The Annals of Probability
2012, Vol. 40, No. 6, 2690–2704
DOI: 10.1214/11-AOP693

DISTRIBUTIONS ON UNBOUNDED MOMENT SPACES AND RANDOM MOMENT SEQUENCES[1]

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In this paper we define distributions on moment spaces corresponding to measures on the real line with an unbounded support. We identify these distributions as limiting distributions of random moment vectors defined on compact moment spaces and as distributions corresponding to random spectral measures associated with the Jacobi, Laguerre and Hermite ensemble from random matrix theory. For random vectors on the unbounded moment spaces we prove a central limit theorem where the centering vectors correspond to the moments of the Marchenko–Pastur distribution and Wigner's semi-circle law.

1. Introduction. For a set $T \subset \mathbb{R}$, let $\mathcal{P}(T)$ denote the set of all probability measures on the Borel field of T with existing moments. For a measure $\mu \in \mathcal{P}(T)$, we denote by

$$m_k(\mu) = \int_T x^k \mu(dx); \qquad k = 0, 1, 2, \dots,$$

the kth moment and define

$$\mathcal{M}(T) = \{\mathbf{m}(\mu) = (m_1(\mu), m_2(\mu), \dots)^T \,|\, \mu \in \mathcal{P}(T)\} \subset \mathbb{R}^{\mathbb{N}} \tag{1.1}$$

as the set of all moment sequences. We denote by Π_n ($n \in \mathbb{N}$) the canonical projection onto the first n coordinates and call $\mathcal{M}_n(T) = \Pi_n(\mathcal{M}(T)) \subset \mathbb{R}^n$ the nth moment space. These moments have found considerable interest in the literature; see Karlin and Studden (1966). Most authors concentrate on the "classical" moment space corresponding to measures on the interval $[0, 1]$; see Karlin and Shapley (1953), Kreĭn and Nudelman (1977), among others. Chang, Kemperman and Studden (1993) equipped it with a uniform distribution in order to understand more fully the shape and the structure. In particular, these authors proved asymptotic normality of an appropriately standardized version of a projection $\Pi_k(\mathbf{m}_n)$ of a uniformly distributed vector $\mathbf{m}_n$ on $\mathcal{M}_n([0, 1])$. Gamboa and Lozada-Chang

Received July 2010; revised May 2011.

[1]Supported in part by the Sonderforschungsbereich Tr/12 (Project C2, Fluctuations and universality of invariant random matrix ensembles).

MSC2010 subject classifications. 60F05, 30E05, 15B52.

Key words and phrases. Gaussian ensemble, Laguerre ensemble, Jacobi ensemble, moment spaces, canonical moments, random moment sequences, Wigner's semicircle law, Marcenko–Pastur distribution.

(2004) considered large deviation principles for random moment sequences on the space $\mathcal{M}_n([0, 1])$, while Lozada-Chang (2005) investigated similar problems for moment spaces corresponding to more general functions defined on a bounded set. More recently, Gamboa and Rouault (2010) discussed random spectral measures related to moment spaces of measures on the interval $[0, 1]$ and moment spaces related to measures defined on the unit circle. Distributions of random moments induced by measures on l_p balls were investigated by Barthe et al. (2010).

The present paper is devoted to the problem of defining probability distributions on unbounded moment spaces. We will investigate these distributions from several perspectives. We introduce a class of general distributions on the moment space corresponding to measures defined on a compact interval. By a limiting argument we will derive canonical distributions on the moment spaces corresponding to measures on the unbounded intervals $T = [0, \infty)$ and $\mathbb{R}$, respectively. We also show that these distributions appear naturally in the study of random spectral measures of the classical Jacobi, Laguerre and Gaussian ensemble. Finally we consider random moment sequences distributed according to the new probability distributions on the unbounded moment spaces. In particular, we prove weak convergence of a centered random moment vector, where the centering vector corresponds to the moments of the Marchenko–Pastur law (in the case of the moment space $\mathcal{M}([0, \infty))$) and to the semi-circle law [for the moment space $\mathcal{M}(\mathbb{R})$]. These measures play a very important role in free probability and quantum probability; see the books of Hiai and Petz (2000) and Hora and Obata (2007).

2. Distributions on unbounded moment spaces. In the present section we will define a class of more general distributions on the nth moment space $\mathcal{M}_n([a, b])$ as considered by Chang, Kemperman and Studden (1993). The motivation for considering this class is twofold. On one hand, we want to introduce distributions on the moment space $\mathcal{M}_k([a, b])$, which are different from the uniform distribution. On the other hand, we want to identify distributions on unbounded moment spaces as limits of distributions on $\mathcal{M}_k([a, b])$, when $b - a \to \infty$.

Let $\mathbf{m}_{k-1} = (m_1, \dots, m_{k-1})^T \in \mathcal{M}_{k-1}([a, b])$ be a given vector of moments of a probability measure on the interval $[a, b]$, then these first $k - 1$ moments impose bounds on the kth moment m_k such that the moment vector $\mathbf{m}_k = (m_1, \dots, m_{k-1}, m_k)^T$ is an element of the kth moment space $\mathcal{M}_k([a, b])$. More precisely, define for $\mathbf{m}_{k-1} \in \mathcal{M}_{k-1}([a, b])$

$$m_k^- = \min\left\{m_k(\mu) \middle| \mu \in \mathcal{P}([a, b]) \text{ with } \int_a^b t^i \, d\mu(t) = m_i \text{ for } i = 1, \dots, k-1\right\},$$

$$m_k^+ = \max\left\{m_k(\mu) \middle| \mu \in \mathcal{P}([a, b]) \text{ with } \int_a^b t^i \, d\mu(t) = m_i \text{ for } i = 1, \dots, k-1\right\},$$

then it follows that $\mathbf{m}_k = (m_1, \dots, m_k)^T \in \operatorname{Int} \mathcal{M}_k([a, b])$ if and only if $m_k^- < m_k < m_k^+$, where $\operatorname{Int} C$ denotes the interior of a set $C \subset \mathbb{R}^k$. Consequently, we define for a point $\mathbf{m}_k \in \operatorname{Int} \mathcal{M}_k([a,b])$ the canonical moment of order $l = 1, \dots, k$ as

$$p_l = p_l(\mathbf{m}_k) = \frac{m_l - m_l^-}{m_l^+ - m_l^-}; \qquad l = 1, \dots, k. \tag{2.1}$$

Note that for $\mathbf{m}_k \in \operatorname{Int} \mathcal{M}_k([a,b])$ we have $p_l \in (0,1); l = 1, \dots, k$; and that p_k describes the relative position of the moment m_k in the set of all possible kth moments with fixed moments $m_1, \dots, m_{k-1}$. For this reason, the canonical moments do not depend on the interval $[a,b]$, that is they are invariant under linear transformations of the measure; see Dette and Studden (1997). Moreover, definition (2.1) defines a one-to-one mapping,

$$\varphi_n : \mathbf{m}_n \mapsto \mathbf{p}_n = (p_1, \dots, p_n)^T \tag{2.2}$$

from the interior of the moment space $\mathcal{M}_n([a,b])$ onto the open cube $(0,1)^n$. It can be shown that for a point $(m_1, \dots, m_{2n-1}) \in \operatorname{Int} \mathcal{M}_{2n-1}([a,b])$ the canonical moments appear in the three-term recurrence relation

$$x P_k(x) = P_{k+1}(x) + b_{k+1} P_k(x) + a_k P_{k-1}(x), \qquad k = 1, \dots, n-1, \tag{2.3}$$

$[P_0(x) = 1, P_1(x) = x - b_1]$ of the monic orthogonal polynomials associated with the vector $(m_1, \dots, m_{2n-1})$; see Chihara (1978). These polynomials are orthogonal with respect to every measure with first moments $m_1, \dots, m_{2n-1}$, and the recursion coefficients in (2.3) are given by

$$\begin{aligned} b_{k+1} &= a + (b-a)\big((1 - p_{2k-1}) p_{2k} + (1 - p_{2k}) p_{2k+1}\big); \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = 0, \dots, n-1, \end{aligned} \tag{2.4}$$

$$a_k = (b-a)^2 (1 - p_{2k-2}) p_{2k-1} (1 - p_{2k-1}) p_{2k}; \qquad k = 1, \dots, n-1, \tag{2.5}$$

where we put $p_{-1} = p_0 = 0$ (note that $a_k > 0; k = 1, \dots, n$). In the case $T = [0, \infty)$ the upper bound m_k^+ does not exist, but we can still define for a point $\mathbf{m}_{k-1} \in \operatorname{Int} \mathcal{M}_{k-1}([0,\infty))$ the lower bound

$$m_k^- = \min\left\{ m_k(\mu) \,\middle|\, \mu \in \mathcal{P}([0,\infty)) \text{ with } \int_0^\infty t^i \, d\mu(t) = m_i \text{ for } i = 1, \dots, k-1 \right\},$$

where $\mathbf{m}_k = (m_1, \dots, m_k)^T \in \operatorname{Int} \mathcal{M}_k([0,\infty))$ if and only if $m_k > m_k^-$. In this case, the analogs of the canonical moments are defined by the quantities

$$z_l = z_l(\mathbf{m}_n) = \frac{m_l - m_l^-}{m_{l-1} - m_{l-1}^-}, \qquad l = 1, \dots, k \tag{2.6}$$

(with $m_0^- = 0$) and related to the coefficients in the three-term recurrence relation (2.3) for the monic orthogonal polynomials by

$$a_k = z_{2k-1} z_{2k}, \qquad b_k = z_{2k-2} + z_{2k-1}. \tag{2.7}$$

Note that (2.6) defines a one-to-one mapping

$$\psi_n : \mathbf{m}_n \mapsto \mathbf{z}_n = (z_1, \dots, z_n)^T \tag{2.8}$$

from the interior of the moment space $\mathcal{M}_n([0,\infty))$ onto $(\mathbb{R}^+)^n$. Finally, in the case $T = \mathbb{R}$, neither m_k^- nor m_k^+ can be defined. Nevertheless, there exists also a one-to-one mapping

$$\xi_n : \mathbf{m}_{2n-1} \mapsto (b_1, a_1, \dots, a_{n-1}, b_n)^T \tag{2.9}$$

from the interior of the $(2n-1)$th moment space $\operatorname{Int} \mathcal{M}_{2n-1}(\mathbb{R})$ onto the space $(\mathbb{R} \times \mathbb{R}^+)^{n-1} \times \mathbb{R}$ of coefficients in the three-term recurrence relation (2.3), which can be considered as the analog of (2.8) and is defined by

$$\int_{\mathbb{R}} x^k P_k(x)\, d\mu(x) = a_1 \cdots a_k, \qquad k = 1, \dots, n-1, \tag{2.10}$$

$$\int_{\mathbb{R}} x^{k+1} P_k(x)\, d\mu(x) = a_1 \cdots a_k (b_1 + \cdots + b_{k+1}), \qquad k = 0, \dots, n-1; \tag{2.11}$$

see, for example, Wall (1948).

In the following sections we will use the canonical moments and corresponding quantities on the interval $[0,\infty)$ and the real line for the definition of distributions on the corresponding moment spaces. The basic idea is to define a general class of distributions on the moment space $\mathcal{M}_n([a,b])$ and to consider the limit as $b - a \to \infty$. To be precise, let for $k \ge 1$ $\tilde{f}_k : (0,1) \longrightarrow \mathbb{R}$ be a nonnegative integrable function with $\int_0^1 \tilde{f}_k(x)\, dx > 0$, then a distribution on the interior of the moment space $\mathcal{M}_n([a,b])$ is given by

$$f_n(\mathbf{m}_n) = c_n \prod_{k=1}^{n} \tilde{f}_k(p_k(\mathbf{m}_n)) \mathbb{1}_{\{m_k^- < m_k < m_k^+\}}, \tag{2.12}$$

where $p_k(\mathbf{m}_n)$ is the kth canonical moment defined in (2.1) and c_n a normalization constant. Our first theorem gives the distribution of the canonical moments corresponding to the random vector $\mathbf{m}_n$ with density f_n defined in (2.12).

THEOREM 2.1. *Suppose that $\mathbf{m}_n$ is a random vector on the moment space $\mathcal{M}_n([a,b])$ with density f_n defined in* (2.12). *Then the canonical moments $p_1(\mathbf{m}_n), \dots, p_n(\mathbf{m}_n)$ are independent and $p_k(\mathbf{m}_n)$ has a density proportional to $\tilde{f}_k(x)(x - x^2)^{n-k} \mathbb{1}_{\{0<x<1\}}$ $(1 \le k \le n)$.*

PROOF. It follows from Theorem 1.4.9 and equation (1.3.6) in Dette and Studden (1997) that the Jacobian determinant of the mapping φ_n defined in (2.2) is given by

$$\begin{aligned} \left| \frac{\partial \varphi_n}{\partial \mathbf{m}_n} \right| &= \prod_{k=1}^{n} \frac{\partial p_k(\mathbf{m}_n)}{\partial m_k} = \prod_{k=1}^{n} (m_k^+ - m_k^-)^{-1} \\ &= (b-a)^{-n(n+1)/2} \prod_{k=1}^{n} \bigl(p_k(1-p_k)\bigr)^{-(n-k)}; \end{aligned} \tag{2.13}$$

considering the product structure of f_n, this gives the asserted distribution. $\square$

For the construction of distributions on the unbounded moment space $\mathcal{M}_n([0,\infty))$, a special case will be of particular interest; that is, $\tilde{f}_k(x) = x^{\gamma_k}(1-x)^{\delta_k}$, where $\gamma = (\gamma_k)_{k\in\mathbb{N}}$, $\delta = (\delta_k)_{k\in\mathbb{N}}$ are sequences of real parameters, such that $\gamma_k, \delta_k > 0$ for all $k \geq 1$. In this case the density on the moment space $\mathcal{M}_n([a,b])$ is given by

$$f_n^{(\gamma,\delta)}(\mathbf{m}_n) = c_n^{[a,b]} \prod_{k=1}^{n} \left(\frac{m_k - m_k^-}{m_k^+ - m_k^-}\right)^{\gamma_k} \left(\frac{m_k^+ - m_k}{m_k^+ - m_k^-}\right)^{\delta_k} \mathbb{1}_{\{m_k^- < m_k < m_k^+\}}, \tag{2.14}$$

where

$$c_n^{[a,b]} = \left[(b-a)^{n(n+1)/2} \prod_{k=1}^{n} \int_0^1 x^{n-k+\gamma_k}(1-x)^{n-k+\delta_k}\,dx\right]^{-1} \tag{2.15}$$

is the normalizing constant. The choice of density (2.14) is motivated by results of Dette and Studden (1995) who showed that the empirical distribution of the (appropriately normalized) roots of the Jacobi polynomials $P_k^{(\gamma_k,\delta_k)}(x)$ converges weakly to a distribution with unbounded support if $\gamma_k \to \infty$ or $\delta_k \to \infty$. It follows from Theorem 2.1 that for the density $f_n^{(\gamma,\delta)}$, the canonical moment p_k has a Beta distribution Beta$(\gamma_k + n - k + 1, \delta_k + n - k + 1)$. In the following we use densities of form (2.14) to construct a distribution on the unbounded moment space $\mathcal{M}_n([0,\infty))$.

THEOREM 2.2. *Let $f_n^{(\gamma^{(d)},\delta^{(d)})}$ denote the density defined in (2.14) on the moment space $\mathcal{M}_n([0,d])$ corresponding to the probability measures on the interval $[0,d]$, where the parameter sequences $\gamma^{(d)} = (\gamma_k^{(d)})_{k\in\mathbb{N}}$, $\delta^{(d)} = (\delta_k^{(d)})_{k\in\mathbb{N}}$ depend on length d and satisfy for all $k \geq 1$ $\gamma_k^{(d)} \to \gamma_k > -1$ and $\delta_k^{(d)}/d \to \delta_k \in \mathbb{R}^+$ as $d \to \infty$. Then for $d \to \infty$ the density $f_n^{(\gamma^{(d)},\delta^{(d)})}$ converges point-wise to the function*

$$g_n^{(\gamma,\delta)}(\mathbf{m}_n) = c_n^{[0,\infty)} \prod_{k=1}^{n} z_k(\mathbf{m}_n)^{\gamma_k} \exp(-\delta_k z_k(\mathbf{m}_n))\mathbb{1}_{\{z_k(\mathbf{m}_n)>0\}}, \tag{2.16}$$

where $z_k(\mathbf{m}_n)$ is given in (2.6) and $c_n^{[0,\infty)} = \prod_{k=1}^{n}(\delta_k^{\gamma_k+n-k+1})/\Gamma(\gamma_k + n - k + 1)$. Moreover, $g_n^{(\gamma,\delta)}$ defines a density on the unbounded moment space $\mathcal{M}_n([0,\infty))$.

PROOF. The fact that $g_n^{(\gamma,\delta)}$ is a density is obvious from the transformation in the proof of Theorem 2.3 below, we prove here only the convergence. For a fixed point $\mathbf{m}_n \in \mathcal{M}_n([0,\infty))$, there exists a $d_0 \in \mathbb{N}$ with $\mathbf{m}_n \in \mathcal{M}_n([0,d])$ for all $d \geq d_0$. Let $\mathbf{p}_n(\mathbf{m}_n)$ denote the vector of canonical moments corresponding to the vector $\mathbf{m}_n$ in the moment space $\mathcal{M}_n([0,d])$. We will show at the end of this proof that

$$p_k(\mathbf{m}_n) = \frac{z_k(\mathbf{m}_n)}{d}\big(1+o(1)\big), \qquad k=1,\dots,n, \tag{2.17}$$

where the quantities $z_k(\mathbf{m}_n)$ are defined in (2.6). Observing this representation and definition (2.14), it follows for $d \to \infty$

$$\begin{aligned} f_n^{(\gamma^{(d)},\delta^{(d)})}(\mathbf{m}_n) &= c_n^{[0,d]} \prod_{k=1}^n \left(\frac{z_k(\mathbf{m}_n)}{d}\right)^{\gamma_k^{(d)}} \left(1-\frac{z_k(\mathbf{m}_n)}{d}\right)^{\delta_k^{(d)}} \big(1+o(1)\big) \\ &= d^{-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} c_n^{[0,d]} \prod_{k=1}^n z_k(\mathbf{m}_n)^{\gamma_k} \exp(-\delta_k z_k(\mathbf{m}_n))\big(1+o(1)\big). \end{aligned}$$

Finally, we obtain from (2.15) for the normalizing constant by a straightforward calculation $d^{-(\gamma_1^{(d)}+\cdots+\gamma_n^{(d)})} c_n^{[0,d]} = c_n^{[0,\infty)}(1+o(1))$, which proves the assertion of the theorem. For the remaining proof of the representation (2.17), let μ be a measure on the interval $[0,d]$ with first moments given by $\mathbf{m}_n$ and let ν denote the measure on the interval $[0,1]$ obtained from μ by the linear transformation $x \mapsto x/d$. We write $p_k(\mu)$ for $p_k(\mathbf{m}_n)$ and $z_k(\mu)$ for $z_k(\mathbf{m}_n)$. Invariance of the canonical moments under linear transformations yields $p_k(\mu) = p_k(\nu)$. The recursion variables of the measure ν can be decomposed as

$$z_k(\nu) = \big(1-p_{k-1}(\nu)\big)p_k(\nu). \tag{2.18}$$

A comparison of the continued fraction expansion of the Stieltjes transform of μ and of ν [see Theorem 3.3.3 in Dette and Studden (1997)] yields $dz_k(\nu) = z_k(\mu)$. With (2.18) we obtain

$$p_k(\mu) = p_k(\nu) = \frac{z_k(\nu)}{1-p_{k-1}(\nu)} = \frac{1}{d}\frac{z_k(\mu)}{1-p_{k-1}(\mu)}$$

for $k>1$. The first canonical moment is given by $p_1(\mathbf{m}_n) = (m_1 - m_1^-)/(m_1^+ - m_1^-) = m_1/d = z_1(\mathbf{m}_n)/d$, and (2.17) follows by an induction argument. □

The following theorem is essential for the asymptotic investigations in Section 3 and gives the distribution of the the vector $\mathbf{z}_n = (z_1,\dots,z_n)^T$ corresponding to a random point in $\mathcal{M}_n([0,\infty))$.

THEOREM 2.3. *Let $\mathbf{m}_n \in \mathcal{M}_n([0,\infty))$ be governed by a law with density $g_n^{(\gamma,\delta)}$. Then the recursion variables $\mathbf{z}_n = \psi_n(\mathbf{m}_n)$ defined by (2.6) are independent and gamma distributed, that is,*

$$z_k \sim \mathrm{Gamma}(\gamma_k + n - k + 1, \delta_k), \qquad k=1,\dots,n.$$

PROOF. By its definition in (2.6), the random variable z_k depends only on the moment $m_1, \dots, m_k$; therefore the Jacobi matrix of the mapping ψ_n is a lower triangular matrix. We obtain for the Jacobian determinant

$$\left|\frac{\partial \mathbf{m}_n}{\partial \mathbf{z}_n}\right| = \prod_{k=1}^{n}\left|\frac{\partial m_k}{\partial z_k}\right| = \prod_{k=1}^{n}(m_{k-1} - m_{k-1}^{-}) = \prod_{k=2}^{n} z_1 \cdots z_{k-1} = \prod_{k=1}^{n} z_k^{n-k},$$

where the third identity follows from the definition of the z_i in (2.6). Considering the second representation of the density in (2.16), this yields the claimed distribution. $\square$

We conclude this section with a discussion of distributions on the moments space corresponding to measures on $\mathbb{R}$. For the sake of brevity we restrict ourselves to moment spaces of odd dimension, that is, $\mathcal{M}_{2n-1}(\mathbb{R})$. To derive a class of distributions on $\mathcal{M}_{2n-1}(\mathbb{R})$ we consider the moment space $\mathcal{M}_{2n-1}([-s, s])$ with $s \to \infty$ and a density of the form (2.14) with parameters varying with s. The proof of the following result is similar to the proof of Theorem 2.2 and therefore omitted.

THEOREM 2.4. *Denote by* $f_{2n-1}^{(\gamma^{(s)},\delta^{(s)})}$ *the density defined in* (2.14) *on the moment space* $\mathcal{M}_{2n-1}([-s, s])$, *where the parameters satisfy*

$$\gamma_{2k-1}^{(s)} = \delta_{2k-1}s^2 + o(1), \qquad \delta_{2k-1}^{s} = \delta_{2k-1}s^2 + o(1),$$
$$\gamma_{2k}^{(s)} = \gamma_k + o(1), \qquad \delta_{2k}^{(s)} = \delta_{2k}s^2 + o(s^2)$$

with $\gamma_k > -1$, $\delta_k > 0$. *Then* $f_{2n-1}^{(\gamma^{(s)},\delta^{(s)})}$ *converges point-wise to the function*

$$\begin{aligned} h_{2n-1}^{(\gamma,\delta)}(\mathbf{m}_{2n-1}) &= \prod_{k=1}^{n}\sqrt{\frac{\delta_{2k-1}}{\pi}}\exp(-\delta_{2k-1}b_k^2(\mathbf{m}_{2n-1})) \\ &\quad \times \prod_{k=1}^{n-1}\frac{\delta_{2k}{}^{\gamma_k+2n-2k}}{\Gamma(\gamma_k+2n-2k)}a_k^{\gamma_k}(\mathbf{m}_{2n-1}) \\ &\quad \times \exp(-\delta_{2k}a_k(\mathbf{m}_{2n-1}))\mathbb{1}_{\{a_k(\mathbf{m}_{2n-1})>0\}}. \end{aligned} \tag{2.19}$$

Moreover, the function $h_{2n-1}^{(\gamma,\delta)}$ *defines a density on the moment space* $\mathcal{M}_{2n-1}(\mathbb{R})$.

The following result is the analog of Theorem 2.3. The proof follows by similar arguments, where the Jacobian can be obtained from equations (2.10) and (2.11).

THEOREM 2.5. *Let* $\mathbf{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ *be a random vector with density* $h_{2n-1}^{(\gamma,\delta)}$ *defined in* (2.19). *Then the random recursion coefficients* $(b_1, a_1, \dots, a_{n-1},$

$b_n)^T = \xi_{2n-1}(\mathbf{m}_{2n-1})$ *in the recurrence relation* (2.3) *for the orthogonal polynomials associated with* $\mathbf{m}_{2n-1}$ *are independent and*

$$b_k \sim \mathcal{N}\left(0, \frac{1}{2\delta_{2k-1}}\right), \qquad a_k \sim \text{Gamma}(\gamma_k + 2n - 2k, \delta_{2k}).$$

REMARK 2.6. It is notable that the introduced distributions on the moment space appear naturally as distributions of moment vectors corresponding to random spectral measures which were recently discussed by Gamboa and Rouault (2009, 2010). To be precise, let $\mathbf{w} = (w_1, \dots, w_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ denote independent random variables. Assume that $\mathbf{w}$ follows a Dirichlet distribution with density

$$\frac{\Gamma(n\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \cdots w_n^{\beta/2-1} \mathbb{1}_{\{w_1,\dots,w_n>0, \sum_{i=1}^n w_i=1\}} \tag{2.20}$$

and that the density of $\boldsymbol{\lambda}$ is given by the Jacobi-ensemble

$$f_J(\lambda) = c_J^{\gamma_0,\delta_0} |\Delta(\lambda)|^\beta \prod_{i=1}^n \lambda_i^{\gamma_0}(1-\lambda_i)^{\delta_0} \mathbb{1}_{\{0<\lambda_i<1\}}, \tag{2.21}$$

where $\gamma_0, \delta_0 > -\frac{1}{2}$; see Killip and Nenciu (2004). We define a random measure by $\mu = \sum_{i=1}^n w_i \delta_{\lambda_i}$, and it follows from Theorem 2.2 and Proposition 5.3 in Killip and Nenciu (2004) (applied to the interval $[-1, 1]$) that μ is the spectral measure of the random tridiagonal matrix

$$J_n = \begin{pmatrix} d_1 & c_1 & & \\ c_1 & d_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ & & c_{n-1} & d_n \end{pmatrix},$$

where $d_k = p_{2k-2}(1 - p_{2k-3}) + p_{2k-1}(1 - p_{2k-2})$ and

$$c_k = \sqrt{p_{2k-1}(1 - p_{2k-2})p_{2k}(1 - p_{2k-1})}$$

and $p_{-1} = p_0 = 0$ and $p_1, \dots, p_{2n-1}$ are independent random variables distributed as

$$p_k \sim \begin{cases} \text{Beta}\left(\frac{2n-k}{4}\beta, \frac{2n-k-2}{4}\beta + \gamma_0 + \delta_0 + 2\right), & k \text{ even}, \\ \text{Beta}\left(\frac{2n-k-1}{4}\beta + \gamma_0 + 1, \frac{2n-k-1}{4}\beta + \delta_0 + 1\right), & k \text{ odd}. \end{cases}$$

This spectral measure is the unique measure with $\langle e_1, J_n^k e_1\rangle = m_k(\mu)$ for all k. The tridiagonal matrix J_n defines monic polynomials $P_1(x), \dots, P_n(x)$ via a recursion (2.3) with recursion coefficients $b_k = d_k$ $(1 \le k \le n)$, $a_k = c_k^2$ $(1 \le k \le n-1)$. Indeed, the polynomial $P_k(x)$ is the characteristic polynomial of the upper left

$(k \times k)$-subblock of the matrix J_n and the kth orthogonal polynomial with respect do the measure $d\mu$. Therefore, we obtain from (2.4) and (2.5) that $\mathbf{p}_{2n-1} = (p_1, \dots, p_{2n-1})^T$ is exactly the vector of canonical moments of the spectral measure μ, and by definition, their joint density is given by

$$f_{\mathbf{p}}(\mathbf{p}_{2n-1}) = c \prod_{k=1}^{n} p_{2k-1}^{(n-k)\beta/2+\gamma_0} (1 - p_{2k-1})^{(n-k)\beta/2+\delta_0} \times \prod_{k=1}^{n-1} p_{2k}^{(n-k)\beta/2-1} (1 - p_{2k})^{(n-k-1)\beta/2+\gamma_0+\delta_0+1}.$$

Since the eigenvalues of the matrix J_n are contained in the interval $(0, 1)$, the moments $\mathbf{m}_{2n-1}(\mu) = \varphi_{2n-1}^{-1}(\mathbf{p}_{2n-1})$ of the spectral measure are in the interior of the moment space $\mathcal{M}_{2n-1}([0, 1])$. The Jacobian of the transformation φ_{2n-1}^{-1} is given by $\prod_{k=1}^{n}(p_k(1 - p_k))^{2n-1-k}$, which gives for the density of the random moment vector $\mathbf{m}_{2n-1}(\mu)$

$$\begin{aligned} f_{\mathbf{m}}(\mathbf{m}_{2n-1}) &= c \prod_{k=1}^{n} p_{2k-1}(\mathbf{m}_{2n-1})^{(\beta/2-2)(n-k)+\gamma_0} \big(1 - p_{2k-1}(\mathbf{m}_{2n-1})\big)^{(\beta/2-2)(n-k)+\delta_0} \\ &\quad \times \prod_{k=1}^{n-1} p_{2k}(\mathbf{m}_{2n-1})^{(\beta/2-2)(n-k)} \big(1 - p_{2k}(\mathbf{m}_{2n-1})\big)^{(\beta/2-2)(n-k-1)+\gamma_0+\delta_0}. \end{aligned}$$

This is a density as in (2.14) with parameters $\gamma_{2k-1} = (\frac{\beta}{2} - 2)(n - k) + \gamma_0$, $\delta_{2k-1} = (\frac{\beta}{2} - 2)(n - k) + \delta_0$, $(1 \le k \le n)$ and $\gamma_{2k} = (\frac{\beta}{2} - 2)(n - k)$, $\delta_{2k} = (\frac{\beta}{2} - 2)(n - k - 1) + \gamma_0 + \delta_0$ $(1 \le k \le n - 1)$. We finally note that densities of the form (2.16) and (2.19) on the moment space $\mathcal{M}_n([0, \infty))$ and $\mathcal{M}_{2n-1}(\mathbb{R})$ are obtained by replacing the Jacobi ensemble by the Laguerre and Hermite ensemble, respectively [see Dette and Nagel (2011) for details].

3. Weak convergence of random moments. In this section we study the probabilistic properties of random vectors on the moment spaces $\mathcal{M}_n([a, b])$, $\mathcal{M}_n([0, \infty))$ and $\mathcal{M}_n(\mathbb{R})$ distributed according to the measures introduced in Section 2. We begin with random moments defined on the moment space corresponding to probability measures on a compact interval. Chang, Kemperman and Studden (1993) and Gamboa and Lozada-Chang (2004) investigated the uniform distribution on $\mathcal{M}_n([a, b])$, and we first demonstrate that weak convergence of random moment vectors can be established for a rather broad class of distributions on $\mathcal{M}_n([a, b])$. An important role in the discussion of moment spaces corresponding to probability measures with bounded support $[a, b]$ plays the arcsine distribution ν with density $d\nu(x) = 1/\pi \sqrt{(x - a)(b - x)} \mathbb{1}_{\{a<x<b\}}\, dx$. The canonical moments

of the arcsine distribution are given by $1/2$ [see Dette and Studden (1997)], and therefore its sequence of moments could be considered as the center of the moment space $\mathcal{M}([a,b])$. The following statements establish the asymptotic properties of the (random) canonical moments corresponding to distributions on the moment space $\mathcal{M}_n([a,b])$ defined in (2.12). Throughout this paper the symbol $\xrightarrow{\mathcal{D}}$ stands for weak convergence.

THEOREM 3.1. *Suppose that the distribution of the random moment vector* $\mathbf{m}_n \in \mathcal{M}_n([a,b])$ *is absolute continuous with density* f_n *defined in* (2.12), *where the point* $\frac{1}{2}$ *is in the support of the measure with density proportional to* $\tilde{f}_k$ *and denote by* $p_k^{(n)}$ *the kth canonical moment of* $\mathbf{m}_n$ $(k = 1, \dots, n)$:

(a) *If* $n \to \infty$, *then almost surely* $p_k^{(n)} \xrightarrow[n\to\infty]{} \frac{1}{2}$.

(b) *If additionally the function* $\tilde{f}_k$ *in the density* (2.12) *is bounded, continuous at* $\frac{1}{2}$ *and positive, then the kth canonical moment corresponding to* $\mathbf{m}_n$ *satisfies* $\sqrt{8n}(p_k^{(n)} - \frac{1}{2}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, I)$.

PROOF. For notational convenience, we consider only the case $[a,b] = [0,1]$. The proof of the first assertion is a consequence of the Laplace method [see, e.g., Pólya and Szegö (1998), page 96]: The canonical moment has density proportional to $\tilde{f}_k(x)(x - x^2)^{n-k}$, which concentrates exponentially fast on any neighborhood of $\frac{1}{2}$. The almost sure convergence follows then with an application of the Borel–Cantelli lemma.

For a proof of part (b), we apply a technique similar to stable convergence; see Aldous and Eagleson (1978) and the references therein. From Chang, Kemperman and Studden (1993) we know that for $\tilde{f}_k \equiv 1$ the convergence holds, that is, $E[I_A(\tilde{p}_k)] \to E[I_A(p)]$ for $\tilde{p}_k = \sqrt{8n}(p_k^{(n)} - \frac{1}{2})$ the normalized canonical moment, p standard normal distributed and A of the form $(-\infty, a]$. Then for any $\tilde{f}_k$ satisfying the assumptions of part (b), the convergence $E[I_A(\tilde{p}_k)\tilde{f}_k(\frac{1}{2} + \frac{1}{\sqrt{8n}}\tilde{p}_k)] \to E[I_A(p)\tilde{f}_k(\frac{1}{2})]$ holds. This implies the convergence of $\tilde{p}_k$ if the density of $p_k^{(n)}$ is multiplied by $\tilde{f}_k$. $\square$

Chang, Kemperman and Studden (1993) showed weak convergence of the vector $\mathbf{m}_k^{(n)}$ of the first k components of a uniformly distributed moment vector $\mathbf{m}_n = (m_1, \dots, m_n)$ on the moment space $\mathcal{M}_n([0,1])$ (i.e., $f_k \equiv 1$), that is,

$$\sqrt{8n}A^{-1}(\mathbf{m}_k^{(n)} - \mathbf{m}_k(\nu)) \xrightarrow[n\longrightarrow\infty]{\mathcal{D}} \mathcal{N}(0, I_k), \tag{3.1}$$

where $\mathbf{m}_k(\nu)$ denotes the vector of the first k moments of the arcsine distribution and A is a $k \times k$ lower triangular matrix with entries

$$a_{i,j} = 2^{-2i+2}\binom{2i}{i-j}, \qquad j \le i.$$

By part (b) of Theorem 3.1, it is easy to see that the weak convergence in (3.1) holds for the more general densities f_n on $\mathcal{M}_n([0,1])$. By the arguments in the proof of Theorem 3.1 it is also apparent that no condition on the density f_n can be dropped, without adding more specific restrictions. In particular, we need $\tilde{f}_k$ to be independent of n, which implies the product structure of f_n. In this sense, the densities as in (2.12) constitute the largest class of densities on the moment space such that the convergence (3.1) holds.

We conclude this paper with a discussion of corresponding results for distributions on the noncompact moment spaces $\mathcal{M}_n([0,\infty))$ and $\mathcal{M}_n(\mathbb{R})$. In this case the analogs of the arcsine distribution in this context are the Marchenko–Pastur and Wigner's semicircle distribution defined by

$$d\eta(x) = \frac{\sqrt{x(4-x)}}{2\pi x} \mathbb{1}_{\{0<x<4\}}\, dx, \qquad d\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{\{-2<x<2\}}\, dx, \tag{3.2}$$

respectively; see Hiai and Petz (2000) or Nica and Speicher (2006). The moments of the Marchenko–Pastur law η are the Catalan numbers c_n defined by $m_n(\eta) = c_n = \frac{1}{n+1}\binom{2n}{n}$ ($n \in \mathbb{N}$), and the moments of the semicircle law ρ are given by $m_{2n}(\rho) = c_n, m_{2n-1}(\rho) = 0$. Our next result establishes the asymptotic properties of the quantities z_k corresponding to a random vector on the moment space $\mathcal{M}_n([0,\infty))$ with density $g_n^{(\gamma,\delta)}$ defined in (2.16). It is a well-known consequence of the asymptotic behavior of the density of the Gamma distribution.

THEOREM 3.2. *Suppose $\mathbf{m}_n$ is a random vector of moments on the moment space $\mathcal{M}_n([0,\infty))$ with density $g_n^{(\gamma,\delta)}$, where the γ_k are fixed, $\delta_1 = \cdots = \delta_n = n$, and let $z_k^{(n)}$ denote the kth component of the vector $\mathbf{z}_n = (z_1^{(n)}, \dots, z_n^{(n)})$. Then the standardized random variable $z_k^{(n)}$ is asymptotically normal distributed, that is,*

$$\sqrt{n}\big(z_k^{(n)} - 1\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

By Theorem 3.2 the vector $\sqrt{n}(\mathbf{z}_k^{(n)} - \mathbf{1}) = \sqrt{n}((z_1^{(n)}, \dots, z_k^{(n)})^T - (1, \dots, 1)^T)$ is asymptotically multivariate normal distributed. In order to derive a corresponding statement of the random vector $\mathbf{m}_k^{(n)} = \psi_k^{-1}(\mathbf{z}_k^{(n)})$ we will use the Delta method and study first the image of the vector $(1, \dots, 1)^T$ under the mapping ψ_k^{-1}.

LEMMA 3.3. *Let $(c_n)_{n\ge 1}$ denote the sequence of Catalan numbers, then*

$$\psi_n(c_1, \dots, c_n) = (1, \dots, 1)^T.$$

PROOF. The proof presented here relies on the combinatorical interpretation of the Catalan numbers and a recursive algorithm given in Skibinsky (1968) to calculate the moments in terms of the variables z_k. The kth Catalan number counts the paths in $\mathbb{N} \times \mathbb{N}$ starting in $(0,0)$ and ending in $(2k, 0)$, where one is only allowed to make steps in the direction $(1, 1)$ or $(1, -1)$. Skibinsky (1968) defines the triangular array $\{g_{i,j}\}_{i,j\geq 0}$ by $g_{i,j} = 0$ for $i > j$, $g_{0,j} = 1$ and the recursion

$$g_{i,j} = g_{i,j-1} + z_{j-i+1} g_{i-1,j}, \qquad 1 \leq i \leq j. \tag{3.3}$$

He showed that $g_{k,k} = m_k$. Consequently, if $z_i = 1$ $(i = 1, 2, \ldots)$, the quantity $g_{k,k}$ is the number of paths through the lattice $\{(i, j)\}_{i,j\geq 0}$, starting in (k, k) and ending in $(0, 0)$, where in each vertex we can only make steps upward or to the left and where we are not allowed to cross the diagonal $\{(i, i)\}$. This number is exactly the kth Catalan number c_k. $\square$

THEOREM 3.4. *If the vector of random moments $\mathbf{m}_n \in \mathcal{M}_n([0, \infty))$ is governed by a law with density $g_n^{(\gamma,\delta)}$, where $\delta_1 = \cdots = \delta_n = n$, and the γ_k are fixed, then the projection $\mathbf{m}_k^{(n)} = \Pi_k^n(\mathbf{m}_n)$ of $\mathbf{m}_n$ onto the first k coordinates satisfies*

$$\sqrt{n} C^{-1}\bigl(\mathbf{m}_k^{(n)} - \mathbf{m}_k(\eta)\bigr) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where the vector $\mathbf{m}_k(\eta) = (c_1, \ldots, c_k)^T$ contains the first k moments of the Marchenko–Pastur distribution, and C is a lower triangular matrix with entries $c_{1,1} = \cdots = c_{k,k} = 1$, and

$$c_{i,j} = \binom{2i}{i-j} - \binom{2i}{i-j-1}, \qquad j < i.$$

PROOF. It suffices to calculate the Jacobi matrix $C = \frac{\partial \psi_k^{-1}}{\partial \mathbf{z}_k}(\mathbf{z}_k^0)$ of the mapping ψ_k^{-1} at $\mathbf{z}_k^0 = (1, \ldots, 1)^T$; then the independence of the recursion variables z_k and Theorem 3.2 yield

$$\sqrt{n}\bigl(\mathbf{m}_k^{(n)} - \mathbf{m}_k(\eta)\bigr) = \sqrt{n} C\bigl(\mathbf{z}_k^{(n)} - \mathbf{z}_k^0\bigr) + o_P(1) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, CC^T).$$

Note that the moment m_i depends only on $z_1, \ldots, z_i$, and consequently C is a lower triangular matrix. To identify the entries of the matrix C we consider the triangular array $\{g_{i,j}\}_{i,j\geq 0}$ defined in (3.3). For a fixed r with $1 \leq r \leq k$, we introduce the notation $u_{i,j} = \frac{\partial g_{i,j}}{\partial z_r}(\mathbf{z}_k^0)$ and obtain a new triangular array $\{u_{i,j}\}_{i,j\geq 0}$. Obviously we have $u_{i,j} = 0$ for $i > j$, and the other values of $u_{i,j}$ are determined by the initial condition $u_{0,j} = 0$ and the recursion

$$u_{i,j} = u_{i,j-1} + u_{i-1,j} + \delta_{r,j-i+1} g_{i-1,j}^0, \qquad 1 \leq i \leq j, \tag{3.4}$$

were $\delta_{i,j}$ denotes the Kronecker symbol, and $g_{i,j}^0$ is the coefficient in the recursion (3.4), if all z_i are equal to 1, that is,

$$g_{i,j}^0 = \binom{i+j}{i} - \binom{i+j}{i-1}.$$

The numbers $g_{i,j}^0$ are sometimes called generalized Catalan numbers; see Finucan (1976). An induction argument shows that the entries in the new triangular array are given by

$$u_{i,j} = \begin{cases} \binom{i+j}{i-1} - \binom{i+j}{i-r-1}, & \text{if } j-i \geq r, \\ \binom{i+j}{j-r} - \binom{i+j}{i-r-1}, & \text{if } 0 \leq j-i < r. \end{cases} \tag{3.5}$$

With this identity we obtain for the entries of the matrix C

$$c_{i,r} = \frac{\partial m_i}{\partial z_r}(\mathbf{z}_k^0) = u_{i,i} = \binom{2i}{i-r} - \binom{2i}{i-r-1}$$

for $1 \leq r \leq i$, which proves the assertion of the theorem. □

By the same arguments as in the compact case, the general class of densities on the unbounded moment space $\mathcal{M}_n([0,\infty))$ for which the result of Theorem 3.4 holds, is $g_n^{(\gamma,\delta)}(\mathbf{m}_n)\prod_{k=1}^n \tilde{g}_k(z_k(\mathbf{m}_n))$. Here, the functions $\tilde{g}_k$ have to be bounded and continuous and positive at 1. An analogous result holds in the remaining case of measures on $\mathbb{R}$.

We now consider the moment space $\mathcal{M}_{2n-1}(\mathbb{R})$ and recall the bijective mapping (2.9) from the interior of the moment space $\mathcal{M}_{2n-1}(\mathbb{R})$ onto $(\mathbb{R}\times\mathbb{R}^+)^{n-1}\times\mathbb{R}$ corresponding to the range for coefficients in the recursive relation of the orthogonal polynomials (2.3). The following results give the weak asymptotics of random recursion coefficients and moments and correspond to Theorem 3.2 and 3.4. The proof is omitted.

THEOREM 3.5. *Let the random vector $\mathbf{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ be governed by a law with density $h_{2n-1}^{(\gamma,\delta)}$ where $\gamma_k > -1$ is fixed and $\delta_k = n$ $(k = 1,\dots,n)$. For fixed k denote by $b_k^{(n)}$ and by $a_k^{(n)}$ the $(2k-1)$th component of the vector $\xi_{2n-1}(\mathbf{m}_{2n-1})$ and the $2k$th component, respectively. Then*

$$\sqrt{2n}\, b_k^{(n)} \sim \mathcal{N}(0,1), \qquad \sqrt{2n}\,(a_k^{(n)} - 1) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

THEOREM 3.6. *Let the vector of random moments $\mathbf{m}_{2n-1} \in \mathcal{M}_{2n-1}(\mathbb{R})$ be governed by a law with density $h_{2n-1}^{(\gamma,\delta)}$ where $\delta_k = n$ $(k = 1,\dots,n)$. For $k \in \mathbb{N}$ denote by $\mathbf{m}_k^{(n)} = \Pi_k^n(\mathbf{m}_{2n-1})$ the projection onto the first k coordinates and by $\mathbf{m}_k(\rho) = \Pi_k(0, c_1, 0, c_2, \dots)$ the vector of the first k moments of the semicircle law defined in (3.2). Then*

$$\sqrt{2n}\, D^{-1}(\mathbf{m}_k^{(n)} - \mathbf{m}_k(\rho)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where D is a $k \times k$ lower triangular matrix with $d_{i,j} = 0$ if $i + j$ is odd, and the remaining entries are given by

$$d_{i,j} = \binom{i}{\frac{i-j}{2}} - \binom{i}{\frac{i-j}{2} - 1}.$$

REMARK 3.7. It follows from Remark 2.6 that the moment density of the Jacobi ensemble is the moment density investigated asymptotically in this section. Although for the random matrix ensembles the parameters γ_k, δ_k depend on n, only minor changes are necessary to obtain a weak convergence result for the first k moments. Note that the canonical moment $p_k^{(n)}$ follows a Beta distribution with parameters behaving like $\frac{\beta}{2}n$. Therefore $\sqrt{4\beta n}(p_k^{(n)} - \frac{1}{2}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, 1)$, and we obtain easily for the vector $\mathbf{m}_k(\mu_n)$ of the first k moments of μ_n of the spectral measure of the Jacobi ensemble defined in (2.21)

$$\sqrt{4\beta n} A^{-1}(\mathbf{m}_k(\mu_n) - \mathbf{m}_k(\nu)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_k(0, I_k),$$

where $\mathbf{m}_k(\nu)$ is the moment vector of the arcsine measure and A is the $k \times k$ matrix in (3.1).

In particular, the moment convergence implies the weak convergence of the spectral measure to the arcsine measure. This is also a consequence of the well-known convergence of the empirical eigenvalue distribution to the arcsine measure, since the (unscaled) moments of the spectral measure have the same asymptotic behavior as the moments of the empirical eigenvalue distribution. Therefore the fluctuations around this limit in terms of the moments are Gaussian. The corresponding results for the Laguerre and Hermite ensemble are omitted for the sake of brevity.

REMARK 3.8. Already in the compact case there are interesting results regarding a functional limit theorem. In particular, Dette and Gamboa (2007) proved the convergence of the standardized range process $(m_{\lfloor nt\rfloor}^+ - m_{\lfloor nt\rfloor}^-)_t$ to a functional of the Brownian motion in the Skorohod topology. With the distributions on the unbounded moment spaces, the question arises whether a corresponding result exist in these cases. Interesting processes are, for example, the moment difference $(m_{\lfloor nt\rfloor} - m_{\lfloor nt\rfloor}^-)_t$ for measures on $[0, \infty)$ and for measures on the whole real line the integrals over orthogonal polynomials as in formulas (2.10) and (2.11). For the sake of brevity, we do not discuss functional limit theorems in the unbounded cases and defer these interesting questions to our future research.

Acknowledgments. The authors are grateful to Martina Stein who typed parts of this paper with considerable technical expertise. The authors would also like to thank two anonymous referees for their constructive comments on an earlier version of this paper.

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