + M^2 + xM(\log n)^2}\right) + 96 \exp\left(-\frac{xc}{4M}\right)$$

proving inequality (49). \square

4.3. *Application to ARCH models.* Theorem 6 applies to the case where $(X_i)_{i \in \mathbf{Z}}$ has an ARCH(∞) structure as described by Giraitis, Kokoszka and Leipus (2000), that is,

$$(52) \quad X_n = \sigma_n \eta_n \quad \text{with } \sigma_n^2 = c + \sum_{j=1}^{\infty} c_j X_{n-j}^2,$$

where $(\eta_n)_{n \in \mathbf{Z}}$ is a sequence of i.i.d. centered random variables such that $\mathbf{E}(\eta_0^2) = 1$, and where $c \geq 0$, $c_j \geq 0$ and $\sum_{j \geq 1} c_j < 1$. Notice that $(X_i)_{i \in \mathbf{Z}}$ is a stationary sequence of martingale differences adapted to the filtration (\mathcal{F}_i) where $\mathcal{F}_i = \sigma(\eta_k, k \leq i)$.

Let $p > 2$ and assume that $\|\eta_0\|_p < \infty$. Notice first that

$$(53) \quad \|\mathbf{E}(X_j^2 | \mathcal{F}_0) - \mathbf{E}(X_0^2)\|_{p/2} = \|\mathbf{E}(\sigma_j^2 | \mathcal{F}_0) - \mathbf{E}(\sigma_j^2)\|_{p/2}.$$

In addition, since $\mathbf{E}(\eta_0^2) = 1$ and $\sum_{j \geq 1} c_j < 1$, the unique stationary solution to (52) is given by Giraitis, Kokoszka and Leipus (2000),

$$(54) \quad \sigma_n^2 = c + c \sum_{\ell=1}^{\infty} \sum_{j_1, \dots, j_\ell=1}^{\infty} c_{j_1} \cdots c_{j_\ell} \eta_{n-j_1}^2 \cdots \eta_{n-(j_1+\dots+j_\ell)}^2.$$

Starting from (53) and using (54), one can prove that

$$\|\mathbf{E}(X_j^2 | \mathcal{F}_0) - \mathbf{E}(X_0^2)\|_{p/2} \leq 2c \|\eta_0\|_p^2 \sum_{\ell=1}^{\infty} \ell \kappa^{\ell-1} \sum_{i=[j/\ell]}^{\infty} c_i,$$

where $\kappa = \|\eta_0\|_p^2 \sum_{j \geq 1} c_j$; see Section 6.6 in Dedecker and Merlevède (2011) for more detailed computations. Therefore, if

$$(55) \quad \|\eta_0\|_p^2 \sum_{j \geq 1} c_j < 1$$

for any $p > 2$ and $\delta \in]0, 1]$, we obtain

$$\sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^\delta \leq c_{p,\delta,X} \sum_{k=1}^n \frac{1}{k^{1+2\delta/p}} \left(\sum_{j=1}^n \sum_{i=j}^{\infty} c_i \right)^\delta,$$

where $c_{p,\delta,X}$ is a positive constant depending only on p, δ, c, κ and $\|\eta_0\|_p$. Applying Theorem 6 for the martingale case, we then get the following corollary:

COROLLARY 25. *Let $X = (X_i)_{i \in \mathbf{Z}}$ be defined by (52) and $S_n = \sum_{i=1}^n X_i$. Let $p > 2$ and assume that (55) is satisfied.*

(1) *If we assume that $\sum_{j \geq n} c_j = O(n^{-b})$ for $b > 1 - 2/p$, then, for any integer n ,*

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll (n\mathbf{E}(X_0^2))^{p/2} + n(\mathbf{E}(|X_0|^p) + b_{p,X}),$$

where $b_{p,X}$ is a positive constant depending on p and on the underlying sequence X , but not depending on n .

(2) *If we assume $\sum_{j \geq n} c_j = O(n^{-b})$ for $b > 0$, then*

$$\limsup_{n \rightarrow \infty} n^{-1/2} \left\| \max_{1 \leq k \leq n} |S_k| \right\|_p \leq a_p (\mathbf{E}(X_0^2))^{1/2} < \infty,$$

where a_p is a constant depending only on p .

4.4. *Application to functions of linear processes.* Let $(a_i)_{i \in \mathbf{Z}}$ be a sequence of real numbers in ℓ^2 and $(\varepsilon_i)_{i \in \mathbf{Z}}$ be a sequence of i.i.d. random variables in L^2 . Define

$$(56) \quad X_k = h\left(\sum_{i \in \mathbf{Z}} a_i \varepsilon_{k-i}\right) - \mathbf{E}\left(h\left(\sum_{i \in \mathbf{Z}} a_i \varepsilon_{k-i}\right)\right).$$

Denote by $w_h(\cdot, M)$ the modulus of continuity of the function h on the interval $[-M, M]$, that is,

$$w_h(t, M) = \sup\{|h(x) - h(y)|, |x - y| \leq t, |x| \leq M, |y| \leq M\}.$$

We shall establish the following result:

COROLLARY 26. *Let $X = (X_k)_{k \in \mathbf{Z}}$ be defined by (56). Assume that h is γ -Hölder on any compact set, with $w_h(t, M) \leq Ct^\gamma M^\alpha$, for some $C > 0$, $\gamma \in]0, 1]$ and $\alpha \geq 0$. Let $p > 2$ and assume that $\mathbf{E}(|\varepsilon_0|^{2\vee(\alpha+\gamma)p}) < \infty$.*

(1) *If p is an even integer, and for $\lambda > p/2 - 2$ if $p > 4$ and $\lambda = 0$ if $p = 4$,*

$$(57) \quad \sum_{i \geq 1} i^{1-2/p} (\log i)^\lambda \left(\sum_{j \geq i} a_j^2\right)^{\gamma/2} < \infty,$$

then, for any integer n ,

$$\mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll (\mathbf{E}(S_n^2))^{p/2} + n(\mathbf{E}(|X_0|^p) + b_{p,X}),$$

where $b_{p,X}$ is a positive constant depending on p and on the underlying sequence X , but not depending on n .

(2) If for some $\eta > 1$

$$(58) \quad \left(\sum_{j \geq i} a_j^2 \right)^{\gamma/2} \ll i^{-\eta},$$

then

$$\limsup_{n \rightarrow \infty} n^{-1/2} \left\| \max_{1 \leq k \leq n} |S_k| \right\|_p \leq a_p \left(\sum_{k \in \mathbf{Z}} \mathbf{E}(X_0 X_k) \right)^{1/2} < \infty,$$

where a_p is a constant depending only on p .

REMARK 27. The proof of the first item of the above result is based on Theorem 17. Our proof reveals that an application of Theorem 6 would involve a more restrictive condition on λ , namely $\lambda > p - 3$.

As a preliminary step in the proof of Corollary 26 we state the following proposition, which is a direct consequence of the proof of Proposition 4.2 and of Theorem 4.2 in Dedecker, Merlevède and Rio (2009), page 988.

PROPOSITION 28. Let $(X_i)_{i \in \mathbf{Z}}$ be as in Corollary 26 and $p \geq 2$. Let $(\varepsilon'_i)_{i \in \mathbf{Z}}$ be an independent copy of $(\varepsilon_i)_{i \in \mathbf{Z}}$, and denote $V_0 = \sum_{i \geq 0} a_i \varepsilon_{-i}$ and

$$M_{1,i} = |V_0| \vee \left| \sum_{0 \leq j < i} a_j \varepsilon_{-j} + \sum_{j \geq i \geq 0} a_j \varepsilon'_{-j} \right|$$

and

$$\tilde{w}_h(i) = \left\| w_h \left(\left| \sum_{k \geq i} a_k \varepsilon_{-k} \right|, M_{1,i} \right) \right\|_p.$$

Then for any $i, j \geq 0$,

$$\|\mathbf{E}_0(X_i)\|_p \leq 2\tilde{w}_h(i)$$

and

$$\begin{aligned} & \|\mathbf{E}_0(X_i X_{j+i}) - \mathbf{E}(X_i X_{j+i})\|_{p/2} \\ & \leq 2\|X_0\|_p \min(\tilde{w}_h(i) + \tilde{w}_h(i+j), \tilde{w}_h(j - \lfloor j/2 \rfloor)). \end{aligned}$$

Moreover, if $w_h(t, M)$ and ε_0 are as in Corollary 26, then for all $i \geq 0$

$$\tilde{w}_h(i) \leq K \left(\sum_{j \geq i} a_j^2 \right)^{\gamma/2},$$

where K is a constant depending on $p, C, \alpha, \gamma, \|\varepsilon_0\|_{2 \vee (\alpha + \gamma)p}$ and on $\sum_{j \geq 0} a_j^2$.

PROOF OF COROLLARY 26. Notice first that by Proposition 28 and relation (58), we get

$$(59) \quad \sum_{k \geq 1} \|\mathbf{E}_0(X_k)\|_p < \infty.$$

For any even integer $p \geq 4$, applying Theorem 17 we obtain, via relation (59) and the triangular inequality, that

$$\begin{aligned} & \mathbf{E}\left(\max_{1 \leq j \leq n} |S_j|^p\right) \\ & \ll n(1 + \mathbf{E}(|X_0|^p)) + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} (\mathbf{E}(S_k^2))^{2/(p-2)} \right)^{p(p-2)/4} \\ & \quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^{2/(p-2)} \right)^{p(p-2)/4}. \end{aligned}$$

Notice now that relation (59) implies the so-called coboundary decomposition; see Theorem 5.4 in Hall and Heyde (1980); namely, there is a constant $K_{p,X}$ such that for all $n \geq 1$

$$S_n = M_n + R_n \quad \text{with } \|R_n\|_p \leq 2 \sum_{k \geq 1} \|\mathbf{E}_0(X_k)\|_p = K_{p,X},$$

where M_n is a martingale in \mathbf{L}^p with stationary differences. It easily follows that for all $1 \leq k \leq n$,

$$\frac{\mathbf{E}(S_k^2)}{k} \leq 4 \frac{\mathbf{E}(S_n^2)}{n} + 2 \frac{\mathbf{E}(R_k^2)}{k} + 4 \frac{\mathbf{E}(R_n^2)}{n} \leq 4 \frac{\mathbf{E}(S_n^2)}{n} + 6 \frac{K_{p,X}}{k}.$$

So,

$$n \left(\sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} (\mathbf{E}(S_k^2))^{2/(p-2)} \right)^{p(p-2)/4} \ll (\mathbf{E}(S_n^2))^{p/2} + K_{p,X}n.$$

Therefore, the first part of the corollary follows if we prove that (57) implies that

$$(60) \quad \sum_{k \geq 1} \frac{1}{k^{1+4/p(p-2)}} \|\mathbf{E}_0(S_k^2) - \mathbf{E}(S_k^2)\|_{p/2}^{2/(p-2)} < \infty.$$

Notice now that by Hölder’s inequality (when $p > 4$) and (38), in order to prove (60), it suffices to prove that for $\lambda > p/2 - 2$ if $p > 4$ and $\lambda = 0$ if $p = 4$,

$$\sum_{k \geq 1} \frac{(\log k)^\lambda}{k^{1+2/p}} \sum_{i=1}^k \sum_{j=0}^{k-i} \|\mathbf{E}_0(X_i X_{j+i}) - \mathbf{E}(X_i X_{j+i})\|_{p/2} < \infty.$$

The first part of Corollary 26 is then a consequence of Proposition 28. The second part follows in the same way by using Theorem 6, simple computations and the fact that under (59), $\lim_{n \rightarrow \infty} n^{-1} \mathbf{E}(S_n)^2 = \sum_{k \in \mathbf{Z}} \mathbf{E}(X_0 X_k)$. \square

4.5. *Application to a stationary reversible Markov chain.* First we want to mention that all our results can be formulated in the Markov chain setting. We assume that $(\zeta_n)_{n \in \mathbb{Z}}$ denotes a stationary Markov chain defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values in a measurable space (E, \mathcal{E}) . The marginal distribution and the transition kernel are denoted by $\pi(A) = \mathbf{P}(\zeta_0 \in A)$ and $Q(\zeta_0, A) = \mathbf{P}(\zeta_1 \in A | \zeta_0)$. In addition Q denotes the operator acting via $(Qf)(\zeta) = \int_E f(s)Q(\zeta, ds)$. Next, let f be a function on E such that $\int_E |f|^p d\pi < \infty$ and $\int_E f d\pi = 0$. Denote by \mathcal{F}_k the σ -field generated by ζ_i with $i \leq k$, $X_i = f(\zeta_i)$, and $S_n(f) = \sum_{i=1}^n X_i$. Notice that any stationary sequence $(Y_k)_{k \in \mathbb{Z}}$ can be viewed as a function of a Markov process $\zeta_k = (Y_i; i \leq k)$, for the function $g(\zeta_k) = Y_k$.

The Markov chain is called reversible if $Q = Q^*$, where Q^* is the adjoint operator of Q . In this setting, an application of Theorem 9 gives the following estimate:

COROLLARY 29. *Let (ζ_n) be a reversible Markov chain. For any even integer $p \geq 4$ and any positive integer n ,*

$$\begin{aligned} \mathbf{E}\left(\max_{1 \leq k \leq n} |S_k(f)|^p\right) &\ll n \mathbf{E}(|f(\zeta_1)|^p) + n \left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k(f))\|_p\right)^p \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+2/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}\right)^{p/2} \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+2/p}} \mathbf{E}(S_k^2(f))\right)^{p/2}. \end{aligned}$$

Moreover, using Theorem 10, we obtain:

COROLLARY 30. *Let (ζ_n) be a reversible Markov chain. For any real number $p > 4$ and any positive integer n ,*

$$\begin{aligned} \mathbf{E}\left(\max_{1 \leq k \leq n} |S_k(f)|^p\right) &\ll n \mathbf{E}(|f(\zeta_1)|^p) \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}^{1/2}\right)^p \\ &\quad + n \left(\sum_{k=1}^n \frac{1}{k^{1+1/p}} \|S_k(f)\|_2\right)^p. \end{aligned}$$

This corollary is also valid for any real $2 < p \leq 4$. For this range, however, according to the Comment 7, Theorem 6 (resp., Corollary 29) gives a better bound for $p \in]2, 4[$ (resp., for $p = 4$).

For a particular example, let $E = [-1, 1]$, and let ν be a symmetric atomless law on E . The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\nu(A),$$

where δ_x denotes the Dirac measure. Assume that $\theta = \int_E |x|^{-1}\nu(dx) < \infty$. Then there is a unique invariant measure

$$\pi(dx) = \theta^{-1}|x|^{-1}\nu(dx),$$

and the stationary Markov chain $(\zeta_i)_i$ is reversible and positively recurrent.

Assume the following assumption on the measure ν : there exists a positive constant c such that for any $x \in [0, 1]$,

$$(61) \quad \frac{d\nu}{dx}(x) \leq cx^{p/2-1}(\log(1 + 1/x))^{-\lambda} \quad \text{for some } \lambda > 0.$$

As an application of Corollary 30 we shall establish:

COROLLARY 31. *Let $p > 2$ be a real number, and let $f(-x) = -f(x)$ for any $x \in E$. Assume that $|f(x)| \leq C|x|^{1/2}$ for any x in E and a positive constant C .*

(1) *Assume in addition that (61) is satisfied for $\lambda > p$. Then for any integer n ,*

$$(62) \quad \mathbf{E}\left(\max_{1 \leq k \leq n} |S_k(f)|^p\right) \ll n^{p/2} \left(\int_0^1 f^2(x)x^{-2}\nu(dx)\right)^{p/2} + n(\mathbf{E}|f(\zeta_0)|^p) + b_{p,\lambda,c,C},$$

where $b_{p,\lambda,c,C}$ is a positive constant depending on p, λ, c and C .

(2) *Assume now (61) is relaxed to*

$$\frac{d\nu}{dx}(x) \leq cx^a \quad \text{for some } a > 0.$$

Then

$$\limsup_{n \rightarrow \infty} n^{-1/2} \left\| \max_{1 \leq k \leq n} |S_k(f)| \right\|_p \leq a_p \left(\int_0^1 f^2(x)x^{-2}\nu(dx)\right)^{1/2},$$

where a_p is a constant depending only on p .

Notice that this example of reversible Markov chain has been considered by Rio [(2009), Section 4] under a slightly more stringent condition on the measure than (61). Corollary 31 then extends Proposition 4.1(b) in Rio (2009) to all real numbers $p > 2$.

PROOF OF COROLLARY 31. To get this result we shall apply Corollary 30. We start by noticing that f being an odd function, we have

$$(63) \quad \mathbf{E}(f(\zeta_k)|\zeta_0) = (1 - |\zeta_0|)^k f(\zeta_0) \quad \text{a.s.}$$

Therefore, for any $j \geq 0$,

$$\mathbf{E}(X_0 X_j) = \mathbf{E}(f(\zeta_0)\mathbf{E}(f(\zeta_j)|\zeta_0)) = \theta^{-1} \int_E f^2(x)(1 - |x|)^j |x|^{-1} \nu(dx).$$

Then, by inequality (36),

$$\begin{aligned} \mathbf{E}(S_k^2(f)) &\leq 2k\theta^{-1} \left(\int_0^1 f^2(x)x^{-1} \nu(dx) + 2 \sum_{j=1}^{k-1} \int_0^1 f^2(x)(1-x)^j x^{-1} \nu(dx) \right) \\ (64) \quad &\leq 2k\theta^{-1} \left(\int_0^1 f^2(x)x^{-1} \nu(dx) + 2 \int_0^1 f^2(x)x^{-2} \nu(dx) \right). \end{aligned}$$

Next, we give an upper bound of the quantity $\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2}$. By using the fact that for any positive k , $\pi Q^k = \pi$ we have

$$\begin{aligned} &\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \\ &\leq \sum_{k=1}^n \left(\int_E \left| (\delta_x Q^k - \pi) \left(f^2 + 2f \sum_{k=1}^{n-k} Q^k f \right) \right|^{p/2} \pi(dx) \right)^{2/p}; \end{aligned}$$

see also inequality (4.12) in Rio (2009). Now, by using the relation (63), one can prove that for any $x \in E$,

$$\left(f^2 + 2f \sum_{k=1}^n Q^k f \right)(x) = f^2(x)(1 + 2(1 - (1 - |x|)^n)(|x|^{-1} - 1));$$

see the computations in Rio (2009) leading to his relation (4.13). Then, since $|f(x)| \leq C|x|^{1/2}$, it follows that

$$\sup_{x \in E} \left| f^2(x) + 2f(x) \sum_{k=1}^n Q^k f(x) \right| \leq 2C^2.$$

Therefore, for any $p > 2$,

$$(65) \quad \|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \leq 4C^2 \sum_{k=1}^n \left(\int_E \|Q^k(x, \cdot) - \pi(\cdot)\| \pi(dx) \right)^{2/p},$$

where $\|\mu(\cdot)\|$ denotes the total variation of the signed measure μ ; see also inequality (4.15) in Rio (2009). We estimate next the coefficients of absolute regularity β_n as defined in (66). Let $a = p/2 - 1$. We shall prove now that under (61), there exists a positive constant K depending on a , c and λ , such that

$$(66) \quad 2\beta_n := \int_E \|Q^n(x, \cdot) - \pi(\cdot)\| \pi(dx) \leq K n^{-a} (\log n)^{-\lambda}.$$

Notice first that by [Lemma 2, page 75, in Doukhan, Massart and Rio (1994)], we have that

$$(67) \quad \beta_n \leq 3 \int_E (1 - |x|)^{[n/2]} \pi(dx).$$

Let $k \geq 2$ be an integer. Clearly, for any $\alpha \in]0, 1[$,

$$(68) \quad \int_0^1 (1-x)^k \pi(dx) \leq c \int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx + c \int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx.$$

Notice now that

$$\int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \leq (\alpha \log k)^{-\lambda} \int_0^1 (1-x)^k x^{a-1} dx.$$

Hence, by the properties of the Beta and Gamma functions,

$$(69) \quad \lim_{k \rightarrow \infty} k^a (\log k)^\lambda \int_0^{k^{-\alpha}} (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \leq \alpha^{-\lambda} a \Gamma(a).$$

On the other hand, we have that

$$\int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx \leq (\log 2)^{-\lambda} (1-k^{-\alpha})^k \int_0^1 x^{a-1} dx,$$

and then, since $\alpha < 1$, we easily obtain

$$(70) \quad \lim_{k \rightarrow \infty} k^a (\log k)^\lambda \int_{k^{-\alpha}}^1 (1-x)^k x^{a-1} (\log(1+1/x))^{-\lambda} dx = 0.$$

Starting from (67) and taking into account (68), (69) and (70), (66) follows. Then, by using inequality (65) combined with (66), we derive that

$$\|\mathbf{E}_0(S_n^2(f)) - \mathbf{E}(S_n^2(f))\|_{p/2} \leq C^2 K_{p,c,\lambda} n^{2/p} (\log n)^{-2\lambda/p},$$

where $K_{p,c,\lambda}$ is a positive constant depending on p, c and λ . Therefore, for any $\delta \in]0, 1[$, there exists a positive constant $b_{p,\lambda,c,C}$ depending on p, λ, c and C , such that

$$(71) \quad \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k^2(f)) - \mathbf{E}(S_k^2(f))\|_{p/2}^{1/2} \leq b_{p,\lambda,c,C} \quad \text{for } \lambda > p.$$

Considering estimates (64) and (71), the first part of Corollary 31 follows from an application of Corollary 30 (taking also into account the comment after its statement for $2 < p \leq 4$). To get the second part, we also apply Corollary 30 but with the upper bound $\beta_n = O(n^{-a})$ instead of (66). \square

5. Application to density estimation. In this section, we estimate the L^p -integrated risk for $p \geq 4$, for the kernel estimator of the unknown marginal density f of a stationary sequence $(Y_i)_{i \geq 0}$.

Applying our Theorem 6, we shall show that if the coefficients of dependence $(\beta_{2,Y}(k))_{k \geq 1}$ (see Definition 32 below) of the sequence $(Y_i)_{i \in \mathbb{Z}}$ satisfy $\beta_{2,Y}(k) =$

$O(n^{-a})$ for $a > p - 1$, then the bound of the \mathbf{L}^p -norm of the random term of the risk is of the same order of magnitude as the optimal one obtained in [Bretagnolle and Huber \(1979\)](#) in the independence setting (see their Corollary 2), provided that the density is bounded and the kernel K satisfies Assumption A_p below.

ASSUMPTION A_p . K is a BV (bounded variation) function such that

$$\int_{\mathbf{R}} |K(u)| du < \infty \quad \text{and} \quad \int_{\mathbf{R}} |K(u)|^p du < \infty.$$

DEFINITION 32. Let $(Y_i)_{i \in \mathbf{Z}}$ be a stationary sequence of real-valued random variables, and let $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$. For any positive i and j , define the random variables

$$b(\mathcal{F}_0, i, j) = \sup_{(s,t) \in \mathbf{R}^2} |\mathbf{P}(Y_i \leq t, Y_j \leq s | \mathcal{F}_0) - \mathbf{P}(Y_i \leq t, Y_j \leq s)|.$$

Define now the coefficient

$$\beta_{2,Y}(k) = \sup_{i \geq j \geq k} \mathbf{E}(b(\mathcal{F}_0, i, j)).$$

PROPOSITION 33. Let $p \geq 4$ and K be any real function satisfying Assumption A_p . Let $(Y_i)_{i \geq 0}$ be a stationary sequence with unknown marginal density f such that $\|f\|_\infty < \infty$. Define

$$X_{k,n}(x) = K(h_n^{-1}(x - Y_k)) \quad \text{and} \quad f_n(x) = \frac{1}{nh_n} \sum_{k=1}^n X_{k,n}(x),$$

where $(h_n)_{n \geq 1}$ is a sequence of positive real numbers.

(1) Assume that there exists a positive constant c such that for some $\eta > 0$ and all $n \geq 1$,

$$(72) \quad \beta_{2,Y}(n) \leq cn^{-(p-1+\eta)}.$$

Then there exist positive constants C_1 and C_2 depending on p, η and c such that for any positive integer n ,

$$\begin{aligned} & \mathbf{E} \int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \\ & \leq C_1 (nh_n)^{-p/2} \|dK\|^{p/2} \|f\|_\infty^{p/2-1} \left(\int_{\mathbf{R}} |K(u)| du \right)^{p/2} \\ (73) \quad & + C_2 (nh_n)^{1-p} \left(\int_{\mathbf{R}} |K(u)|^p du + \|dK\| \int_{\mathbf{R}} |K(u)|^{p-1} du \right. \\ & \quad \left. + \|dK\|^2 \int_{\mathbf{R}} |K(u)|^{p-2} du \right), \end{aligned}$$

where $\|dK\|$ is the total variation norm of the measure dK .

(2) Assume now, in addition, that $nh_n \rightarrow \infty$. Then we also have the following asymptotic result:

$$\begin{aligned} \limsup_{n \rightarrow \infty} (nh_n)^{-p/2} \mathbf{E} \int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \\ \leq a_{p,c,\eta} \|dK\|^{p/2} \|f\|_{\infty}^{p/2-1} \left(\int_{\mathbf{R}} |K(u)| du \right)^{p/2}, \end{aligned}$$

where $a_{p,c,\eta}$ is a constant depending only on p, c and η .

The bound obtained in Proposition 33 can be also compared to the one obtained in Theorem 3.3 in Viennet (1997) under the assumption that the strong β -mixing coefficients in the sense of Volkonskiĭ and Rozanov (1959) of the sequence $(Y_i)_{i \in \mathbf{Z}}$, denoted by $\beta_{\infty}(k)$, satisfy $\sum_{k \geq 1} k^{p-2} \beta_{\infty}(k) < \infty$. Our condition is then comparable to the one imposed by Viennet (1997) but less restrictive in the sense that many processes are such that the sequence $\beta_{2,Y}(n)$ tends to zero as $n \rightarrow \infty$ which is not the case for $\beta_{\infty}(n)$; see the examples given in Dedecker and Prieur (2007). Notice also that, for $p = 2$, inequality (73) is proven in Dedecker and Prieur [(2005), Proposition 3] under the summability of the weak β -dependence coefficients. In addition, when $p = 3$, under the condition $\sum_{k \geq 1} k \beta_{2,Y}(k) < \infty$, it can be proven that inequality (73) still holds by applying the Rosenthal-type inequality in Dedecker (2001) [see also Proposition 3 in Dedecker and Prieur (2007) for a version of the Dedecker’s inequality in case of stationary sequences]; however, his inequality does not lead to (73) when $p > 3$.

When the random variables Y_i are a function of an i.i.d. sequence, namely, $Y_i = g(\dots, \varepsilon_{i-1}, \varepsilon_i)$ where g is a measurable function, and $(\varepsilon_n, n \in \mathbf{Z})$ are i.i.d. random variables, Wu, Huang and Huang [(2010), Theorem 1] obtained an upper bound of similar order as in (73) (for $p > 1$) under conditions imposed to the so-called “predictive dependent measures.” When, in addition, the variable ε_0 has a density with bounded derivatives up to order 2, the method is especially effective for short memory linear processes with independent innovations; see Section 4.1 in Wu, Huang and Huang (2010). Our Proposition 33 complements the above cited results since the coefficients $\beta_{2,Y}(n)$ can be estimated without assuming that ε_0 has a density. For instance, we obtain upper bound (73) for $Y_i = \sum_{k \geq 0} 2^{-k-1} \varepsilon_{i-k}$, and the ε_k ’s are i.i.d. Bernoulli random variables with parameter $1/2$; see Section 6.1 of Dedecker and Prieur (2007) for computations of $\beta_{2,Y}(n)$. In addition, our Proposition 33 applies even for situations when the variables Y_i are not assumed to be a function of an i.i.d. sequence. We refer, for instance, to Dedecker and Prieur [(2009), Theorem 3.1] who gave an upper bound of the coefficients $\beta_{2,Y}(n)$ of the Markov chain associated to an intermittent map.

If we assume now that f has a derivative of order s , where $s \geq 1$ is an integer and that the following bound holds for the bias term:

$$(74) \quad \int_{\mathbf{R}} |f(x) - \mathbf{E}(f_n(x))|^p dx \leq M h_n^{sp} \|f^{(s)}\|_p^p,$$

where M is a constant depending on the kernel K , then the choice of $(nh_n)^{p/2} \times h_n^{sp} = O(1)$ leads to the following estimate:

$$(75) \quad \mathbf{E} \int_{\mathbf{R}} |f_n(x) - f(x)|^p dx = O(n^{-sp/(2s+1)}).$$

We mention that (74) holds for any Parzen kernel of order s ; see Section 4 in **Bretagnolle and Huber (1979)**. We also mention that if we only assume that $\sum_{k \geq 1} k^{p-2} \beta_{2,Y}(k) < \infty$ instead of (72) in Proposition 33, then inequality (73) is valid with $(nh_n)^{1-p} n^\varepsilon$ (for any $\varepsilon > 0$) replacing $(nh_n)^{1-p}$ in the second term of the right-hand side. In this situation, bound (74) combined with a choice of h_n of order $n^{-1/(1+2s)}$ still leads to estimate (75).

PROOF OF PROPOSITION 33. Setting $X_{i,n}(x) = K((x - Y_i)/h_n) - \mathbf{E}(K((x - Y_i)/h_n))$, we have that

$$(76) \quad \mathbf{E} \int_{\mathbf{R}} |f_n(x) - \mathbf{E}(f_n(x))|^p dx \leq (nh_n)^{-p} \int_{\mathbf{R}} \mathbf{E} \left| \sum_{i=1}^n X_{i,n}(x) \right|^p dx.$$

Starting from (76) and applying Proposition 21 to the stationary sequence $(X_{i,n}(x))_{i \in \mathbf{Z}}$, Proposition 33 follows provided we establish the following bounds (in what follows C is a positive constant which may vary from line to line, and that may depend on p, c and η but not on n):

$$(77) \quad \int_{\mathbf{R}} \mathbf{E} |X_{1,n}(x)|^p dx \leq 2^{p+1} h_n \int_{\mathbf{R}} |K(u)|^p du,$$

$$(78) \quad \int_{\mathbf{R}} \left(\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \right)^{p/2} dx$$

$$\leq Ch_n^{p/2} \|dK\|^{p/2} \|f\|_\infty^{(p/2)-1} \left(\int_{\mathbf{R}} |K(u)| du \right)^{p/2},$$

and that for $\varepsilon > 0$ small enough,

$$(79) \quad \sum_{j=1}^n j^{p-2+\varepsilon} \int_{\mathbf{R}} \|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} dx$$

$$\leq Ch_n \|dK\| \int_{\mathbf{R}} |K(u)|^{p-1} du$$

and

$$(80) \quad \sum_{j=1}^n j^{p-2+\varepsilon} \sup_{i \geq j} \int_{\mathbf{R}} \|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} dx$$

$$\leq Ch_n \|dK\|^2 \int_{\mathbf{R}} |K(u)|^{p-2} du.$$

In what follows, we shall prove these bounds. Notice first that

$$\int_{\mathbf{R}} \mathbf{E}|X_{1,n}(x)|^p dx \leq 2^{p+1} \int_{\mathbf{R}} \int_{\mathbf{R}} |K((x - y)h_n^{-1})|^p f(y) dx dy$$

proving (77) by the change of variables $u = (x - y)h_n^{-1}$. To prove (78), we first apply item (1) of Lemma 35 implying that

$$\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \leq \|dK\| \mathbf{E}(\tilde{b}(\mathcal{F}_0, n)|K((x - Y_0)/h_n)|),$$

where $\tilde{b}(\mathcal{F}_0, n) = \sum_{j=0}^{n-1} b(\mathcal{F}_0, j, j)$. An application of Hölder’s inequality as done in Viennet [(1997), page 474], then gives

$$\begin{aligned} & \int_{\mathbf{R}} \left(\sum_{j=0}^{n-1} |\mathbf{E}(X_{0,n}(x)X_{j,n}(x))| \right)^{p/2} dx \\ & \leq h_n^{p/2} \|f\|_{\infty}^{p/2-1} \mathbf{E}(\tilde{b}(\mathcal{F}_0, n))^{p/2} \left(\int_{\mathbf{R}} |K(u)| du \right)^{p/2}. \end{aligned}$$

This proves (78) since $\mathbf{E}(\tilde{b}(\mathcal{F}_0, n))^{p/2} \leq C \sum_{k=1}^n k^{p-2} \beta_{2,Y}(k)$ and

$$\sum_{k=1}^n k^{p-2} \beta_{2,Y}(k) = O(1)$$

by condition (72).

We turn now to the proof of (79). With this aim we notice that

$$\|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} = \mathbf{E}(Z_0(x)\mathbf{E}_0(X_{j,n}(x))) = \mathbf{E}(Z_0(x)X_{j,n}(x)),$$

where $Z_0(x) = |X_{0,n}|^{p/2} |\mathbf{E}_0(X_{j,n}(x))|^{p/2-1} \text{sign}(\mathbf{E}_0(X_{j,n}(x)))$. Then, by using item (1) of Lemma 35, we derive that

$$\begin{aligned} (81) \quad & \|X_{0,n}(x)\mathbf{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} = \text{Cov}(Z_0(x), K((x - Y_j)/h_n)) \\ & \leq \|dK\| \mathbf{E}(b(\mathcal{F}_0, j, j)|Z_0(x)|). \end{aligned}$$

Notice now that by using the elementary inequality: $x^\alpha y^{1-\alpha} \leq x + y$ valid for $\alpha \in [0, 1]$ and nonnegative x and y , we get that $|Z_0(x)| \leq (|X_{0,n}(x)| + |\mathbf{E}_0(X_{j,n}(x))|)^{p-1}$. Therefore, some computations involving Jensen’s inequality lead to

$$(82) \quad \int_{\mathbf{R}} |Z_0(x)| dx \leq 4^p h_n \int_{\mathbf{R}} |K(u)|^{p-1} du,$$

where $\|dK\|$ is the total variation norm of the measure dK . Starting from (81), we end the proof of (79) by taking into account (82) and the fact that

$$\sum_{j=1}^n j^{p-2+\varepsilon} \mathbf{E}(b(\mathcal{F}_0, j, j)) \leq \sum_{j=1}^n j^{p-2+\varepsilon} \beta_{2,Y}(j)$$

is convergent by condition (72) for any $\varepsilon < \eta$.

It remains to prove (80). We first write that

$$\|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} = \mathbf{E}((Z'_0)^{(0)}(x)X_{i,n}(x)X_{j,n}(x)),$$

where the notation $X^{(0)}$ stands for $X^{(0)} = X - \mathbf{E}(X)$ and

$$Z'_0(x) = |\mathbf{E}_0(B_{i,j}(x))|^{p/2-1} \text{sign}(\mathbf{E}_0(B_{i,j}(x)))$$

with $B_{i,j}(x) = X_{i,n}(x)X_{j,n}(x) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))$. Since the variables $X_{i,n}(x)$ and $X_{j,n}(x)$ are centered, an application of item (2) of Lemma 35 then gives

$$\begin{aligned} &\|\mathbf{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbf{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} \\ &\leq \|dK\|^2 \mathbf{E}(|Z_0(x)|(b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j))). \end{aligned}$$

Notice now that since $p/2 - 1 \geq 1$, we can easily get

$$\int_{\mathbf{R}} |Z'_0(x)| dx \leq c_p h_n \int_{\mathbf{R}} |K(u)|^{p-2} du,$$

where c_p is a positive constant depending on p . In addition,

$$\sum_{j=1}^n j^{p-2+\varepsilon} \sup_{i \geq j} \mathbf{E}(b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j)) \leq 3 \sum_{j=1}^n j^{p-2+\varepsilon} \beta_{2,Y}(j),$$

which is convergent by condition (72) for any $\varepsilon < \eta$. Then (80) holds and so does the proposition. \square

APPENDIX

This section is devoted to some technical lemmas. The next lemma gives estimates for terms of the type $\mathbf{E}(X_0^u X_1^{p-u})$.

LEMMA 34. *Let p and u be real numbers such that $0 \leq u \leq p - 2$. Let X_0 and X_1 be two positive identically distributed random variables. With the notation $a^p = \mathbf{E}(X_0^p)$, $\mathbf{E}_0(X_1) = \mathbf{E}(X_1|X_0)$ the following estimates hold:*

$$(83) \quad \mathbf{E}(X_0^u X_1^{p-u}) \leq a^{p-2u/(p-2)} \|\mathbf{E}_0(X_1^2)\|_{p/2}^{u/(p-2)}$$

and

$$(84) \quad \mathbf{E}(X_0^{p-1} X_1) \leq a^{p-1} \|\mathbf{E}_0(X_1^2)\|_{p/2}^{1/2}.$$

PROOF. Inequality (83) is trivial for $u = 0$. To prove it for $u = p - 2$, it suffices to write that $\mathbf{E}(X_0^{p-2} X_1^2) = \mathbf{E}(X_0^{p-2} \mathbf{E}_0(X_1^2))$, and then to use Hölder's inequality.

We prove now inequality (83) for $0 < u < p - 2$. Select $x = (p/2 - 1)/u = (p - 2)/2u$. Notice that $2x > 1$ and $p - u - 1/x > 0$ since $u < p - 2$. Then, since the variables are identically distributed,

$$\begin{aligned} \mathbf{E}(X_0^u X_1^{p-u}) &= \mathbf{E}(X_0^u X_1^{1/x} X_1^{p-u-1/x}) \leq \|X_0^u X_1^{1/x}\|_{2x} \|X_1^{p-u-1/x}\|_{2x/(2x-1)} \\ &\leq (\mathbf{E}(X_0^{p-2} X_1^2))^{u/(p-2)} (a^p)^{1-u/(p-2)}. \end{aligned}$$

Now, again by Hölder’s inequality applied with $x = p/(p - 2)$ and $1 - 1/x = 2/p$,

$$\begin{aligned} \mathbf{E}(X_0^{p-2} X_1^2) &= \mathbf{E}(X_0^{p-2} \mathbf{E}_0(X_1^2)) \leq (\mathbf{E}(X_0^p))^{(p-2)/p} (\mathbf{E}(\mathbf{E}_0(X_1^2))^{p/2})^{2/p} \\ &= a^{(p-2)} \|\mathbf{E}_0(X_1^2)\|_{p/2}. \end{aligned}$$

Overall

$$\begin{aligned} \mathbf{E}(X_0^u X_1^{p-u}) &\leq a^u \|\mathbf{E}_0(X_1^2)\|_{p/2}^{u/(p-2)} (a^p)^{1-u/(p-2)} \\ &= a^{p-2u/(p-2)} \|\mathbf{E}_0(X_1^2)\|_{p/2}^{u/(p-2)} \end{aligned}$$

ending the proof of inequality (83).

To prove inequality (84), we use Hölder’s inequality which entails that

$$\mathbf{E}(X_0^{p-1} X_1) \leq \mathbf{E}(X_0^{p-1} \mathbf{E}_0^{1/2}(X_1^2)) \leq a^{p-1} \|\mathbf{E}_0(X_1^2)\|_{p/2}^{1/2}. \quad \square$$

The next lemma gives covariance-type inequalities in terms of beta coefficients as defined in Definition 32.

LEMMA 35. *Let Z be a \mathcal{F}_0 -measurable real-valued random variable, and let h and g be two BV functions. Denote by $\|dh\|$ (resp., $\|dg\|$) the total variation norm of the measure dh (resp., dg). Denote $Z^{(0)} = Z - \mathbf{E}(Z)$, $h^{(0)}(Y_i) = h(Y_i) - \mathbf{E}(h(Y_i))$ and $g^{(0)}(Y_j) = g(Y_j) - \mathbf{E}(g(Y_j))$. Define the random variables $b(\mathcal{F}_0, i, j)$ as in Definition 32. Then:*

- (1) $|\mathbf{E}(Z^{(0)} h^{(0)}(Y_i))| = |\text{Cov}(Z, h(Y_i))| \leq \|dh\| \mathbf{E}(|Z| b(\mathcal{F}_0, i, i)).$
- (2) $|\mathbf{E}(Z^{(0)} h^{(0)}(Y_i) g^{(0)}(Y_j))| \leq \|dh\| \|dg\| \mathbf{E}(|Z| (b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j))).$

PROOF. Item (1) has been proven by Dedecker and Prieur (2005) [see item (2) of their Proposition 1]. Item (2) needs a proof. We first notice that

$$h^{(0)}(X) g^{(0)}(Y) = \int \int (\mathbf{1}_{X \leq t} - F_X(t)) (\mathbf{1}_{Y \leq s} - F_Y(s)) dh(t) dg(s).$$

Therefore

$$\begin{aligned} &\mathbf{E}(Z^{(0)} h^{(0)}(Y_i) g^{(0)}(Y_j)) \\ &= \mathbf{E}\left(Z \int \int (\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)} - \mathbf{E}(\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)})) dh(t) dg(s) \right) \\ &= \mathbf{E}\left(Z \int \int \mathbf{E}(\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)} - \mathbf{E}(\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)}) | \mathcal{F}_0) dh(t) dg(s) \right), \end{aligned}$$

which proves item (2) by noticing that

$$\begin{aligned} & |\mathbf{E}(\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)} - \mathbf{E}(\mathbf{1}_{Y_i \leq t}^{(0)} \mathbf{1}_{Y_j \leq s}^{(0)} | \mathcal{F}_0))| \\ & \leq b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j). \end{aligned} \quad \square$$

The next lemma gives inequalities for $|x + y|^p$ for different ranges of p , where $p \geq 2$ is a real number.

LEMMA 36. (1) *Let x and y be two real numbers and $2 \leq p \leq 3$. Then*

$$(85) \quad |x + y|^p \leq |x|^p + |y|^p + p|x|^{p-1} \operatorname{sign}(x)y + \frac{p(p-1)}{2}|x|^{p-2}y^2.$$

(2) *Let x and y be two real numbers and $3 < p \leq 4$. Then*

$$(86) \quad \begin{aligned} |x + y|^p & \leq |x|^p + |y|^p + p|x|^{p-1} \operatorname{sign}(x)y + \frac{p(p-1)}{2}|x|^{p-2}y^2 \\ & + \frac{2p}{(p-2)}|x||y|^{p-1}. \end{aligned}$$

(3) *Let x and y be two positive real numbers and $p \geq 1$ any real number. Then*

$$(87) \quad (x + y)^p \leq x^p + y^p + 4^p(x^{p-1}y + xy^{p-1}).$$

(4) *Let x and y be two real numbers and p an even positive integer. Then*

$$(88) \quad (x + y)^p \leq x^p + y^p + p(x^{p-1}y + xy^{p-1}) + 2^p(x^2y^{p-2} + x^{p-2}y^2).$$

PROOF. Inequality (85) was established in Rio [(2009), Relation (3.3)] by using Taylor's expansion with integral rest for evaluating the difference $|x + y|^p - |x|^p$. To prove inequality (86), we also use the Taylor integral formula of order 2 that gives

$$(89) \quad \begin{aligned} |x + y|^p - |x|^p & = p|x|^{p-1} \operatorname{sign}(x)y + C_p^2|x|^{p-2}y^2 \\ & + 2C_p^2y^2 \int_0^1 (1-t)(|x + ty|^{p-2} - |x|^{p-2}) dt, \end{aligned}$$

where $C_p^2 = p(p-1)/2$. Notice now that, for $3 < p \leq 4$,

$$|x + ty|^{p-2} \leq \frac{x^2 + 2|x||ty| + y^2}{(|x| + |ty|)^{4-p}} \leq |x|^{p-2} + 2|x||ty|^{p-3} + |ty|^{p-2}.$$

Hence

$$\begin{aligned} & 2C_p^2y^2 \int_0^1 (1-t)(|x + ty|^{p-2} - |x|^{p-2}) dt \\ & \leq 2C_p^2|y|^p \int_0^1 (1-t)t^{p-2} dt + 4C_p^2|x||y|^{p-1} \int_0^1 (1-t)t^{p-3} dt \end{aligned}$$

$$\begin{aligned}
 &= 2|y|^p C_p^2 \frac{\Gamma(p-1)}{\Gamma(p+1)} + 4|x||y|^{p-1} C_p^2 \frac{\Gamma(p-2)}{\Gamma(p)} \\
 &= |y|^p + \frac{2p}{(p-2)} |x||y|^{p-1}.
 \end{aligned}$$

Starting from (89), inequality (86) follows.

Inequality (87) was observed by Shao (1995), page 957. We shall establish now (88). We start by noticing that for any a, b two positive real numbers and $2 \leq k \leq p-2$ we have

$$(90) \quad a^{p-k} b^k \leq \max(a^p (b/a)^{p-2}, a^p (b/a)^2) \leq a^2 b^{p-2} + a^{p-2} b^2.$$

Now for p an even positive integer and x and y two real numbers, the Newton binomial formula gives

$$\begin{aligned}
 (x+y)^p &= x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k x^{p-k} y^k \\
 &\leq x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k |x|^{p-k} |y|^k.
 \end{aligned}$$

Whence, by (90) and the fact that $\sum_{k=0}^p C_p^k = 2^p$, inequality (88) follows. \square

LEMMA 37. *Let $(V_i)_{i \geq 0}$ be a sequence of nonnegative numbers such that $V_0 = 0$ and for all $i, j \geq 0$,*

$$(91) \quad V_{i+j} \leq C(V_i + V_j),$$

where $C \geq 1$ is a constant not depending on i and j . Then:

(1) *For any integer $r \geq 1$, any integer n satisfying $2^{r-1} \leq n < 2^r$ and any real $q \geq 0$,*

$$\sum_{i=0}^{r-1} \frac{1}{2^{iq}} V_{2^i} \leq C 2^{q+2} (2^{q+1} - 1)^{-1} \sum_{k=1}^n \frac{V_k}{k^{1+q}}.$$

(2) *For any positive integers k and m and any real $q > 0$,*

$$\sum_{j=1}^k \frac{1}{j^q} V_{jm} \leq 2^{q+1} C q^{-1} m^{q-1} \sum_{\ell=1}^m \frac{1}{(\ell+m)^q} V_\ell + 2 C q^{-1} m^{q-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_\ell.$$

(3) *Let $0 < \delta \leq \gamma \leq 1$. Then for any real $q \geq 0$,*

$$\left(\sum_{k=1}^n \frac{1}{k^{1+q\gamma}} V_k^\gamma \right)^{1/\gamma} \leq 2^{1/\delta-1/\gamma} C^{(\gamma-\delta)/\delta} \left(\sum_{k=1}^n \frac{1}{k^{1+q\delta}} V_k^\delta \right)^{1/\delta}.$$

REMARK 38. If $(V_i)_{i \geq 0}$ satisfies (91) with $C = 1$, then the sequence is said to be subadditive.

PROOF OF LEMMA 37. Condition (91) implies that for any integer k and any integer $0 \leq j \leq k$,

$$(92) \quad V_k \leq C(V_j + V_{k-j}) \quad \text{and then that} \quad (k+1)V_k \leq 2C \sum_{j=1}^k V_j.$$

Therefore for $2^{r-1} \leq n < 2^r$,

$$\sum_{i=0}^{r-1} \frac{1}{2^{iq}} V_{2^i} \leq 2C \sum_{j=1}^{2^{r-1}} V_j \sum_{i: 2^i \geq j} \frac{1}{2^{i(q+1)}}$$

proving item (1). To prove item (2), using again (92), it suffices to notice that

$$\begin{aligned} \sum_{j=1}^k \frac{1}{j^q} V_{jm} &\leq 2C \sum_{j=1}^k \frac{1}{(1+jm)^{jq}} \sum_{\ell=1}^{jm} V_\ell \\ &\leq 2Cm^{-1}(2m)^q \sum_{j=1}^k j^{-q-1} \sum_{\ell=1}^m \frac{1}{(\ell+m)^q} V_\ell \\ &\quad + 2Cq^{-1}m^{q-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_\ell. \end{aligned}$$

To prove item (3), we first notice that (91) entails that

$$V_{i+j}^\gamma \leq C^\gamma (V_i^\gamma + V_j^\gamma) \quad \text{and then that} \quad (k+1)V_k^\gamma \leq 2C^\gamma \sum_{j=1}^k V_j^\gamma.$$

Then for any real $q \geq 0$,

$$(93) \quad k^{-q(\gamma-\delta)} V_k^{\gamma-\delta} \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \left(\sum_{j=1}^k j^{-(1+q\gamma)} V_j^\gamma \right)^{1-\delta/\gamma}.$$

Writing that $k^{-(1+q\gamma)} V_k^\gamma = (k^{-(1+q\delta)} V_k^\delta) (k^{-q(\gamma-\delta)} V_k^{\gamma-\delta})$ and using (93), the following inequality holds:

$$\sum_{k=1}^n \frac{1}{k^{1+q\gamma}} V_k^\gamma \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \left(\sum_{k=1}^n k^{-(1+q\delta)} V_k^\delta \right) \left(\sum_{j=1}^n j^{-(1+q\gamma)} V_j^\gamma \right)^{1-\delta/\gamma}$$

proving item (3). \square

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REFERENCES

- BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. [MR0233396](#)
- BRADLEY, R. C. (2007). *Introduction to Strong Mixing Conditions*. Kendrick Press, Heber City, UT.
- BRETAGNOLLE, J. and HUBER, C. (1979). Estimation des densités: Risque minimax. *Z. Wahrsch. Verw. Gebiete* **47** 119–137. [MR0523165](#)
- BURKHOLDER, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probab.* **1** 19–42. [MR0365692](#)
- DE LA PEÑA, V. H. and GINÉ, E. (1999). *Decoupling: From Dependence to Independence, Randomly Stopped Processes. U-Statistics and Processes. Martingales and Beyond*. Springer, New York. [MR1666908](#)
- DEDECKER, J. (2001). Exponential inequalities and functional central limit theorems for a random fields. *ESAIM Probab. Stat.* **5** 77–104 (electronic). [MR1875665](#)
- DEDECKER, J., MERLEVÈDE, F. and RIO, E. (2009). Rates of convergence for minimal distances in the central limit theorem under projective criteria. *Electron. J. Probab.* **14** 978–1011. [MR2506123](#)
- DEDECKER, J. and MERLEVÈDE, F. (2011). Rates of convergence in the central limit theorem for linear statistics of martingale differences. *Stochastic Process. Appl.* **121** 1013–1043. [MR2775105](#)
- DEDECKER, J. and PRIEUR, C. (2005). New dependence coefficients. Examples and applications to statistics. *Probab. Theory Related Fields* **132** 203–236. [MR2199291](#)
- DEDECKER, J. and PRIEUR, C. (2007). An empirical central limit theorem for dependent sequences. *Stochastic Process. Appl.* **117** 121–142. [MR2287106](#)
- DEDECKER, J. and PRIEUR, C. (2009). Some unbounded functions of intermittent maps for which the central limit theorem holds. *ALEA Lat. Am. J. Probab. Math. Stat.* **5** 29–45. [MR2475605](#)
- DEDECKER, J., DOUKHAN, P., LANG, G., LEÓN, J. R., LOUHICHI, S. and PRIEUR, C. (2007). *Weak Dependence: With Examples and Applications. Lecture Notes in Statistics* **190**. Springer, New York. [MR2338725](#)
- DOUKHAN, P., MASSART, P. and RIO, E. (1994). The functional central limit theorem for strongly mixing processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **30** 63–82. [MR1262892](#)
- GIRAITIS, L., KOKOSZKA, P. and LEIPUS, R. (2000). Stationary ARCH models: Dependence structure and central limit theorem. *Econometric Theory* **16** 3–22. [MR1749017](#)
- GORDIN, M. I. (1969). The central limit theorem for stationary processe. *Soviet. Math. Dokl.* **10** 1174–1176.
- GORDIN, M. and PELIGRAD, M. (2011). On the functional central limit theorem via martingale approximation. *Bernoulli* **17** 424–440. [MR2797997](#)
- GOUÉZEL, S. (2010). Almost sure invariance principle for dynamical systems by spectral methods. *Ann. Probab.* **38** 1639–1671. [MR2663640](#)
- HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York. [MR0624435](#)
- HITCZENKO, P. (1990). Best constants in martingale version of Rosenthal’s inequality. *Ann. Probab.* **18** 1656–1668. [MR1071816](#)
- KEVEI, P. and MASON, D. M. (2011). A note on a maximal Bernstein inequality. *Bernoulli* **17** 1054–1062.
- KWAPIEŃ, S. and WOYCZYŃSKI, W. A. (1991). Semimartingale integrals via decoupling inequalities and tangent processes. *Probab. Math. Statist.* **12** 165–200. [MR1199772](#)

- MERLEVÈDE, F., PELIGRAD, M. and RIO, E. (2009). Bernstein inequality and moderate deviations under strong mixing conditions. In *IMS Collections. High Dimensional Probability V* 273–292. Beachwood, OH.
- MÓRICZ, F. A., SERFLING, R. J. and STOUT, W. F. (1982). Moment and probability bounds with quasisuperadditive structure for the maximum partial sum. *Ann. Probab.* **10** 1032–1040. [MR0672303](#)
- PELIGRAD, M. (1985). Convergence rates of the strong law for stationary mixing sequences. *Z. Wahrsch. Verw. Gebiete* **70** 307–314. [MR0799152](#)
- PELIGRAD, M. (1989). The r -quick version of the strong law for stationary φ -mixing sequences. In *Almost Everywhere Convergence (Columbus, OH, 1988)* (G. A. Edgar and L. Sucheston, eds.) 335–348. Academic Press, Boston, MA. [MR1035254](#)
- PELIGRAD, M. and GUT, A. (1999). Almost-sure results for a class of dependent random variables. *J. Theoret. Probab.* **12** 87–104. [MR1674972](#)
- PELIGRAD, M. and UTEV, S. (2005). A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33** 798–815. [MR2123210](#)
- PELIGRAD, M., UTEV, S. and WU, W. B. (2007). A maximal L_p -inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.* **135** 541–550 (electronic). [MR2255301](#)
- RIO, E. (2000). *Théorie Asymptotique des Processus Aléatoires Faiblement Dépendants. Mathématiques et Applications* **31**. Springer, Berlin. [MR2117923](#)
- RIO, E. (2009). Moment inequalities for sums of dependent random variables under projective conditions. *J. Theoret. Probab.* **22** 146–163. [MR2472010](#)
- ROSENBLATT, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Natl. Acad. Sci. USA* **42** 43–47. [MR0074711](#)
- ROSENTHAL, H. P. (1970). On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.* **8** 273–303. [MR0271721](#)
- SHAO, Q. M. (1988). A moment inequality and its applications. *Acta Math. Sin.* **31** 736–747. [MR1000416](#)
- SHAO, Q. M. (1995). Maximal inequalities for partial sums of ρ -mixing sequences. *Ann. Probab.* **23** 948–965. [MR1334179](#)
- UTEV, S. A. (1991). Sums of random variables with ϕ -mixing. *Siberian Adv. Math.* **1** 124–155. [MR1128381](#)
- UTEV, S. and PELIGRAD, M. (2003). Maximal inequalities and an invariance principle for a class of weakly dependent random variables. *J. Theoret. Probab.* **16** 101–115. [MR1956823](#)
- VIENNET, G. (1997). Inequalities for absolutely regular sequences: Application to density estimation. *Probab. Theory Related Fields* **107** 467–492. [MR1440142](#)
- VOLKONSKIĬ, V. A. and ROZANOV, Y. A. (1959). Some limit theorems for random functions. I. *Theory Probab. Appl.* **4** 178–197. [MR0121856](#)
- WITTMANN, R. (1985). An application of Rosenthal’s moment inequality to the strong law of large numbers. *Statist. Probab. Lett.* **3** 131–133. [MR0801858](#)
- WU, W. B. (2007). Strong invariance principles for dependent random variables. *Ann. Probab.* **35** 2294–2320. [MR2353389](#)
- WU, W. B., HUANG, Y. and HUANG, Y. (2010). Kernel estimation for time series: An asymptotic theory. *Stochastic Process. Appl.* **120** 2412–2431. [MR2728171](#)
- WU, W. B. and WOODROOFE, M. (2004). Martingale approximations for sums of stationary processes. *Ann. Probab.* **32** 1674–1690. [MR2060314](#)
- WU, W. B. and ZHAO, Z. (2008). Moderate deviations for stationary processes. *Statist. Sinica* **18** 769–782. [MR2411619](#)
- ZHAO, O. and WOODROOFE, M. (2008). On martingale approximations. *Ann. Appl. Probab.* **18** 1831–1847. [MR2462550](#)

ZINN, J. (1985). Comparison of martingale difference sequences. In *Probability in Banach Spaces, V (Medford, Mass., 1984). Lecture Notes in Math.* **1153** 453–457. Springer, Berlin. [MR0821997](#)

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