RANDOM WALKS DRIVEN BY LOW MOMENT MEASURES

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We study the decay of convolution powers of probability measures without second moment but satisfying some weaker finite moment condition. For any locally compact unimodular group G and any positive function $\varrho : G \to [0, +\infty]$, we introduce a function $\Phi_{G,\varrho}$ which describes the fastest possible decay of $n \mapsto \phi^{(2n)}(e)$ when ϕ is a symmetric continuous probability density such that $\int \varrho \phi$ is finite. We estimate $\Phi_{G,\varrho}$ for a variety of groups G and functions ϱ. When ϱ is of the form $\varrho = \rho \circ \delta$ with $\rho : [0, +\infty) \to [0, +\infty)$, a fixed increasing function, and $\delta : G \to [0, +\infty)$, a natural word length measuring the distance to the identity element in G, $\Phi_{G,\varrho}$ can be thought of as a group invariant.

1. Introduction. Throughout this work, G is a locally compact unimodular group equipped with its Haar measure λ, and $L^p(G) = L^p(G, \lambda)$, $1 \le p \le \infty$, is the space of (classes of) p integrable measurable functions. When convenient, we write $\lambda(dx) = dx$.

Sometimes, but not always, we will assume that G is also compactly generated. When that is the case, we let U be an open relatively compact set which is symmetric and contains a compact generating neighborhood of the identity element e. For any element x in G, we set $|x| = \inf\{n : x \in U^n\}$ (with the convention that $U^0 = \{e\}$) and $V(n) = \lambda(U^n)$. The function V is called the volume growth function of the group G. The rough behavior of both $x \mapsto |x|$ and $n \mapsto V(n)$ is essentially independent of the choice of U; for example, see [31]. The case when G is a finitely generated group equipped with a finite symmetric generating set, and its counting measure is of course included here, and the results we obtain are particularly interesting in this case.

Given a Borel probability measure μ on G, we let $\mu^{(n)}$ be the n-fold convolution power of μ and let $\check{\mu}$ be the measure defined by $\check{\mu}(A) = \mu(A^{-1})$ for any Borel set A. Recall that $\mu^{(n)}$ is the law of the random walk driven by μ and started at e. We call a measure symmetric if $\mu = \check{\mu}$. Since G is unimodular, we have $\check{\lambda} = \lambda$. It follows that a measure having a density ϕ w.r.t. the Haar measure λ is symmetric if and only if ϕ is symmetric, that is, $\phi = \check{\phi}$ where $\check{\phi}(x) = \phi(x^{-1})$; see, for example, [7], Exercise 5, page 89. Throughout the paper, we denote by R_ϕ the operator of

convolution by the function $\phi \in L^2(G)$ on the right, that is, $R_\phi f = f * \phi$ (say, for compactly supported continuous function f). When ϕ is in $L^1(G)$, R_ϕ also denotes the extension of this operator to $L^2(G)$ [and, more generally, $L^p(G)$]. When $\phi = \check{\phi} \in L^1(G)$, R_ϕ is a bounded self-adjoint operator on $L^2(G)$.

1.1. *The decay of convolution powers.* A probability measure μ on a compactly generated group G is said to have finite second moment if $\mu(|\cdot|^2) < \infty$. A fundamental result concerning symmetric random walks on groups asserts that there exists a nonincreasing positive function Φ_G such that, for any symmetric probability measure $d\mu = \phi \, d\lambda$ with finite second moment and continuous density ϕ whose support contains a generating compact neighborhood of the identity, we have

$$(1.1) \qquad \mu^{(2n)}(U) \simeq \phi^{(2n)}(e) \simeq \Phi_G(n);$$

see [13, 20]. Here, $f(n) \simeq g(n)$ means that there are constants $c_i \in (0, \infty)$ such that, for all n, $c_1 f(c_2 n) \le g(n) \le c_3 f(c_4 n)$. Clearly, in the above estimates, the implied constants c_i are allowed to depend on μ and G.

The following list provides some examples of explicit computation of Φ_G, assuming that G is compactly generated. More accurately, it is the equivalence class of Φ_G under the equivalence relation $\simeq$ which is computed.

- If G is such that $V(n) \simeq n^D$, then $\Phi_G(n) \simeq n^{-D/2}$. Every nilpotent group has these properties for some integer D; see [29, 31] and the references therein.
- If G is polycyclic (or linear solvable) and has exponential volume growth, then $\Phi_G(n) \simeq \exp(-n^{1/3})$; see [1, 22, 30, 31].
- The group G is nonamenable if and only if $\Phi_G(n) \simeq \exp(-n)$ (this is a formulation of Kesten's celebrated theorem regarding amenability and random walks).
- Let M, N be two finitely generated groups, and let G be the wreath product $G = M \wr N = (\sum_{n \in N} M_n) \rtimes N$. This is the semidirect product of N with of the direct sum of countably many copies of M indexed by N where the action of N is by index translation; see, for example, [21] for a precise definition.
  - Assume N satisfies $V_N(n) \simeq n^d$ for some $d \ge 1$ and M is nontrivial. Then we have

$$\Phi_G(n) \simeq \begin{cases} \exp(-n^{d/(d+2)}), & \text{if } M \text{ is finite;} \\ \exp(-[n^d (\log n)^2]^{1/(d+2)}), & \text{if } V_M(n) \simeq n^b, b \ge 1; \\ \exp(-n^{(d+1)/(d+3)}), & \text{if } M \in \mathcal{PE}, \end{cases}$$

  where $\mathcal{PE}$ stands for polycyclic with exponential volume growth.
  - Assume that $N \in \mathcal{PE}$ and M is nontrivial, finite or polycyclic. Then we have

$$\Phi_G(n) \simeq \exp(-n(\log n)^{-2});$$

see [9, 10, 21, 26] for details and further results.

- Let $N = \mathbb{Z}^d$, M be nontrivial, and $k \geq 2$ be an integer. Set $G = M \wr (M \wr (\cdots (M \wr N) \cdots))$ where k successive wreath products are taken. Then

$$\Phi_G(n) \simeq \begin{cases} \exp(-n(\log_{k-1} n)^{-2/d}), & \text{if } M \text{ is finite,} \\ \exp(-n[(\log_{k-1} n)/\log_k n]^{-2/d}), & \text{if } V_M(n) \simeq n^b, b \geq 1. \end{cases}$$

Here, $\log_1(x) = \log(e + x)$ and $\log_k(x) = \log(e + \log_{k-1}(x))$, $k \geq 2$; see [9, 10, 26].

The article [25] gives an overview. Many further behaviors are possible for the function Φ_G, but a complete classification of the possible behaviors is not known. In fact, the very existence of such a classification seems highly unlikely, and there are (uncountably) many amenable finitely generated groups G for which the behavior of Φ_G is unknown. Still, Definition 1.1 means that on any such group, we know that all random walks driven by a symmetric measure with generating support and finite second moment have comparable probability of return.

This work focuses on the probability of return of random walks driven by measures that may fail to have a finite second moment but satisfy some finite moment condition. Namely, consider a nonnegative, nondecreasing function $\rho : [0, +\infty) \to [0, +\infty)$. For any finitely generated group G equipped with a word length $|\cdot|$ as above, let ρ_G be the function

$$\rho_G : G \to [0, +\infty), \qquad x \mapsto \rho_G(x) = \rho(|x|).$$

We will abuse notation and write ρ for ρ_G when convenient. We say that a probability measure μ on G has finite ρ_G-moment if

$$\mu(\rho_G) = \sum_{g \in G} \rho_G(g)\mu(g) < \infty.$$

Since we are mostly interested in measures without second moment, the following are some of the natural choices for ρ:

- Small powers: $\rho_\alpha(t) = (1 + t)^\alpha$, $\alpha \in (0, 2)$.
- Regularly varying functions of index $\alpha \in (0, 2)$, for example,

$$\rho(t) = (1 + t)^\alpha [\log(e + t)]^\beta, \qquad \beta \in \mathbb{R}.$$

- Slowly varying increasing functions including:
  - $\rho_{c,\alpha}^{\exp}(t) = \exp(c[\log(1 + t)]^\alpha)$, $\alpha \in (0, 1)$ and $c > 0$;
  - $\rho_\alpha^{\log}(t) = [\log(e + t)]^\alpha$, $\alpha \in (0, \infty)$.

We consider the following natural question. What can be said about the decay of $\phi^{(2n)}(e)$ when $d\mu = \phi\, d\lambda$ is a symmetric measure having a finite ρ_G-moment for one of the functions ρ mentioned above?

1.2. *Group invariants associated with random walks and moment conditions.* In general, requiring that a symmetric measure μ has a finite moment of some sort is not enough to determine the behavior of the convolution powers of that measure. The following definition introduces the notion of "fastest decay" allowed by a given moment condition.

DEFINITION 1.1 (Fastest decay under ϱ-moment).   Let G be a locally compact unimodular group. Fix a measurable function $\varrho : G \to [0, +\infty]$. Fix a compact symmetric neighborhood Ω of e in G such that $\lambda(\Omega) \geq 1$ and $\sup_{\Omega^2}\{\varrho\} > 0$. For $K > 1$, let $\mathcal{S}_{G,\varrho}^{\Omega,K}$ be the set of all symmetric continuous probability densities ϕ on G with the properties that $\|\phi\|_\infty \leq K$ and $\int \phi \varrho \, d\lambda \leq K \sup_{\Omega^2}\{\varrho\}$. Set

$$\Phi_{G,\varrho}^{\Omega,K} : n \mapsto \Phi_{G,\varrho}^{\Omega,K}(n) := \inf\{\phi^{(2n)}(e) : \phi \in \mathcal{S}_{G,\varrho}^{\Omega,K}\}.$$

In words, $\Phi_{G,\varrho}^{\Omega,K}$ provides the best lower bound valid for all convolution powers of probability measures with density in $\mathcal{S}_{G,\varrho}^{\Omega,K}$.

Let $\phi_0 = \lambda(\Omega)^{-1}\mathbf{1}_\Omega$. Then $\phi_0^{(2)} \in \mathcal{S}_{G,\varrho}^{\Omega,K}$ so that $\Phi_{G,\varrho}^{\Omega,K}$ takes finite values. Clearly, $n \mapsto \Phi_{G,\varrho}^{\Omega,K}(n)$ is nonincreasing because $n \mapsto \phi^{(2n)}(e)$ is nonincreasing when ϕ is symmetric. By definition, $\Phi_{G,a\varrho}^{\Omega,K} = \Phi_{G,\varrho}^{\Omega,K}$ for any $a > 0$. A priory, it is possible that $\Phi_{G,\varrho}^{\Omega,K} \equiv 0$, but in many cases, this possibility can be ruled out so that $\Phi_{G,\varrho}^{\Omega,K}$ is actually meaningful and contains information. As indicated below, the choice of Ω and K in this definition is mostly irrelevant.

The following proposition contains basic (but not entirely obvious) properties of $\Phi_{G,\varrho}^{\Omega,K}$ that indicate that Definition 1.1 is quite reasonable. Because of this proposition, we will often omit the reference to Ω and K in $\Phi_{G,\varrho}^{\Omega,K}$ and write

$$\Phi_{G,\varrho}^{\Omega,K} = \Phi_{G,\varrho}.$$

PROPOSITION 1.2.   *Let G be a locally compact unimodular group. Let $\varrho : G \to [0, +\infty]$ be a measurable function and fix a compact symmetric neighborhood Ω of e in G such that $\lambda(\Omega) \geq 1$ and $\sup_{\Omega^2}\{\varrho\} > 0$. Fix $K > 1$.*

- *If there exists a constant C such that, for all $x, y \in G$, $\varrho(xy) \leq C(\varrho(x) + \varrho(y))$ then, for each integer n, $\Phi_{G,\varrho}^{\Omega,K}(n) > 0$.*
- *For any symmetric continuous probability density ϕ with finite ϱ-moment, that is, such that $\int \varrho \phi \, d\lambda < \infty$, there are a positive constant $c = c(\phi)$ and an integer $k = k(\phi)$ such that, $\forall n$, $\phi^{(2n)}(e) \geq c\Phi_{G,\varrho}^{\Omega,K}(kn)$.*
- *For $i = 1, 2$, fix constants $K_i > 1$ and compact symmetric neighborhoods Ω_i of e in G with $\lambda(\Omega_i) \geq 1$. Let ϱ_i, $i = 1, 2$, be nonnegative measurable functions on*

G such that $a\varrho_1 \le \varrho_2 \le A\varrho_1$ for some $a, A \in (0, \infty)$ and $\sup_{\Omega_i^2}\{\varrho_i\} \in (0, \infty)$. Then, we have

$$\Phi_{G,\varrho_1}^{\Omega_1, K_1} \simeq \Phi_{G,\varrho_2}^{\Omega_2, K_2}.$$

For general ϱ, we do not expect to be able to give a precise bound on $\Phi_{G,\varrho}$, even in the case of Abelian groups such as $\mathbb{Z}^d$.

A more reasonable question is to try to understand $\Phi_{G,\varrho}$ when $\varrho = \rho_G$ and ρ belongs to a specific family of examples such as the families ρ_α, $\rho_{c,\alpha}^{\exp}$, or $\rho_\alpha^{\log}$ mentioned above. Indeed, in such cases, the function Φ_{G,ρ_G} (or, perhaps, its equivalence class under the equivalence relation $\simeq$) can be thought of as a group invariant describing the fastest possible decay of the probability of return of a random walk driven by a symmetric measure with finite ρ_G-moment. In this restricted context, one may hope to estimate Φ_{G,ρ_G} in terms of the function Φ_G in (1.1) and the function ρ. Further, it is an interesting natural question to ask whether or not all/some of the invariants Φ_{G,ρ_G} are actually already determined by Φ_G. This appears to be a rather subtle question.

Another interesting question raised by Definition 1.1 is the question of describing classes of measures that are in $\mathcal{S}_{G,\varrho}^{\Omega, K}$ and approach the extremal behavior described by Φ_{G,ρ_G}. What is the typical "shape" of an almost optimal density? For instance, should we expect these densities to include densities that are roughly "radial" in terms of the given word-length $|\cdot|$? Can we obtain almost extremal densities as convex combinations of the convolution powers of the uniform probability on a compact symmetric generating neighborhood of the identity element in G?

Let us observe that determining the exact behavior of Φ_{G,ρ_G} is a delicate task, even for $G = \mathbb{Z}$ and $\rho(x) = (1 + |x|)^\alpha$, $\alpha \in (0, 2)$. Hence, it is useful and natural to introduce simplified invariants by comparing $\Phi_{G,\varrho}$ to certain scales of functions. The following definition introduces a sample of such simplified invariants.

DEFINITION 1.3. For G and ϱ as in Definition 1.1, define:

(1) The power decay invariant,

$$\text{power}(G, \varrho) = \inf\Big\{\gamma \in (0, \infty) : \sup_n\{n^\gamma \Phi_{G,\varrho}(n)\} = \infty\Big\}.$$

(2) The exponential-polylog decay invariant,

$$\text{exp-plg}(G, \varrho) = \inf\Big\{\gamma \in (0, \infty) : \inf_n\{(\log(e + n))^{-\gamma} \log(1/\Phi_{G,\varrho}(n))\} = 0\Big\}.$$

Computing this quantity is of interest when $\text{power}(G, \varrho) = \infty$.

(3) The exponential-power decay invariant,

$$\text{exp-pow}(G, \varrho) = \inf\Big\{\gamma \in (0, 1] : \inf_n\{n^{-\gamma} \log(1/\Phi_{G,\varrho}(n))\} = 0\Big\}.$$

Again, computing this quantity is of interest when $\text{exp-plg}(G, \varrho) = \infty$.

1.3. *A sample of illustrative results.*   Throughout this subsection we assume that G is compactly generated and that Φ_G is the function given by (1.1) (up to the equivalence relation $\simeq$). With the notation introduced above, we can state a number of theorems that illustrate the type of results we obtain in this work. Recall the following notation:

- $\rho_\alpha(t) = (1+t)^\alpha$, $\alpha \in (0,2)$.
- $\rho_{c,\alpha}^{\exp}(t) = \exp(c[\log(1+t)]^\alpha)$, $\alpha \in (0,1)$ and $c > 0$;
- $\rho_\alpha^{\log}(t) = [\log(e+t)]^\alpha$, $\alpha \in (0,\infty)$.

THEOREM 1.4.   *If G has polynomial volume growth of degree D, that is,* $V(n) \simeq n^D$, *then*

$$\forall \alpha \in (0,2) \qquad \mathrm{power}(G, \rho_\alpha) = D/\alpha$$

and

$$\forall \alpha \in (0,1) \qquad \mathrm{exp\text{-}plg}(G, \rho_{c,\alpha}^{\exp}) = 1/\alpha.$$

As we shall see, we run into difficulties when estimating $\mathrm{exp\text{-}pow}(G, \rho_\alpha^{\log})$. Assuming G has polynomial volume growth, we are only able to obtain the estimates

$$\frac{1}{\alpha+1} \le \mathrm{exp\text{-}pow}(G, \rho_\alpha^{\log}) \le \frac{1}{\alpha}, \qquad \alpha > 1.$$

This indicates that our techniques need to be improved in order to treat low moment conditions. Indeed, on $\mathbb{Z}$ (and other Abelian groups), simple Fourier analysis techniques yield

$$\mathrm{exp\text{-}pow}(\mathbb{Z}, \rho_\alpha^{\log}) = \frac{1}{\alpha+1}, \qquad \alpha > 0;$$

see [3].

THEOREM 1.5.   *Assume that the group G has the property that*

$$\forall n \qquad \Phi_G(n) \ge \exp(-cn^\gamma)$$

for some $c \in (0,\infty)$ and $\gamma \in (0,1)$. Then, for any $\alpha \in (0,2)$, there exists $c_1 \in (0,\infty)$ such that

$$\forall n \qquad \Phi_{G,\rho_\alpha}(n) \ge \exp(-c_1 n^{\gamma_\alpha}) \qquad \text{where } \gamma_\alpha = \frac{\gamma}{\gamma + (\alpha/2)(1-\gamma)}.$$

So, for instance, for any finitely generated polycyclic group with exponential volume growth, we have $\gamma = 1/3$, and thus the probability of return of a random walk driven by a symmetric measure μ with finite first moment [i.e., $\mu(|\cdot|) < \infty$] is bounded below by

$$\mu^{(2n)}(e) \ge \exp(-c_1 n^{1/2}).$$

As indicated by the following results, the lower bound stated in Theorem 1.5 is essentially sharp in a number of cases.

THEOREM 1.6. *Assume that the group G has exponential volume growth and satisfies* $\forall n, \Phi_G(n) \geq \exp(-cn^{1/3})$. *Then, for each* $\alpha \in (0, 2)$,

$$\text{exp-pow}(G, \rho_\alpha) = \frac{1}{1 + \alpha}$$

and $\text{exp-pow}(G, \rho_{c,\beta}^{\exp}) = \text{exp-pow}(G, \rho_\alpha^{\log}) = 1, \beta \in (0, 1), c > 0, \alpha > 2.$

Note that the statement that $\text{exp-pow}(G, \rho_{c,\beta}^{\exp}) = \text{exp-pow}(G, \rho_\alpha^{\log}) = 1$ for the groups considered in Theorem 1.6 is crude. More detailed results are described in the core of the paper. For instance, $\text{exp-pow}(G, \rho_{c,\beta}^{\exp}) = 1$ can be refined to the much more informative statement that, for any fixed $c > 0$ and $\beta \in (0, 1)$, there exist $c_1, c_2 \in (0, \infty)$ such that

$$-n \exp(-c_1 (\log n)^\beta) \leq \log \Phi_{G, \rho_{c,\beta}^{\exp}}(n) \leq -n \exp(-c_2 (\log n)^\beta)$$

for all n large enough.

The case of the lamplighter groups $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$, the simplest wreath products, is particularly interesting.

THEOREM 1.7. *For* $G_d = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, d = 1, 2, \ldots,$ *and for* $\alpha \in (0, 2)$,

$$\text{exp-pow}(G_d, \rho_\alpha) = \frac{d}{d + \alpha}.$$

PROOF. The upper bound follows from Theorem 1.5. The lower bound requires an ad hoc argument explained in Section 5. □

For the next result, recall that a group G is meta-Abelian if it contains a normal Abelian subgroup A such that G/A is Abelian. From the view point of group theory, meta-Abelian groups are only "one step" removed from being Abelian.

THEOREM 1.8. *Let G be a finitely generated meta-Abelian group. Then either G has polynomial volume growth and there is an integer D such that*

$$\forall \alpha \in (0, 2) \qquad \text{power}(G, \rho_\alpha) = D/\alpha$$

or there exists an integer d such that

$$\forall \alpha \in (0, 2) \qquad \frac{1}{1 + \alpha} \leq \text{exp-pow}(G, \rho_\alpha) \leq \frac{d}{d + \alpha}.$$

PROOF.   Being solvable, finitely generated meta-Abelian groups either have polynomial volume growth or exponential volume growth; see [18, 32]. In the polynomial volume growth case, apply Theorem 1.4. For any group with exponential volume growth, Theorem 4.10 gives the lower bound exp-pow$(G, \rho_\alpha) \geq \frac{1}{1+\alpha}$. By [21], any meta-Abelian group has $\Phi_G(n) \geq \exp(-Cn^{d/d+2})$ for some integer $d \geq 1$. Thus the upper bound exp-pow$(G, \rho_\alpha) \leq \frac{d}{d+\alpha}$ follows from Theorem 1.5. $\square$

1.4. *Methodology.*   We close this introduction by describing in broad terms the techniques we will use to prove the results described above. For the purpose of this discussion, we focus on the problem of estimating the rate of decay of convolution powers of symmetric measures having a continuous density and a finite ρ_α-moment, $\alpha \in (0, 2)$ $[\rho_\alpha(x) = (1 + |x|)^\alpha]$. We start with a quick review of classical results in the context of the lattice $\mathbb{Z}^d$. In this context, the literature focuses on local limit theorems, that is, results that describe the precise asymptotic behavior of $\phi^{(2n)}(x)$. For instance, if ϕ is a symmetric probability density which has generating support and finite second moment, $\phi^{(2n)}(0) \sim c(d, \mu)n^{-d/2}$ (e.g., [28], P9, Section 7). For $\alpha \in (0, 2)$, the simple condition of having a finite ρ_α-moment is not sufficient for the validity of a local limit theorem, even on $\mathbb{Z}$.

For a symmetric probability density ϕ on $\mathbb{Z}$, set $G(k) = \sum_{|i| \geq k} \phi(k)$, $H(k) = k^{-2} \sum_{|i| \leq k} i^2 \phi(i)$. Then ϕ is in the domain of attraction of a symmetric stable law of index α if $\lim_\infty H/G = \alpha/(2 - \alpha)$. In such a case, a local limit theorem holds stating that $\phi^{(2n)}(e) \sim c(\alpha, \mu)a_n$ with a_n defined by $Q(a_n) = 1/n$, $Q = G + H$; see, for example, [11, 12, 14, 15]. All classical discussions of such results make heavy use of Fourier transform techniques. It is easy to use these techniques to see that if a symmetric probability density ϕ with generating support on $\mathbb{Z}^d$ has finite ρ_α-moment for some $\alpha \in (0, 2)$, then we must have $\phi^{(2n)}(0) \geq c(d, \mu)n^{-d/\alpha}$ and

$$\Phi_{\mathbb{Z}^d, \rho_\alpha}(n) \geq c(d, \alpha)n^{-d/\alpha}.$$

As laws that are in the domain of attraction of a symmetric stable law of index $\beta > \alpha$ have finite ρ_α-moment, we also get that

$$\forall \beta > \alpha \qquad \Phi_{\mathbb{Z}^d, \rho_\alpha}(n) \leq c_\beta n^{-d/\beta}.$$

Hence power$(\mathbb{Z}^d, \rho_\alpha) = d/\alpha$. Note that determining the exact behavior of $n \mapsto \Phi_{\mathbb{Z}, \rho_\alpha}(n)$ appears to be a somewhat subtle problem and will not be discussed here.

Both the Fourier transform and explicit examples such as symmetric stable laws are not available on most noncommutative groups so that the arguments outlined above must be replaced by different ideas. Our approach is as follows:

(1) Our lower bounds on Φ_{G, ρ_α} are obtained and expressed via the function Φ_G given by (1.1). This function Φ_G describes the decay of convolution powers of symmetric, nondegenerate densities with finite second moment on the group G.

To transfer the information contained in this function Φ_G and make it relevant to the study of the convolution powers of measures with finite ρ_α-moment, we will use a sort of interpolation argument, the comparison of Dirichlet forms and the notion of von Neumann trace. Each one of these ingredients plays a crucial role in obtaining our lower bounds.

Section 2 contains the proof of Proposition 1.2 as well as an interesting and important variation on Definition 1.1. It also develops the key interpolation argument which leads to the comparison of important quadratic forms including the Dirichlet forms of the probability measures we want to study.

The role of the notion of von Neumann trace is explained in the Appendix where related needed material is described. The results developed in the Appendix are the tools that allow us to turn the comparison of quadratic forms obtained in Section 2 into lower bound for Φ_{G,ρ_α}.

(2) To obtain upper bounds on Φ_{G,ρ_α}, it suffices to exhibit some probability densities satisfying the desired moment condition and whose convolution powers can be estimated. On a general noncommutative group, this is not necessarily an easy task. One possible technique—discrete subordination—uses Bernstein functions to produce probability densities on G that include laws that can be thought of as analogs of symmetric stable laws. The decay of the convolution powers of these laws can be precisely expressed and controlled in terms of the group invariant Φ_G at (1.1), and this technique is quite interesting in its own right. This idea, which the authors developed for the purpose of the present paper, is presented in detail in [4]. We will use some of the results of [4]. However, the moment properties of these subordinated laws are directly related to the rate of escape of the basic simple random walk on the underlying group. In particular, for groups with a rate of escape that is faster than the classical $\sqrt{n}$, the moment conditions satisfied by these subordinated laws are not what one would expect from a simplistic analogy with the classical case of $\mathbb{Z}$. For instance, on a finitely generated group with linear rate of escape, the "symmetric stable law" of exponent $\beta \in (0, 2)$ [by definition, the law obtained via discrete $(\beta/2)$-subordination from the law of simple random walk] will only have a finite ρ_α-moment for $\beta > 2\alpha$ (instead of $\beta > \alpha$ in the classical case). What this means is that, in general, upper bounds obtained by using [4] will not match closely the lower bounds discussed in (1) above. They will only do so if there exists a simple random walk on G that has a rate of escape of type $\sqrt{n}$ as in the classical case of $\mathbb{Z}$. This is a subtle requirement since it is not known whether or not all random walks associated with finite symmetric generating sets on a given finitely generated group have the same rate of escape.

(3) There is a more elementary way to produce probability distributions with finite ρ_α-moment and whose convolution powers can be estimated. This technique goes back to [24, 29]. It is revisited in Section 4.2. It works well for groups where the invariant Φ_G behaves precisely as predicted by the available upper bounds based on volume growth (e.g., polycyclic groups). It does not work well for wreath products such as $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$, $d \geq 2$. For such groups neither of the techniques in

(2) or (3) produce upper bounds on Φ_{G,ρ_α} matching the lower bounds obtained via (1). Nevertheless, Section 5 shows that the lower bounds obtained via (1) are essentially tight even in the case of these wreath products. This requires an ad hoc argument that takes advantage of the precise structure of these groups.

2. Comparisons of Dirichlet forms.
This section develops the key technique that we use to obtain lower bounds on the functions $\Phi_{G,\varrho}$ introduced in Definition 1.1, namely, comparison of Dirichlet forms. The first subsection contains simple results that show that the object introduced in Definition 1.1, $\Phi_{G,\varrho}$, has some nice stability properties. The second subsection develops a somewhat sophisticated comparison between certain quadratic forms. It plays a central role in our results.

It is useful to introduce the following somewhat subtle modification of Definition 1.1 in which a "weak moment condition," $W(\varrho, \mu) < \infty$, replaces the "strong moment condition" $\mu(\varrho) < \infty$. For any probability measure μ and $\varrho: G \to [0, \infty)$, $W(\varrho, \mu)$ is defined by

$$W(\varrho, \mu) = \sup_{s>0}\{s\mu(\varrho > s)\}.$$

DEFINITION 2.1. Let G be a locally compact unimodular group. Fix a measurable function $\varrho: G \to [0, +\infty]$. Fix a compact symmetric neighborhood Ω of e in G such that $\lambda(\Omega) \geq 1$ and $\sup_{\Omega^2}\{\varrho\} > 0$. For $K > 1$, let $\widetilde{\mathcal{S}}_{G,\varrho}^{\Omega,K}$ be the set of all symmetric continuous probability densities ϕ on G with the properties that $\|\phi\|_\infty \leq K$ and $W(\varrho, \phi\,d\lambda) \leq K \sup_{\Omega^2}\{\varrho\}$. Set

$$\widetilde{\Phi}_{G,\varrho}^{\Omega,K}: n \mapsto \widetilde{\Phi}_{G,\varrho}^{\Omega,K}(n) := \inf\{\phi^{(2n)}(e): \phi \in \widetilde{\mathcal{S}}_{G,\varrho}^{\Omega,K}\}.$$

Obviously,

$$(2.1) \qquad\qquad \Phi_{G,\varrho}^{\Omega,K} \geq \widetilde{\Phi}_{G,\varrho}^{\Omega,K}.$$

In the classical case of $\mathbb{R}^d$ or $\mathbb{Z}^d$ with $\varrho = \rho_\alpha(|\cdot|) = (1 + |\cdot|)^\alpha$, as long as $\alpha \in (0, 2)$, we have

$$\widetilde{\Phi}_{\mathbb{Z}^d,\rho_\alpha}^{\Omega,K}(n) \simeq n^{-d/\alpha},$$

whereas it is not easy to estimate $\Phi_{\mathbb{Z}^d,\rho_\alpha}^{\Omega,K}(n)$ precisely (see the comments made in the Introduction). Interestingly enough, for $\alpha = 2$, we have (see [14])

$$\Phi_{\mathbb{Z}^d,\rho_2}^{\Omega,K}(n) \simeq n^{-d/2}, \qquad \widetilde{\Phi}_{\mathbb{Z}^d,\rho_2}^{\Omega,K}(n) \simeq (n\log n)^{-d/2}.$$

2.1. *Some basic stability results for $\Phi_{G,\varrho}$.* By definition, a continuous symmetric probability density ϕ such that

$$\|\phi\|_\infty \leq K \quad \text{and} \quad \int \varrho\phi\,d\lambda \leq K \sup_{\Omega^2}\{\varrho\}$$

must satisfy

$$\phi^{(2n)}(e) \geq \Phi_{G,\varrho}^{\Omega,K}(n).$$

It is natural to ask what can be said of a symmetric probability density $\phi \in L^2(G)$ such that $\int \varrho\phi \, d\lambda < \infty$. This section gives a reassuring answer to this question and proves the results stated in Proposition 1.2. We need the following elementary fact.

PROPOSITION 2.2.   *Let G be a locally compact unimodular group. Assume that $\varrho : G \to [0, \infty]$ is a measurable function with the property that there exists $C \in [1, \infty)$ such that*

$$\forall x, y \in G \qquad \varrho(xy) \leq C(\varrho(x) + \varrho(y)).$$

If μ is a probability measure satisfying $\mu(\varrho) < \infty$, then

$$\mu^{(n)}(\varrho) \leq nC^{n-1}\mu(\varrho), \qquad n = 1, 2, \ldots.$$

Further, we have

$$W(\varrho, \mu^{(n)}) \leq n(2C)^{n-1}W(\varrho, \mu).$$

PROOF.   By definition of the convolution product, for any two measures μ, ν,

$$\mu * \nu(\varrho) = \int_{G \times G} \varrho(xy) \, d\mu(x) \, d\nu(y).$$

If μ, ν are probability measures, since $\varrho(xy) \leq C(\varrho(x) + \varrho(y))$, we obtain

$$\mu * \nu(\varrho) \leq C(\mu(\varrho) + \nu(\varrho)).$$

The inequality $\mu^{(n)}(\varrho) \leq nC^{n-1}\mu(\varrho)$ follows by induction. To obtain the inequality regarding $W(\varrho, \mu^{(n)})$ observe that

$$\{(x, y) : \varrho(x, y) > s\} \subset \{(x, y) : \varrho(x) > s/(2C)\} \cup \{(x, y) : \rho(y) > s/(2C)\}.$$

Hence, for any two probability measures μ, ν, we have

$$\mu * \nu(\{\varrho > s\}) = \int_{\{(x,y) : \varrho(xy) > s\}} d\mu(x) \, d\nu(y)$$

$$\leq \mu(\{\varrho > s/2C\}) + \nu(\{\varrho > s/(2C)\}).$$

This yields $W(\varrho, \mu * \nu) \leq 2C(W(\varrho, \mu) + W(\varrho, \nu))$ and the desired result follows by induction.   □

COROLLARY 2.3.   *Let ϱ, Ω, K be as in Definition 1.1. Assume that ϱ tends to infinity at infinity and satisfies*

$$\forall x, y \in G \qquad \varrho(xy) \leq C(\varrho(x) + \varrho(y)).$$

Then $\Phi_{G,\varrho}^{\Omega,K}(n) > 0$ and $\widetilde{\Phi}_{G,\varrho}^{\Omega,K}(n) > 0$, for all Ω, K, n.

PROOF. We prove the result for $\Phi_{G,\varrho}^{\Omega,K}$ (the case of $\widetilde{\Phi}_{G,\varrho}^{\Omega,K}$ is similar). Let $\varrho_0 = \sup_{\Omega^2}\{\varrho\}$. Let ϕ be a symmetric continuous probability density in $\mathcal{S}_{G,\varrho}^{\Omega,K}$, that is, such that $\|\phi\|_\infty \leq K$ and $\int \phi\varrho\,d\lambda \leq K\varrho_0$. Then $\int \phi^{(2n)}\varrho\,d\lambda \leq 2KnC^{2n-1}\varrho_0$. Further, for any N,

$$\int_{\varrho>N} \phi^{(2n)}\,d\lambda \leq N^{-1}\int \phi^{(2n)}\varrho\,d\lambda \leq \frac{2KnC^{2n-1}\varrho_0}{N}.$$

Since $\phi^{(2n)}$ attains its maximum at e, we obtain that

$$\phi^{(2n)}(e) \geq \frac{1}{\lambda(\varrho \leq N)}\int_{\varrho\leq N}\phi^{(2n)}\,d\lambda \geq \frac{1 - 2KnC^{2n-1}\varrho_0/N}{\lambda(\varrho \leq N)}.$$

As ϱ tends to infinity at infinity, $\lambda(\varrho \leq N)$ is finite for any finite N. Hence, for $N = 4KnC^{2n-1}\varrho_0$, we obtain a uniform positive lower bound on $\phi^{(2n)}(e)$ for all $\phi \in \mathcal{S}_{G,\varrho}^{\Omega,K}$. $\quad\square$

PROPOSITION 2.4. *Let G be a locally compact unimodular group. Let ϱ, Ω, K be as in Definition* 1.1. *Let ϕ be a symmetric continuous probability density.*

- *Assume that $\int \varrho\phi\,d\lambda < \infty$. Then there exist $c_1 = c_1(\phi) > 0, c_2 = c_2(\phi) \in \mathbb{N}$ such that*

$$\forall n = 1, 2, \ldots \qquad \phi^{(2n)}(e) \geq c_1\Phi_{G,\varrho}^{\Omega,K}(c_2 n).$$

- *Assume that $W(\varrho, \phi\,d\lambda) < \infty$. Then there exist $c_1 = c_1(\phi) > 0, c_2 = c_2(\phi) \in \mathbb{N}$ such that*

$$\forall n = 1, 2, \ldots \qquad \phi^{(2n)}(e) \geq c_1\widetilde{\Phi}_{G,\varrho}^{\Omega,K}(c_2 n).$$

PROOF. The proofs of the two statements are similar and we only give the proof under the condition $\int \varrho\phi\,d\lambda < \infty$. Let $\phi_0 = \lambda(\Omega)^{-1}\mathbf{1}_\Omega$. Obviously, since $\lambda(\Omega) \geq 1$, we have $\|\phi_0^{(2)}\|_\infty \leq 1$ and $\int \varrho\phi_0^{(2)}\,d\lambda \leq \sup_{\Omega^2}\{\varrho\}$. By hypothesis,

$$M = \max\left\{\|\phi\|_\infty, \left(\sup_{\Omega^2}\{\varrho\}\right)^{-1}\int \varrho\phi\,d\lambda\right\} < +\infty.$$

If $M \leq K$, the result is clear. If not then $M > K > 1$. In this case, set $\alpha = (K-1)/(M-1) \in (0, 1)$, and observe that the symmetric continuous probability density $\phi_1 = \alpha\phi + (1-\alpha)\phi_0^{(2)}$ satisfies $\|\phi_1\|_\infty \leq K$ and $\int \varrho\phi_1\,d\lambda \leq K\sup_{\Omega^2}\{\varrho\}$. Thus,

$$\forall n \qquad \phi_1^{(2n)}(e) \geq \Phi_{G,\varrho}^{\Omega,K}(n).$$

Further, by construction, the Dirichlet forms $\mathcal{E} = \mathcal{E}_{\phi\,d\lambda}$ and $\mathcal{E}_1 = \mathcal{E}_{\phi_1\,d\lambda}$ [see (2.5)] satisfy $\mathcal{E} \leq (1/\alpha)\mathcal{E}_1$. In terms of the convolution operator R_ϕ (convolution on the right by ϕ) acting on $L^2(G)$, this is equivalent to say that

$$I - R_\phi \leq (1/\alpha)(I - R_{\phi_1}).$$

By Corollary A.10, this implies that $\phi^{(2n)}(e) \geq c_1\phi_1^{(2c_2n)}(e)$, for some $c_1 > 0$ and $c_2 \in \mathbb{N}$. $\square$

REMARK 2.5. If, for all $x, y \in G$, $\varrho(xy) \leq C(\varrho(x) + \varrho(y))$, then any symmetric probability density $\phi \in L^2(G)$ (not necessarily continuous) with finite ϱ-moment satisfy

$$\phi^{(2n)}(e) \geq c_1 \Phi_{G,\varrho}^{\Omega,K}(c_2n)$$

for some $c_1 = c_1(\phi) > 0$ and $c_2 = c_2(\phi) \in \mathbb{N}$. Indeed, it suffices to apply the previous result to $\phi * \phi$ which is continuous and also has finite ϱ-moment.

PROPOSITION 2.6. *Let G be a locally compact unimodular group. For $i = 1$, 2, fix constants $K_i > 1$ and compact neighborhoods Ω_i of e in G with $\lambda(\Omega_i) \geq 1$. Let $\varrho_i : G \to [0, \infty)$, $i = 1, 2$, be measurable functions such that $a\varrho_1 \leq \varrho_2 \leq A\varrho_1$ for some $a, A \in (0, \infty)$ and $\sup_{\Omega_i^2}\{\varrho_i\} \in (0, \infty)$. Then, we have*

$$\Phi_{G,\varrho_1}^{\Omega_1,K_1} \simeq \Phi_{G,\varrho_2}^{\Omega_2,K_2}, \qquad \widetilde{\Phi}_{G,\varrho_1}^{\Omega_1,K_1} \simeq \widetilde{\Phi}_{G,\varrho_2}^{\Omega_2,K_2}.$$

PROOF. We treat the case of the function Φ. The case of $\widetilde{\Phi}$ is similar. Set $M_i = \sup_{\Omega_i^2}\{\varrho_i\} \in (0, \infty)$. Let $\phi_1 \in \mathcal{S}_{G,\varrho_1}^{\Omega_1,K_1}$. Let $\phi_0 = \lambda(\Omega_2)^{-1}\mathbf{1}_{\Omega_2}$ and set $\phi_2 = \alpha\phi_1 + (1 - \alpha)\phi_0^{(2)}$, for some $\alpha \in (0, 1]$ to be chosen later. This continuous symmetric probability density satisfies

$$\|\phi_2\|_\infty \leq \alpha K_1 + (1 - \alpha) \quad \text{and} \quad \int \phi_2\varrho_2 \, d\lambda \leq \alpha A K_1 M_1 + (1 - \alpha)M_2.$$

It follows that, for α close enough to 1, we have $\phi_2 \in \mathcal{S}_{G,\varrho_2}^{\Omega_2,K_2}$. Indeed, picking $\alpha = \min\{1, (K_2 - 1)/(K_1 - 1), (K_2 - 1)M_2/AK_1|M_2 - M_1|\}$ will work.

As in the previous proof, setting $\mathcal{E}_i = \mathcal{E}_{\phi_i \, d\lambda}$ [see (2.5)], we find that $\mathcal{E}_1 \leq (1/\alpha)\mathcal{E}_2$. By Corollary A.10, this implies that there exists $c > 0$ and an integer k such that

$$\phi_1^{(2n)}(e) \geq c\phi_2^{(2kn)}(e).$$

Further c and k depend only K_1, K_2, M_1, M_2 and A. Hence

$$\forall n \qquad \Phi_{G,\varrho_1}^{\Omega_1,K_1}(n) \geq c\Phi_{G,\varrho_2}^{\Omega_2,K_2}(kn)$$

as desired. Using the symmetry of the hypotheses, the reverse inequality holds as well. $\square$

2.2. *An abstract interpolation/comparison result.* The results developed in this key section make use of a given nonnegative self-adjoint operator (A, D_A) on $L^2(G)$ with associated semigroup $H_t = e^{-tA}$, $t \geq 0$, which is assumed to be, in some sense, well understood. In applications, H_t will actually be a symmetric Markov semigroup, and $\|A^{1/2}f\|_2^2$ will be a Dirichlet form. We assume that A (and thus also H_t) commutes with left translations in G. Namely, for $f \in L^2(G)$ and $h \in G$, set $\tau_h f = f(h\cdot) \in L^2(G)$. We assume that A has the property that $f \in D_A$ implies $\tau_h f \in D_A$ and $A(\tau_h f) = \tau_h(Af)$ for any $h \in G$.

As mentioned above, we think of the semigroup H_t as a basic object which is well understood. The key idea is that we then also understand quite well the semigroups generated by certain functions $\psi(A)$ of A. The class of functions ψ of interest to us here is the class of those functions that admit the Laplace-type representation

$$(2.2) \qquad \psi(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda s} \omega(s)\, ds \qquad \text{with } \omega \geq 0.$$

The simplest example of such function is $\psi : \lambda \mapsto \lambda^\alpha$, $\alpha \in (0, 1)$, which is obtained by picking $\omega(s) = c_\alpha s^{1-\alpha}$, $c_\alpha = 1/\Gamma(2-\alpha)$. By spectral theory the $L^2(G)$-domain of $\psi(A)^{1/2}$ is the set of functions $f \in L^2(G)$ such that

$$(2.3) \qquad \|\psi(A)^{1/2}f\|_2^2 = \int_0^\infty \|AH_{s/2}f\|_2^2\omega(s)\, ds < \infty.$$

It is easy to see that $\|\psi(A)^{1/2}f\|_2^2 < \infty$ whenever $f \in D_A$ and $\omega(s) \leq C(1+s)$ (in fact, $f \in D_{A^{1/2}}$ suffices). It follows that $\psi(A)^{1/2}$ is densely defined and self-adjoint whenever $\omega(s) \leq C(1+s)$.

Next, we introduce a key assumption about (A, D_A). This assumption is expressed in term of a given positive (measurable) function $\delta : G \mapsto [0, \infty)$. It captures a fundamental relation between the L^2-variation of f and $\|A^{1/2}f\|_2$. Namely, setting

$$f_h(x) = f(xh), \qquad f \in L^2(G), x, h \in G,$$

we assume that there exists a constant $C_0 \in [1, \infty)$ such that

$$(2.4) \quad \forall f \in D_A, \forall h \in G \qquad \left(\int_G |f_h - f|^2\, d\lambda \right)^{1/2} \leq C_0 \delta(h) \|A^{1/2}f\|_2.$$

Finally, for any probability measure μ on G, we set

$$(2.5) \quad \forall f \in L^2(G) \qquad \mathcal{E}_\mu(f, f) = \frac{1}{2} \int_G \int_G |f(xy) - f(x)|^2\, d\lambda(x)\, d\mu(y).$$

When μ is symmetric, $\mathcal{E}_\mu$ is the Dirichlet form associated with μ and $\mathcal{E}_\mu(f, f) = \langle f - f * \mu, f \rangle$.

THEOREM 2.7. *Referring to the setting and notation introduced above, consider a pair of nonnegative increasing functions ω, ψ related by* (2.2). *Assume that $s \mapsto \omega(s)/s$ is decreasing, and set*

$$(2.6) \qquad \xi(t) = \int_0^t \left(\frac{s}{\omega(s)} \right)^{1/2} \frac{ds}{s}, \qquad \zeta(t) = t^{1/2} \int_t^\infty \frac{ds}{s\omega(s)^{1/2}}.$$

Let $\rho : [0, \infty) \to [1, \infty)$ be an increasing function such that, for all $t \geq 0$,

$$(2.7) \qquad \frac{t \max\{\xi(t^2), \zeta(t^2)\}}{\omega(t^2)^{1/2}} \leq C_1^2 \rho(t).$$

Then, if A satisfies (2.4) *and μ is such that $\mu(\rho \circ \delta) < \infty$, we have*

$$\mathcal{E}_\mu(f, f) \leq 8 C_0^2 C_1 \mu(\rho \circ \delta) \|\psi(A)^{1/2} f\|_2^2, \qquad f \in D_A.$$

REMARK 2.8. When μ is symmetric, $\mathcal{E}_\mu$ is a Dirichlet form. In general, $f \mapsto \|\psi(A)^{1/2} f\|_2^2$ is not a Dirichlet form. If we assume that $-A$ is the infinitesimal generator of a symmetric Markov semigroup, then $f \mapsto \|\psi(A)^{1/2} f\|_2^2$ is a Dirichlet form if we assume that ψ is a Bernstein function; see [4, 17]. This will not play an important role in this paper but [4], Theorem 2.5, shows that it is often possible to choose ψ to be a Bernstein function.

REMARK 2.9. The functions ξ, ζ are always greater or equal to $(t/\omega(t))^{1/2}$. The typical functions ω of interest to us are such that $\omega(s) \geq \eta s^{1-\varepsilon}$ in $(0, 1)$ with $\varepsilon \in (0, 1)$ and $\omega(s) \simeq s/\beta(s)$ at infinity with β an increasing regularly varying function of index in $[0, 1]$. If β has index in $(0, 1)$, then

$$\xi(t) \simeq \zeta(t) \simeq \left(\frac{t}{\omega(t)} \right)^{1/2} \qquad \text{at infinity}$$

and (2.7) can be replaced by

$$\frac{t^2}{\omega(t^2)} \leq C_1^2 \rho(t).$$

If, instead, β is slowly varying then it is still the case that $\zeta(t) \simeq (t/\omega(t))^{1/2}$ at infinity but, for t large enough,

$$\left(t/\omega(t) \right)^{1/2} \ll \xi(t) \leq C \log(e + t)\left(t/\omega(t) \right)^{1/2}.$$

In this case, $\max\{\zeta(t^2), \xi(t^2)\} = \xi(t^2)$ for t large enough.

If β has index 1 and is of the form $\beta(t) = t/\ell(t)$ with $\ell(t)$ slowly varying at infinity then $\omega(t) \simeq \ell(t)$, $\xi(t) \simeq (t/\omega(t))^{1/2}$ but $(t/\omega(t))^{1/2} \ll \zeta(t)$ and, in fact, $\zeta(t)$ might be infinite unless further assumptions are made on ℓ.

PROOF OF THEOREM 2.7.   Let $f \in D_A$ and write

$$g(h) = \|f_h - f\|_2,$$

$$f_h - f = ([H_t f]_h - H_t f) + ([f - H_t f]_h) - (f - H_t f).$$

Since $f - H_t f = \int_0^t A H_s f \, ds$, we have

$$(2.8) \qquad \|([f - H_t f]_h) - (f - H_t f)\|_2 \leq 2 \int_0^t \|A H_s f\|_2 \, ds.$$

Using (2.4), we also have

$$(2.9) \qquad \|[H_t f]_h - H_t f\|_2 \leq C_0 \delta(h) \|A^{1/2} H_t f\|_2.$$

Further,

$$(2.10) \quad \|A^{1/2} H_t f\|_2 = \left\| \int_t^\infty A^{1/2} A H_s f \, ds \right\|_2 \leq \int_t^\infty (es)^{-1/2} \|A H_{s/2} f\|_2 \, ds.$$

Here we have used the inequalities

$$\|A^{1/2} H_s f\|_2 \leq \|A^{1/2} H_{s/2}\|_{2 \to 2} \|H_{s/2} f\|_2$$

and (by spectral theory)

$$\|A^{1/2} H_s\|_{2 \to 2} \leq \max_{a > 0} \{a^{1/2} e^{-sa}\} = (2es)^{-1/2}.$$

Putting together inequalities (2.8), (2.9) and (2.10) yields

$$g(h) \leq 2 \int_0^t \|A H_s f\|_2 \, ds + C_0 \delta(h) \int_t^\infty (es)^{-1/2} \|A H_{s/2} f\|_2 \, ds.$$

Pick $t = \tau(h) = \max\{1, \delta(h)^2\}$, set $\theta = \max\{\xi, \zeta\}$ and write

$$\frac{g(h)}{\theta \circ \tau(h)} \leq 2 C_0 \int_0^\infty K(h, s)([s\omega(s)]^{1/2} \|A H_{s/2} f\|_2) \frac{ds}{s},$$

where K is the kernel on $G \times (0, \infty)$ given by

$$K(h, s) = \frac{s^{1/2}}{\theta \circ \tau(h) \omega(s)^{1/2}} \big(\mathbf{1}_{(0, \tau(h))}(s) + \delta(h) s^{-1/2} \mathbf{1}_{[\tau(h), \infty)}(s) \big).$$

Consider this kernel as defining an integral operator

$$K : L^2\left((0, \infty), \frac{ds}{s}\right) \to L^2(G, [\theta \circ \tau]^2 \, d\mu), \qquad u \mapsto Ku,$$

$$Ku(h) = \int_0^\infty K(h, s) u(s) \frac{ds}{s}.$$

Assuming that this operator is bounded with norm N_*, we obtain

$$\mathcal{E}_\mu(f, f) = \int_G |g|^2 \, d\mu$$

(2.11)
$$\leq 4C_0^2 N_*^2 \int_0^\infty \|AH_{s/2}f\|_2^2 \omega(s) \, ds$$

$$= 4C_0^2 N_*^2 \|\psi(A)^{1/2} f\|_2^2.$$

A standard interpolation argument gives

$$N_*^2 \leq \left(\sup_{h \in G} \int_0^\infty K(h, s) \frac{ds}{s} \right) \left(\sup_{s>0} \int_G K(\cdot, s)[\theta \circ \delta]^2 \, d\mu \right)$$

and we have

$$\int_0^\infty K(h, s) \frac{ds}{s} = \frac{1}{\theta(\tau(h))} \int_0^{\tau(h)} \frac{ds}{[s\omega(s)]^{1/2}} + \frac{\delta(h)}{\theta(\tau(h))} \int_{\tau(h)}^\infty \frac{ds}{s\omega(s)^{1/2}},$$

$$\int_G K(\cdot, s)[\theta \circ \tau]^2 \, d\mu = \frac{s^{1/2}}{\omega(s)^{1/2}} \int_{\{\tau > s\}} [\theta \circ \tau] \, d\mu + \frac{1}{\omega(s)^{1/2}} \int_{\{\tau \leq s\}} \delta[\theta \circ \tau] \, d\mu.$$

By the definitions of ξ, ζ and θ,

$$\sup_{h \in G} \left\{ \int_0^\infty K(h, s) \frac{ds}{s} \right\} \leq 2.$$

Further, since we assume that $s \mapsto \omega(s)$ is increasing and $s \mapsto \omega(s)/s$ decreasing, (2.7) yields

$$\sup_{s>0} \left\{ \int_G K(\cdot, s)[\theta \circ \tau]^2 \, d\mu \right\} \leq C_1 \int \rho \circ \delta \, d\mu.$$

This gives the desired result. $\quad\square$

This proof admits the following result as a corollary.

THEOREM 2.10. *Referring to the setting and notation introduced above, consider a pair of smooth nonnegative increasing functions ω, ψ related by (2.2). Fix $\alpha \in (0, 1)$ and assume that ω is smoothly regularly varying of index $1 - \alpha$ at infinity and bounded below by $\omega(t) \geq \eta t^{1-\varepsilon}$ at 0 for some $\eta > 0$ and $\varepsilon \in (0, 1)$. Set*

(2.12)
$$\rho(t) = (1 + t^2/\omega(t^2)).$$

Assume that A satisfies (2.4) and that μ satisfies

(2.13)
$$W(\rho, \mu) = \sup_{s>0} \{s\mu(\{\rho \circ \delta > s\})\} < \infty.$$

Then we have

$$\mathcal{E}_\mu(f, f) \leq C_0^2 C(\omega) W(\rho, \mu) \|\psi(A)^{1/2} f\|_2^2, \qquad f \in D_A.$$

PROOF. We follow the proof of Theorem 2.7. Taking into account that $\theta(t) \simeq c_\alpha(t/\omega(t))^{1/2}$ and that we have set $\rho(t) = 1 + t^2/\omega(t^2)$, the proof of Theorem 2.7 shows that we need to estimate

$$\int_G K(\cdot, s^2)[\rho \circ \delta]^2 \, d\mu = \rho(s)^{1/2} \int_{\{\delta > s\}} [\rho \circ \delta]^{1/2} \, d\mu$$

$$+ \frac{1}{\omega(s^2)^{1/2}} \int_{\{\delta \leq s\}} \delta[\rho \circ \delta]^{1/2} \, d\mu,$$

uniformly over the range $s > 1$. Setting $v(s) = \mu(\delta > s)$, we have

$$\rho(t)v(t) \leq W(\rho, \mu)$$

and

$$\rho(s)^{1/2} \int_{\{\delta > s\}} [\rho \circ \delta]^{1/2} \, d\mu = \rho(s)^{1/2} \int_s^\infty \rho^{1/2}(t) \, d[-v(t)]$$

$$\leq \frac{\rho(s)^{1/2}}{2} \int_s^\infty \rho'(t)\rho(t)^{-1/2} v(t) \, dt + \rho(s)v(s)$$

$$\leq W(\rho, \mu)\left(1 + \frac{\rho(s)^{1/2}}{2} \int_s^\infty \rho'(t)\rho(t)^{-3/2} \, dt\right)$$

$$\leq 2W(\rho, \mu).$$

Further, using the fact that ω is regularly varying with positive index $1 - \alpha$, we have $t\rho'(t) \sim 2\alpha\rho(t)$ and

$$\frac{1}{\omega(s^2)^{1/2}} \int_{\{\tau \leq s\}} \delta[\rho \circ \delta]^{1/2} \, d\mu \leq \frac{1}{\omega(s^2)^{1/2}} \int_0^s t\rho(t)^{1/2} d[-v(t)]$$

$$\leq \frac{1}{\omega(s^2)^{1/2}} \int_0^s (\rho(t)^{1/2} + t\rho'(t)\rho(t)^{-1/2}) v(t) \, dt$$

$$\leq \frac{C(\omega)W(\rho, \mu)}{\omega(s^2)^{1/2}} \int_0^s \rho(t)^{-1/2} \, dt$$

$$\leq \frac{C(\omega)W(\rho, \mu)}{\omega(s^2)^{1/2}} \int_0^s \frac{\omega(t^2)^{1/2} \, dt}{t}$$

$$\leq C'(\omega)W(\rho, \mu).$$

This gives the desired result. $\square$

REMARK 2.11. The case when ω is a slowly varying increasing function corresponds to moment conditions that are close to a finite second moment. In this case, the use of Theorem 2.7 is limited by the fact that it involves the possibly infinite quantity

$$\zeta(t) = t^{1/2} \int_t^\infty \frac{ds}{s\omega(s)^{1/2}}.$$

We can improve the result by using a slightly different proof. Namely, using the same notation as in the proof of Theorem 2.7, we write

$$g(h) \le 2C_0 \int_0^\infty \mathbf{K}(h, s)([s\omega(s)]^{1/2}\|AH_{s/2}f\|_2)\frac{ds}{s},$$

where $\mathbf{K}$ is the kernel on $G \times (0, \infty)$ given by

$$\mathbf{K}(h, s) = \frac{s^{1/2}}{\omega(s)^{1/2}}\left(\mathbf{1}_{(0,\tau(h))}(s) + \delta(h)s^{-1/2}\mathbf{1}_{[\tau(h),\infty)}(s)\right).$$

Next, we use the Hilbert–Schmidt norm $\int_G \int_0^\infty |\mathbf{K}(h, s)|^2 \frac{ds}{s} d\mu(h)$ to estimate the norm of $\mathbf{K}: L^2((0, \infty), \frac{ds}{s}) \to L^2(G, d\mu)$. We have

$$\int_G \int_0^\infty |\mathbf{K}(h, s)|^2 \frac{ds}{s} d\mu(h) = \int_G \left(\int_0^{\tau(h)} \frac{ds}{\omega(s)} + \delta(h)^2 \int_{\tau(h)}^\infty \frac{ds}{s\omega(s)}\right) d\mu(h)$$

$$\le \int_G \left(\widetilde{\xi}^2(\tau(h)) + \widetilde{\zeta}^2(\tau(h))\right) d\mu(h),$$

where

$$\widetilde{\xi}(t) = \left(\int_0^t \frac{ds}{\omega(s)}\right)^{1/2} \quad \text{and} \quad \widetilde{\zeta}(t) = \left(t \int_t^\infty \frac{ds}{s\omega(s)}\right)^{1/2}.$$

This implies that the conclusion of Theorem 2.7 holds under the hypothesis that $\rho(t) \ge \widetilde{\zeta}^2(t^2) + \widetilde{\zeta}^2(t^2)$.

For instance, consider the case when $\omega(t) = [\log(e + t)]^\alpha$, $\alpha > 0$. In this case, $\psi(t) \sim t[\log(e + 1/t)]^\alpha$. On the one hand, we have $\zeta(t) = \infty$ if $\alpha \le 2$ and $\zeta(t) \simeq t^{1/2}[\log(e + t)]^{1-\alpha/2}$ if $\alpha > 2$. This means that Theorem 2.7 requires $\alpha > 2$ and $\rho(t) \ge C(1 + t)^2[\log(e + t)]^{1-\alpha}$.

On the other hand, we have $\widetilde{\zeta}(t) \simeq t^{1/2}[\log(e + t)]^{(1-\alpha)/2}$ if $\alpha > 1$. This means that the variation explained above requires only $\alpha > 1$, with the same ρ, that is, $\rho(t) \ge C(1 + t)^2[\log(e + t)]^{1-\alpha}$.

2.3. *Two fundamental examples*.

First example. Let G be a unimodular Lie group, and let (A, D_A) be the (unique) self-adjoint extension of a Hörmander sum of squares

$$A = \sum_1^k X_i^2 \qquad \text{acting on } \mathcal{C}_c^\infty(G),$$

where $\{X_i, i = 1, \ldots, k\}$ is a fixed set of left-invariant vector fields which generates the Lie algebra of G (Hörmander condition). Then, it is known that $H_t f = f * \mu_t$ where $(\mu_t)_{t>0}$ is a convolution semigroup of probability measures, and each μ_t

admits a smooth positive density $x \mapsto h_t(x)$ with respect to the Haar measure λ; see, for example, [31], Chapter 3. Further, as t tends to infinity, we have

$$h_t(e) \simeq \Phi_G(t)$$

$$\simeq \begin{cases} e^{-t}, & \text{if } G \text{ is not amenable,} \\ e^{-t^{1/3}}, & \text{if } G \text{ is amenable with exponential volume growth,} \\ t^{-D/2}, & \text{for some integer } D, \text{ otherwise.} \end{cases}$$

For each integer D, the last case occurs exactly when G has polynomial volume growth of degree D. The value $h_t(e)$ is the maximal value of the function h_t on G, and, furthermore, it equals the norm of the linear operator $H_t \colon L^1(G) \to L^\infty(G)$ as well as the square of the norm of $H_{t/2} \colon L^2(G) \to L^\infty(G)$. In this case, we set

$$\delta(x) = \sup\left\{ f(x) - f(e) \colon f \in \mathcal{C}_c^\infty(G), \sum_1^k |X_i f|^2 \leq 1 \right\}.$$

This distance is the sub-Riemannian distance naturally associated with the set of left-invariant vector fields $\{X_1, \ldots, X_k\}$, and $\delta(x)$ is finite for all $x \in G$ because we assume that the X_i's generate the Lie algebra (this is a special case of one of the fundamental theorem of sub-Riemannian geometry, often referred to as Chow's theorem); see [19] for a detailed discussion. Further, it is a simple matter ([31], Lemma VII.1.1) to see that $\mathcal{E}_A(f, f) = \int \sum_1^k |X_i f|^2 d\lambda$ and

$$\int |f_h - f|^2 d\lambda \leq \delta(h)^2 \int \sum_1^k |X_i f|^2 d\lambda, \qquad f \in \mathcal{C}_c^\infty(G), h \in G.$$

This shows that (2.4) holds true in this case since $\int \sum_1^k |X_i f|^2 d\lambda = \|A^{1/2} f\|_2^2$.

Second example. Let G be a compactly generated unimodular group, and set $Af = f - f * \phi_0$ where ϕ_0 is continuous, symmetric, compactly supported probability density on G with the property that $\phi_0 > 0$ on a compact generating neighborhood of the identity. Then

$$\mathcal{E}_A(f, f) = (1/2) \int_G \|f_h - f\|_2^2 \phi_0(h) \, d\lambda(h)$$

and $H_t f = f * h_t$ where

$$h_t = e^{-t} \sum_0^\infty \frac{t^n}{n!} \phi_0^{(n)}.$$

In particular, if G is a finitely generated group with finite symmetric generating set S containing the identity, we can set $\phi_0 = (\#S)^{-1} \mathbf{1}_S$. In any case, for $t \geq 1$,

$$h_t(e) \simeq \phi_0^{(2t)}(e) \simeq \Phi_G(t).$$

As explained in the Introduction, many different behaviors are possible for the function Φ_G, depending on G. Assuming that U is a symmetric neighborhood of the identity which contains a generating compact set, that $\inf_{U^3}\{\phi_0\} > 0$, and setting

$$\delta(x) = \inf\{n : x \in U^n\},$$

[31], Proposition VII.3.2, gives that (the discrete case of this inequality is a bit simpler and more elementary)

$$\int |f_h - f|^2 \, d\lambda \leq C(U, \phi_0)\delta(h)^2 \mathcal{E}_A(f, f), \qquad f \in L^2(G), h \in G.$$

Again, this shows that (2.4) holds true in this setting.

3. Applications: Main lower bounds on $\Phi_{G,\rho}$. Let G be as in the second example of Section 2.3. Keep the notation introduced there. In the applications we have in mind, we are given a continuous increasing function $\rho : (0, \infty) \to [1, \infty)$ and set $\rho_G = \rho \circ \delta$. Our main aim is to estimate the functions Φ_{G,ρ_G} and $\widetilde{\Phi}_{G,\rho_G}$ introduced in Definitions 1.1 and 2.1. Hence, we consider a (otherwise arbitrary) symmetric continuous probability density ϕ on G with the property that $\|\phi\|_\infty \leq K$ and $\int \rho_G \phi \, d\lambda \leq K \sup_{\Omega^2}\{\rho\}$ or $W(\rho_G, \phi) \leq K \sup_{\Omega^2}\{\rho\}$. Here $K > 1$ and Ω are as in Definitions 1.1 and 2.1.

In order to apply Theorem 2.7, we have to find an increasing function ω compatible with ρ in the sense that the pair ρ, ω satisfies the various hypotheses of Theorem 2.7. The function ψ associated to ρ via ω is then defined by (2.2).

The following examples are of particular interest:

- If $\rho(s) = \rho_{2\alpha}(s) = (1 + s)^{2\alpha}$, $\alpha \in (0, 1]$, then we can take

$$\omega(s) = \Gamma(2 - \alpha)^{-1}s^{1-\alpha} \quad \text{and} \quad \psi(s) = s^\alpha.$$

- If $\rho(s) = (1 + s^2)^\alpha \ell(1 + s^2)^\alpha$ with $\alpha \in (0, 1)$ and ℓ smooth, positive and slowly varying at infinity, then we can take

$$\omega(s) = \frac{1 + s}{[(1 + s)\ell(1 + s)]^\alpha} \qquad \text{at infinity}$$

and

$$\psi(s) \simeq [(1 + s)/\ell(1 + 1/s)]^\alpha \qquad \text{at } 0.$$

- If $\rho(s) = \rho_{c,\alpha}^{\exp}(s) = \exp(c[\log(1 + s)]^\alpha)$, $\alpha \in (0, 1)$, $c > 0$, then we can take

$$\omega(s) = s \exp(-c_1[\log(1 + s)]^\alpha)$$

for some $c_1 > 0$ (see Remark 2.9) and we have [see (2.2)]

$$\psi(s) \sim \exp(-c_1[\log(1 + 1/s)]^\alpha) \qquad \text{at } 0.$$

- If $\rho(s) = \rho_\alpha^{\log}(s) = [\log(e + s)]^\alpha$, $\alpha > 1$, then we can take

$$\omega(s) = s[\log(e + s)]^{1-\alpha}$$

for some $c_1 > 0$ (see Remark 2.9), and this gives

$$\psi(s) \sim [\log(e + 1/s)]^{1-\alpha} \qquad \text{at } 0.$$

These computations indicate that Theorem 2.7 is too weak to provide results when $\rho(s) = \rho_\alpha^{\log}(s) = [\log(e + s)]^\alpha$ and $\alpha \le 1$.

- The previous two cases can be generalized as follows. Assume that

$$\rho(s) \ge c \log(e + s)\big(1 + \eta(s^2)\big),$$

where η is a positive increasing function such that $\eta(s) \sim s$ at 0 and η is slowly varying at infinity. Set

$$\omega(s) = s/\eta(s) \quad \text{and} \quad \psi(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda s} \omega(s)\, ds.$$

By [5], Theorem 1.7.1, we have $\psi(\lambda) \simeq c/\eta(1/\lambda)$. Further, referring to the notation used in Theorem 2.7, we then have

$$\frac{t \max\{\xi(t^2), \zeta(t^2)\}}{\omega(t^2)^{1/2}} \le C\rho(t).$$

We are now ready to state and prove lower bounds on the functions Φ_{G,ρ_G} and $\widetilde{\Phi}_{G,\rho_G}$ of Definitions 1.1 and 2.1 for some groups G and functions ρ. We will use the notation ρ_α, $\rho_{c,\alpha}^{\exp}$, $\rho_\alpha^{\log}$ recalled above. If ρ is a real function, and G is a compactly generated group with world length $\delta(x) = |x| = \inf\{n : x \in U^n\}$ for some fixed symmetric relatively compact generating neighborhood of the identity, we set $\Phi_{G,\rho} = \Phi_{G,\rho_G}$ where $\rho_G = \rho \circ \delta$.

We state four theorems that cover various cases of particular interest. The proofs of these results all follow the same outline based on Theorems 2.7 and 2.10 together with Corollary A.10 and the results of Section A.4, Theorems A.6 and A.7. The main line of reasoning described in the proof of Theorem 3.1 below is also used for the proofs of Theorems 3.2, 3.3 and 3.4. The results presented in the Appendix play a crucial role in these proofs.

THEOREM 3.1. *Let G be a locally compact, compactly generated unimodular group such that $\Phi_G(n) \simeq n^{-D/2}$ at infinity, for some integer D.*

(1) *Assume that $\rho(s) \ge [(1 + s^2)\ell(1 + s^2)]^\alpha$ with $\alpha \in (0, 1)$ and ℓ smooth positive slowly varying at infinity with de Bruijn conjugate $\ell^\#$. Then there exist $c = c_\rho \in (0, \infty)$ and an integer $N = N_\rho$ such that*

$$\forall n > N \qquad \Phi_{G,\rho}(n) \ge \widetilde{\Phi}_{G,\rho}(n) \ge c[n^{1/\alpha}\ell^\#(n^{1/\alpha})]^{-D/2}.$$

(2) *Assume that $\rho(s) \geq \log(e+s)\ell(1+s^2)$ and ℓ smooth positive increasing and slowly varying at infinity and such that $\log \ell^{-1}(t) \simeq t^\gamma \omega(t)^{1+\gamma}$ at infinity, with $\gamma \geq 0$ and ω slowly varying with de Bruijn conjugate $\omega^{\#}$. Then there exist $C = C_\rho \in (0, \infty)$ and an integer $N = N_\rho$ such that*

$$\forall n > N \qquad \log \Phi_{G,\rho}(n) \geq -C[n^\gamma/\omega^{\#}(n)]^{1/(1+\gamma)}.$$

PROOF. We will use the following notation which is consistent with the notation used in the Appendix. Let $\psi : [0, 2] \to [0, 2]$ be a continuous increasing function with continuous derivative such that with $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(2) < 2$. Fix a symmetric probability density $\phi_0 \in L^2(G)$ and assume that its support is a compact generating neighborhood of the identity element (we assume that G is compactly generated). This implies that $\phi_0^{(2n)}(e) \simeq \Phi_G(n)$; see [13, 20, 31]. We set $T = R_{\phi_0}$ and [see (A.5)]

$$T_\psi = I - \psi(I - T).$$

Let $\mathcal{E}_0$ denote Dirichlet form $\mathcal{E}_0(f, f) = \langle (I - T)f, f \rangle$ associated with ϕ_0. By Section 2.3 and Theorems 2.7 and 2.10, if $d\mu = \phi \, d\lambda$ is a symmetric probability with continuous density satisfying $\mu(\rho \circ \delta) < \infty$, then

(3.1) $$\mathcal{E}_\mu(f, f) \leq C \|\psi(I - T)^{1/2} f\|_2^2.$$

Here ψ is chosen such that the condition of Theorems 2.7 and 2.10 relating ψ to ρ (via ω) are satisfied. See the explicit examples discussed at the beginning of this section.

By Corollary A.10, (3.1) implies [τ is the natural semifinite trace on the von Neumann $V(G)$; see the Appendix]

$$\phi^{(2n)}(e) \geq C\big(e^{-cn} + \tau\big(T_\psi^{2[cn]}\big)\big).$$

Now, depending on the behavior of ψ near 0, the trace $\tau(T_\psi^{2n})$ can be estimated using the results of Section A.4, Theorems A.6 and A.7; see also Example A.2. This gives the announced lower bounds on $\phi^{(2n)}(e)$. $\square$

EXAMPLE 3.1. The second statement in Theorem 3.1 can be illustrated by the following two examples:

(1) $\log \Phi_{G, \rho_{c,\alpha}^{\exp}}(n) \geq -C_{D,c,\alpha}(\log n)^{1/\alpha}$, $\alpha \in (0, 1)$, $c > 0$.
(2) $\log \Phi_{G, \rho_\alpha^{\log}}(n) \geq -C_{D,\alpha} n^{1/\alpha}$, $\alpha > 1$.

THEOREM 3.2. *Let G be a locally compact, compactly generated unimodular group such that $\log \Phi_G(n) \geq -Cn^\gamma$ at infinity, for some $\gamma \in (0, 1)$ and $C \in (0, \infty)$. Fix $\alpha \in (0, 1)$ and $\rho(s) \simeq [(1 + s^2)\ell(1 + s^2)]^\alpha$ with ℓ smooth positive slowly varying at infinity. Then, there exist $C_\rho \in (0, \infty)$ and an integer N_ρ such that*

$$\forall n > N_\rho \qquad \log \Phi_{G,\rho}(n) \geq \log \widetilde{\Phi}_{G,\rho}(n) \geq -C_\rho n^{\gamma\alpha}/\ell_*^{\#}\big(n^{(1-\gamma)\gamma\alpha/\gamma}\big)^\alpha,$$

where $\ell_*^{\#}$ is the de Bruijn conjugate of

$$\ell_*(s) = \ell^{\#}(s)^{\gamma_\alpha}, \qquad \gamma_\alpha = \frac{\gamma}{\gamma + \alpha(1 - \gamma)}.$$

This theorem with $\ell \equiv 1$ implies Theorem 1.5.

PROOF OF THEOREM 3.2.   The given ρ calls for using $\psi(s) = [s/\ell(s)]^\alpha$ in Theorem 2.7. Note that $\psi^{-1}(t) \simeq t^{1/\alpha}/\ell^{\#}(1/t^{1/\alpha})$.

Using Theorem A.7 and the same notation and line of reasoning as in the proof of Theorem 3.1, we obtain that if $d\mu = \phi \, d\lambda$ is a symmetric probability with continuous density satisfying $\mu(\rho \circ \delta) < \infty$, then

$$\log \phi^{(2n)}(e) \geq -C_1 n/\pi_\psi(n)$$

with

$$C\pi_\psi^{-1}(Ct) \geq t\psi^{-1}(1/t)^{-\gamma/(1-\gamma)} \geq ct^{(\alpha(1-\gamma)+\gamma)/\alpha(1-\gamma)}\ell^{\#}(t^{1/\alpha})^{\gamma/(1-\gamma)}.$$

This can be written as (for a different constant C)

$$C\pi_\psi^{-1}(Ct) \geq t^{(\alpha(1-\gamma)+\gamma)/\alpha(1-\gamma)}\ell_*(t^{1/\alpha})^{(\alpha(1-\gamma)+\gamma)/(1-\gamma)}$$

with $\ell_*(s) = \ell^{\#}(s)^{\gamma/(\alpha(1-\gamma)+\gamma)}$. This gives

$$c\pi_\psi(ct) \leq t^{\alpha(1-\gamma)/(\alpha(1-\gamma)+\gamma)}\ell_*^{\#}(t^{(1-\gamma)/(\alpha(1-\gamma)+\gamma)})^\alpha$$

and

$$\log \Phi_{G,\rho}(n) \geq -Cn^{\gamma_\alpha}/\ell_*^{\#}(n^{(1-\gamma)/(\alpha(1-\gamma)+\gamma)})^\alpha$$

with $\gamma_\alpha = \gamma/(\alpha(1-\gamma)+\gamma)$, as desired. $\square$

EXAMPLE 3.2.   Assume that ℓ satisfies $\ell(t^a) \simeq \ell(t)$ for all $a > 0$. Then $\ell^{\#} = 1/\ell$ and $(1/\ell)^{\#} \simeq \ell$. Hence $\ell_* \simeq (1/\ell)^{\gamma/(\alpha(1-\gamma)+\gamma)}$ and $\ell_*^{\#} \simeq \ell^{\gamma/(\alpha(1-\gamma)+\gamma)}$. Hence we get

$$-\log \Phi_{G,\rho}(n) \leq C[n/\ell^\alpha(n)]^{\gamma_\alpha}.$$

This is consistent with Example A.3.

THEOREM 3.3.   *Let G be a locally compact, compactly generated unimodular group such that $\log \Phi_G(n) \geq -Cn^\gamma/\ell(n)$ at infinity, for some $\gamma \in (0, 1]$, $C \in (0, \infty)$ and slowly varying function ℓ satisfying $\ell(t^a) \simeq \ell(t)$ for all $a > 0$. Assume that $\alpha \in (0, 1)$ and $\rho(s) = (1 + s)^{2\alpha}$. Then, we have*

$$\log \Phi_{G,\rho}(n) \geq \log \widetilde{\Phi}_{G,\rho}(n) \geq -C_\rho[n/\ell(n)^{\alpha/\gamma}]^{\gamma_\alpha}, \qquad \gamma_\alpha = \frac{\gamma}{\gamma + \alpha(1 - \gamma)}.$$

THEOREM 3.4. *Let G be a locally compact, compactly generated unimodular group such that $\log \Phi_G(n) \geq -Cn/\pi(n)$ with π continuous increasing and satisfying $\pi(t) \leq t^{1-\varepsilon}$ for $t \geq 1$. Assume that $\rho(s) \geq c \log(e+s)\ell(s^2)$ with $c > 0$ and ℓ smooth positive increasing and slowly varying at infinity. Then there exist $c_1, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$\log \Phi_{G,\rho}(n) \geq -C_1 n/\ell(\pi(c_1 n^\varepsilon)).$$

EXAMPLE 3.3. If $\Phi_G(n) \geq c \exp(-Cn^\gamma)$ with $\gamma \in (0, 1)$, this yields:

(1) $\log \Phi_{G,\rho_{c,\alpha}^{\exp}}(n) \geq -C_{\gamma,c,\alpha} n \exp(-c_{\gamma,c,\alpha}[\log n]^\alpha)$, $\alpha \in (0, 1)$, $c > 0$.
(2) $\log \Phi_{G,\rho_\alpha^{\log}}(n) \geq -C_{\gamma,\alpha} n[\log n]^{-(\alpha-1)}$, $\alpha > 1$.

If, instead, $\Phi_G(n) \geq c \exp(-Cn/\ell(n))$ with ℓ increasing slowly varying and satisfying $\ell(t^a) \simeq \ell(t)$ for all $a >$, we obtain:

(1) $\log \Phi_{G,\rho_{c,\alpha}^{\exp}}(n) \geq -C_{\gamma,c,\alpha} n \exp(-c_{\gamma,c,\alpha}[\log \ell(n)]^\alpha)$, $\alpha \in (0, 1)$, $c > 0$.
(2) $\log \Phi_{G,\rho_\alpha^{\log}}(n) \geq -C_{\gamma,\alpha} n[\log \ell(n)]^{-(\alpha-1)}$, $\alpha > 1$.

PROOF OF THEOREMS 3.3 AND 3.4. In each case, we use either Theorem 2.7 or Theorem 2.10 together with either Theorems A.6 or A.7, and Corollary A.10. □

4. Upper bounds on $\Phi_{G,\varrho}$. The aim of this section is to obtain upper bounds on the function $\Phi_{G,\varrho}$ (and its variant $\widetilde{\Phi}_{G,\varrho}$) under various conditions on the group G and the function ϱ. To obtain such upper bounds, we only need to exhibit an example of a symmetric probability density ϕ such that $\int \varrho\phi \, d\lambda < \infty$ (or $\sup_{s>0}\{s \int_{\{\varrho>s\}} \phi \, d\lambda\}$, in the case of $\widetilde{\Phi}_{G,\varrho}$) and for which we can obtain an upper bound on $n \mapsto \phi^{(2n)}(e)$. Of course, to obtain good upper bounds, we need to identify probability densities with the desired moment condition and for which $n \mapsto \phi^{(2n)}(e)$ presents an almost optimal decay. This question—which densities produce the optimal decay?—is quite interesting in its own right. For instance, when G is finitely generated with finite symmetric generating set S and ϱ is of the form $\varrho = \rho_G = \rho(|\cdot|)$, and $|\cdot| = |\cdot|_S$ is the word-length based on the generating set S, should we expect to find a probability density with nearly optimal decay among "radial densities" of the form $\phi(x) = f(|x|)$?

4.1. *Φ_G-based upper bounds*: *subordination*. The lower bounds on $\Phi_{G,\varrho}$ (and $\widetilde{\Phi}_{G,\varrho}$) obtained in Section 3 for certain $\varrho = \rho \circ \delta$ are all based on lower bounds on the function Φ_G. It is thus natural to seek upper bounds of the same nature. These applications of Theorem 2.7 start with a symmetric compactly supported continuous density ϕ (with generating support) and involve comparison with the behavior of certain operators T_ψ of the form $T_\psi = I - \psi(I - R_\phi)$ where the function ψ is chosen appropriately, depending on ρ.

It would be very nice to identify a class of functions ψ so that $T_\psi = R_{\phi_\psi}$ where ϕ_ψ is, itself, a symmetric probability density. As already noted after (A.5), this is certainly the case when ψ is a Bernstein function satisfying $\psi(0) = 0$, $\psi(1) = 1$; see, for example, [17, 27] for an access to the literature on Bernstein functions. A Bernstein function is a smooth positive function $\psi : (0, \infty) \to (0, \infty)$ such that $(-1)^k \frac{d^k \psi}{dt^k} \leq 0$ and two good and important examples of Bernstein functions are $\psi_\alpha : s \mapsto s^\alpha$, $\alpha \in (0, 1]$ and $\psi_\alpha^{\log} : s \mapsto [\log_2(1 + s^{-1/\alpha})]^{-\alpha}$. Further, for any smooth positive increasing regularly varying function ψ_1 of index α in $[0, 1)$ at 0 such that $x \mapsto x\psi_1'(x)$ is also regularly varying of index α, there exists a Bernstein function ψ such that $\psi \sim \psi_1$; see [4], Theorem 2.5.

In order to obtain upper bounds on the functions Φ_{G,ρ_G} and $\widetilde{\Phi}_{G,\rho_G}$, it suffices to find a Bernstein function ψ such that the probability density ϕ_ψ satisfies the required moment condition and to estimate $\phi_\psi^{(2n)}(e)$. The companion paper [4] develops this idea, and we will simply quote the relevant results.

We start with results concerning groups with polynomial volume growth $V(n) \simeq n^D$. By [16], these groups satisfy $\Phi_G(n) \simeq n^{-D/2}$. In fact, thanks to [16] and deep results of Guivarc'h, Gromov and Losert, groups of polynomial volume growth are exactly those groups that satisfy $\Phi_G(n) \simeq n^{-D/2}$ for some integer D. An alternative and self-contained proof of the theorems discussed below is given in the next section.

THEOREM 4.1 ([4]). *Assume that G is a compactly generated locally compact group with polynomial volume growth $V(n) \simeq n^D$.*

(1) *Assume that $\rho(s) \simeq g(1 + s^2)$ where $g(s) = [s\ell(s)]^\alpha$ where $\alpha \in (0, 1)$, and ℓ is a positive slowly varying function at infinity with de Bruijn conjugate $\ell^\#$. Then there exist $C \in (0, \infty)$ and an integer N such that*

$$\forall n > N \qquad \widetilde{\Phi}_{G,\rho_G}(n) \leq C[n^{1/\alpha}\ell^\#(n^{1/\alpha})]^{-D/2}.$$

Further, for any slowly varying function ℓ_1 with de Bruijn conjugate $\ell_1^\#$ such that $\sum_1^\infty \frac{\ell(n)^\alpha}{n\ell_1(n)^\alpha} < \infty$, there exist $C(\ell_1) \in (0, \infty)$ and an integer $N(\ell_1)$ such that

$$\forall n > N(\ell_1) \qquad \Phi_{G,\rho_G}(n) \leq C(\ell_1)[n^{1/\alpha}\ell_1^\#(n^{1/\alpha})]^{-D/2}.$$

(2) *Assume that $\rho(t) \simeq g(1 + s^2)$ where $g(s) = \widehat{\ell}(s)$ and $\widehat{\ell}(t) = 1/\int_t^\infty \frac{du}{u\ell(u)}$ where ℓ is a positive increasing slowly varying function at infinity. Assume further that $\log \widehat{\ell}^{-1}(t) \simeq t^\gamma \omega(t)^{1+\gamma}$ at infinity, with $\gamma \geq 0$ and ω slowly varying with de Bruijn conjugate $\omega^\#$. Then there exist $c, C \in (0, \infty)$ and an integer N such that*

$$\forall n > N \qquad \widetilde{\Phi}_{G,\rho_G}(n) \leq C\exp\big(-c[n^\gamma/\omega^\#(n)]^{1/(1+\gamma)}\big).$$

Further, for any slowly varying function ℓ_1 such that

$$\sum_1^\infty \frac{\widehat{\ell}(n)}{n\ell_1(n)} < \infty \quad and \quad \log \widehat{\ell}_1^{-1}(t) \simeq t^{\gamma_1}\omega_1(t)^{1+\gamma_1}$$

with $\gamma_1 \geq 0$ and ω_1 slowly varying at infinity, there exist $c = c(\ell_1), C = C(\ell_1) \in (0, \infty)$ and an integer $N = N(\ell_1)$ such that

$$\forall n > N \qquad \Phi_{G,\rho_G}(n) \leq C \exp\left(-c[n^{\gamma_1}/\omega_1^{\#}(n)]^{1/(1+\gamma_1)}\right).$$

Putting together the results of Theorems 3.1 and 4.1, we obtain the following results which imply Theorem 1.4.

THEOREM 4.2. *Assume that G is a compactly generated locally compact group with polynomial volume growth $V(n) \simeq n^D$.*

(1) *Assume that $\rho(s) \simeq g(1 + s^2)$ where $g(s) = [s\ell(s)]^{\alpha}$ where $\alpha \in (0, 1)$ and ℓ is a positive slowly varying function at infinity with de Bruijn conjugate $\ell^{\#}$. Then*

$$\widetilde{\Phi}_{G,\rho_G}(n) \simeq [n^{1/\alpha}\ell^{\#}(n^{1/\alpha})]^{-D/2}.$$

(2) *For any $\alpha \in (0, 1)$ and $c > 0$, there are constants c_1, c_2, C_1, C_2 (depending on G, α and c) such that*

$$\forall n \qquad c_1 \exp(-C_1[\log n]^{1/\alpha}) \leq \Phi_{G,\rho_{c,\alpha}^{\exp}}(n) \leq C_2 \exp(-c_2[\log n]^{1/\alpha}).$$

(3) *For any $\beta > \alpha > 1$, there are constants c_1, c_2, C_1, C_2 (depending on G, α and β) such that*

$$\forall n \qquad c_1 \exp(-C_1 n^{1/\alpha}) \leq \Phi_{G,\rho_{\alpha}^{\log}}(n) \leq C_2 \exp(-c_2 n^{1/(\beta+1)}).$$

Our next result concerns groups with volume growth faster than polynomial and moment of the type $\rho_{\alpha}(s) = (1 + s)^{\alpha}$. No classifications of either volume growth or the behavior of Φ_G are known for such groups. The upper bounds in the following theorem cannot be obtained by the methods developed in the next section. This theorem follows immediately from Theorems 3.2 and 3.3 and [4], Theorem 5.3.

THEOREM 4.3. *Assume that G is a compactly generated locally compact group and that there exist $0 \leq \bar{\gamma} \leq \gamma \leq 1$ and positive slowly varying functions $\eta, \bar{\eta}$, both satisfying $\eta(t^a) \simeq \eta(t)$ for all $a > 0$, such that, for n large enough,*

$$-n^{\gamma}/\eta(n) \leq \log \Phi_G(n) \leq -n^{\bar{\gamma}}/\bar{\eta}(n).$$

For any $s > 0$, set $\gamma_s = \gamma/[s(1 - \gamma) + \gamma], \bar{\gamma}_s = \bar{\gamma}/[s(1 - \bar{\gamma}) + \bar{\gamma}]$.

Assume further that there exists a symmetric continuous probability density ϕ with compact support, positive on a generating compact set and such that

$$(4.1) \qquad \forall n, s \qquad \int \mathbf{1}_{\{|x| \geq n^{\theta}s\}}\phi^{(n)} \, d\lambda \leq C \exp(-cs^q)$$

for some $C, \theta, q > 0$.

(1) *For any $\alpha \in (0, \min\{2, 1/\theta\})$, there exist $c_1, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1[n/\eta(n)^{\alpha/2\gamma}]^{\gamma_{\alpha/2}} \leq \log \widetilde{\Phi}_{G,\rho_\alpha}(n) \leq -c_1[n/\bar{\eta}(n)^{\alpha\theta/\bar{\gamma}}]^{\bar{\gamma}_{\alpha\theta}}.$$

(2) *For any $\alpha \in (0, \min\{2, 1/\theta\})$ and $\varepsilon > 0$, there exist $c_1, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1[n/\eta(n)^{\alpha/2\gamma}]^{\gamma_{\alpha/2}} \leq \log \Phi_{G,\rho_\alpha}(n) \leq -c_\varepsilon[n/[\bar{\eta}(n)(\log n)^{1+\varepsilon}]^{\alpha\theta/\bar{\gamma}}]^{\bar{\gamma}_{\alpha\theta}}.$$

EXAMPLE 4.1. Assume that $G = F \wr H$ where F is a nontrivial finite group, and H is polycylic with exponential volume growth. Then $\Phi_G(n) \simeq \exp(-n/(\log n)^2)$. Condition (4.1) is trivially verified with $\theta = 1$. For $\alpha \in (0, 2)$, Theorem 4.3(1) yields

$$-C_1 n/[\log n]^\alpha \leq \Phi_{G,\rho_\alpha}(n) \leq -c_1 n/[\log n]^{2\alpha}$$

for all n large enough. We conjecture that the lower bound is correct.

We now state two corollaries of Theorem 4.3. The first corollary gives a result that is widely applicable whereas the second corollary requires a precise understanding of the most basic random walks on the group G. In particular, the hypothesis (4.2) made in Corollary 4.5 requires a classical $\sqrt{n}$ rate of escape for simple random walk on G.

COROLLARY 4.4. *Assume that G is a compactly generated locally compact group and that there exist $0 \leq \bar{\gamma} \leq \gamma \leq 1$ such that, for n large enough,*

$$-n^\gamma \leq \log \Phi_G(n) \leq -n^{\bar{\gamma}}.$$

For any $s > 0$, set $\gamma_s = \gamma/[s(1-\gamma)+\gamma]$, $\bar{\gamma}_s = \bar{\gamma}/[s(1-\bar{\gamma})+\bar{\gamma}]$.

(1) *For any $\alpha \in (0, 1)$, there exist $c_1, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1 n^{\gamma_{\alpha/2}} \leq \log \widetilde{\Phi}_{G,\rho_\alpha}(n) \leq -c_1 n^{\bar{\gamma}_\alpha}.$$

(2) *For any $\alpha \in (0, 1)$ and $\varepsilon > 0$, there exist $c_\varepsilon, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1 n^{\gamma_{\alpha/2}} \leq \log \Phi_{G,\rho_\alpha}(n) \leq -c_\varepsilon n^{\bar{\gamma}_\alpha}/(\log n)^{(1+\varepsilon)\bar{\gamma}_\alpha \alpha/\bar{\gamma}}.$$

COROLLARY 4.5. *Assume that G is a compactly generated locally compact group and that there exist $\gamma \in (0, 1)$ and a positive slowly varying function η satisfying $\eta(t^a) \simeq \eta(t)$ for all $a > 0$, such that*

$$\log \Phi_G(n) \simeq -n^\gamma/\eta(n).$$

For any $s > 0$, set $\gamma_s = \gamma / [s(1 - \gamma) + \gamma]$. Assume further that there exists a symmetric continuous probability density ϕ with compact support, positive on a generating compact set, and such that

$$(4.2) \qquad \forall n, s \qquad \int \mathbf{1}_{\{|x| \geq n^{1/2}s\}} \phi^{(n)} \, d\lambda \leq C \exp(-cs^q)$$

for some $C, q > 0$.

(1) *For any $\alpha \in (0, 2)$, there exist $c_1, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1 [n/\eta(n)^{\alpha/2\gamma}]^{\gamma\alpha/2} \leq \log \widetilde{\Phi}_{G,\rho_\alpha}(n) \leq -c_1 [n/\eta(n)^{\alpha/2\gamma}]^{\gamma\alpha/2}.$$

(2) *For any $\alpha \in (0, 2)$ and $\varepsilon > 0$, there exist $c_\varepsilon, C_1 \in (0, \infty)$ such that, for all n large enough,*

$$-C_1 [n/\eta(n)^{\alpha/2\gamma}]^{\gamma\alpha/2} \leq \log \Phi_{G,\rho_\alpha}(n) \leq -c_\varepsilon [n/[\eta(n)(\log n)^{1+\varepsilon}]^{\alpha/2\gamma}]^{\gamma\alpha/2}.$$

EXAMPLE 4.2.   Let the group G be either the group $\mathrm{Sol} = \mathbb{Z} \ltimes_A \mathbb{Z}^2$ where $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, or the wreath product $F \wr \mathbb{Z}$ where F is any finite group. By [23], these groups satisfy (4.2). Further, these groups have exponential volume growth and satisfy $\Phi_G(n) \simeq \exp(-n^{1/3})$; see, for example, [31] and the references therein. Hence Corollary 4.5 applies. In particular, for any $\alpha \in (0, 2)$, we have

$$\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq \exp(-n^{1/(1+\alpha)}).$$

Using a different argument, we shall see in the next section that this result also holds for all polycyclic groups.

The final two results of this section concern groups with super-polynomial volume growth and slowly varying moment condition.

THEOREM 4.6.   *Assume that G is a compactly generated locally compact group and that there exist $0 < \bar{\gamma} \leq \gamma < 1$ and $c, C \in (0, \infty)$ such that, for n large enough,*

$$-Cn^\gamma \leq \log \Phi_G(n) \leq -cn^{\bar{\gamma}}.$$

Let $\rho(t) \simeq \log(e + t)\ell(t)$ where ℓ is a continuous increasing slowly varying function at infinity. Let ρ_1 be a slowly varying function such that $\sum_1^\infty \frac{\rho(n)}{n\rho_1(n)} < \infty$, set $\widehat{\rho}_1(t) = 1/\int_t^\infty \frac{ds}{s\rho_1(s)}$ and fix $\varepsilon \in (0, 1)$. Then there are $C_1(\varepsilon), c_1(\rho_1) \in (0, \infty)$ such that, for all n large enough,

$$-C(\varepsilon)n/\ell(n^{\varepsilon\gamma/2}) \leq \log \Phi_{G,\rho}(n) \leq -c(\rho_1)n/\widehat{\rho}_1(n^{\bar{\gamma}}).$$

Theorem 4.6 applies to a very large collection of groups. For instance, it applies to all polycyclic groups with exponential volume growth since such groups have $\Phi_G(n) \simeq \exp(-n^{1/3})$. It also applies to groups with volume growth satisfying $cn^a \leq \log V(n) \leq Cn^b$ with $0 < a \leq b < 1$ since these volume estimates imply $-C_1 n^b \leq \log \Phi_G(n) \leq -c_1 n^{a/(a+2)}$.

The following two examples provide a proof of the assertions made in Theorem 1.6 that concern $\rho_\alpha^{\log}$ and $\rho_{c,\alpha}^{\exp}$.

EXAMPLE 4.3.   We can apply Theorem 4.6 when

$$\rho(t) = \rho_\alpha^{\log}(t) = [\log(e + t)]^\alpha, \qquad \alpha > 1.$$

In this case we can take $\ell \simeq \rho_{\alpha-1}^{\log}$ and $\rho_1 \simeq \rho_{\beta+1}^{\log}$ with $\beta > \alpha$. Then $\widehat{\rho}_1 \simeq \rho_\beta^{\log}$ and the conclusion is that for any $\beta > \alpha$, there are constants $C_2, c_\beta \in (0, \infty)$ such that, for all n large enough,

$$-C_2 n/[\log n]^{\alpha-1} \leq \log \Phi_{G, \rho_\alpha^{\log}}(n) \leq -c_\beta n/[\log n]^\beta.$$

EXAMPLE 4.4.   Theorem 4.6 gives a good result when

$$\rho(t) = \rho_{c,\alpha}^{\exp}(t) = \exp(c[\log(1 + t)]^\alpha), \qquad \alpha \in (0, 1), c > 0.$$

Indeed, in this case we can obviously write $\rho(t) = \log(e + t)\ell(t)$ with $\ell \leq \rho_{c,\alpha}^{\exp}$, and we can take $\rho_1 = \rho_{c_2,\alpha}^{\exp}$ for any fixed constant $c_2 > c$. The conclusion is that there are constants c_3, C_3 such that, for all n large enough,

$$-C_3 n \exp(-c_3[\log n]^\alpha) \leq \log \Phi_{G, \rho_{c,\alpha}^{\exp}}(n) \leq -c_3 n \exp(-C_3[\log n]^\alpha).$$

THEOREM 4.7.   *Assume that G is a compactly generated locally compact group and that there exist two continuous increasing functions $\pi, \bar{\pi}$ such that, for all n large enough,*

$$-n/\pi(n) \leq \log \Phi_G(n) \leq -cn/\bar{\pi}(n).$$

Assume that $\pi(t) \leq t^{1-\varepsilon}$ for some $\varepsilon \in (0, 1)$ Let $\rho(t) \simeq \log(e + t)\ell(t)$ where ℓ is a continuous increasing slowly varying function at infinity. Let ρ_1 be a slowly varying function such that $\sum_1^\infty \frac{\rho(n)}{n\rho_1(n)} < \infty$ and set $\widehat{\rho}_1(t) = 1/\int_t^\infty \frac{ds}{s\rho_1(s)}$. Then there are $C_1, c_1(\rho_1) \in (0, \infty)$ such that, for all n large enough,

$$-C_1 n/\ell(\pi(n^{\varepsilon/2})) \leq \log \Phi_{G, \rho}(n) \leq -c(\rho_1)n/\widehat{\rho}_1(\bar{\pi}(n)).$$

EXAMPLE 4.5.   Assume that $G = F \wr H$ where F is a nontrivial finite group and H is polycyclic with exponential volume growth. Then $\Phi_G(n) \simeq \exp(-n/(\log n)^2)$. Hence, for any $\alpha > 1$ and $\beta > \alpha$, we obtain

$$-C_1 n/[\log(\log n)]^{\alpha-1} \leq \Phi_{G, \rho_\alpha^{\log}}(n) \leq -c_\beta n/[\log(\log n)]^\beta$$

for all n large enough.

4.2. *Volume-based upper bounds.* Let G be a locally compact unimodular group equipped with its Haar measure λ (this group may well not be compactly generated). Consider the problem of studying the decay of convolution powers of probability measures of the form

$$(4.3) \qquad \mu = \sum_1^\infty p_i \mu_i,$$

where $p_i \geq 0$, $\sum_1^\infty p_i = 1$ and

$$\mu_i = \phi_i \, d\lambda, \qquad \|\phi_i\|_\infty = \beta_i, \qquad \phi_i \geq 0, \qquad \mu_i(G) = 1.$$

In words, μ is a convex linear combination of the probability measures μ_i, $i = 1, 2, \ldots$, and these measures are assumed to have bounded densities. It was observed in [24, 29] that interesting upper bounds for convolution powers of such measures can sometimes be obtained by elementary means. This is developed further below.

Set

$$\sigma_k = \sum_{i > k} p_i, \qquad k = 0, 1, \ldots, \sigma_{-1} = +\infty,$$

and

$$b_k = \min_{i \leq k}\{\beta_i\}, \qquad k = 1, 2, \ldots, b_0 = b_1,$$

and consider the function F on $(0, \infty)$ [this function depends only on $(\sigma_i)_0^\infty$ and $(b_i)_0^\infty$] defined by

$$F(s) = b_k \qquad \text{if } \sigma_k < s \leq \sigma_{k-1}.$$

The following result is quite versatile and surprisingly sharp when applied to low moment measures.

PROPOSITION 4.8. *Referring to the notation introduced above and assuming that $b_i \to 0$, the density $\phi^{(n)} = d\mu^{(n)}/d\lambda$ of the nth convolution power $\mu^{(n)}$ of μ satisfies*

$$\|\phi^{(n)}\|_\infty \leq \int_0^\infty e^{-ns} \, dF(s) = \sum_{i=1}^\infty e^{-n\sigma_i}(b_i - b_{i+1}).$$

REMARK 4.9. One important class of examples is obtained by considering a given increasing sequence of compact sets B_i with $\bigcup_1^\infty B_i = G$ and setting

$$d\mu_i = d\lambda_{B_i} = \frac{1}{\lambda(B_i)} \mathbf{1}_{B_i} \, d\lambda.$$

In this case, $b_i = \beta_i = 1/\lambda(B_i)$.

PROOF OF PROPOSITION 4.8.   Write

$$\phi^{(n)} = \left(\sum_{i=1}^{\infty} p_i \phi_i\right)^{(n)} = \sum_{k=1}^{\infty}\left(\left(\sum_{i \le k} p_i \phi_i\right)^{(n)} - \left(\sum_{i \le k-1} p_i \phi_i\right)^{(n)}\right)$$

$$= \sum_{k \ge 1}\left(\left(\sum_{i_j \le k} p_{i_1}\phi_{i_1} * \cdots * p_{i_n}\phi_{i_n}\right) - \left(\sum_{i_j \le k-1} p_{i_1}\phi_{i_1} * \cdots * p_{i_n}\phi_{i_n}\right)\right)$$

$$= \sum_{k \ge 1}\left(\sum_{\max\{i_1,\ldots,i_n\}=k} p_{i_1}\phi_{i_1} * \cdots * p_{i_n}\phi_{i_n}\right).$$

Next we use Minkowski inequality and the estimate

$$\|f_1 * \cdots * f_n\|_\infty \le \min\{\|f_i\|_\infty\}$$

for functions f_i with L^1-norm at most 1. This estimate holds on G because we assume unimodularity of G. It yields

$$\|\phi^{(n)}\|_\infty \le \sum_{k \ge 1} b_k \sum_{\max\{i_1,\ldots,i_n\}=k} p_{i_1} \cdots p_{i_n}$$

$$= \sum_{k \ge 1} b_k[(1-\sigma_k)^n - (1-\sigma_{k-1})^n]$$

$$= \sum_{k \ge 1}(1-\sigma_k)^n[b_k - b_{k+1}]$$

$$\le \sum_{k \ge 1} e^{-n\sigma_k}[b_k - b_{k+1}]. \qquad \square$$

We now give some simple applications when G is locally compact, compactly generated and unimodular (we assume that G is noncompact). Fix a symmetric open set U that contains a generating compact neighborhood of the identity element, and set $|x| = \inf\{n : x \in U^n\}$, with $|e| = 0$. Thus $|\cdot|$ induces a familiar word distance on G when G is finitely generated. Observe that we have $\lambda(U^{4n}) \ge 2\lambda(U^n)$. Indeed, if $|z| = 3n$ (such a z does indeed exist!), then the sets U^n and zU^n are disjoint and contained in U^{4n}. We consider the probability densities $\phi_i = \lambda(B_i)^{-1}\mathbf{1}_{B_i}$ with $B_i = U^{4^i}$ and set $b_i = \lambda(B_i)^{-1}$. Since $\lambda(B_i) \ge 2\lambda(B_{i-1})$, we have $b_i \ge b_i - b_{i+1} \ge b_i/2$. Set $\phi = \sum_1^\infty p_i \phi_i$ with $\sum_1^\infty p_i = 1$ and $\sigma_k = \sum_{i>k} p_i$. Fix a nondecreasing function $\rho : (0,\infty) \to (0,\infty)$, and set $\rho_G = \rho(|\cdot|)$. With this notation, we have

(4.4) $\forall n \qquad \phi^{(2n)}(e) \le \sum_{k \ge 1} e^{-2n\sigma_k} b_k$

and

(4.5) $\displaystyle\int_G \rho_G \phi \, d\lambda = \int_G \sum_{k \ge 1} p_k \rho_G \phi_k \, d\lambda \le \sum_{k \ge 1} \rho(4^k) p_k.$

Further, we also have

$$s \int_{\{\rho_G \geq s\}} \phi \, d\lambda \leq s \sum_{\rho(4^{k-1}) \geq s} p_k.$$

Hence, assuming that ρ is a doubling function, we have

$$(4.6) \qquad W(\rho, \phi \, d\lambda) = \sup_{s>0} \left\{ s \int_{\{\rho_G \geq s\}} \phi \, d\lambda \right\} \leq C(\rho) \sup_k \{\rho(4^k)\sigma_k\}.$$

These two estimates allow us to derive upper bounds on Φ_{G,ρ_G} in terms of the volume growth of the group G. Indeed, for a given ρ, (4.5) tells us how to pick $(p_i)_1^\infty$ so that $d\mu = \phi \, d\lambda$ satisfies $\mu(\rho_G) < \infty$. For this choice of $(p_i)_1^\infty$, (4.4) yields an upper bound on $\phi^{(2n)}(e)$ [hence on $\Phi_{G,\rho_G}(n)$] in terms of (a lower bound on) the volume growth which determines the sequence $(b_i)_1^\infty$. This approach yields an alternative proof of Theorem 4.1 (polynomial volume growth) as well as new results in the super-polynomial volume case.

ALTERNATIVE PROOF OF THEOREM 4.1. We give the details only for $\widetilde{\Phi}_{G,\rho}$. The proofs concerning $\Phi_{G,\rho}$ are similar. Recall that Theorem 4.1 deals with the case when $V(n) \simeq n^D$, and ρ is comparable to either (a) a regularly varying function with positive index $\alpha \in (0, 2)$ or (b) a slowly varying of the form $\rho(t) \simeq 1/\int_t^\infty \frac{ds}{s\ell(s)}$ with ℓ positive and slowly varying. Further, in case (b), we assume that $\log \rho^{-1}(t) \simeq t^\gamma \omega(t)^{1+\gamma}$ for some $\gamma \in [0, \infty)$ and positive slowly varying function ω.

In case (a), set $p_i = c\rho(4^i)^{-1}$. In case (b) set $p_i = c\ell(4^i)^{-1}$. Then it is easy to check that

$$\sigma_i = \sum_{k>i} p_k \simeq \rho(4^i)^{-1}.$$

Using (4.6), this implies that $\phi = \sum_1^\infty p_i \phi_i$ satisfies the moment condition

$$W(\rho, \phi \, d\lambda) \leq \sup_i \{\rho(4^i)\sigma_i\} < \infty.$$

Further

$$(4.7) \qquad \phi^{(2n)}(e) \leq C_1 \sum_i e^{-c_1 n/\rho(4^i)} 4^{-iD}.$$

In case (a) where $\rho(t) \simeq (1+t)^{2\alpha} \ell(s^2)^\alpha$ with $\alpha \in (0, 1)$ and ℓ positive and slowly varying, observe that $\rho^{-1}(1/u) \simeq u^{-1/2\alpha} \ell^{\#}(1/u^{1/\alpha})^{1/2}$ for small u and write

$$\phi^{(2n)}(e) \leq C_1 \sum_i e^{-c_1 n/\rho(4^i)} 4^{-iD} \leq C_2 \int_1^\infty e^{-c_1 n/\rho(s)} \frac{ds}{s^{1+D}}$$

$$\leq C_3 \int_0^1 e^{-c_1 nu} \left(\frac{1}{\rho^{-1}(1/u)} \right)^D \frac{du}{u}$$

$$\leq C_4 \int_0^\infty e^{-c_1 nu} \left(\frac{u^{1/\alpha}}{\ell^{\#}(1/u^{1/\alpha})} \right)^{D/2} \frac{du}{u} \simeq C_5 [n^{1/\alpha} \ell^{\#}(n^{1/\alpha})]^{-D/2}.$$

This yields the desired result, namely,

$$\widetilde{\Phi}_{G,\rho}(n) \le C[n^{1/\alpha}\ell^{\#}(n^{1/\alpha})]^{-D/2}$$

for case (a).

In case (b), write

$$\phi^{(2n)}(e) \le C_1 \sum_i e^{-c_1 n/\rho(4^i)-(D/2)\log 4^{-i}} 2^{-Di}$$

$$\le C_2 \exp\left(-c_2 \inf_{s>0}\{n/\rho(s) + \log(e+s)\}\right).$$

Using the hypothesis concerning ρ^{-1}, observe that

$$\inf_{s>0}\{n/\rho(s) + \log(e+s)\} = \inf_{s>0}\{ns + \log(e + \rho^{-1}(1/s))\}$$

$$\simeq \inf_{s>0}\{ns + s^{-\gamma}\omega(1/s)^{1+\gamma}\}$$

$$\simeq n^{\gamma/(1+\gamma)}/\omega^{\#}(n^{1/(1+\gamma)}).$$

As stated in Theorem 4.1(2) and under the hypotheses of case (b), this yields

$$\widetilde{\Phi}_{G,\rho}(n) \le C\exp(-cn^{\gamma/(1+\gamma)}/\omega^{\#}(n^{1/(1+\gamma)}))$$

as desired. $\square$

THEOREM 4.10 (The super polynomial case).   *Assume that* $\lambda(U^n) \ge \exp(cn^\theta)$ *for some* $c, \theta > 0$. *Then*:

(1) *Fix* $\alpha \in (0,1)$, *a positive slowly varying function* ℓ *at infinity, and set* $\rho(s) = [(1+s^2)\ell(1+s^2)]^\alpha$. *Then there exists* $C \in (0, \infty)$ *such that, for all n large enough,*

$$\widetilde{\Phi}_{G,\rho_\alpha}(n) \le C\exp(-cn^{\theta/(\theta+2\alpha)}/\ell_\bullet^{\#}(n^{2/(\theta+2\alpha)})^\alpha),$$

where $\ell_\bullet = [\ell^{\#}]^{\theta/(\theta+2\alpha)}$, *and* $\ell_\bullet^{\#}$ *is its de Bruijn conjugate.*

(2) *Fix* $\alpha \in (0,2)$. *For all* $\beta > \alpha$, *there are constants* $C_\beta, c_\beta > 0$ *such that, for all n large enough,* $\Phi_{G,\rho_\alpha}(n) \le C_\beta \exp(-c_\beta n^{\theta/(\theta+\beta)})$.

(3) *For any fixed* $\alpha > 0$ *we have* $\widetilde{\Phi}_{G,\rho_\alpha^{\log}}(n) \le C\exp(-cn/[\log n]^\alpha)$. *Further, for all* $\beta > \alpha$, *there are constants* $C_\beta, c_\beta > 0$ *such that, for all n large enough,* $\Phi_{G,\rho_\alpha^{\log}}(n) \le C_\beta \exp(-c_\beta n/[\log n]^\beta)$.

(4) *For any fixed* $\alpha \in (0,1)$ *and* $c > 0$, *there is a constant* $C_1 > 0$ *such that, for all n large enough,* $\Phi_{G,\rho_{c,\alpha}^{\exp}}(n) \le C_1 \exp(-n/\exp(C_1[\log n]^\alpha)))$.

PROOF.   We prove statement (1). The variations needed for the other statements are straightforward. We have $b_i \le \exp(-c4^{i\theta})$. Fulfilling the desired moment conditions forces the choice of the sequence $(p_i)_1^\infty$. For instance, in the first

case, we take $p_i = c\rho(4^i)^{-1}$ so that $\sigma_i \simeq \rho(4^i)^{-1}$. Hence

$$\phi^{(2n)}(e) \leq C_1 \sum_i e^{-c_1(n/\rho(4^i)+4^{i\theta})}$$

$$\leq C_2 \exp\left(-c_2 \inf_{s>0}\{n/\rho(s) + s^\theta\}\right).$$

Write

$$\inf_{s>0}\{n/\rho(s) + s^\theta\} = \inf_{s>0}\{ns + \rho^{-1}(1/s)^\theta\}.$$

A good approximation of the infimum is obtained by picking $s = s_n$ such that $n = (1/s_n)\rho^{-1}(1/s_n)^\theta$. At infinity, $\rho^{-1}(t) = t^{1/2\alpha}\ell^\#(t^{1/\alpha})^{1/2}$ and thus, at 0,

$$(1/t)[\rho^{-1}(1/t)]^\theta = t^{-(2\alpha+\theta)/2\alpha}\ell^\#(1/t^{1/\alpha})^{\theta/2}.$$

Setting $\ell_\bullet = [\ell^\#]^{\theta/(\theta+2\alpha)}$, we have $s_n \simeq n^{-2\alpha/(2\alpha+\theta)}[\ell_\bullet^\#(n^{2/(2\alpha+\theta)})]^{-\alpha}$. Finally,

$$\phi^{(2n)}(e) \leq C \exp(-c_3 n^{\theta/(\theta+2\alpha)}/\ell_\bullet^\#(n^{2/(\theta+2\alpha)})^\alpha). \qquad \square$$

REMARK 4.11. Note that the hypotheses in Theorem 4.10 and in Theorem 4.3 are notably different. Theorem 4.3 is based on hypotheses regarding the behavior of ϕ_G whereas Theorem 4.10 assumes $V(n) \geq \exp(cn^\theta)$. We note that the hypothesis $V(n) \geq \exp(cn^\theta)$ implies $\Phi_G(n) \leq \exp(-cn^{\theta/(2+\theta)})$ [31]. If $V(n) \geq \exp(cn^\theta)$ and $\Phi_G(n) \geq \exp(-Cn^{\theta/(2+\theta)})$, then the upper bound of Theorem 4.10(1) matches precisely the lower bound of Theorem 3.2.

The next theorem treats the case of groups that have exponential volume growth (i.e., $\theta = 1$) and such that $\Phi_G(n) \simeq \exp(-n^{1/3})$. (This is the case, e.g., if G is polycyclic with exponential volume growth.) This result contains the part of Theorem 1.6 dealing with $\rho_\alpha, \alpha \in (0,2)$.

THEOREM 4.12. *Assume that G has exponential volume growth and satisfies $\phi_G(n) \simeq \exp(-n^{1/3})$.*

(1) *Fix $\alpha \in (0,1)$, a positive slowly varying function ℓ at infinity, and set $\rho(s) = [(1+s^2)\ell(1+s^2)]^\alpha$. Then we have*

$$\widetilde{\Phi}_{G,\rho}(n) \simeq \exp(-n^{1/(1+2\alpha)}/\ell_\bullet^\#(n^{2/(1+2\alpha)})^\alpha),$$

where $\ell_\bullet = [\ell^\#]^{1/(1+2\alpha)}$, and $\ell_\bullet^\#$ is its de Bruijn conjugate.

(2) *Fix $\alpha \in (0,2)$. For all $\beta > \alpha$, there are constants $C_\beta, c_\beta > 0$ such that, for all n large enough,*

$$C_\beta \exp(-cn^{1/(1+\alpha)}) \leq \Phi_{G,\rho_\alpha}(n) \leq C_\beta \exp(-c_\beta n^{1/(1+\beta)}).$$

5. The case of the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. The wreath product construction provides important examples of groups whose behavior differs from linear groups. The simplest family of wreath products is $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. An element of this group is a pair (η, k) with $\eta \in \bigoplus_{i \in \mathbb{Z}^d} (\mathbb{Z}/2\mathbb{Z})_i$ (algebraic sum) and $k \in \mathbb{Z}^d$. In the popular lamplighter interpretation, k is the position of the lamplighter, and $\eta = (\eta_i)_{i \in \mathbb{Z}^d}$ is a configuration of lamps that can be on ($\eta_i = 1$) or off ($\eta_i = 0$). Only finitely many lamps can be on. The product is given by $(\eta, k)(\eta', k') = (\eta'', k'')$ where $k'' = k + k'$ (addition in $\mathbb{Z}^d$) and $\eta_i'' = \eta_i + \eta_{i-k}'$ (addition in $\mathbb{Z}/2\mathbb{Z}$). In other words, $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$ is the semidirect product of $\bigoplus_{i \in \mathbb{Z}^d} (\mathbb{Z}/2\mathbb{Z})_i$ by $\mathbb{Z}^d$ where the action of $\mathbb{Z}^d$ on $\bigoplus_{i \in \mathbb{Z}^d} (\mathbb{Z}/2\mathbb{Z})_i$ is by translation of the indices. These groups have exponential volume growth.

The aim of this section is to prove the following theorem.

THEOREM 5.1. *For any integer $d \geq 1$ and $\alpha \in (0, 2)$, we have*

$$\widetilde{\Phi}_{(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \rho_\alpha}(n) \simeq \exp(-n^{d/(d+\alpha)}).$$

Further, for any $\beta > \alpha$, there are constants $c, C, c_\beta, C_\beta \in (0, \infty)$ such that, for all n large enough,

$$c \exp(-Cn^{d/(d+\alpha)}) \leq \Phi_{(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \rho_\alpha}(n) \leq C_\beta \exp(-c_\beta n^{d/(d+\beta)}).$$

We shall see in the proof given below that the lower bounds stated in this theorem follow from Theorem 3.2. The interesting part are the upper bounds. These upper bounds are interesting because they do not follow from the results in Sections 4.1 and 4.2.

PROOF OF THEOREM 5.1. We can identify $\mathbb{Z}^d$ as a subgroup of $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$ in an obvious way, and we can also identify $\mathbb{Z}/2\mathbb{Z}$ with $(\mathbb{Z}/2\mathbb{Z})_0$ in $\bigoplus_{i \in \mathbb{Z}^d} (\mathbb{Z}/2\mathbb{Z})_i \subset (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. Hence, any probability measure on $\mathbb{Z}/2\mathbb{Z}$ or on $\mathbb{Z}^d$ can be interpreted as a measure on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. Following the notation used in [21], if ν is a measure supported on $(\mathbb{Z}/2\mathbb{Z})_0$, and μ a measure supported on $\mathbb{Z}^d$, we set $q = \nu * \mu * \nu$ in $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. In [21], it is observed that a famous large deviation theorem, due to Donsker and Varadhan [8] and concerning the range of certain random walks on $\mathbb{Z}^d$, implies that

$$q^{(2n)}(e) \simeq \exp(-n^{d/(d+2)}),$$

when ν is the uniform measure on $\mathbb{Z}/2\mathbb{Z}$, and μ is any symmetric measure on $\mathbb{Z}^d$ with finite generating support. By [20], this implies that

$$(5.1) \qquad \Phi_{(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d}(n) \simeq \exp(-n^{d/(d+2)}).$$

Here, we are interested in determining the behavior of

$$\widetilde{\Phi}_{(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \rho_\alpha} \quad \text{and} \quad \Phi_{(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \rho_\alpha}, \qquad \alpha \in (0, 2).$$

First, consider how the results obtained so far in this paper apply in this case. Theorem 3.2 readily gives the lower bound

$$(5.2) \qquad \Phi_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d,\rho_\alpha}(n) \geq \widetilde{\Phi}_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d,\rho_\alpha}(n) \geq \exp(-C(d,\alpha)n^{d/(d+\alpha)}),$$

because if $\gamma = d/(d+2)$, then $\gamma_{\alpha/2} := \gamma/(\gamma + (\alpha/2)(1-\gamma)) = d/(d+\alpha)$. We are faced with the problem of deciding whether or not this is sharp. Can we find measures with finite ρ_α-moment and whose convolution powers decay as rapidly as permitted by this lower bound?

For this purpose, we have so far discussed two methods: (a) the use of subordination as developed in [4] and (b) direct computation based on volume estimates (see Theorem 4.10).

The direct computation of Theorem 4.10 provides the upper bounds

$$(5.3) \qquad \widetilde{\Phi}_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d,\rho_\alpha}(n) \leq \exp(-Cn^{1/(1+\alpha)})$$

and

$$(5.4) \qquad \Phi_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d,\rho_\alpha}(n) \leq \exp(-C(\beta)n^{1/(1+\beta)}), \qquad \beta > \alpha.$$

When $d = 1$ (and only in this case), these upper bounds show that the lower bounds stated in (5.2) are essentially sharp. In particular, we get

$$\text{exp-pow}((\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}, \rho_\alpha) = 1/(1+\alpha), \qquad \alpha \in (0,2).$$

For $d \geq 2$, (5.3) and (5.4) fail to match (5.2) for a good reason: Theorem 4.10 is based solely on a volume hypothesis and thus cannot provide more subtle information that is based on the particular structure of these wreath products.

The subordination technique of [4] fails to give good upper bounds for a different reason related to the fact that, for simple random walks on wreath products such as $(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d$ with $d > 1$, the rate of escape to infinity is much faster than $\sqrt{n}$. See the discussion in [4].

Thus, the two techniques used earlier in this paper to provide upper bounds on Φ_{G,ρ_α} and $\widetilde{\Phi}_{G,\rho_\alpha}$ both fail to match the lower bound (5.2) when $d \geq 2$. The following argument shows that (5.2) is sharp nonetheless. For each $\alpha \in (0,2)$ let μ_α be the probability measure on $\mathbb{Z}^d$ given by

$$\mu_\alpha(k) = \frac{c(d,\alpha)}{(1+\|k\|^2)^{(d+\alpha)/2}}, \qquad k \in \mathbb{Z}^d, \|k\|^2 = \sum_1^d k_i^2.$$

The theorem of Donsker and Varadhan ([8], Theorem 1) implies that, for any fixed s and n large enough,

$$E(e^{-sD_n^\#}) \simeq \exp(-n^{d/(d+\alpha)}).$$

Here $D_n^\#$ is the number of visited sites up to time n for the random walk on $\mathbb{Z}^d$ driven by μ_α.

For any fixed $\beta \in (0, 2)$, this, together with [21], Theorem 3.1, implies that the measure $q_\beta = \nu * \mu_\beta * \nu$ on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$ satisfies

$$q^{(2n)}(e) \simeq \exp(-n^{d/(d+\beta)}).$$

It is plain that the measure q_α has finite weak-ρ_α-moment $W(\rho_\alpha, q_\alpha) < \infty$, and that q_β has finite ρ_α-moment if and only if $\beta > \alpha$. To check this, notice that q_α is almost entirely concentrated on $\mathbb{Z}^d$ inside $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. Thus these measures provide witnesses to the fact that

$$\widetilde{\Phi}_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d, \rho_\alpha}(n) \leq C_1 \exp(-c_1 n^{d/(d+\alpha)})$$

and that, for each $\beta > \alpha$, $\alpha \in (0, 2)$,

$$\Phi_{(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}^d, \rho_\alpha}(n) \leq C_\beta \exp(-c_\beta n^{d/(d+\beta)}).$$

These are the desired upper bounds. $\square$

In particular it follows that

$$\text{exp-pow}\big((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \rho_\alpha\big) = \frac{d}{d+\alpha}, \qquad d = 1, 2, \ldots, \alpha \in (0, 2).$$

It is interesting to note that the optimal measure q_α that we have exhibited above is spread out only in a very small part of the group, that is, in the directions of the lamplighter moves $\mathbb{Z}^d$.

APPENDIX: ULTRACONTRACTIVITY, FUNCTIONAL CALCULUS AND VON NEUMANN TRACE

A.1. Spectral theory. Let T be a self-adjoint operator acting on a Hilbert space H. We denote by $E^T_{\mathcal{I}}$ its spectral projector associated with the open set $\mathcal{I} \subset \mathbb{R}$, and by $E^T_s = E^T_{(-\infty,s)}$ the associated (left-continuous) spectral resolution of T so that

$$T = \int_{-\infty}^{+\infty} s \, dE^T_s.$$

In the cases of interest to us, T is actually a bounded operator so that $E^T_{(a,b)} = 0$ if $\max\{a, -b\}$ is larger than $\|T\|$. For any continuous function $m : \mathbb{R} \to \mathbb{C}$, the operator $m(T)$ with domain $D_f = \{u \in H : \int |m(s)|^2 \, d\langle E^T_s u, u\rangle < \infty\}$ is defined by

$$m(T) = \int_{-\infty}^{+\infty} m(s) \, dE^T_s,$$

where this integral is obtained as the strong limit of finite Riemann sums. Further, note that if m is real valued, then $m(T)$ is self-adjoint and

$$E^{m(T)}_{(a,b)} = E^T_{m^{-1}(a,b)}.$$

A.2. The von Neumann algebra $V(G)$. We will make fundamental use of the notion of von Neumann trace for certain operators in the von Neumann algebra $V(G)$ generated by the right translations $r_g : f \mapsto f(\cdot g)$ acting on $L^2(G)$. By construction, $V(G)$ is equipped with a faithful semifinite normal trace τ defined as follows. Let S be a nonnegative Hermitian element in $V(G)$ [i.e., a self-adjoint element satisfying $\langle Su, u \rangle \geq 0$ for every $u \in L^2(G)$]. If $S^{1/2} = R_a$ for some $a \in L^2(G)$, set $\tau(S) = \|a\|_2^2$. Otherwise, set $\tau(S) = +\infty$. See [7], page 97. Since $S^{1/2} = R_a$, R_a is self-adjoint. This is equivalent to say that the function $a \in L^2(G)$ satisfies $a = \check{a}$ [where $\check{a}(x) = \bar{a}(x^{-1})$, $x \in G$]. Hence

$$\tau(S) = \int_G |a|^2 \, d\lambda = a * a(e).$$

Note that, as the convolution of two functions in $L^2(G)$, the function $a * a$ is bounded and continuous (i.e., admits a continuous representative) and that S acts on $\phi \in \mathcal{C}_c(G)$ by $S\phi = \phi * [a * a]$.

Let S, T be two Hermitian nonnegative elements in $V(G)$ such that $S \leq T$. Then $\tau(S) \leq \tau(T)$. In particular, if T has finite trace and spectral decomposition

$$T = \int_0^\infty s \, dE_s^T,$$

then $E_{(s,+\infty)}^T$ is in $V(G)$ and has finite trace for all $s > 0$ since $sE_{(s,+\infty)}^T \leq T$. Note that, in general (i.e., when G is not countable), $E_\infty^T = I$ does not have finite trace.

If T is Hermitian of the form $T = R_{a*\check{a}}$, then

$$\tau(T) = a * \check{a}(e) = \int_0^{+\infty} s \, d[-\tau(E_{(s,\infty)}^T)] = \int_0^{+\infty} \tau(E_{(s,\infty)}^T) \, ds.$$

This follows from the well-known properties of spectral resolutions and the fact that τ is a normal trace [this means that τ has the property that, for any positive Hermitian T and any increasing filtering set $\mathcal{F}$ of positive Hermitian elements with supremum T, $\sup_{\mathcal{F}} \tau(S) = \tau(T)$].

Strictly speaking, the trace τ is defined only on nonnegative Hermitian elements. However, the set of Hermitian nonnegative elements with finite trace is the positive part of a two-sided ideal $\mathfrak{m}$ of $V(G)$, and there is a unique linear form defined on this two-sided ideal which coincides with the trace on nonnegative Hermitian elements. Abusing notation, we denote this extension by $\tau : \mathfrak{m} \to \mathbb{R}$. If $a, b \in L^2(G)$ and $R_a, R_b \in V(G)$, then $R_a R_b \in \mathfrak{m}$ and $\tau(R_a * R_b) = b * a(e)$. See [7], Theorem 1, page 97. In particular, if $\phi = \check{\phi} \in L^1(G) \cap L^2(G)$ and $T = R_\phi$, then, for any $n = 2, 3, \ldots$, $T^n = R_{\phi^{(n)}}$ has finite trace and

(A.1) $$\phi^{(n)}(e) = \tau(T^n).$$

A.3. Ultracontractivity. Let $\phi = \check{\phi} \in L^1(G) \cap L^2(G)$ be a symmetric probability density. Let $T = R_\phi : f \mapsto f * \phi$ be the operator of right convolution by ϕ acting on $L^2(G)$. This is an Hermitian element of $V(G)$ with norm at most 1. Its powers T^n, $n \geq 2$, are of finite trace and the function

$$n \mapsto \tau(T^n) = \phi^{(n)}(e)$$

is of interest to us because it quantifies the ultracontractivity of the operators T^{2n}, $n \geq 1$. Indeed, we have

$$\sup_{\|f\|_1 \leq 1} \{\|T^{2n} f\|_\infty\} = \|T^{2n}\|_{1 \to \infty} = \phi^{(2n)}(e).$$

We assume throughout that $\phi^{(2n)}(e) \to 0$, which simply means that ϕ is not supported on a compact subgroup of G. As a consequence $\|T^n f\|_\infty \to 0$ for any $f \in L^2(G)$. In particular, there are no nontrivial functions in $L^2(G)$ such that $Tf = \pm f$.

Let E_s^T, E_s^{I-T}, $s \in \mathbb{R}$, be the left-continuous spectral resolutions of T and $I - T$ and note that

$$T^n = \int_0^2 (1-s)^n \, dE_s^{I-T}, \qquad E_{(1-b,1-a)}^T = E_{(a,b)}^{I-T}, \qquad 0 \leq a < b \leq \infty,$$

with $\lim_{s \searrow 0} E_s^{I-T} = E_{[1,\infty)}^T = 0$ because there are no $L^2(G)$-solutions of $Tu = u$. Note also that the projection valued measure dE_s^{I-T} could have an atom at $s = 1$ [corresponding to $L^2(G)$-functions satisfying $Tu = 0$] but that this atom is irrelevant to the integral formula $m(T) = \int_0^2 m(1-s) \, dE_s^{I-T}$ as long as m is continuous and satisfies $m(0) = 0$. Observe further that $(1-s)^2 E_s^{I-T} \leq T^2$ for $s \in [0,1]$ so that E_s^{I-T} has finite trace for all $s \in [0,1)$. Similarly $E_{(s,2)}^{I-T}$ has finite trace for $s \in (1,2)$.

Using this fact we define the nondecreasing, nonnegative functions $N_\phi : [0, 1) \to [0, +\infty)$ by

$$(A.2) \qquad N_\phi(s) = \tau(E_s^{I-T}) = \tau(E_{(1-s,\infty)}^T), \qquad s \in (0,1).$$

The following lemma is proved in [4]. It indicates that the part of the spectrum of T near -1 does not play a crucial role in estimating $\phi^{(2n)}(e)$ [this uses the fact that $\phi^{(k)}(e) \geq 0$].

LEMMA A.1 (See, e.g., [4], Proposition 3.1). *Assume that ϕ is a symmetric probability density in $L^2(G)$. Then*

$$(A.3) \qquad \int_0^1 (1-s)^{2n} \, dN_\phi(s) \leq \phi^{(2n)}(e) \leq 2 \int_0^1 (1-s)^{2(n-1)} \, dN_\phi(s).$$

Thanks to this Laplace transform type relation, the behavior of $n \mapsto \phi^{(2n)}(e)$ as n tends to infinity and the behavior of $N_\phi(s)$ as s tends to 0 are related to each other. The following statements are appropriate versions of classical results. See [2, 4, 5] for details.

For $\theta = 0$ or $+\infty$, we let $\mathcal{R}_\alpha(\theta)$ be the set of regularly varying functions of index α at θ. If ℓ is a slowly varying function at infinity, we let $\ell^\#$ be its de Bruijn conjugate. See [5], Theorem 1.5.13. For simple applications, we observe that if $\ell(x) \sim \ell(x\ell(x))$ at infinity, then $\ell^\# \sim 1/\ell$. For instance, this applies to $\ell(x) = (\log x)^\beta$, $\beta \in \mathbb{R}$. See [5], Corollary 2.3.4. In the following result, φ and N are abstract functions but, applications we have in mind, $\varphi(k) = \phi^{(2k)}(e)$ and $N = N_\phi$ as in Lemma A.1.

PROPOSITION A.2. *Let the nondecreasing function $N : (0, 1) \to (0, +\infty)$ and nonincreasing function $\varphi : \{1, 2, \ldots\} \to (0, +\infty)$ be related by*

$$\forall k > k_0 \qquad c \int_0^1 (1 - s)^k \, dN(s) \leq \varphi(k) \leq C \int_0^1 (1 - s)^{k-k_0} \, dN(s)$$

for some $k_0, c, C \in (0, \infty)$.

(1) *Fix $\alpha > 0$, and let ℓ be a slowly varying function at infinity. There exists a $c_1 \in (0, 1)$ such that $\varphi(k)k^\alpha \ell(k) \geq c_1$ [resp., $\varphi(k)k^\alpha \ell(k) \leq c_1$] for all k large enough if and only if there exists a constant $c_2 \in (0, 1)$ such that $N(s)s^{-\alpha}\ell(1/s) \geq c_2$ [resp., $N(s)s^{-\alpha}\ell(1/s) \leq c_2$] for all $s > 0$ small enough.*

(2) *Fix $\alpha \in (0, 1)$, and let ℓ be a slowly varying function at infinity. There exists a constant $c_1 \in (0, 1)$ such that*

$$[-\log \varphi(k)][k^\alpha / \ell(k^{1-\alpha})]^{-1} \geq c_1 \qquad (resp., \leq c_1) \qquad \text{for large enough } k,$$

if and only if there exists a constant $c_2 \in (0, 1)$ such that

$$[-\log N(s)][s^\alpha \ell^\#(1/s)]^{1/(1-\alpha)} \geq c_2 \qquad (resp., \leq c_2)$$

$$\text{for small enough } s > 0.$$

(3) *Let M, π and $t \mapsto t/\pi(t)$ be continuous increasing functions on $(0, \infty)$ which tend to infinity at infinity and such that*

(A.4) $$\pi^{-1}(t) \simeq tM(t) \qquad \text{at infinity.}$$

The following two properties are equivalent:

(a) *there exists $c_1 \in (0, \infty)$ such that*

$$-\log N(s) \geq c_1 M(c_1/s) \qquad [resp., -\log N(s) \leq c_1 M(c_1/s)]$$

for all s small enough;

(b) *there exists $c_2 \in (0, \infty)$ such that*

$$-\log \varphi(n) \geq c_2 n / \pi(n/c_2) \qquad [resp., -\log \varphi(n) \leq c_2 n / \pi(n/c_2)]$$

for all n large enough.

EXAMPLE A.1. The reason behind considering these elaborate statements is the nature of the known results concerning $\phi^{(2n)}(e)$ when ϕ is symmetric compactly supported. Here is a small selection of specific examples of interest.

(1) The properties

$$\phi^{(2n)}(e) \simeq n^{-D/2} \qquad \text{at infinity}$$

and

$$N_\phi(s) \simeq s^{D/2} \qquad \text{at zero}$$

are equivalent. These properties hold when ϕ is compactly supported, and G has polynomial volume growth of degree D.

(2) The properties

$$\phi^{(2n)}(e) \simeq \exp(-n^{1/3}) \qquad \text{at infinity}$$

and

$$N_\phi(s) \simeq \exp(-1/s^{1/2}) \qquad \text{at zero}$$

are equivalent. These properties hold whenever ϕ is compactly supported with generating support and G is virtually polycyclic with exponential volume growth.

(3) The properties

$$\phi^{(2n)}(e) \simeq \exp\left(-n^{d/(d+2)}[\log n]^{2/(d+2)}\right) \qquad \text{at infinity}$$

and

$$N_\phi(s) \simeq \exp(-s^{-d/2}[\log 1/s]) \qquad \text{at zero}$$

are equivalent. They hold, for instance, when $G = \mathbb{Z} \wr \mathbb{Z}^d$ (the lamplighter group with street map $\mathbb{Z}^d$ and lamps in $\mathbb{Z}$).

See [1, 2, 9, 10, 21, 31]. Remarkably enough, the first two types of behaviors are the only possibilities for unimodular amenable Lie groups and for finitely generated amenable discrete subgroups of Lie groups; see, for example, [26] and the references therein.

A.4. Functional calculus. Let $T = R_\phi : f \mapsto f * \phi$ be a convolution operator with a symmetric probability density $\phi \in L^2(G)$. Consider a function $\psi : [0, 2] \to [0, 2]$ that is increasing, continuous with continuous derivative and which satisfies $\psi(0) = 0$, $\psi(1) = 1$, $\psi(2) < 2$. With such a function we associate the operator

$$\psi(I - T) = \int_0^2 \psi(s) \, dE_s^{I-T}$$

and

(A.5) $$T_\psi = I - \psi(I - T), \qquad T = R_\phi.$$

LEMMA A.3. *Let $\phi \in L^2(G)$ be a symmetric probability density. Let $\psi : [0, 2] \to [0, 2]$ satisfies $\psi(0) = 0$, $\psi(1) = 1$, $\psi(2) < 2$ and assume that ψ is increasing and continuous with continuous derivative. Then T_ψ defined at (A.5) is in $V(G)$, and T_ψ^n has finite trace for all $n \geq 2$. Further, if $\phi = \xi * \xi$ with $\xi = \check{\xi} \in L^2(G) \cap L^1(G)$, then $T = R_{\phi_\psi}$ with $\phi_\psi = (\phi_\psi)^\vee \in L^2(G)$ and R_{ϕ_ψ} bounded on $L^2(G)$.*

PROOF. Note that the operators $\psi(I - T)$ and T_ψ belong to the von Neumann algebra $V(G)$. Further, from the elementary fact that $|1 - \psi(s)| \leq C|1 - s|$ on $[0, 2]$, for some $C \in (0, \infty)$, we deduce that T_ψ^{2k} is a Hermitian nonnegative element in $V(G)$ which is dominated by

$$CR_\phi^2 = C^2 \int_0^2 |1 - s|^2 \, dE_s^{I-T}.$$

This last Hermitian element has finite trace equal to $C\tau(R_\phi^2) = C\phi^{(2)}(e)$. Hence, T_ψ^{2k} has finite trace for $n \geq 2$. This implies that $T_\psi^{(2k+1)}$ has (extended) finite trace.

If $\phi = \xi * \xi$, then T is Hermitian nonnegative and of finite trace. Further T_ψ is also Hermitian nonnegative and dominated by CT. Hence $T_\psi = R_a^2$ with $a \in L^2(G)$, R_a bounded on $L^2(G)$ and $\check{a} = a$. In particular, $T_\psi = R_{\phi_\psi}$ with $\phi_\psi = a * a \in L^2(G)$. This function is not a probability density, in general. It is a probability density when ψ is a Bernstein function; see, for example, [4], Section 3.4, and [17], Section 3.9. $\square$

LEMMA A.4. *Let $\phi \in L^2(G)$ be a symmetric probability density such that $\lim_{n \to \infty} \phi^{(2n)}(e) = 0$. Let $\psi : [0, 2] \to [0, 2]$ be nonnegative increasing, continuous with continuous derivative and such that $\psi(0) = 0$, $\psi(1) = 1$, $\psi(2) < 2$. Then the operator $T_\psi \in V(G)$ defined at (A.5) is such that T_ψ^n has finite trace and, setting*

(A.6) $$N_\phi^\psi = N_\phi \circ \psi^{-1},$$

we have

(A.7) $$\tau(T_\psi^n) = \int_0^1 (1 - s)^n \, dN_\phi^\psi(s) + O(a^n), \qquad a = \psi(2) - 1 \in [0, 1).$$

REMARK A.5. The hypothesis $\psi(2) < 2$ insures that the contribution coming from the spectrum of $I - R_\phi$ that lies in the interval $(1, 2)$ is exponentially small. If R_ϕ is nonnegative [as a Hermitian operator on $L^2(G)$], the value of ψ in the interval $(1, 2)$ becomes completely irrelevant and

$$\tau(T_\psi^n) = \int_0^1 (1 - s)^n \, dN_\phi^\psi(s).$$

PROOF OF LEMMA A.4. Since $R_\phi^2 = \int_0^2 |1 - \psi(s)|^2 \, dE_s^{I-T}$ has finite trace equal to $\phi^{(2)}(e)$, the the nondecreasing functions $N_\phi(s) = \tau(E_s^{I-T})$ [see definition (A.2)] and $N_\phi^\sharp(s) = \tau(E_{(2-s,2)}^{I-T})$ are finite for all $s \in (0, 1)$. Further, since $\phi^{(2n)}(e) \to 0$, we have $N_\phi(0) = 0$ [i.e., there are no $L^2(G)$ solutions to $Tf = f$]. Hence,

$$\tau(T_\psi^n) = \int_0^1 (1 - \psi(s))^n \, dN_\phi(s) + \int_0^1 (1 - \psi(2 - s))^n \, dN_\phi^\sharp(s).$$

The second integral is bounded by

$$\left| \int_0^1 (1 - \psi(2 - s))^n \, dN_\phi^\sharp(s) \right| \leq \int_0^1 |1 - \psi(2 - s)|^2 \, dN_\phi^\sharp(s) |\psi(2) - 1|^{n-2}.$$

Since $T^2 = \int_0^2 |1 - \psi(s)|^2 \, dE_s^{I-T}$ has finite trace $\phi^{(2)}(e)$ and $|1 - \psi(2 - s)| \leq C|1 - s|$, we obtain that

$$\int_0^1 |1 - \psi(2 - s)|^2 \, dN_\phi^\sharp(s) \leq C\phi^{(2)}(e).$$

This yields the desired estimate since, by hypothesis, $|\psi(2) - 1| < 1$. □

To illustrate this lemma, we treat the following simple test case.

THEOREM A.6. *Let $\phi \in L^2(G)$ be a symmetric positive probability density such that*

$$\phi^{(2n)}(e) \simeq n^{-D/2} \qquad \text{at infinity.}$$

Let $\psi : [0, 2] \to [0, 2]$ be nonnegative increasing, continuous with continuous derivative and such that $\psi(0) = 0$, $\psi(1) = 1$, $\psi(2) < 2$. Assume further that $\psi(s) \simeq (s/\ell(1/s))^\alpha$ at 0, where $\alpha \in (0, \infty)$ and ℓ a positive function, slowly varying at infinity with de Bruijn conjugate $\ell^\#$. Then

$$\tau(T_\psi^n) \simeq [n^{1/\alpha} \ell^\#(n^{1/\alpha})]^{-D/2} \qquad \text{at infinity.}$$

PROOF. This follows easily from Proposition A.2 and Lemma A.4, together with [5], Proposition 1.5.15. □

Similar considerations, together with the arguments developed in [2], Lemma 2.3, Proposition 2.5, yield the following result which is most useful when dealing with super-polynomial behaviors.

THEOREM A.7. *Let $\phi \in L^2(G)$ be a symmetric positive probability density. Let $\pi : (0, \infty) \to (0, \infty)$ be such that π and $t \mapsto t/\pi(t)$ are continuous increasing functions which tend to infinity at infinity. Let $\psi : [0, 2] \to [0, 2]$ be nonnegative increasing, continuous with continuous derivative and such that $\psi(0) = 0$, $\psi(1) = 1$, $\psi(2) < 2$. Set*

$$(\text{A.8}) \qquad \pi_\psi^{-1}(t) = t\psi^{-1}(1/t)\pi^{-1}\big(1/\psi^{-1}(1/t)\big).$$

(1) *Assume that there exists $c_1 \in (0, \infty)$ such that, for n large enough,*

$$-\log \phi^{(2n)}(e) \geq c_1 n/\pi(n).$$

Then there exists $c_2 \in (0, \infty)$ such that, for n large enough,

$$-\log \tau(T_\psi^n) \geq c_2 n/\pi_\psi(c_2 n).$$

(2) *Assume that there exists $C_1 \in (0, \infty)$ such that, for n large enough,*

$$-\log \phi^{(2n)}(e) \leq C_1 n/\pi(n).$$

Then there exists $C_2 \in (0, \infty)$ such that, for n large enough,

$$-\log \tau(T_\psi^n) \leq C_2 n/\pi_\psi(n/C_2).$$

PROOF. Let us observe that for a bijection π, the two properties (a) π and $t \mapsto t/\pi(t)$ are increasing, and (b) $t \mapsto \pi^{-1}(t)/t$ is increasing, are equivalent. Further, given that ψ is positive increasing, property (a) for π implies (b) for π which implies (b) for π_ψ which finally implies (a) for π_ψ. The result now easily follows from Proposition A.2 and Lemma A.4. □

It is useful to illustrate Theorem A.7 with some concrete examples. Note that Theorem A.7 allows us to treat upper and lower bounds separately. For simplicity, we write down the examples in the context of the rough equivalence $\simeq$.

EXAMPLE A.2. Assume that $-\log \phi^{(2n)}(e) \simeq \log n$ and that $\psi(t) \simeq 1/\ell(1/t)$ where ℓ is an increasing slowly varying function tending to infinity at infinity and such that

$$\log \ell^{-1}(t) \simeq t^\gamma \omega(t)^{1+\gamma},$$

where $\gamma \in [0, \infty)$ and ω is a slowly varying function at infinity with de Bruijn conjugate $\omega^{\#}$. Then

$$-\log \tau(T_\psi^n) \simeq n^{\gamma/(1+\gamma)}/\omega^{\#}\big(n^{1/(1+\gamma)}\big).$$

EXAMPLE A.3. Assume that

$$-\log \phi^{(2n)}(e) \simeq n^\gamma, \qquad \gamma \in (0, 1),$$

and that

$$\psi(t) \simeq t^{\alpha}/\ell(1/t), \qquad \alpha \in [0, \infty),$$

where ℓ is an increasing slowly varying function at infinity such that, for every $a > 0$, $\ell(t^a) \simeq \ell(t)$. Then

$$-\log \tau(T_{\psi}^n) \simeq [n/\ell(n)]^{\gamma_{\alpha}}, \qquad \gamma_{\alpha} = \frac{\gamma}{\gamma + \alpha(1 - \gamma)}.$$

EXAMPLE A.4.   Assume that

$$-\log \phi^{(2n)}(e) \leq n/\pi(n)$$

with π positive increasing.

- Assume that $\pi(t) = t^{1-\gamma}\ell(t)$ with $\gamma \in (0, 1]$ and ℓ slowly varying and satisfying $\ell(t^a) \simeq \ell(t)$ for all $a > 0$. Then, for any $\alpha \in (0, 1)$ and $\psi(t) = t^{\alpha}$, we have

$$-\log \tau(T_{\psi}^n) \leq [n/\ell(n)^{\alpha/\gamma}]^{\gamma_{\alpha}}, \qquad \gamma_{\alpha} = \frac{\gamma}{\gamma + \alpha(1 - \gamma)}.$$

  The cases $\gamma = 1$ and $\gamma \in (0, 1)$ should be treated separately using slightly different arguments. See [4], Theorem 3.4, for a similar computation.
- Assume that π is regularly varying of index less than 1. Then for any positive increasing slowly varying ℓ, $\psi = 1/\ell(1/t)$, and any $\varepsilon \in (0, 1)$, we have (see [4], Theorem 3.4, for a similar computation)

$$-\log \tau(T_{\psi}^n) \leq C_{\varepsilon} n/\ell(\pi(C_{\varepsilon} n^{\varepsilon})).$$

A.5. Trace and comparison.   Let T_1, T_2 be self-adjoint contractions that belong to a von Neumann algebra V equipped with a faithful semifinite normal trace τ. For $i = 1, 2$, let $E_s^{I-T_i}$, $s \in [0, \infty)$, be the (left-continuous) spectral projectors of $I - T_i$, so that $T_i = \int_0^{\infty}(1 - s)\,dE_s^{I-T_i}$. The following result is crucial for our purpose. It is the von Neumann version of a classical finite-dimensional spectral comparison theorem. We set

$$N_i(s) = \tau(E_s^{I-Ti}), \qquad s > 0, i = 1, 2.$$

Note that it can well be the case that $N_i(s) = \infty$.

PROPOSITION A.8.   *Referring to the above setting and notation, let T_1, T_2 be self-adjoint contractions that belong to the von Neumann algebra V equipped with a faithful semifinite normal trace τ. Assume that*

$$(I - T_1) \leq C(I - T_2)$$

and that T_2 is nonnegative. Then we have

(A.9) $\forall s \in [0, 1) \qquad N_2(s) \leq N_1(Cs).$

PROOF. Recall that, for any bounded self-adjoint operator $S \in V$, $E^S_{(a,b)}$ denotes the spectral projector associated to S and the interval (a, b). By convention, the left-continuous spectral resolution of S is $E^S_s = E^S_{(-\infty,s)}$ so that $S = \int_{-\infty}^{+\infty} s \, dE^S_s$ and $E^S_{(a,b)} = \int_{(a,b)} dE^S_s$. According to [6], Lemma 3, if S_1, S_2 are nonnegative self-adjoint operators such that $S_2 \leq S_1$ then (allowing for the possibility that the traces in question are infinite)

(A.10) $$\forall s > 0 \qquad \tau\big(E^{S_2}_{(s,\infty)}\big) \leq \tau\big(E^{S_1}_{(s,\infty)}\big).$$

By hypothesis, we have $I - T_1 \leq C(I - T_2)$, which we write

$$T_2 \leq I - C^{-1}(I - T_1).$$

Applying (A.10) to $S_2 = T_2$, $S_1 = I - C^{-1}(I - T_1)$ (T_2 is nonnegative by hypothesis and this implies that S_2, S_1 are also nonnegative) and using the simple fact that

$$E^{I-C^{-1}(I-T_1)}_{(s,\infty)} = E^{T_1}_{(1-C(1-s),\infty)},$$

we obtain

$$\forall s > 0 \qquad \tau\big(E^{T_2}_{(s,\infty)}\big) \leq \tau\big(E^{T_1}_{(1-C(1-s),\infty)}\big).$$

Translating this inequality in terms of the spectral functions

$$N_i(s) = \tau\big(E^{I-T_i}_{(-\infty,s)}\big) = \tau\big(E^{T_i}_{(1-s,\infty)}\big),$$

we obtain

$$\forall s \in [0, 1) \qquad N_2(s) \leq N_1(Cs). \qquad \square$$

COROLLARY A.9. *Referring to the above setting and notation, assume that T_1, T_2 are nonnegative and that there exist an integer k_0 and a constant $C \geq 1$ such that*

$$\tau(T_1^{k_0}), \qquad \tau(T_2^{k_0}) < \infty \quad and \quad I - T_1 \leq C(I - T_2).$$

Then, for all $n \geq k_0$,

$$\tau(T_2^n) \leq 2C^2 \tau\big(T_1^{\lfloor n/2C \rfloor}\big) + 2e^{-(n/16C)+k_0/8}\big(\tau(T_2^{k_0}) + 2C^2 \tau(T_1^{k_0})\big).$$

PROOF. We have

$$\tau(T_i^n) = \int_0^1 (1-s)^n \, dN_i(s)$$

$$= n \int_0^\varepsilon (1-s)^{n-1} N_i(s) \, ds + (1-\varepsilon)^n N_i(\varepsilon)$$

$$+ \int_\varepsilon^1 (1-s)^n \, dN_i(s).$$

Since $(1-s)^{k_0} N_i(s) \leq \tau(T_i^{k_0})$ and $\int_0^1 (1-s)^{k_0} \, dN_i(s) = \tau(T_i^{k_0})$, we obtain that

$$\left| \tau(T_i^n) - n \int_0^\varepsilon (1-s)^{n-1} N_i(s) \, ds \right| \leq 2(1-\varepsilon)^{n-k_0} \tau(T_i^{k_0})$$

for any real $n \geq k_0$. Now, set $c = 1/8C$, and use Proposition A.8 and the elementary inequality $(1-s) \leq (1-Cs)^{1/2C}$, $s \in [0, c]$, to write

$$n \int_0^c (1-s)^{n-1} N_2(s) \, ds \leq n \int_0^c (1-Cs)^{(n-1)/2C} N_1(Cs) \, ds$$

$$\leq Cn \int_0^{1/8} (1-s)^{(n-1)/2C} N_1(s) \, ds$$

$$\leq 2C^2 (n/2C) \int_0^{1/8} (1-s)^{(n/2C)-1} N_1(s) \, ds.$$

It thus follows that

$$\tau(T_2^n) \leq 2C^2 \tau(T_1^{n/2C}) + 2\left(1 - \frac{1}{8C}\right)^{n-k_0} \tau(T_2^{k_0}) + 4C^2 \left(1 - \frac{1}{8}\right)^{(n/2C)-k_0} \tau(T_1^{k_0}).$$

This yields the desired result. □

In applications of Corollary A.9, one may want to relax the hypothesis that T_1, T_2 are nonnegative. This is possible thanks to the following result.

COROLLARY A.10. *Referring to the above setting and notation, assume that there exist an integer k_0 and a constant $C \geq 1$ such that*

$$\tau(T_1^{k_0}), \qquad \tau(T_2^{k_0}) < \infty \quad and \quad I - T_1 \leq C(I - T_2).$$

Assume further that $\tau(T_2^k) \geq 0$ for all $k \geq k_0$. Then there are constants C_1, C_2 depending only on upper bounds on C, $\tau(T_1^{k_0})$, $\tau(T_2^{k_0})$ and such that

$$\tau(T_2^{2n}) \leq C_1\big(\tau\big(T_1^{2\lfloor n/C_2 \rfloor}\big) + e^{-n/C_2}\big) \qquad for\ all\ n\ large\ enough.$$

PROOF. Set $S = \frac{1}{2}(T_2^2 + T_2^3) = \frac{1}{2}T_2^2(T_2 + I)$. This is a Hermitian nonnegative contraction. Further $\tau(S^{2n}) = 2^{-2n} \sum_0^{2n} \binom{2n}{i} \tau(T_2^{6n-i})$. Since $\ell \mapsto \tau(T_2^{2\ell})$ is decreasing (e.g., by spectral theory) and $\tau(T_2^{2\ell+1}) \geq 0$ (by hypothesis), we have

$$\tau(S^{2n}) \geq \frac{1}{2^{2n}} \sum_{k \in 2\mathbb{N} \cap [2n, 6n]} \binom{2n}{k} \tau(T_2^k) \geq \frac{1}{2}\tau(T_2^{6n}).$$

This shows that it suffices to estimate $\tau(S^{2n})$ by $\tau(T_1^{2\lfloor cn \rfloor})$ for some $c > 0$. This will follow from Corollary A.9 applied to the Hermitian nonnegative contractions

$T = T_1^2$, $S = \frac{1}{2}(T_2^3 + T_2^2)$, if we can prove that $I - T \le 4C(I - S)$. This last inequality follows immediately from the hypothesis $I - T_1 \le C(I - T_2)$ because $I - T = I - T_1^2 \le 2(I - T_1)$ and $I - T_2 \le 2(I - S)$. The last two inequalities follows from spectral theory and the elementary inequalities $1 - s^2 \le 2(1 - s)$ and $1 - s \le 2 - s^3 - s^2$, $s \in [-1, 1]$. $\square$

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