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THE LOCAL QUANTIZATION BEHAVIOR OF ABSOLUTELY CONTINUOUS PROBABILITIES

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For a large class of absolutely continuous probabilities P it is shown that, for $r > 0$, for n-optimal $L^r(P)$-codebooks α_n, and any Voronoi partition $V_{n,a}$ with respect to α_n the local probabilities $P(V_{n,a})$ satisfy $P(V_{a,n}) \approx n^{-1}$ while the local L^r-quantization errors satisfy $\int_{V_{n,a}} \|x-a\|^r \, dP(x) \approx n^{-(1+r/d)}$ as long as the partition sets $V_{n,a}$ intersect a fixed compact set K in the interior of the support of P.

1. Introduction. The theory of quantization of probability distributions has its origin in electrical engineering and image processing where it plays a decisive role in digitizing analog signals and compressing digital images (see Gray and Neuhoff [11]). More recently, it has also found many applications in numerical integration (see, e.g., [2, 3, 13, 14]) and mathematical finance (see, e.g., [15] for a survey).

Optimal (vector) quantization deals with the best approximation of an $\mathbb{R}^d$-valued random vector X with probability distribution P by $\mathbb{R}^d$-valued random vectors which attain only finitely many values. If $r > 0$ and $\int \|x\|^r \, dP < \infty$ and $n \in \mathbb{N}$, then the nth-level $L^r(P)$-quantization error is defined to be

$$\begin{aligned} e_{n,r} &= e_{n,r}(P) \\ &= \inf\biggl\{\biggl(\int \|x-q(x)\|^r \, dP(x)\biggr)^{1/r} \Bigm| q:\mathbb{R}^d \to \mathbb{R}^d \text{ Borel measurable} \\ &\qquad\qquad \text{with } \operatorname{card}(q(\mathbb{R}^d)) \le n\biggr\}, \end{aligned} \tag{1.1}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^d$ and $\operatorname{card}(A)$ stands for the cardinalility of A.

It is known that the above infimum remains unchanged if the Borel functions $q:\mathbb{R}^d \to \mathbb{R}^d$ are chosen to be projections onto their range $\alpha := q(\mathbb{R}^d) \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha) \le n$ which obey a nearest neighbor rule, that is,

$$q(x) = \sum_{a\in\alpha} a 1_{V_{n,a}}(x),$$

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where $(V_{n,a})_{a\in\alpha}$ is a Voronoi partition of $\mathbb{R}^d$ with respect to α, that is, a Borel partition such that each of the partition sets $V_{n,a}$ is contained in the Voronoi cell $W(a|\alpha_n) := \{x \in \mathbb{R}^d \,|\, \|x-a\| = \min_{b\in\alpha} \|x-b\|\}$.

If $d(x,\alpha) := \min_{a\in\alpha} \|x-a\|$ denotes the distance of x to the set α, then

$$e_{n,r} = \inf\left\{\left(\int d(x,\alpha)^r \, dP(x)\right)^{1/r} \Big| \alpha \subset \mathbb{R}^d \text{ and } \operatorname{card}(\alpha) \le n\right\}.$$

The above infimum is in fact a minimum which is attained at an optimal "code-book" α_n (see [8], Theorem 4.12). If P is absolutely continuous with density $h \ge 0$ and $\int \|x\|^{r+\delta} \, dP(x) < \infty$ for some $\delta > 0$, then

$$\lim_{n\to\infty} n^{1/d} e_{n,r}(P) = Q_r(P)^{1/r} \tag{1.2}$$

for a positive real constant $Q_r(P)$ (see Zador [17, 18], Bucklew and Wise [1] and Graf and Luschgy [8], Theorem 6.2). Thus, the sharp asympotics of the sequence $(e^r_{n,r})_{n\in\mathbb{N}}$ is completely elucidated up to the numerical value of the constant $Q_r(P)$.

A famous conjecture of Gersho [7] states that the bounded Voronoi-cells of L^r-optimal codebooks α_n have asymptotically the same L^r-inertia and a normalized shape close to that of a fixed polyhedron H as n tends to infinity.

In particular, this conjecture suggests that the local L^r-quantization errors ($=L^r$-local inertia) satisfy

$$\int_{V_{n,a}} \|x-a\|^r \, dP(x) \sim \frac{1}{n} e^r_{n,r}, \qquad a \in \alpha_n, \tag{1.3}$$

where $a_n \sim b_n$ abbreviates $a_n = \varepsilon_n b_n$ with $\lim_{n\to\infty} \varepsilon_n = 1$.

So far, this last statement has only been proved for certain parametric classes of one-dimensional distributions P (see Fort and Pagès [6]).

In the present paper, we will investigate the asymptotic behavior for $n \to \infty$ of $P(W(a|\alpha_n))$ and $\int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x)$ for a large class of distributions on $\mathbb{R}^d$ including the nonsingular normal distributions. To derive a conjecture for the asymptotic size of $P(W(a|\alpha_n))$, one can use the following heuristics. The empirical measure theorem (see [8], Theorem 7.5) states that the empirical probabilities $\frac{1}{n}\sum_{a\in\alpha_n} \delta_a$ weakly converge as $n \to \infty$ to the "point density measure"

$$P_r = \frac{1}{\int h^{d/(r+d)} \, d\lambda^d} h^{d/(r+d)} \lambda^d,$$

where λ^d denotes the d-dimensional Lebesgue measure. Thus we obtain, at least for bounded continuous densities h and an arbitrary bounded continuous function

$f:\mathbb{R}^d \to \mathbb{R}$, that

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{a\in\alpha_n} \frac{1}{n}\Big(\int h^{d/(r+d)}\, d\lambda^d\Big) h^{r/(r+d)}(a) \int f\, d\delta_a \\
&\quad = \int h^{d/(r+d)}\, d\lambda^d \lim_{n\to\infty}\Big(\frac{1}{n}\sum_{a\in\alpha_n} h^{r/(r+d)}(a) f(a)\Big) \\
&\quad = \int h^{d/(r+d)}\, d\lambda^d \int h^{r/(r+d)}(x) f(x)\, dP_r(x) \\
&\quad = \int f(x)\, dP(x),
\end{aligned} \tag{1.4}
$$

so that

$$
\sum_{a\in\alpha_n}\Big(\frac{1}{n}\int h^{d/(r+d)}\, d\lambda^d\Big) h^{r/(r+d)}(a)\delta_a \overset{(\mathbb{R}^d)}{\Longrightarrow} P,
$$

where $\overset{(\mathbb{R}^d)}{\Longrightarrow}$ denotes the weak convergence of finite measures on $\mathbb{R}^d$. Since it is well known that $\sum_{a\in\alpha_n} P(V_{n,a})\delta_a \overset{(\mathbb{R}^d)}{\Longrightarrow} P$ as well (see [13, 14] but also [2, 3] or [8], Equation (7.6)), it is reasonable to conjecture that

$$
P(V_{n,a}) \sim \frac{1}{n}\Big(\int h^{d/(r+d)}\, d\lambda^d\Big) h^{r/(r+d)}(a). \tag{1.5}
$$

We were not able to prove this asymptotical behavior of $P(V_{n,a})$ in its sharp and general form. But we will show that, for a large class of absolutely continuous distributions P, there are real constants $c_1, c_2, c_3, c_4 > 0$ only depending on P such that $\forall K \subseteq \mathbb{R}^d$, compact, $\exists n_K \in \mathbb{N}, \forall n \geq n_K, \forall a \in \alpha_n$

$$
\begin{aligned}
&K \cap W(a|\alpha_n) \neq \varnothing \\
&\quad\Longrightarrow \quad \frac{c_1}{n}\big(\operatorname{essinf} h_{|W_0(a|\alpha_n)}\big)^{r/(r+d)} \\
&\qquad\qquad \leq P(V_{n,a}) \leq \frac{c_2}{n}\big(\operatorname{esssup} h_{|W(a|\alpha_n)}\big)^{r/(r+d)},
\end{aligned} \tag{1.6}
$$

where

$$
W_0(a|\alpha_n) = \big\{x \in \mathbb{R}^d \,|\, \|x-a\| < d(x, \alpha_n \setminus \{a\})\big\} \tag{1.7}
$$

and

$$
\frac{c_3}{n} e^r_{n,r} \leq \int_{V_{n,a}} \|x-a\|^r\, dP(x) \leq \frac{c_4}{n} e^r_{n,r}. \tag{1.8}
$$

The proofs mainly rely on the following two ingredients:

- A "differentiated Zador's theorem"

$$
e^r_{n,r} - e^r_{n+1,r} \approx n^{-(1+r/d)} \tag{1.9}
$$

[where $a_n \approx b_n$ means that the sequence $(\frac{a_n}{b_n})$ is bounded and bounded away from 0] and

• Two *micro–macro inequalities* which relate the *pointwise distance* of a quantizer to the global *mean quantization error* induced on a distribution P by this quantizer:

For $b \in (0, \frac{1}{2})$ fixed, there is a constant $c_5 > 0$ with

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d \qquad c_5(e^r_{n,r} - e^r_{n+1,r}) \geq d(x, \alpha_n)^r P(B(x, bd(x, \alpha_n))) \tag{1.10}$$

and

$$\forall n \geq 2 \qquad e^r_{n-1,r} - e^r_{n,r} \leq \int_{V_{n,a}} \big(d(x, \alpha_n \setminus \{a\})^r - \|x - a\|^r\big)\, dP(x). \tag{1.11}$$

We have stated and established these inequalities in earlier papers: see especially [10]; for a preliminary version of (1.11), see [9] and for a one-sided first version of (1.9), see Lemma 3.2 in [16]. They were somewhat hidden as technical tools inside proofs but their full impact will become clear here.

The remaining part of the Introduction contains a sketch of the contents of the paper. In Section 2, we indicate the proofs of the above micro–macro inequalities and the (weak) asymptotics of quantization error differences. In Section 3, we show that absolutely continuous probabilities P on $\mathbb{R}^d$, which have a peakless, connected and compact support as well as a density which is bounded and bounded away from 0 on the support, have asymptotically uniform local quantization errors (Theorem 3.1). In Section 4, we show that absolutely continuous probabilities whose densities are the composition of a decreasing function on $\mathbb{R}_+$ and a norm or a quasi-concave function outside a compact set satisfy a sharpened first micro–macro inequality of the following type:

There exist a constant $c > 0$ such that, for every $K \subset \mathbb{R}^d$ compact,

$$\exists n_K \in \mathbb{N}, \forall n \geq n_K, \forall x \in K \qquad cn^{-1/d} h(x)^{-1/(r+d)} \geq d(x, \alpha_n).$$

Assuming this inequality, we derive asymptotic estimates for the probabilities of the quantization cells and local quantization errors (Theorem 4.1). Section 5 deals with the local quantization behavior of certain Borel probabilities P in the interior of their support. The results are stated for arbitrary absolutely continuous probabilities with density h satisfying the moment condition $\int \|x\|^{r+\delta} h(x)\, d\lambda(x) < +\infty$ for some $\delta > 0$. They are particularly useful if the density h is bounded and bounded away from 0 on each compact subset of the interior of the support of P. Under these very general assumptions, the results are quite similar to those given in Section 4 but the given constants are a little bit less effective (Theorem 5.1).

ADDITIONAL NOTATION. For $x \in \mathbb{R}^d$ and $\rho > 0$ $B(x, \rho) = B_{\|\cdot\|}(x, \rho) = \{y \in \mathbb{R}^d \mid \|y - x\| < \rho\}$ denotes the open ball with center x and radius ρ. $\|\cdot\|_2$ will denote the canonical Euclidean norm on $\mathbb{R}^d$.

$\mathring{A}$ denotes the interior of a set $A \subset \mathbb{R}^d$.

2. Important inequalities in quantization. In the following, $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^d$ and P is always an absolutely continuous Borel probability on $\mathbb{R}^d$ which has density h with respect to the d-dimensional Lebesgue measure λ^d. Let $r \in (0, +\infty)$ be fixed. We always assume that there is a $\delta > 0$ with $\int \|x\|^{r+\delta}\, dP(x) < +\infty$. For every $n \in \mathbb{N}$, let $e_{n,r}$ denote the nth-level $L^r(P)$-quantization error. Then we have

$$e_{n,r}^r = e_{n,r}^r(P) = \inf\left\{\int d(x,\alpha)^r\, dP(x) \,\middle|\, \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \le n\right\}. \tag{2.1}$$

For each $n \in \mathbb{N}$, we choose an arbitrary n-optimal set $\alpha_n \subset \mathbb{R}^d$, that is, a set $\alpha_n \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha_n) \le n$ and

$$e_{n,r}^r = \int d(x,\alpha_n)^r\, dP(x). \tag{2.2}$$

It is well known that, under the above conditions, such a set exists and satisfies

$$\operatorname{card}(\alpha_n) = n. \tag{2.3}$$

In this section, we will state the fundamental inequalities which relate the behavior of the distance function $d(\cdot, \alpha_n)$ to the difference $e_{n,r}^r - e_{n+1,r}^r$ of successive rth powers of the quantization errors. Using these inequalities, we will be able to determine the (weak) asymptotics of $e_{n,r}^r - e_{n+1,r}^r$.

2.1. *Micro–macro inequalities.*

PROPOSITION 2.1 (First micro–macro inequality). *For every $b \in (0, \frac{1}{2})$, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}^d$,*

$$e_{n,r}^r - e_{n+1,r}^r \ge (2^{-r} - b^r) d(x,\alpha_n)^r P(B(x, bd(x,\alpha_n))). \tag{2.4}$$

PROOF. The proof can be found as part of the proof of Theorem 2 in [10]. $\square$

REMARKS. (a) Inequality (2.4) holds for arbitrary Borel probabilities P on $\mathbb{R}^d$ for which $\int \|x\|^r\, dP(x) < \infty$. P need not be absolutely continuous.

(b) By the differentiation theorem for absolutely continuous measures $P = h\lambda^d$ and the fact (see [5]) that $\lim_{n\to\infty} d(x,\alpha_n) = 0$ for every $x \in \operatorname{supp}(P)$, we know that, for λ^d-a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{P(B(x, bd(x,\alpha_n)))}{\lambda^d(B(x, bd(x,\alpha_n)))} = h(x). \tag{2.5}$$

Having this in mind, we can rephrase (2.4) as follows:

$$\begin{gathered} \forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d \\ c_5(e_{n,r}^r - e_{n+1,r}^r) \ge d(x,\alpha_n)^{r+d} \frac{P(B(x, bd(x,\alpha_n)))}{\lambda^d(B(x, bd(x,\alpha_n)))}, \end{gathered} \tag{2.6}$$

where

$$c_5 = [(2^{-r} - b^r)b^d \lambda^d(B(0,1))]^{-1}$$

(with the convention $0 \cdot \text{undefined} = 0$).

Suppose that there is a constant $c_9 > 0$ such that

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall x \in \mathbb{R}^d \qquad \frac{P(B(x, bd(x, \alpha_n)))}{\lambda^d(B(x, bd(x, \alpha_n)))} \geq c_9 h(x). \tag{2.7}$$

Then, for $c_{10} = c_5 c_9^{-1}$, we have

$$\forall n \geq n_0, \forall x \in \mathbb{R}^d \qquad c_{10}(e_{n,r}^r - e_{n+1,r}^r) \geq d(x, \alpha_n)^{r+d} h(x). \tag{2.8}$$

PROPOSITION 2.2 (Second micro–macro inequality). *One has*

$$\begin{aligned} &\forall n \geq 2, \forall a \in \alpha_n \\ &e_{n-1,r}^r - e_{n,r}^r \leq \int_{W_0(a|\alpha_n)} \big(d(x, \alpha_n \setminus \{a\})^r - \|x - a\|^r\big)\, dP(x), \end{aligned} \tag{2.9}$$

where $W_0(a|\alpha_n)$ *is defined by* (1.7).

PROOF. The proof is part of the proof of [10], Theorem 2. □

REMARK. Inequality (2.9) holds for arbitrary Borel probabilities P on $\mathbb{R}^d$ with $\int \|x\|^r \, dP(x) < +\infty$.

2.2. *A differentiated version of Zador's theorem.* To use the preceding propositions for concrete calculations, it is essential to know the asymptotic behavior of the error differences $e_{n,r}^r - e_{n+1,r}^r$. We have the following result in that direction.

PROPOSITION 2.3. *If P is absolutely continuous on $\mathbb{R}^d$, then*

$$e_{n,r}^r - e_{n+1,r}^r \approx n^{-(1+r/d)}.$$

PROOF. In the proof of Theorem 2 in [10], it is shown that there is a constant $c_{11} > 0$ such that

$$\forall n \in \mathbb{N} \qquad e_{n,r}^r - e_{n+1,r}^r \leq c_{11} n^{-(1+r/d)}.$$

To obtain the lower bound for $e_{n,r}^r - e_{n+1,r}^r$, we proceed as follows.

It follows from (2.5) and Egorov's theorem (see [4], Proposition 3.1.3) that there exists a real constant $c > 0$ and a Borel set $A \subset \{h > c\}$ of finite and positive Lebesgue measure such that

$$\text{the convergence of } \frac{P(B(x, bd(x, \alpha_n)))}{\lambda^d(B(x, bd(x, \alpha_n)))} \text{ to } h \qquad \text{is uniform in } x \in A.$$

Hence, there exists an $n_0 \in \mathbb{N}$ with

$$\forall n \geq n_0, \forall x \in A \qquad \frac{P(B(x, bd(x, \alpha_n)))}{\lambda^d(B(x, bd(x, \alpha_n)))} > \frac{1}{2}c. \tag{2.10}$$

Combining (2.6) and (2.10) and integrating both sides of the resulting inequality with respect to the Lebesgue measure on A yields

$$\begin{aligned} c_5(e_{n,r}^r - e_{n+1,r}^r) &\geq \frac{1}{\lambda^d(A)} \frac{1}{2} c \int_A d(x, \alpha_n)^{r+d} \, d\lambda^d(x) \\ &\geq \frac{1}{2} c e_{n,r+d}^{r+d}(\lambda^d(\cdot|A)), \end{aligned}$$

where $\lambda^d(\cdot|A)$ denotes the normalized Lebesgue measure on A. By Zador's theorem (see (1.2) or [8], Theorem 6.2), we have

$$\liminf_{n\to\infty} n^{1+r/d} e_{n,r+d}^{r+d}(\lambda^d(\cdot|A)) > 0,$$

so that $\liminf_{n\to\infty} n^{1+r/d}(e_{n,r}^r - e_{n+1,r}^r) > 0$. $\square$

REMARK. It would be interesting to know the sharp asymptotic behavior of $e_{n,r}^r - e_{n+1,r}^r$. We conjecture that

$$\lim_{n\to\infty} n^{1+r/d}(e_{n,r}^r - e_{n+1,r}^r) = \frac{d}{r} Q_r(P) = \frac{d}{r} Q_r([0,1]^d) \|h\|_{d/(d+r)},$$

where $Q_r([0,1]^d) \in (0, \infty)$ is as in [8], Theorem 6.2.

3. Uniform local quantization rate for absolutely continuous distributions with peakless connected compact support. As before, P is an absolutely continuous probability with density h. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of optimal codebooks of order $r \in (0, \infty)$ for P. We will investigate the asymptotic size of

$$W(a|\alpha_n), \qquad P(W(a|\alpha_n)) \quad \text{and} \quad \int_{W(a|\alpha_n)} \|x - a\|^r \, dP(x)$$

under some compactness and regularity assumptions on supp(P) and P.

The main result of this section is stated below. Its proof, which heavily relies on the following two paragraphs devoted to upper and lower bounds, respectively, is postponed to the end of this section.

THEOREM 3.1. *Suppose that P is an absolutely continuous Borel probability on $\mathbb{R}^d$ whose density is essentially bounded, whose support is connected and compact, and which is "peakless" in the following sense*:

$$\exists c > 0, \exists s_0 > 0, \forall s \in (0, s_0), \forall x \in \mathrm{supp}(P) \qquad P(B(x,s)) \geq c\lambda^d(B(x,s)).$$

Let (α_n) *be a sequence of codebooks which are optimal of order* $r \in (0, \infty)$. *For* $a \in \alpha_n$, *let us define the inradius and the circumradius of the Voronoi cell* $W(a|\alpha_n)$ *by*

$$\underline{s}_{n,a} = \sup\{s > 0, B(a,s) \subset W(a|\alpha_n)\}$$

and

$$\overline{s}_{n,a} = \inf\{s > 0, W(a|\alpha_n) \cap \operatorname{supp}(P) \subset B(a,s)\},$$

respectively. Then

$$\frac{1}{n} \preccurlyeq \min_{a \in \alpha_n} P(W_0(a|\alpha_n)) \le \max_{a \in \alpha_n} P(W(a|\alpha_n)) \preccurlyeq \frac{1}{n}, \tag{3.1}$$

$$\begin{aligned}\frac{e_{n,r}^r}{n} &\preccurlyeq \min_{a \in \alpha_n} \int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) \\ &\le \max_{a \in \alpha_n} \int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \preccurlyeq \frac{e_{n,r}^r}{n}\end{aligned} \tag{3.2}$$

and

$$n^{-1/d} \preccurlyeq \min_{a \in \alpha_n} \underline{s}_{n,a} \le \max_{a \in \alpha_n} \overline{s}_{n,a} \preccurlyeq n^{-1/d}. \tag{3.3}$$

[*Here* $a_n \preccurlyeq b_n$ *means that* $(\frac{a_n}{b_n})$ *is bounded from above*.]

REMARKS. The name "peakless" given to the above assumption illustrates that a subset of $\mathbb{R}^d$ that satisfies this condition cannot have infinitely thin peaks (or spines) on its boundary and that the existence of such peaks or spine is the only way to make the assumption fail.

Inequality (3.3) was proved by Gruber in [12], Theorem 3(ii), under an additional continuity assumption on h, but with a more general distortion measure.

3.1. *Upper bounds*. The following proposition is essentially contained in Graf and Luschgy [9] (Proposition 3.3 and the following remark). It has been independently proved by Gruber [12], Theorem 3(ii).

PROPOSITION 3.1. *Suppose that* $\operatorname{supp}(P)$ *is compact and that there exist constants* $c_{12} > 0$ *and* $s_0 > 0$ *such that*

$$\forall s \in (0, s_0), \forall x \in \operatorname{supp}(P) \qquad P(B(x,s)) \ge c_{12} \lambda^d (B(x,s)). \tag{3.4}$$

Then there is a constant $c_{13} < +\infty$ *such that*

$$\forall n \in \mathbb{N}, \forall x \in \operatorname{supp}(P) \qquad d(x, \alpha_n) \le c_{13} n^{-1/d}. \tag{3.5}$$

PROOF. Let $b \in (0, \frac{1}{2})$ be fixed. Since $K := \mathrm{supp}(P)$ is compact it follows from [5], Proposition 1, that $\lim_{n\to\infty} \max_{x\in K} d(x, \alpha_n) = 0$. Thus, there is an $n_0 \in \mathbb{N}$ with

$$\forall n \geq n_0, \forall x \in K \qquad d(x, \alpha_n) < s_0$$

and, hence, by (3.4)

$$\forall n \geq n_0, \forall x \in K \qquad P(B(x, bd(x, \alpha_n))) \geq c_{12}\lambda^d(B(x, bd(x, \alpha_n))). \tag{3.6}$$

By Proposition 2.3, there exists a constant $c_{11} > 0$ such that

$$\forall n \in \mathbb{N} \qquad e^r_{n,r} - e^r_{n+1,r} \leq c_{11} n^{-(1+r/d)}. \tag{3.7}$$

Combining (2.6), (3.6) and (3.7), yields

$$c_{12}^{-1} c_{11} c_5 n^{-(1+r/d)} \geq d(x, \alpha_n)^{r+d}$$

for every $x \in K$ and every $n \geq n_0$. Inequality (3.5) follows by setting

$$c_{13} = \max\{(c_{12}^{-1} c_{11} c_5)^{1/(r+d)}, \max\{d(x, \alpha_n) n^{1/d}, x \in K, n \in \{1, \dots, n_0\}\}\}. \qquad \square$$

PROPOSITION 3.2 (Upper-bounds). *Suppose that the assumptions of Proposition 3.1 are satisfied and that, in addition, h is essentially bounded. Then there exist constants $c_{14}, c_{15} \in (0, \infty)$ such that*

$$\forall n \in \mathbb{N}, \forall a \in \alpha_n \qquad \begin{cases} P(W(a|\alpha_n)) \leq \dfrac{c_{14}}{n}, \\ \displaystyle\int_{W(a|\alpha_n)} \|x - a\|^r \, dP(x) \leq c_{15} n^{-(1+r/d)}. \end{cases} \tag{3.8}$$

PROOF. By Proposition 3.1, we have, for every $n \in \mathbb{N}$ and every $a \in \alpha_n$,

$$W(a|\alpha_n) \cap \mathrm{supp}(P) = \{x \in \mathrm{supp}(P) \,|\, \|x - a\| = d(x, \alpha_n)\} \subseteq B(a, c_{13} n^{-1/d}),$$

which implies

$$\begin{aligned} P(W(a|\alpha_n)) \leq P(B(a, c_{13} n^{-1/d})) &= \int_{B(a, c_{13} n^{-1/d})} h \, d\lambda^d \\ &\leq \|h\|_{\mathbb{R}^d} \lambda^d(B(0,1)) c_{13}^d \frac{1}{n}, \end{aligned}$$

where $\|h\|_B = \operatorname{esssup} h_{|B}$. Likewise, we obtain

$$\begin{aligned} \int_{W(a|\alpha_n)} \|x - a\|^r \, dP(x) &\leq \int_{B(a, c_{13} n^{-1/d})} \|x - a\|^r \, dP(x) \\ &\leq (c_{13} n^{-1/d})^r P(B(a, c_{13} n^{-1/d})). \end{aligned}$$

Setting $c_{14} = \|h\|_{\mathbb{R}^d} \lambda^d(B(0,1)) c_{13}^d$ and $c_{15} = c_{14} c_{13}^r$ yields (3.8). $\square$

REMARK. Thus, assumption (3.4) is satisfied if $\operatorname{supp}(P)$ is peakless, that is,

$$\begin{aligned}&\exists c > 0, \exists s_1 > 0, \forall s \in (0, s_1), \forall x \in \operatorname{supp}(P)\\ &\lambda^d\big(B(x,s) \cap \operatorname{supp}(P)\big) \geq c\lambda^d(B(x,s)),\end{aligned} \tag{3.9}$$

and h is essentially bounded away from 0 on $\operatorname{supp}(P)$, that is,

$$\exists \underline{t} > 0, h(x) \geq \underline{t} \qquad \text{for } \lambda^d\text{-a.e. } x \in \operatorname{supp}(P).$$

As an example, (3.9) holds for finite unions of compact convex sets with positive λ^d-measure (see [8], Example 12.7 and Lemma 12.4).

3.2. *Lower bounds.*

LEMMA 3.1. *If* $\operatorname{supp}(P)$ *is connected then, for every* $n \geq 2$ *and every* $a \in \alpha_n$,

$$d(a, \alpha_n \setminus \{a\}) \leq 2\sup\big(\{\|y - a\|, y \in W(a|\alpha_n) \cap \operatorname{supp}(P)\}\big). \tag{3.10}$$

PROOF. Let $n \geq 2$ be fixed. First, we will show that

$$\forall a \in \alpha_n \qquad W(a|\alpha_n) \cap \bigcup_{b \in \alpha_n \setminus \{a\}} W(b|\alpha_n) \cap \operatorname{supp}(P) \neq \varnothing. \tag{3.11}$$

Let $a \in \alpha_n$. Since the nonempty closed sets (see [8], Theorem 4.1) $W(a|\alpha_n) \cap \operatorname{supp}(P)$ and $\bigcup_{b \in \alpha_n \setminus \{a\}} W(b|\alpha_n) \cap \operatorname{supp}(P)$ cover the connected set $\operatorname{supp}(P)$, claim (3.11) follows.

By (3.11), there exists $b \in \alpha_n \setminus \{a\}$ with $W(a|\alpha_n) \cap W(b|\alpha_n) \cap \operatorname{supp}(P) \neq \varnothing$. Let z be a point in this set. Then $\|z - a\| = d(z, \alpha_n) = \|z - b\|$ and

$$\begin{aligned}d(a, \alpha_n \setminus \{a\}) &\leq \|a - b\| \leq \|a - z\| + \|z - b\|\\ &\leq 2\|z - a\| \leq 2\sup\{\|y - a\|, y \in W(a|\alpha_n) \cap \operatorname{supp}(P)\}. \qquad \square\end{aligned}$$

PROPOSITION 3.3 (Lower bounds I). *Suppose that* $\operatorname{supp}(P)$ *is compact and connected, that P satisfies* (3.4) *and is absolutely continuous with an essentially bounded probability density h.*

Then there exist constants $c_{16}, c_{17} > 0$ *such that*

$$\forall n \geq 2, \forall a \in \alpha_n \qquad d(a, \alpha_n \setminus \{a\}) \geq c_{16} n^{-1/d} \tag{3.12}$$

and

$$\forall n \in \mathbb{N}, \forall a \in \alpha_n \qquad P(W_0(a|\alpha_n)) \geq \frac{c_{17}}{n}. \tag{3.13}$$

PROOF. Let $n \geq 2$ and $a \in \alpha_n$ be arbitrary. By the second micro–macro inequality (2.9), we have

$$\begin{aligned} & e^r_{n-1,r} - e^r_{n,r} \\ & \leq \int_{W_0(a|\alpha_n)} \bigl(d(x, \alpha_n \setminus \{a\})^r - \|x-a\|^r\bigr)\, dP(x) \\ & \leq \int_{W_0(a|\alpha_n)} \bigl((\|x-a\| + d(a, \alpha_n \setminus \{a\}))^r - \|x-a\|^r\bigr)\, dP(x). \end{aligned} \tag{3.14}$$

By Proposition 2.3, there exists a real constant $c > 0$ with

$$cn^{-(1+r/d)} \leq e^r_{n-1,r} - e^r_{n,r}. \tag{3.15}$$

CASE 1 ($r \geq 1$). Combining (3.14) and (3.15) and using the mean value theorem for differentiation yields

$$\begin{aligned} cn^{-(1+r/d)} \leq \int_{W_0(a|\alpha_n)} & r\bigl(\|x-a\| + d(a, \alpha_n \setminus \{a\})\bigr)^{r-1} \\ & \times d(a, \alpha_n \setminus \{a\})\, dP(x). \end{aligned} \tag{3.16}$$

Using Lemma 3.1 and (3.5), we know that

$$\forall x \in W(a|\alpha_n) \cap \operatorname{supp}(P) \qquad \|x-a\| + d(a, \alpha_n \setminus \{a\}) \leq 3c_{13} n^{-1/d}. \tag{3.17}$$

Combining (3.16) and (3.17) yields

$$r^{-1} c (3c_{13})^{-(r-1)} n^{-1-1/d} \leq d(a, \alpha_n \setminus \{a\}) P(W_0(a|\alpha_n)). \tag{3.18}$$

Since $P(W_0(a|\alpha_n)) \leq P(W(a|\alpha_n)) \leq c_{14} n^{-1}$ by (3.8), we deduce

$$c_{14}^{-1} r^{-1} c (3c_{13})^{-(r-1)} n^{-1/d} \leq d(a, \alpha_n \setminus \{a\})$$

and, hence, (3.12) with $c_{16} = c_{14}^{-1} r^{-1} c (3c_{13})^{-(r-1)}$.

Since $d(a, \alpha_n \setminus \{a\}) \leq 2c_{13} n^{-1/d}$, we deduce from (3.18) that

$$(2c_{13})^{-1} r^{-1} c (3c_{13})^{-(r-1)} n^{-1} \leq P(W_0(a|\alpha_n))$$

and, hence, (3.13) with $c_{17} = (2c_{13})^{-1} r^{-1} c (3c_{13})^{-(r-1)}$.

CASE 2 ($r < 1$). In this case, we have

$$\bigl(\|x-a\| + d(a, \alpha_n \setminus \{a\})\bigr)^r \leq \|x-a\|^r + d(a, \alpha_n \setminus \{a\})^r$$

for all $x \in W_0(a|\alpha_n)$. Combining this inequality with (3.14) and (3.15) yields

$$cn^{-(1+r/d)} \leq d(a, \alpha_n \setminus \{a\})^r P(W_0(a|\alpha_n)).$$

Since $P(W_0(a|\alpha_n)) \leq c_{14}/n$ by (3.8), we deduce

$$(c_{14}^{-1} c)^{1/r} n^{-1/d} \leq d(a, \alpha_n \setminus \{a\})$$

and hence, (3.12) with $c_{16} = (c_{14}^{-1}c)^{1/r}$.

Since $d(a, \alpha_n \setminus \{a\})^r \le (3c_{13})^r n^{-r/d}$, we obtain

$$(3c_{13})^{-r} cn^{-1} \le P(W_0(a|\alpha_n))$$

and, hence, (3.13) with $c_{17} = (3c_{13})^{-r} c$. □

COROLLARY 3.1. *Let the assumptions of Proposition 3.3 be satisfied. Then there exists a constant $c_{18} > 0$ such that*

$$\forall n \in \mathbb{N}, \forall a \in \alpha_n \qquad B(a, c_{18} n^{-1/d}) \subset W_0(a|\alpha_n). \tag{3.19}$$

PROOF. Set $c_{18} = \frac{1}{2} c_{16}$. For $n = 1$ and $a \in \alpha_n$, the assertion is obviously true since $W_0(a|\alpha_1) = \mathbb{R}^d$. Now let $n \ge 2$ and let $a \in \alpha_n$ be arbitrary. We will show that

$$B(a, c_{18} n^{-1/d}) \subset W_0(a|\alpha_n).$$

Let $x \in \mathbb{R}^d$ with $\|x - a\| < c_{18} n^{-1/d}$. By (3.12), we know that

$$\|x - a\| < \tfrac{1}{2} d(a, \alpha_n \setminus \{a\})$$

and, hence, for every $b \in \alpha_n \setminus \{a\}$:

$$\begin{aligned}\|x - b\| &\ge \|a - b\| - \|x - a\| \\ &\ge d(a, \alpha_n \setminus \{a\}) - \|x - a\| > \tfrac{1}{2} d(a, \alpha_n \setminus \{a\}) \\ &> \|x - a\|.\end{aligned}$$

This implies $x \in W_0(a|\alpha_n)$. □

PROPOSITION 3.4 (Lower bounds II). *Let the assumptions of Proposition 3.3 be satisfied. Then there exists a real constant $c_{19} > 0$ such that*

$$\forall n \in \mathbb{N}, \forall a \in \alpha_n \qquad \int_{W_0(a|\alpha_n)} \|x - a\|^r \, dP(x) \ge c_{19} n^{-(1+r/d)}. \tag{3.20}$$

PROOF. Let $n \in \mathbb{N}$ and $a \in \alpha_n$ be arbitrary. By (3.13), we have $P(W_0(a|\alpha_n)) > 0$. Let $s_a = \inf\{s > 0 | P(B(a, s)) \ge \frac{1}{2} P(W_0(a|\alpha_n))\}$. Since $s \mapsto P(B(a, s))$ is continuous with $\lim_{s \downarrow 0} P(B(a, s)) = 0$ and $\lim_{s \uparrow +\infty} P(B(a, s)) = 1$, we deduce

$$P(B(a, s_a)) = \tfrac{1}{2} P(W_0(a|\alpha_n)). \tag{3.21}$$

This implies

$$\begin{aligned}\int_{W_0(a|\alpha_n)} \|x - a\|^r \, dP(x) &\ge \int_{W_0(a|\alpha_n) \setminus B(a, s_a)} \|x - a\|^r \, dP(x) \\ &\ge s_a^r P\big(W_0(a|\alpha_n) \setminus B(a, s_a)\big) \\ &\ge s_a^r \big(P(W_0(a|\alpha_n)) - P(B(a, s_a))\big) \\ &= \frac{1}{2} s_a^r P(W_0(a|\alpha_n)).\end{aligned} \tag{3.22}$$

On the other hand, since h is essentially bounded we have

$$\begin{aligned} P(W_0(a|\alpha_n)) &= 2P(B(a, s_a)) \\ &\le 2\lambda^d(B(a, s_a))\|h\|_{\mathbb{R}^d} \\ &= 2\lambda^d(B(0,1))s_a^d\|h\|_{\mathbb{R}^d}. \end{aligned}$$

Hence,

$$s_a^r \ge \left(\frac{1}{2\lambda^d(B(0,1))\|h\|_{\mathbb{R}^d}}\right)^{r/d} P(W_0(a|\alpha_n))^{r/d}. \tag{3.23}$$

Setting $c = \frac{1}{2}(\frac{1}{2\lambda^d(B(0,1))\|h\|_{\mathbb{R}^d}})^{r/d}$ and combining (3.22) and (3.23) yields

$$\int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) \ge cP(W_0(a|\alpha_n))^{1+r/d}. \tag{3.24}$$

Since $P(W_0(a|\alpha_n)) \ge c_{17}\frac{1}{n}$ by (3.12), we deduce

$$\int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) \ge cc_{17}^{1+r/d} n^{-(1+r/d)}$$

and, hence, the conclusion (3.20) of the proposition with $c_{19} = cc_{17}^{1+r/d}$. □

PROOF OF THEOREM 3.1. The result is a combination of the results in Propositions 3.1–3.4, Corollary 3.1 and Zador's theorem which says that $\lim_{n\to\infty} \frac{e_{n,r}^r}{n^{-r/d}}$ exists in $(0, +\infty)$ (see, e.g., [8], Theorem 6.2). □

4. The local quantization rate for a class of absolutely continuous probabilities with unbounded support. In this section, we propose extensions of the results of Section 3 to distributions with an unbounded support which requires to have a control of the behavior of the distribution at infinity, even if our results are only *locally* uniform.

First, we introduce in item (c) of the definition below a class of probability density functions satisfying the "Peakless Sublevel Tail Property" (PSTP) for which a sharpened version of the *micro–macro* inequality (2.6) holds [see (4.4) further on]. This improved inequality is in fact the key to get the main results of this section (Proposition 4.3 and Theorem 4.1).

Although the PSTP may look rather technical and will not be shown to be necessary for the results in the unbounded framework, it seems clear from the case of compactly supported distribution, that one needs a restrictive condition of this nature for the conclusions in the case of distributions with unbounded support. The "Peakless Sublevel Property" (PSP) [item (a) in the definition below] is in some way the "core" of the PSTP and the "Convex Sublevel Approximation Property" (CSAP) [item (b) in the definition below] is simply a tractable criterion for the PSP.

DEFINITION 4.1.

(a) A Borel measurable map $f:\mathbb{R}^d \to \mathbb{R}$ satisfies the *peakless sublevel property* (PSP) outside $\overline{B}(0, R)$, $R > 0$, if there are real constants $s_0, c_f > 0$ such that

$$\begin{aligned} &\forall x \in \mathbb{R}^d \setminus \overline{B}(0, R), \forall s \in (0, s_0) \\ &\lambda^d\big(\{f \leq f(x)\} \cap B(x, s)\big) \geq c_f \lambda^d(B(x, s)). \end{aligned} \tag{4.1}$$

(b) A Borel measurable map $f:\mathbb{R}^d \to \mathbb{R}$ has the *convex sublevel approximation property* (CSAP) outside $\overline{B}(0, R)$, $R > 0$, if there is a bounded convex set $C \subset \mathbb{R}^d$ with nonempty interior such that

$$\forall x \in \mathbb{R}^d \setminus \overline{B}(0, R), \exists \varphi_x:\mathbb{R}^d \to \mathbb{R}^d, \text{Euclidean motion}, \exists a_x \geq 1$$
$$\text{such that } x \in \varphi_x(a_x C) \subset \{f \leq f(x)\}.$$

[By Euclidean motion, we mean an affine transform of the form $\varphi(y) = Ay + b$, A orthogonal matrix and $b \in \mathbb{R}^d$.]

(c) A probability distribution P has the *peakless sublevel tail property* (PSTP) outside $\overline{B}(0, R)$, $R > 0$, if:
  (i) P is absolutely continuous with an essentially bounded density h,
  (ii) h is bounded away from 0 on compacts sets, that is,

$$\forall \rho > 0, \exists c_\rho > 0 \qquad \text{such that } h(x) \geq c_\rho \qquad \text{for all } x \in \overline{B}(0, \rho). \tag{4.2}$$

  (iii) There exist a function $f:\mathbb{R}^d \to I$, I interval of $\mathbb{R}$, having the PSP and a nonincreasing function $g: I \to (0, +\infty)$ such that

$$\forall x \in \mathbb{R}^d \qquad \|x\| \geq R \quad \Longrightarrow \quad h(x) = g \circ f(x).$$

  Note that $\operatorname{supp}(P) = \mathbb{R}^d$.

PROPOSITION 4.1. *If $f:\mathbb{R}^d \to \mathbb{R}^d$ has the CSAP outside $\overline{B}(0, R)$, then it has the PSP outside $\overline{B}(0, R)$.*

PROOF. Let $s_0 > 0$ be arbitrary. By [8], Example 12.7, there exists a constant $\widetilde{c} > 0$ such that

$$\forall x \in C, \forall s \in (0, s_0) \qquad \lambda^d\big(C \cap B_{\|\cdot\|_2}(x, s)\big) \geq \widetilde{c}\lambda^d\big(B_{\|\cdot\|_2}(x, s)\big). \tag{4.3}$$

There exists a constant $\kappa \in (0, \infty)$ such that

$$\frac{1}{\kappa}\|\cdot\|_2 \leq \|\cdot\| \leq \kappa\|\cdot\|_2.$$

Now let $x \in \mathbb{R}^d$ with $\|x\| \geq R$ and let $s \in (0, s_0)$ be arbitrary. Then we have

$$\begin{aligned}
&\lambda^d(\{f \leq f(x)\} \cap B(x,s)) \\
&\quad \geq \lambda^d\left(\varphi_x(a_x C) \cap B_{\|\cdot\|_2}\left(x, \frac{s}{\kappa}\right)\right) \\
&\quad = \lambda^d\left(a_x C \cap \varphi_x^{-1}\left(B_{\|\cdot\|_2}\left(x, \frac{s}{\kappa}\right)\right)\right) \\
&\quad = a_x^d \lambda^d\left(C \cap \frac{1}{a_x}\varphi_x^{-1}\left(B_{\|\cdot\|_2}\left(x, \frac{s}{\kappa}\right)\right)\right) \\
&\quad = a_x^d \lambda^d\left(C \cap B_{\|\cdot\|_2}\left(\frac{1}{a_x}\varphi_x^{-1}(x), \frac{s}{a_x\kappa}\right)\right) \\
&\quad \geq \widetilde{c} a_x^d \lambda^d\left(B_{\|\cdot\|_2}\left(\frac{1}{a_x}\varphi_x^{-1}(x), \frac{s}{a_x\kappa}\right)\right) \qquad \text{owing to (4.3)} \\
&\quad = \widetilde{c} a_x^d \frac{1}{\kappa^d a_x^d} s^d \lambda^d(B_{\|\cdot\|_2}(0,1)) \\
&\quad = \widetilde{c}\kappa^{-d} \frac{\lambda^d(B_{\|\cdot\|_2}(0,1))}{\lambda^d(B(0,1))} \lambda^d(B(x,s)).
\end{aligned}$$ □

EXAMPLES.

(a) *If $\|\cdot\|_0$ is any norm on $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ is defined by $f(x) = \|x\|_0$. Then f has the CSAP outside $\overline{B}(0, R)$, for every $R > 0$.*

In particular, every nonsingular normal distribution has the PSTP outside $\overline{B}(0, R)$ for every $R > 0$ and more generally, this is the case for hyper-exponential distributions of the forms

$$h(x) = K\|x\|_2^a e^{-c\|x\|_2^b}, \qquad a, b, c, K > 0.$$

for large enough $R > 0$ (in fact this is true for any norm).

PROOF. Let $R > 0$ be arbitrary. Then there is an $\widetilde{R} > 0$ with

$$\overline{B}_{\|\cdot\|_0}(0, \widetilde{R}) \subset \overline{B}(0, R).$$

Let $C = \overline{B}_{\|\cdot\|_0}(0, \widetilde{R})$. Then C is convex with nonempty interior. Let $x \in \mathbb{R}^d \setminus \overline{B}_{\|\cdot\|_0}(0, \widetilde{R})$ be arbitrary. Set $\varphi_x = id_{\mathbb{R}^d}$ and $a_x = \frac{1}{\widetilde{R}}\|x\|_0 \geq 1$. Then

$$x = \varphi_x\left(a_x \widetilde{R} \frac{x}{\|x\|_0}\right) \in \varphi_x(a_x C) = \overline{B}_{\|\cdot\|_0}(0, \|x\|_0) = \{f \leq f(x)\}.$$ □

(b) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be semi-concave outside $\overline{B}(0, R)$ in the following sense:*

$\exists \theta > 1, \exists L > 0, \exists \varrho : \mathbb{R}^d \setminus \overline{B}(0, R) \to \mathbb{R}_+ \setminus \{0\}, \exists \delta : \mathbb{R}^d \setminus \overline{B}(0, R) \to \mathbb{R}^d \setminus \{0\}$

such that:

(i) $\forall x \in \mathbb{R}^d \setminus \overline{B}(0, R)$, $\frac{\varrho(x)}{\|\delta(x)\|_2} \le L$,

(ii) $\forall x \in \mathbb{R}^d \setminus \overline{B}(0, R), \forall y \in B(x, (\frac{1}{L})^{1/(\theta-1)})$, $f(y) \le f(x) + \delta(x) \cdot (y - x) + \varrho(x)\|y - x\|_2^{\theta}$, *where* $w \cdot z$ *denotes the standard scalar product of* w, $z \in \mathbb{R}^d$.

Then f *has the CSAP outside* $\overline{B}(0, R)$.

PROOF. Set $C = \{y = (y_1, \dots, y_d) \in \mathbb{R}^d | y_1 + L\|y\|_2^{\theta} \le 0\}$. We will show that C is a bounded convex set with nonempty interior. For $\lambda \in [0, 1]$ and $y, \widetilde{y} \in C$ we have

$$\begin{aligned}\big(\lambda y_1 + (1-\lambda)\widetilde{y}_1\big) &+ L\|\lambda y + (1-\lambda)\widetilde{y}\|_2^{\theta} \\ &\le \lambda y_1 + (1-\lambda)\widetilde{y}_1 + L\big(\lambda\|y\|_2 + (1-\lambda)\|\widetilde{y}\|_2\big)^{\theta}.\end{aligned}$$

Since $\theta > 1$, we have

$$\big(\lambda\|y\|_2 + (1-\lambda)\|\widetilde{y}\|_2\big)^{\theta} \le \lambda\|y\|_2^{\theta} + (1-\lambda)\|\widetilde{y}\|_2^{\theta},$$

which yields

$$\lambda y + (1-\lambda)\widetilde{y} \in C.$$

Thus, C is convex. For $y \in C$, we have

$$\begin{aligned}0 &\ge y_1 + L\|y\|_2^{\theta} \ge -\|y\|_2 + L\|y\|_2^{\theta} \\ &= \|y\|_2(L\|y\|_2^{\theta-1} - 1),\end{aligned}$$

hence $\|y\|_2 \le (\frac{1}{L})^{1/(\theta-1)}$, so that C is bounded.

There exists a $t > 0$ with $-t + Lt^{\theta} = t(Lt^{\theta-1} - 1) < 0$. For $y = (-t, 0, \dots, 0)$ this implies $y_1 + L\|y\|_2^{\theta} < 0$. Hence, there exists a neighborhood of y which is contained in C, that is, the interior of C is not empty.

Now let $x \in \mathbb{R}^d$ with $\|x\| > R$ be arbitrary. Set $u = \frac{\delta(x)}{\|\delta(x)\|_2}$. Let ψ_x be a rotation which maps $e_1 = (1, 0, \dots, 0)$ onto u. Define $\varphi_x : \mathbb{R}^d \to \mathbb{R}^d$ by $\varphi_x(y) = \psi_x(y) + x$. Then φ_x is a Euclidean motion. Set $a_x = 1$. Since $0 \in C$ we have $x \in \varphi_x(C) = \varphi_x(a_x C)$. For $y \in \varphi_x(a_x C) = \varphi_x(C)$ there is a $z \in C$ with $y = \varphi_x(z)$, hence

$$\begin{aligned}\delta(x) \cdot (y - x) + \varrho(x)\|y - x\|_2^{\theta} &= \delta(x) \cdot \psi_x(z) + \varrho(x)\|\psi_x(z)\|_2^{\theta} \\ &= \|\delta(x)\|_2 u \cdot \psi_x(z) + \varrho(x)\|\psi_x(z)\|_2^{\theta} \\ &= \|\delta(x)\|_2 e_1 \cdot z + \varrho(x)\|z\|_2^{\theta} \\ &= \|\delta(x)\|_2\bigg(z_1 + \frac{\varrho(x)}{\|\delta(x)\|_2}\|z\|_2^{\theta}\bigg) \\ &\le \|\delta(x)\|_2(z_1 + L\|z\|_2^{\theta}) \le 0\end{aligned}$$

since $z \in C$. Moreover, $\|\varphi_x(z) - x\|_2 = \|\psi_x(z)\|_2 = \|z\|_2$ and $-\|z\|_2 + L\|z\|_2^\theta \le 0$ implies $\|z\|_2 \le (\frac{1}{L})^{1/(\theta-1)}$, that is, $y = \psi_x(z) \in B(x, (\frac{1}{L})^{1/(\theta-1)})$.

By (ii), this yields

$$f(y) \le f(x) + \delta(x) \cdot (y - x) + \varrho(x)\|y - x\|_2^\theta \le f(x)$$

and, hence,

$$\varphi_x(a_x C) \subseteq \{f \le f(x)\}. \qquad \square$$

(c) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function and let $R > 0$ be such that there exist real constants $\alpha \in (0, 1)$, $\beta > 0$ and $c \in (0, +\infty)$ satisfying*:

(i) $\forall x, y \in \mathbb{R}^d, [x, y] := \{x + t(y - x), t \in [0, 1]\} \subset \mathbb{R}^d \setminus \overline{B}(0, R) \Longrightarrow \|\operatorname{grad} f(x) - \operatorname{grad} f(y)\| \le c\|x - y\|^\alpha(1 + \|x\|^\beta + \|y\|^\beta)$.

(ii) $\inf_{\|x\| \ge R} \frac{\|\operatorname{grad} f(x)\|}{1+\|x\|^\beta} > 0$.

Then f is semi-concave outside of $\overline{B}(0, R + 1)$.

PROOF. For every $x, y \in \mathbb{R}^d$ with $\|x\| > R$ and $\|x - y\| \le 1$, we have

$$\|y\|^\beta \le (\|x\| + \|y - x\|)^\beta \le (\|x\| + 1)^\beta = \|x\|^\beta \left(1 + \frac{1}{\|x\|}\right)^\beta$$

so that

$$\begin{aligned} 1 + \|x\|^\beta + \|y\|^\beta &\le 1 + \|x\|^\beta \left(\left(1 + \frac{1}{R}\right)^\beta + 1\right) \\ &\le \left(\left(1 + \frac{1}{R}\right)^\beta + 1\right)(\|x\|^\beta + 1). \end{aligned}$$

Let $\kappa \in (0, \infty)$ such that $\frac{1}{\kappa}\|\cdot\|_2 \le \|\cdot\| \le \kappa\|\cdot\|_2$.

Let $\theta = 1 + \alpha$. Define $\varrho : \mathbb{R}^d \to \mathbb{R}_+ \setminus \{0\}$ by $\varrho(x) = \kappa^2 c((1 + \frac{1}{R})^\beta + 1)(\|x\|^\beta + 1)$ and $\delta : \mathbb{R}^d \to \mathbb{R}^d$ by $\delta(x) = \operatorname{grad} f(x)$. Since $M := \inf_{\|x\| \ge R} \frac{\|\operatorname{grad} f(x)\|}{1+\|x\|^\beta} > 0$, we have $\delta(x) \ne 0$ for all $x \in \mathbb{R}^d \setminus \overline{B}(0, R)$. Moreover,

$$\frac{\varrho(x)}{\|\delta(x)\|_2} \le \frac{\varrho(x)}{(1/\kappa)\|\delta(x)\|} \le \kappa^3 c\left(\left(1 + \frac{1}{R}\right)^\beta + 1\right)\frac{1}{M} \le L,$$

where $L = \max\{1, \kappa^3 c((1 + \frac{1}{R})^\beta + 1)\frac{1}{M}\}$. Let $x \in \mathbb{R}^d \setminus \overline{B}(0, R + 1)$ and $y \in B(x, (\frac{1}{L})^{1/(\theta-1)})$ be arbitrary. Since $L \ge 1$ we have $[x, y] \subset \mathbb{R}^d \setminus \overline{B}(0, R)$ and, by the mean value theorem of differentiation,

$$\begin{aligned} f(y) - f(x) &= (\operatorname{grad} f(x)) \cdot (y - x) \\ &\quad + \big(\operatorname{grad} f\big(x + t(y - x)\big) - \operatorname{grad} f(x)\big) \cdot (y - x) \end{aligned}$$

for some $t \in [0, 1]$. By our assumption, we obtain

$$\begin{aligned}
&\big(\operatorname{grad} f\big(x + t(y - x)\big) - \operatorname{grad} f(x)\big) \cdot (y - x) \\
&\quad \le \big\|\operatorname{grad} f\big(x + t(y - x)\big) - \operatorname{grad} f(x)\big\|_2 \|y - x\|_2 \\
&\quad \le \kappa^2 \big\|\operatorname{grad} f\big(x + t(y - x)\big) - \operatorname{grad} f(x)\big\| \|y - x\| \\
&\quad \le \kappa^2 c t^\alpha \|y - x\|^\alpha \big(1 + \|x\|^\beta + \|x + t(y - x)\|^\beta\big) \|y - x\|.
\end{aligned}$$

Since $\|x + t(x - y) - x\| = t\|x - y\| \le (\frac{1}{L})^{1/(\theta-1)} \le 1$, we deduce

$$\begin{aligned}
&\big(\operatorname{grad} f\big(x + t(y - x)\big) - \operatorname{grad} f(x)\big) \cdot (y - x) \\
&\quad \le \kappa^2 c \left(\left(1 + \frac{1}{R}\right)^\beta + 1\right) (\|x\|^\beta + 1) \|y - x\|^\theta \\
&\quad \le \varrho(x) \|y - x\|^\theta.
\end{aligned}$$

It follows that

$$f(y) \le f(x) + \delta(x) \cdot (y - x) + \varrho(x) \|y - x\|^\theta.$$

Thus, f is semi-concave outside the ball $\overline{B}(0, R + 1)$. $\square$

As always in this manuscript α_n is an n-optimal codebook for P of order $r > 0$, where we assume $\int \|x\|^{r+\delta} \, dP(x) < \infty$ for some $\delta > 0$.

Our first aim is to prove another variant of the first micro–macro inequality for distributions P having the PSTP.

PROPOSITION 4.2. *Let P, with density h, have the PSTP outside $\overline{B}(0, R)$ for a given $R > 0$. There exists a constant $c_{21} > 0$ such that*

$$\begin{aligned}
&\forall K \subset \mathbb{R}^d, \textit{compact}, \exists n_K \in \mathbb{N} \textit{ such that } \forall n \ge n_K, \forall x \in K \\
&\qquad c_{21} n^{-1/d} h(x)^{-1/(r+d)} \ge d(x, \alpha_n).
\end{aligned} \tag{4.4}$$

PROOF. Let $K \subset \mathbb{R}^d$ be compact. Since $\operatorname{supp}(P) = \mathbb{R}^d$, Proposition 2.2 in [5] implies

$$\lim_{n \to \infty} \max_{y \in K} d(y, \alpha_n) = 0.$$

Let f and g be as in Definition 4.1(c)(iii) and let $s_0 > 0$ be related to f by Definition 4.1(a). Choose $n_K \in \mathbb{N}$, so that

$$\forall n \ge n_K \qquad \max_{y \in K} d(y, \alpha_n) < \min(s_0, R).$$

Let $n \ge n_K$ and let $x \in K$ be arbitrary. By (2.6), we know that

$$c_5(e_{n,r}^r - e_{n+1,r}^r) \ge d(x, \alpha_n)^{r+d} \frac{P(B(x, b d(x, \alpha_n)))}{\lambda^d(B(x, b d(x, \alpha_n)))}. \tag{4.5}$$

Since $\overline{B}(0,2R)$ is bounded and convex, there exists a constant $\widetilde{c} > 0$ with

$$\forall s \in (0, s_0), \forall y \in \overline{B}(0,2R) \qquad \lambda^d\big(\overline{B}(0,2R) \cap B(y,s)\big) \geq \widetilde{c}\lambda^d(B(y,s)).$$

If $x \in \overline{B}(0,2R)$, by Definition 4.1(c)(ii) there exists a lower bound $c_{2R} > 0$ of h on $\overline{B}(0,2R)$, so that

$$\begin{aligned} P(B(x, bd(x,\alpha_n))) &\geq c_{2R}\lambda^d\big(\overline{B}(0,R) \cap B(x, bd(x,\alpha_n))\big) \\ &\geq c_{2R}\widetilde{c}\lambda^d(B(x, bd(x,\alpha_n))), \end{aligned}$$

hence $c_5(e^r_{n,r} - e^r_{n+1,r}) \geq c_{2R}\widetilde{c}d(x,\alpha_n)^{r+d}$ and consequently

$$c_5(e^r_{n,r} - e^r_{n+1,r}) \geq c_{2R}\widetilde{c}\frac{1}{\|h\|_{\overline{B}(0,2R)}}h(x)d(x,\alpha_n)^{r+d} \tag{4.6}$$

for every $x \in \overline{B}(0, 2R)$. If $x \notin \overline{B}(0,2R)$ and $y \in B(x, bd(x,\alpha_n)) \cap \{f \leq f(x)\}$, then we have

$$y \notin \overline{B}(0,R) \quad \text{and} \quad h(y) = g(f(y)) \geq g(f(x)) = h(x)$$

since g is nonincreasing and we obtain

$$\begin{aligned} P(B(x, bd(x,\alpha_n))) &\geq P\big(B(x, bd(x,\alpha_n)) \cap \{f \leq f(x)\}\big) \\ &= \int_{\{f \leq f(x)\} \cap B(x, bd(x,\alpha_n))} h(y)\, d\lambda^d(y) \\ &\geq h(x)\lambda^d\big(\{f \leq f(x)\} \cap B(x, bd(x,\alpha_n))\big) \\ &\geq c_f h(x)\lambda^d(B(x, bd(x,\alpha_n))) \end{aligned}$$

since f has the PSP. Hence,

$$c_5(e^r_{n,r} - e^r_{n+1,r}) \geq c_f h(x) d(x,\alpha_n)^{r+d}. \tag{4.7}$$

Note that, by Proposition 2.3, there exists a constant $c_{11} > 0$ such that

$$\forall n \in \mathbb{N} \qquad e^r_{n,r} - e^r_{n+1,r} \leq c_{11}n^{-(1+r/d)}.$$

Setting $c_{21} = (c_{11}c_5 \max\{c_f^{-1}, (c_{2R}\widetilde{c})^{-1}\})^{1/(r+d)}$ and combining the last inequality with (4.6) and (4.7) yields the conclusion of the proposition. $\square$

REMARK. Note at this stage that the results established in the rest of this section depend only on properties (4.2) and (4.4), not directly on PSP.

Our next aim is to give an upper and a lower bound for $P(W(a|\alpha_n))$ and the local quantization error $\int_{W(a|\alpha_n)} \|x - a\|^r\, dP(x)$, provided all the $W(a|\alpha_n)$ intersect a given compact set. The following lemma provides an essential tool for the proof. Here and in the rest of the paper, we set

$$\overline{s}_{n,a} = \sup\{\|x - a\|, x \in W(a|\alpha_n)\},$$

which can be considered as the *radius* of the Voronoi cell $W(a|\alpha_n)$.

LEMMA 4.1. *Let $K \subset \overset{\circ}{\overbrace{\mathrm{supp}(P)}}$ be an arbitrary compact set and let $\varepsilon > 0$ be arbitrary. Then there exists an $n_{K,\varepsilon} \in \mathbb{N}$ such that*

$$\forall n \geq n_{K,\varepsilon}, \forall a \in \alpha_n \qquad W(a|\alpha_n) \cap K \neq \varnothing \quad \Rightarrow \quad \overline{s}_{n,a} \leq \varepsilon. \tag{4.8}$$

PROOF. Let $\varepsilon > 0$. Since $K \subset \overset{\circ}{\overbrace{\mathrm{supp}(P)}}$, one may assume without loss of generality that ε is small enough so that the ε-neighborhood $K_\varepsilon := \{y \in \mathbb{R}^d | d(y, K) \leq \varepsilon\}$ is included in supp P. Since K is compact and contained in supp(P), [5], Proposition 2.2 implies $\lim_{n\to\infty} \max_{x\in K} d(x, \alpha_n) = 0$. Hence, there exists an $n_0 \in \mathbb{N}$ with

$$\forall x \in K, \forall n \geq n_0 \qquad d(x, \alpha_n) < \frac{\varepsilon}{2}. \tag{4.9}$$

Now assume that (4.8) does not hold for $\frac{\varepsilon}{2}$ in the place of ε. Then there exist sequences $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and (a_k) with $n_k \uparrow \infty$, $a_k \in \alpha_{n_k}$ with

$$W(a_k|\alpha_{n_k}) \cap K \neq \varnothing,$$

and $\overline{s}_{n_k,a_k} > \frac{\varepsilon}{2}$. Without loss of generality, we assume $n_k > n_0$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, there is an $\widetilde{x}_k \in W(a_k, \alpha_{n_k})$ with $\|\widetilde{x}_k - a_k\| > \frac{\varepsilon}{2}$. Set $x_k = a_k + \frac{\varepsilon}{2\|\widetilde{x}_k - a_k\|}(\widetilde{x}_k - a_k)$. Then we have $\|x_k - a_k\| = \frac{\varepsilon}{2}$ and, since $W(a_k, \alpha_{n_k})$ is star shaped with center a_k (see [8], Proposition 1.2), we deduce that $x_k \in [a_k, \widetilde{x}_k] \subset W(a_k|\alpha_{n_k})$. Now let $z_k \in W(a_k|\alpha_{n_k}) \cap K$. Then $\|z_k - a_k\| < \frac{\varepsilon}{2}$ owing to (4.9) and $\|x_k - a_k\| = \frac{\varepsilon}{2}$, so that $x_k \in K_\varepsilon$.

Since K_ε is compact there exists a convergent subsequence of (x_k), whose limit we denote by $x_\infty \in K_\varepsilon$. Then we have

$$\begin{aligned} d(x_\infty, \alpha_{n_k}) &\geq d(x_k, \alpha_{n_k}) - \|x_k - x_\infty\| \\ &= \|x_k - a_k\| - \|x_k - x_\infty\| \\ &= \frac{\varepsilon}{2} - \|x_k - x_\infty\| \end{aligned}$$

so that $\limsup_{k\to\infty} d(x_\infty, \alpha_{n_k}) \geq \frac{\varepsilon}{2}$.

Since $x_\infty \in K_\varepsilon \subset \mathrm{supp}(P)$, we know that $\lim_{n\to\infty} d(x_\infty, \alpha_n) = 0$ (see [8], Lemma 6.1 and [5], Proposition 2.2) and obtain a contradiction. □

DEFINITION 4.2. For a compact set $K \subset \mathbb{R}^d$, let

$$\alpha_n(K) = \{a \in \alpha_n | W(a|\alpha_n) \cap K \neq \varnothing\}.$$

PROPOSITION 4.3. *Let P satisfy the micro–macro inequality* (4.4). *There are constants $c_{22}, c_{23}, c_{24}, c_{25} > 0$ such that, for every compact set $K \subset \mathbb{R}^d$ and every*

$\varepsilon > 0$, there exists an $n_{K,\varepsilon} \in \mathbb{N}$ such that, for every $n \geq n_{K,\varepsilon}$, and every $a \in \alpha_n(K)$ the Voronoi cell $W(a|\alpha_n)$ is contained in K_ε and

$$P(W(a|\alpha_n)) \leq c_{22}\big(\|h\|_{W(a|\alpha_n)}\big)^{r/(r+d)}\frac{1}{n}, \tag{4.10}$$

$$\int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \leq c_{23}\bigg(1+\log\frac{\|h\|_{W(a|\alpha_n)}}{\operatorname{essinf} h_{|W(a|\alpha_n)}}\bigg)n^{-(1+r/d)}, \tag{4.11}$$

$$P(W_0(a|\alpha_n)) \geq c_{24}\big(\operatorname{essinf} h_{|W(a|\alpha_n)}\big)^{r/(r+d)}\frac{1}{n}, \tag{4.12}$$

$$\int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) \geq c_{25}\bigg(\frac{\operatorname{essinf} h_{|W(a|\alpha_n)}}{\|h\|_{W(a|\alpha_n)}}\bigg)^{\max(r,1)} n^{-(1+r/d)}. \tag{4.13}$$

PROOF. Let $K \subset \mathbb{R}^d$ be compact and $\varepsilon > 0$ be arbitrary. By Lemma 4.1 and Proposition 4.2, there exists an $n_{K,\varepsilon} \in \mathbb{N}$ with $n_{K,\varepsilon} \geq 2$ such that

$$\forall n \geq n_{K,\varepsilon}, \forall a \in \alpha_n(K) \qquad W(a|\alpha_n) \subset K_\varepsilon \tag{4.14}$$

and

$$\forall n \geq n_{K,\varepsilon}, \forall x \in K_\varepsilon \qquad c_{21} n^{-1/d} h(x)^{-1/(r+d)} \geq d(x, \alpha_n). \tag{4.15}$$

Now let $n \geq n_{K,\varepsilon}$ and let $a \in \alpha_n(K)$ be fixed. Set $\bar{t}_{n,a} = \|h\|_{W(a|\alpha_n)}$ and $\underline{t}_{n,a} = \operatorname{essinf} h_{|W(a|\alpha_n)}$. Since $W(a|\alpha_n) \subset K_\varepsilon$ by (4.14), inequality (4.15) implies

$$\forall t > 0, \forall x \in \{h > t\} \cap W(a|\alpha_n) \qquad \|x-a\| \leq c_{21} n^{-1/d} t^{-1/(r+d)}. \tag{4.16}$$

This yields

$$\begin{aligned} \lambda^d\big(\{h > t\} \cap W(a|\alpha_n)\big) &\leq \lambda^d\big(B\big(a, c_{21} n^{-1/d} t^{-1/(r+d)}\big)\big) \\ &= \lambda^d(B(0,1)) c_{21}^d t^{-d/(r+d)} n^{-1}. \end{aligned} \tag{4.17}$$

Now we will prove (4.10). Observing that $\lambda^d(\{h > t\} \cap W(a|\alpha_n)) = 0$ for $t > \bar{t}_{n,a}$ we deduce

$$\begin{aligned} P(W(a|\alpha_n)) &= \int_{W(a|\alpha_n)} h \, d\lambda^d \\ &= \int_0^\infty \lambda^d\big(\{h > t\} \cap W(a|\alpha_n)\big) \, dt \\ &= \int_0^{\bar{t}_{n,a}} \lambda^d\big(\{h > t\} \cap W(a|\alpha_n)\big) \, dt \\ &\leq \bigg(\int_0^{\bar{t}_{n,a}} t^{-d/(r+d)} \, dt\bigg) \lambda^d(B(0,1)) c_{21}^d n^{-1} \qquad \text{owing to (4.17)} \\ &\leq \lambda^d(B(0,1)) \frac{r+d}{r} c_{21}^d \big(\|h\|_{W(a|\alpha_n)}\big)^{r/(r+d)} \frac{1}{n}, \end{aligned}$$

which proves (4.10) with $c_{22} = \lambda^d(B(0,1))\frac{r+d}{r}c_{21}^d$.

Next, we will show (4.11). Using again $\lambda^d(\{h > t\} \cap W(a|\alpha_n)) = 0$ for $t > \bar{t}_{n,a}$, we get

$$\begin{aligned}\int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) &= \int_{W(a|\alpha_n)} \|x-a\|^r h(x)\, d\lambda^d(x) \\ &= \int_0^\infty \int_{\{h>t\}\cap W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x)\, dt \qquad (4.18) \\ &= \int_0^{\bar{t}_{n,a}} \int_{\{h>t\}\cap W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x)\, dt.\end{aligned}$$

For $t \le \underline{t}_{n,a}$, we have $h(y) \ge t$ for λ^d-a.e. $y \in W(a|\alpha_n)$ so that

$$\int_{\{h>t\}\cap W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x) = \int_{W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x).$$

By (4.14) and (4.15), we have, for λ^d-a.e. $x \in W(a|\alpha_n)$,

$$\|x-a\| = d(x,\alpha_n) \le c_{21} n^{-1/d} h(x)^{-1/(r+d)} \le c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)}$$

so that

$$\lambda^d\bigl(W(a|\alpha_n) \setminus B\bigl(a, c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)}\bigr)\bigr) = 0.$$

Consequently,

$$\begin{aligned}&\int_0^{\underline{t}_{n,a}} \int_{\{h>t\}\cap W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x)\, dt \\ &\quad \le \int_0^{\underline{t}_{n,a}} \int_{B(a, c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)})} \bigl(c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)}\bigr)^r \, d\lambda^d(x)\, dt \qquad (4.19) \\ &\quad = c_{23} n^{-(1+r/d)},\end{aligned}$$

where $c_{23} = c_{21}^{r+d}\lambda^d(B(0,1))$. Using (4.16) and the same argument as before, we obtain

$$\begin{aligned}&\int_{\underline{t}_{n,a}}^{\bar{t}_{n,a}} \int_{\{h>t\}\cap W(a|\alpha_n)} \|x-a\|^r \, d\lambda^d(x)\, dt \\ &\quad \le \int_{\underline{t}_{n,a}}^{\bar{t}_{n,a}} \int_{B(a, c_{21} n^{-1/d} t^{-1/(r+d)})} c_{21}^r t^{-r/(r+d)} n^{-r/d} \, dP(x)\, dt \\ &\quad \le c_{23} n^{-(1+r/d)} \int_{\underline{t}_{n,a}}^{\bar{t}_{n,a}} t^{-1}\, dt \qquad (4.20) \\ &\quad = c_{23} n^{-(1+r/d)} \log\left(\frac{\bar{t}_{n,a}}{\underline{t}_{n,a}}\right).\end{aligned}$$

Combining (4.19) and (4.20) with (4.18) yields (4.11).

Now we will prove (4.12). It follows from the second micro–macro inequality (Proposition 2.2) and Proposition 2.3 that there exists a real constant $c > 0$ (independent of n and a) such that

$$cn^{-(1+r/d)} \leq \int_{W_0(a|\alpha_n)} \big(d(x, \alpha_n \setminus \{a\})^r - \|x - a\|^r\big)\, dP(x). \tag{4.21}$$

Since (4.59) implies that $\overline{W_0(a|\alpha_n)}$ is compact and nonempty there exists a $z \in \partial W_0(a|\alpha_n)$. Obviously this z satisfies

$$\|z - a\| = d(z, \alpha_n \setminus \{a\})$$

and, therefore,

$$d(a, \alpha_n \setminus \{a\}) \leq \|a - z\| + d(z, \alpha_n \setminus \{a\}) = 2\|z - a\|. \tag{4.22}$$

This implies that, for every $x \in \overline{W_0(a|\alpha_n)}$,

$$\begin{aligned} d(x, \alpha_n \setminus \{a\}) &\leq \|x - a\| + d(a, \alpha_n \setminus \{a\}) \\ &\leq \|x - a\| + 2\|z - a\| = d(x, \alpha_n) + 2d(z, \alpha_n). \end{aligned}$$

Since $d_{\alpha_n} := d(\cdot, \alpha_n)$ is continuous and every nonempty relatively open subset of $\overline{W_0(a|\alpha_n)}$ has positive Lebesgue measure, we deduce

$$\max\{d(y, \alpha_n) : y \in \overline{W_0(a|\alpha_n)}\} = \operatorname{esssup} d_{\alpha_n|\overline{W_0(a|\alpha_n)}}.$$

By (4.14) and (4.15) this yields

$$\begin{aligned} d(x, \alpha_n \setminus \{a\}) &\leq 3 \operatorname{esssup} d_{\alpha_n|\overline{W_0(a|\alpha_n)}} \\ &\leq 3c_{21} n^{-1/d} \operatorname{esssup}\big(h_{|\overline{W_0(a|\alpha_n)}}\big)^{-1/(r+d)} \\ &= 3c_{21} n^{-1/d} \big(\operatorname{essinf} h_{|\overline{W_0(a|\alpha_n)}}\big)^{-1/(r+d)} \\ &\leq 3c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+a)} \end{aligned}$$

and, therefore,

$$\int_{W_0(a|\alpha_n)} d(x, \alpha_n \setminus \{a\})^r \, dP(x) \leq 3^r c_{21}^r n^{-r/d} (\underline{t}_{n,a})^{-r/(r+d)} P(W_0(a|\alpha_n)). \tag{4.23}$$

Using (4.21), we deduce

$$c3^{-r} c_{21}^{-r} (\underline{t}_{n,a})^{r/(r+d)} n^{-1} \leq P(W_0(a|\alpha_n))$$

and, hence, (4.12) with $c_{24} = c3^{-r} c_{21}^{-r}$.

Now we will prove (4.13). It follows from (4.21) that

$$cn^{-(1+r/d)} \leq \int_{W_0(a|\alpha_n)} \big((\|x - a\| + d(a, \alpha_n \setminus \{a\}))^r - \|x - a\|^r\big)\, dP(x). \tag{4.24}$$

CASE 1 ($r \geq 1$). Using the mean value theorem for differentiation yields

$$\begin{aligned}&cn^{-(1+r/d)}\\&\quad\leq \int_{W_0(a|\alpha_n)} r\big(\|x-a\| + d(a,\alpha_n \setminus \{a\})\big)^{r-1} d(a,\alpha_n \setminus \{a\})\,dP(x).\end{aligned} \tag{4.25}$$

By (4.22), (4.14) and (4.15), we know that

$$\|x-a\| + d(a,\alpha_n \setminus \{a\}) \leq 3c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)}. \tag{4.26}$$

Combining (4.25) and (4.26) yields

$$cn^{-(1+r/d)} \leq d(a,\alpha_n \setminus \{a\}) r\big(3c_{21} n^{-1/d} (\underline{t}_{n,a})^{-1/(r+d)}\big)^{r-1} P(W_0(a|\alpha_n)). \tag{4.27}$$

By (4.10), we have

$$P(W_0(a|\alpha_n)) \leq c_{22} \overline{t}_{n,a}^{r/(r+d)} \frac{1}{n}$$

and, hence,

$$c_{22}^{-1} c r^{-1} (3c_{21})^{1-r} \underline{t}_{n,a}^{(r-1)/(r+d)} \overline{t}_{n,a}^{-r/(r+d)} n^{-1/d} \leq d(a,\alpha_n \setminus \{a\}). \tag{4.28}$$

Set $\widetilde{c} = c_{22}^{-1} c r^{-1} (3c_{21})^{1-r}$. Then we deduce

$$B\left(a, \frac{\widetilde{c}}{2} \underline{t}_{n,a}^{(r-1)/(r+d)} \overline{t}_{n,a}^{-r/(r+d)} n^{-1/d}\right) \subset W_0(a|\alpha_n). \tag{4.29}$$

It follows that

$$\begin{aligned}&\int_{B(a,(\widetilde{c}/2)\underline{t}_{n,a}^{(r-1)/(r+d)} \overline{t}_{n,a}^{-r/(r+d)} n^{-1/d})} \|x-a\|^r h(x)\,d\lambda^d(x)\\&\qquad\leq \int_{W_0(a|\alpha_n)} \|x-a\|^r \,dP(x).\end{aligned} \tag{4.30}$$

Since $h(x) \geq \underline{t}_{n,a}$, for λ^d-a.e. $x \in B(a, \frac{\widetilde{c}}{2} \underline{t}_{n,a}^{(r-1)/(r+d)} \overline{t}_{n,a}^{-r/(r+d)} n^{-1/d})$ and

$$\int_{B(a,\varrho)} \|x-a\|^r \,d\lambda^d(x) = \varrho^{r+d} \int_{B(0,1)} \|u\|^r \,d\lambda^d(u)$$

for every $\varrho > 0$, the left-hand side of (4.30) is greater or equal to

$$\begin{aligned}&\underline{t}_{n,a} \int_{B(0,1)} \|x\|^r \,d\lambda^d(x) \left(\frac{\widetilde{c}}{2} \underline{t}_{n,a}^{(r-1)/(r+d)} \overline{t}_{n,a}^{-r/(r+d)}\right)^{r+d} n^{-(1+r/d)}\\&\quad= \int_{B(0,1)} \|u\|^r \,d\lambda^d(u) \left(\frac{\widetilde{c}}{2}\right)^{r+d} \underline{t}_{n,a}^{r} \overline{t}_{n,a}^{-r} n^{-(1+r/d)}.\end{aligned}$$

Inequality (4.13) follows by setting $c_{25} = \int_{B(0,1)} \|u\|^r \,d\lambda^d(u) (\frac{\widetilde{c}}{2})^{r+d}$.

CASE 2 ($r < 1$). In this case, we have

$$\big(\|x-a\| + d(a, \alpha_n \setminus \{a\})\big)^r \le \|x-a\|^r + d(a, \alpha_n \setminus \{a\})^r$$

for all $x \in W_0(a|\alpha_n)$, so that, by (4.24),

$$\begin{aligned} cn^{-(1+r/d)} &\le \int_{W_0(a|\alpha_n)} d(a, \alpha_n \setminus \{a\})^r \, dP(x) \\ &\le d(a, \alpha_n \setminus \{a\})^r P(W_0(a|\alpha_n)). \end{aligned} \tag{4.31}$$

By (4.10), we know that

$$P(W_0(a|\alpha_n)) \le c_{22} (\overline{t}_{n,a})^{r/(r+d)} \frac{1}{n}$$

and, hence,

$$c^{1/r} c_{22}^{-1/r} \overline{t}_{n,a}^{-1/(r+d)} n^{-1/d} \le d(a, \alpha_n \setminus \{a\}). \tag{4.32}$$

As above this implies, for $\widetilde{c} = c^{1/r} c_{22}^{-1/r}$,

$$\underline{t}_{n,a} \int_{B(0,1)} \|x\|^r \, d\lambda^d(x) \bigg(\frac{\widetilde{c}}{2}\bigg)^{r+d} \frac{\underline{t}_{n,a}}{\overline{t}_{n,a}} n^{-(1+r/d)} \le \int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x)$$

and (4.13) follows. □

THEOREM 4.1. *Let P satisfy the micro–macro inequality* (4.4). *Then there are constants $c_{22}, c_{23}, c_{24}, c_{25} > 0$ such that, for every compact set $K \subset \mathbb{R}^d$, the following hold*:

$$\limsup_{n\to\infty} n \max_{a \in \alpha_n(K)} P(W(a|\alpha_n)) \le c_{22} \Big(\inf_{\varepsilon>0} \|h\|_{K_\varepsilon}\Big)^{r/(r+d)}, \tag{4.33}$$

$$\begin{aligned} &\limsup_{n\to\infty} n^{1+r/d} \max_{a \in \alpha_n(K)} \int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \\ &\quad \le c_{23} \bigg(1 + \log\bigg(\inf_{\varepsilon>0} \frac{\|h\|_{K_\varepsilon}}{\operatorname{essinf} h_{|K_\varepsilon}}\bigg)\bigg), \end{aligned} \tag{4.34}$$

$$\liminf_{n\to\infty} n \min_{a \in \alpha_n(K)} P(W_0(a|\alpha_n)) \ge c_{24} \sup_{\varepsilon>0} (\operatorname{essinf} h_{|K_\varepsilon})^{r/(r+d)}, \tag{4.35}$$

$$\begin{aligned} &\liminf_{n\to\infty} n^{(1+r/d)} \min_{a \in \alpha_n(K)} \int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \\ &\quad \ge c_{25} \sup_{\varepsilon>0} \bigg(\frac{\operatorname{essinf} h_{|K_\varepsilon}}{\|h\|_{K_\varepsilon}}\bigg)^{\max(1,r)}. \end{aligned} \tag{4.36}$$

PROOF. The theorem follows immediately from Proposition 4.3. □

COROLLARY 4.1. *For every $x \in \mathbb{R}^d$, let $a_{n,x} \in \alpha_n$ satisfy $x \in W(a_{n,x}|\alpha_n)$. Then*

$$\limsup_{n\to\infty} n P(W(a_{n,x}|\alpha_n)) \le c_{22}\Big(\limsup_{y\to x} h(y)\Big)^{r/(r+d)}, \tag{4.37}$$

$$\begin{aligned}&\limsup_{n\to\infty} n^{1+r/d}\int_{W(a_{n,x}|\alpha_n)} \|x-a\|^r\,dP(x)\\ &\quad\le c_{23}\left(1+\log\lim_{\varepsilon\downarrow 0}\frac{\sup h(B(x,\varepsilon))}{\inf h(B(x,\varepsilon))}\right),\end{aligned} \tag{4.38}$$

$$\liminf_{n\to\infty} n P(W_0(a_{n,x},|\alpha_n)) \ge c_{24}\Big(\liminf_{y\to x} h(y)\Big)^{r/(r+d)}, \tag{4.39}$$

$$\begin{aligned}&\liminf_{n\to\infty} n^{1+r/d}\int_{W_0(a_{n,x}|\alpha_n} \|x-a\|^r\,dP(X)\\ &\quad\ge c_{25}\left(\lim_{\varepsilon\downarrow 0}\frac{\inf h(B(x,\varepsilon))}{\sup h(B(x,\varepsilon))}\right)^{\max(1,r)}.\end{aligned} \tag{4.40}$$

Moreover, if h is continuous, then $\limsup_{y\to x} h(y) = h(x) = \liminf_{y\to x} h(y)$ *and*

$$\lim_{\varepsilon\downarrow 0}\frac{\sup h(B(x,\varepsilon))}{\inf h(B(x,\varepsilon)} = \lim_{\varepsilon\downarrow 0}\frac{\inf h(B(x,\varepsilon))}{\sup h(B(x,\varepsilon))} = 1.$$

PROOF. The corollary follows from Theorem 4.9 if one sets $K = \{x\}$. □

REMARKS. (a) For certain one-dimensional distribution functions, sharper versions of the above corollary have been proved by Fort and Pagès ([6], Theorem 6).

(b) If $R > 0$ and the density h has the form $h(x) = g(\|x\|_0)$ for all $x \notin B(0, R)$, where $g : [0, +\infty) \to (0, +\infty)$ is a decreasing function and $\|\cdot\|_0$ is an arbitrary norm on $\mathbb{R}^d$ then there exists a constant $c > 0$ and an $m = m(c) \in \mathbb{N}$ such that

$$\forall n \ge m, \forall x \in \mathbb{R}^d \qquad cn^{-1/d}h(x)^{-1/(r+d)} \ge d(x, \alpha_n).$$

This can be used to show that there is a $\widetilde{c} > 0$ with

$$\forall n \ge m, \forall a \in \alpha_n \qquad P(W(a|\alpha_n)) \le \widetilde{c}\big(\|h\|_{W(a|\alpha_n)}\big)^{r/(r+d)}\frac{1}{n}.$$

Under additional assumptions on g (g regularly varying), one can also give a similar upper bound for the local L^s-quantization errors, $s \in (0, r)$.

5. The local quantization behavior in the interior of the support. In this section, we will show that weaker versions of the results in Section 4 still hold without assuming the strong version of the first micro–macro inequality as stated in (4.4). We have to restrict our investigations to compact sets in the interior of the support of the probability in question and also obtain weaker constants in the corresponding inequalities for the local probabilities and quantization errors.

Let $r \in (0, \infty)$ be fixed. In this section P is always an absolutely continuous Borel probability on $\mathbb{R}^d$ with density h. We assume that there is a $\delta > 0$ with $\int \|x\|^{r+\delta}\, dP(x) < +\infty$. As before, α_n is an n-optimal codebook for P of order r. For $n \in \mathbb{N}$ and $a \in \alpha_n$ set $\overline{s}_{n,a} = \sup\{\|x - a\|, x \in W(a|\alpha_n)\}$ and $\underline{s}_{n,a} = \sup\{s > 0, B(a, s) \subset W(a|\alpha_n)\}$.

Moreover, we assume that h is essentially bounded and that $\operatorname{essinf} h_{|K} > 0$ for every compact set $K \subset \overbrace{\operatorname{supp}(P)}^{\circ}$, where $\overset{\circ}{B}$ denotes the interior of the set $B \subset \mathbb{R}^d$. For the use in the first micro–macro inequality, we fix a $b \in (0, \frac{1}{2})$.

LEMMA 5.1. *There exists a constant $c_{26} > 0$ such that, for every $n \in \mathbb{N}$ and $a \in \alpha_n$,*

$$c_{26} n^{-1/d} \bigl(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\bigr)^{-1/(r+d)} \geq \overline{s}_{n,a}. \tag{5.1}$$

PROOF. By the first micro–macro inequality (2.6) and Proposition 2.3 there exists a constant $c > 0$ with

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d \qquad c n^{-(1+r/d)} \geq d(x, \alpha_n)^{r+d} \frac{P(B(x, bd(x, \alpha_n)))}{\lambda^d(B(x, bd(x, \alpha_n)))}. \tag{5.2}$$

Now let $n \in \mathbb{N}$ and $a \in \alpha_n$ be arbitrary.

It follows from (5.2) that

$$\forall x \in W(a|\alpha_n) \qquad \|x - a\|^{r+d} \frac{P(B(x, b\|x - a\|))}{\lambda^d(B(x, b\|x - a\|))} \leq c n^{-(1+r/d)}. \tag{5.3}$$

For $x \in W(a|\alpha_n)$ and $y \in B(x, bd(x, \alpha_n))$, we have

$$\|y - a\| < \|y - x\| + \|x - a\| \leq b\|x - a\| + \|x - a\| \leq (1 + b)\overline{s}_{n,a}$$

so that

$$B(x, b\|x - a\|) \subseteq B\bigl(a, (1 + b)\overline{s}_{n,a}\bigr). \tag{5.4}$$

This yields

$$\begin{aligned} P\bigl(B(x, b\|x - a\|)\bigr) &= \int_{B(x,b\|x-a\|)} h\, d\lambda^d \\ &\geq \operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})} \lambda^d\bigl(B(x, b\|x - a\|)\bigr) \end{aligned} \tag{5.5}$$

owing to (5.4). Thus, (5.3) implies

$$\|x - a\|^{r+d} \operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})} \leq c n^{-(1+r/d)}. \tag{5.6}$$

Since $x \in W(a|\alpha_n)$ was arbitrary, we deduce

$$\overline{s}_{n,a}^{r+d} \operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})} \leq c n^{-(1+r/d)}$$

and, hence, (5.1) with $c_{26} = c^{1/(r+d)}$. $\square$

LEMMA 5.2. *There exist real constants $c_{27}, c_{28} > 0$ such that, for every $n \in \mathbb{N}$ and $a \in \alpha_n$,*

$$P(W(a|\alpha_n)) \leq c_{27} \frac{\|h\|_{B(a,\overline{s}_{n,a})}}{(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})})^{d/(r+d)}} n^{-1} \tag{5.7}$$

and

$$\int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \leq c_{28} \frac{\|h\|_{B(a,\overline{s}_{n,a})}}{\operatorname{essinf} h_{|B(a(1+b),\overline{s}_{n,a})}} n^{-(1+r/d)}. \tag{5.8}$$

PROOF. Let $n \in \mathbb{N}$ and $a \in \alpha_n$ be arbitrary. Then (5.1) implies

$$\begin{aligned}
P(W(a|\alpha_n)) &\leq P(B(a,\overline{s}_{n,a})) \leq \|h\|_{B(a,\overline{s}_{n,a})} \lambda^d(B(a,\overline{s}_{n,a})) \\
&\leq \lambda^d(B(0,1)) \|h\|_{B(a,\overline{s}_{n,a})} \overline{s}_{n,a}^d \\
&\leq \lambda^d(B(0,1)) c_{26}^d \|h\|_{B(a,\overline{s}_{n,a})} \big(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\big)^{-d/(r+d)} n^{-1}.
\end{aligned}$$

Thus (5.7) follows for $c_{27} = \lambda^d(B(0,1)) c_{26}^d$.

Similarly (5.1) implies

$$\begin{aligned}
&\int_{W(a|\alpha_n)} \|x-a\|^r \, dP(x) \\
&\quad \leq \int_{B(a,\overline{s}_{n,a})} \|x-a\|^r \, dP(x) \\
&\quad \leq \|h\|_{B(a,\overline{s}_{n,a})} \int_{B(a,\overline{s}_{n,a})} \|x-a\|^r \, d\lambda^d(x) \\
&\quad \leq \lambda^d(B(0,1)) \|h\|_{B(a,\overline{s}_{n,a})} \overline{s}_{n,a}^{r+d} \\
&\quad \leq \lambda^d(B(0,1)) c_{26}^{r+d} \|h\|_{B(a,\overline{s}_{n,a})} \big(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\big)^{-1} n^{-(1+r/d)}
\end{aligned}$$

still owing to (5.1).

Thus, (5.8) follows for $c_{28} = \lambda^d(B(0,1)) c_{26}^{r+d}$. □

LEMMA 5.3. *There exists real constants $c_{29}, c_{30} > 0$ such that, for every $n \geq 2$ and every $a \in \alpha_n$,*

$$\underline{s}_{n,a} \geq c_{29} \frac{(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})})^{1-1/(r+d)}}{\|h\|_{B(a,\overline{s}_{n,a})}} n^{-1/d} \qquad \text{for } r \geq 1 \tag{5.9}$$

and

$$\underline{s}_{n,a} \geq c_{30} \left(\frac{(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})})^{d/(r+d)}}{\|h\|_{B(a,\overline{s}_{n,a})}} \right)^{1/r} n^{-1/d} \qquad \text{for } 0 < r < 1. \tag{5.10}$$

PROOF. By the second micro–macro inequality (Proposition 2.2) combined with Proposition 2.3, there is a constant $c > 0$ such that

$$\forall n \geq 2 \qquad cn^{-(1+r/d)} \leq \int_{W_0(a|\alpha_n)} \big(d(x, \alpha_n \setminus \{a\})^r - \|x-a\|^r\big)\, dP(x).$$

CASE 1 ($r \geq 1$). As in (4.24) and (4.25), we deduce

$$\begin{aligned} cn^{-(1+r/d)} &\leq \int_{W_0(a|\alpha_n)} r\big(\|x-a\| + d(a, \alpha_n \setminus \{a\})\big)^{r-1} \\ &\qquad\qquad \times d(a, \alpha_n \setminus \{a\})\, dP(x). \end{aligned} \tag{5.11}$$

Since $n \geq 2$ there exists an $\widetilde{a} \in \alpha_n \setminus \{a\}$ with

$$W(a|\alpha_n) \cap W(\widetilde{a}|\alpha_n) \neq \varnothing.$$

Let $z \in W(a|\alpha_n) \cap W(\widetilde{a}|\alpha_n)$ be arbitrary. Then we have

$$\|z-a\| = d(z, \alpha_n) = \|z - \widetilde{a}\|$$

and, hence

$$d(a, \alpha_n \setminus \{a\}) \leq \|a - \widetilde{a}\| \leq \|a - z\| + \|z - \widetilde{a}\| = 2\|z-a\|$$

so that

$$d(a, \alpha_n \setminus \{a\}) \leq 2\overline{s}_{n,a}.$$

It follows from (5.11) that

$$\begin{aligned} cn^{-(1+r/d)} &\leq r(3\overline{s}_{n,a})^{r-1} d(a, \alpha_n \setminus \{a\}) P(W_0(a|\alpha_n)) \\ &\leq r(3\overline{s}_{n,a})^{r-1} d(a, \alpha_n \setminus \{a\}) \|h\|_{B(a,\overline{s}_{n,a})} \lambda^d(B(0,1)) \overline{s}_{n,a}^d \\ &= r3^{r-1} \overline{s}_{n,a}^{r+d-1} \lambda^d(B(0,1)) \|h\|_{B(a,\overline{s}_{n,a})} d(a, \alpha_n \setminus \{a\}). \end{aligned} \tag{5.12}$$

This implies

$$cr^{-1}3^{1-r}(\lambda^d(B(0,1)))^{-1}\big(\|h\|_{B(a,\overline{s}_{n,a})}\big)^{-1} \overline{s}_{n,a}^{1-(r+d)} n^{-(1+r/d)} \leq d(a, \alpha_n \setminus \{a\})$$

and, hence, by (5.1)

$$\begin{aligned} &cr^{-1}3^{1-r}(\lambda^d(B(0,1)))^{-1}\big(\|h\|_{B(a,\overline{s}_{n,a})}\big)^{-1} c_{26}^{1-(r+d)} \\ &\quad \times \big(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\big)^{-(1-(r+d))/(r+d)} n^{-1/d} \\ &\qquad \leq d(a, \alpha_n \setminus \{a\}). \end{aligned}$$

Since $\underline{s}_{n,a} = \frac{1}{2} d(a, \alpha_n \setminus \{a\})$ this leads to (5.9) with

$$c_{29} = \tfrac{1}{2} cr^{-1} 3^{1-r} (\lambda^d(B(0,1)))^{-1} c_{26}^{1-(r+d)}.$$

CASE 2 ($r \le 1$). As in (4.31), we have

$$\begin{aligned} cn^{-1+r/d} &\le d(a, \alpha_n \setminus \{a\})^r P(W_0(a|\alpha_n)) \\ &\le d(a, \alpha_n \setminus \{a\})^r \|h\|_{B(a,\overline{s}_{n,a})} \lambda^d(B(0,1)) \overline{s}_{n,a}^d \end{aligned}$$

and, hence, by (5.1)

$$\begin{aligned} &cn^{-(1+r/d)} \big(\|h\|_{B(a,\overline{s}_{n,a})}\big)^{-1} (\lambda^d(B(0,1)))^{-1} c_{26}^{-d} n \big(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\big)^{d/(r+d)} \\ &\qquad \le d(a, \alpha_n \setminus \{a\})^r, \end{aligned}$$

which implies

$$\begin{aligned} &c^{1/r} \big(\|h\|_{B(a,\overline{s}_{n,a})}\big)^{-1/r} (\lambda^d(B(0,1)))^{-1/r} c_{26}^{-d/r} \\ &\quad \times \big(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\big)^{d/(r(r+d))} n^{-1/d} \\ &\qquad \le d(a, \alpha_n \setminus \{a\}). \end{aligned}$$

Since $\underline{s}_{n,a} = \frac{1}{2} d(a, \alpha_n \setminus \{a\})$ this leads to

$$c_{30} \bigg(\frac{(\operatorname{essinf} h_{b(a,(1+b)\overline{s}_{n,a})})^{d/(r+d)}}{\|h\|_{B(a,\overline{s}_{n,a})}} \bigg)^{1/r} n^{-1/d} \le \underline{s}_{n,a}$$

with

$$c_{30} = \tfrac{1}{2} c^{1/r} (\lambda^d(B(0,1)))^{-1/r} c_{26}^{-d/r}.$$

□

LEMMA 5.4. *There exist constants $c_{31}, c_{32}, c_{33}, c_{34} > 0$ such that, for every $n > 2$ and $a \in \alpha_n$,*

$$\begin{aligned} &P(W_0(a|\alpha_n)) \\ &\ge \begin{cases} c_{31} \bigg(\dfrac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \bigg)^d \big(\operatorname{essinf} h_{B(a,(1+b)\overline{s}_{n,a})}\big)^{r/(r+d)} n^{-1}, \\ \qquad \textit{for } r \ge 1, \\ c_{32} \bigg(\dfrac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \bigg)^{d/r} \big(\operatorname{essinf} h_{|B(a,(1+b)\underline{s}_{n,a})}\big)^{r/(r+d)} n^{-1}, \\ \qquad \textit{for } 0 < r < 1, \end{cases} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} &\int_{W_0(a|\alpha_n)} \|x - a\|^r \, dP(x) \\ &\ge \begin{cases} c_{33} \bigg(\dfrac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \bigg)^{r+d} n^{-(1+r/d)}, & \textit{for } r \ge 1, \\ c_{34} \bigg(\dfrac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \bigg)^{1+d/r} n^{-(1+r/d)}, & \textit{for } 0 < r < 1. \end{cases} \end{aligned} \tag{5.14}$$

PROOF. First, we will prove (5.13). We have

$$P(W_0(a|\alpha_n)) \geq P(B(a, \underline{s}_{n,a})) = \int_{B(a,\underline{s}_{n,a})} h\, d\lambda^d$$
$$\geq \operatorname{essinf} h_{|B(a,\underline{s}_{n,a})} \lambda^d(B(0,1)) \underline{s}_{n,a}^d$$
$$\geq \operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})} \lambda^d(B(0,1)) \underline{s}_{n,a}^d.$$

Using (5.9), we obtain

$$P(W_0(a|\alpha_n)) \geq \lambda^d(B(0,1)) c_{29}^d \left(\frac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \right)^d \times \left(\operatorname{essinf} h_{|B(a,(1+b)\underline{s}_{n,a})}\right)^{r/(r+d)} n^{-1}$$

for $r \geq 1$ and using (5.10) we get

$$\begin{aligned} P(W_0(a|\alpha_n)) &\geq \lambda^d(B(0,1)) c_{30}^d \left(\|h\|_{B(a,\overline{s}_{n,a})}\right)^{-d/r} \\ &\quad \times \left(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\right)^{(d/(r+d))(d/r)+1} n^{-1} \\ &= \lambda^d(B(0,1)) c_{30}^d \left(\frac{\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}}{\|h\|_{B(a,\overline{s}_{n,a})}} \right)^{d/r} \\ &\quad \times \left(\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})}\right)^{r/(r+d)} n^{-1} \end{aligned}$$

for $0 < r < 1$. With $c_{31} = \lambda^d(B(0,1)) c_{29}^d$ and $c_{32} = \lambda^d(B(0,1)) c_{30}^d$ we deduce (5.13).

Now we will prove (5.14). We have

$$\begin{aligned} \int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) &\geq \int_{B(a,\underline{s}_{n,a})} \|x-a\|^r \operatorname{essinf} h_{|B(a,\underline{s}_{a,n})} \, d\lambda^d(x) \\ &\geq \left(\operatorname{essinf} h_{|B(a,\underline{s}_{n,a})}\right) \int_{B(a,\underline{s}_{n,a})} \|x-a\|^r \, d\lambda^d(x). \end{aligned}$$

Now

$$\int_{B(a,\underline{s}_{n,a})} \|x-a\|^r \, d\lambda^d(x) = \underline{s}_{n,a}^{r+d} \int_{B(0,1)} \|x\|^r \, d\lambda^d(x)$$

so that

$$\int_{W_0(a|\alpha_n)} \|x-a\|^r \, dP(x) \geq \int_{B(0,1)} \|x\|^r \, d\lambda^d(x) \operatorname{essinf} h_{|B(a,\underline{s}_{n,a})} \underline{s}_{n,a}^{r+d}.$$

Using Lemma 5.3, we obtain (5.14) with $c_{33} = \int_{B(0,1)} \|x\|^r \, d\lambda^d(x) c_{29}^{r+d}$ and $c_{34} = \int_{B(0,1)} \|x\|^r \, d\lambda^d(x) c_{30}^{r+d}$. $\square$

LEMMA 5.5. *Let* $K \subset \overset{\circ}{\overbrace{\mathrm{supp}(P)}}$ *be an arbitrary compact set and let*

$$\varepsilon \in \big(0, d\big(K, \mathbb{R}^d \setminus \overset{\circ}{\overbrace{\mathrm{supp}(P)}}\big)\big)$$

be arbitrary [*where* $d(K, \varnothing) = \infty$]. *Then there exists an* $n_{K,\varepsilon} \in \mathbb{N}$ *such that*

$$\forall n \geq n_{K,\varepsilon}, \forall a \in \alpha_n(K) \qquad \overline{s}_{n,a} \leq \varepsilon, \tag{5.15}$$

where $\alpha_n(K) = \{a \in \alpha_n | W(a|\alpha_n) \cap K \neq \varnothing\}$.

PROOF. The proof is identical to that of Lemma 4.1. $\square$

THEOREM 5.1. *Let* P *be an absolutely continuous Borel probability measure on* $\mathbb{R}^d$ *with density* h *and* $\int \|x\|^{r+\delta}\,dP(x) < \infty$ *for some* $\delta > 0$. *Then there exist constants* $c_{27}, c_{28}, c_{31}, c_{32}, c_{33}, c_{34} > 0$ *such that, for every compact* $K \subset \overset{\circ}{\overbrace{\mathrm{supp}(P)}}$, *the following hold*:

$$\begin{aligned}&\limsup_{n\to\infty} n \max_{a\in\alpha_n(K)} P(W(a|\alpha_n))\\ &\qquad \leq c_{27} \inf_{\varepsilon>0} \frac{\|h\|_{K_\varepsilon}}{(\operatorname{essinf} h_{K_\varepsilon})^{d/(r+d)}},\end{aligned} \tag{5.16}$$

$$\begin{aligned}&\limsup_{n\to\infty} n^{1+r/d} \max_{a\in\alpha_n(K)} \int_{W(a|\alpha_n)} \|x-a\|^r\,dP(x)\\ &\qquad \leq c_{28} \inf_{\varepsilon>0} \frac{\|h\|_{K_\varepsilon}}{\operatorname{essinf} h_{|K_\varepsilon}},\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\liminf_{n\to\infty} n \min_{a\in\alpha_n(K)} P(W_0(a|\alpha_n))\\ &\qquad \geq \begin{cases} c_{31} \inf_{\varepsilon>0} \left(\dfrac{\operatorname{essinf} h_{|K_\varepsilon}}{\|h\|_{K_\varepsilon}}\right)^{d} (\operatorname{essinf} h_{|K_\varepsilon})^{r/(r+d)}, \\ \qquad \textit{for } r \geq 1, \\ c_{32} \inf_{\varepsilon>0} \left(\dfrac{\operatorname{essinf} h_{|K_\varepsilon}}{\|h\|_{K_\varepsilon}}\right)^{d/r} (\operatorname{essinf} h_{|K_\varepsilon})^{r/(r+d)}, \\ \qquad \textit{for } 0 < r < 1,\end{cases}\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}&\liminf_{n\to\infty} n^{1+r/d} \min_{a\in\alpha_n(K)} \int_{W_0(a|\alpha_n)} \|x-a\|^r\,dP(x)\\ &\qquad \geq \begin{cases} c_{33} \inf_{\varepsilon>0} \left(\dfrac{\operatorname{essinf} h_{|K_\varepsilon}}{\|h\|_{K_\varepsilon}}\right)^{r+d}, \\ \qquad \textit{for } r \geq 1, \\ c_{34} \inf_{\varepsilon>0} \left(\dfrac{\operatorname{essinf} h_{|K_\varepsilon}}{\|h\|_{K_\varepsilon}}\right)^{1+d/r}, \\ \qquad \textit{for } 0 < r < 1.\end{cases}\end{aligned} \tag{5.19}$$

PROOF. Let $\varepsilon > 0$ satisfy $\varepsilon < d(K, \mathbb{R}^d \setminus \overset{\circ}{\overbrace{\operatorname{supp}(P)}})$. By Lemma 5.5 there exists an $n_{K,\varepsilon} \in \mathbb{N}$ such that

$$\forall n \geq n_{K,\varepsilon}, \forall a \in \alpha_n(K) \qquad \overline{s}_{n,a} < \frac{\varepsilon}{2(1+b)}.$$

This implies

$$\forall n \geq n_{K,\varepsilon}, \forall a \in \alpha_n(K) \qquad B\big(a, (1+b)\overline{s}_{n,a}\big) \subset K_\varepsilon$$

and, therefore,

$$\|h\|_{B(a,(1+b)\overline{s}_{n,a})} \leq \|h\|_{K_\varepsilon}$$

as well as

$$\operatorname{essinf} h_{|B(a,(1+b)\overline{s}_{n,a})} \geq \operatorname{essinf} h_{|K_\varepsilon}$$

for all $n \geq n_{K,\varepsilon}$ and all $a \in \alpha_n(K)$.

These inequalities combined with Lemma 5.2 and Lemma 5.4 yield the assertions of the theorem. □

REMARK. The above theorem yields estimates for the asymptotics of the local cell probabilities and quantization errors only if the density h is essentially bounded and bounded away from 0 on each compact subset of the interior of the support of P.

COROLLARY 5.1. *For every $x \in \mathbb{R}^d$, let $a_{n,x} \in \alpha_n$ satisfy $x \in W(a_{n,x}|\alpha_n)$. Assume that $x \in \overset{\circ}{\overbrace{\operatorname{supp}(P)}}$ and h is continuous at x. Then*

$$\begin{aligned}\min(c_{31}, c_{32}) h(x)^{r/(r+d)} &\leq \liminf_{n\to\infty} n P(W_0(a_{n,x}|\alpha_n)) \\ &\leq \limsup_{n\to\infty} n P(W(a_{n,x}|\alpha_n)) \leq c_{27} h(x)^{r/(r+d)}\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\min(c_{33}, c_{34}) &\leq \liminf_{n\to\infty} n^{1+r/d} \int_{W(a_{n,x}|\alpha_n)} \|y - a_{n,x}\|^r \, dP(y) \\ &\leq \limsup_{n\to\infty} n^{1+r/d} \int_{W(a_{n,x}|\alpha_n)} \|y - a_{n,x}\|^r \, dP(y) \leq c_{28}.\end{aligned} \tag{5.21}$$

PROOF. Set $K = \{x\}$ in Theorem 5.1. □

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