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GENERALIZED SELF-INTERSECTION LOCAL TIME FOR A SUPERPROCESS OVER A STOCHASTIC FLOW

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This paper examines the existence of the self-intersection local time for a superprocess over a stochastic flow in dimensions $d \leq 3$, which through constructive methods, results in a Tanaka-like representation. The superprocess over a stochastic flow is a superprocess with dependent spatial motion, and thus Dynkin's proof of existence, which requires multiplicity of the log-Laplace functional, no longer applies. Skoulakis and Adler's method of calculating moments is extended to higher moments, from which existence follows.

1. Introduction. Superprocesses (or critical branching particle systems), originally studied by Watanabe (1968) and Dawson (1977, 1993) were first shown by Dynkin (1988) to have a self-intersection local time (SILT). In particular, Dynkin was able to show existence of the self-intersection local time for super Brownian motion in $\mathbb{R}^d$, $d \leq 7$, provided the SILT is defined over a region that is bounded away from the diagonal. When the region contains any part of the diagonal, through renormalization, the SILT for super Brownian motion has been shown by Adler and Lewin (1992) to exist in $d \leq 3$, and further renormalization processes have been found to establish existence in higher dimensions by Rosen (1992) and Adler and Lewin (1991). In regards to non-Gaussian superprocesses, the SILT has been shown to exist for certain α-stable processes by Adler and Lewin (1991), and more recently, encompassing more α values, by Mytnik and Villa (2007). Of important note, as the L^2-limit of an appropriate approximating process, Adler and Lewin have shown the existence of a class of renormalized SILTs (indexed on $\lambda > 0$) for the super Brownian motion in dimensions $d = 4$ and 5 and for the super α-stable processes for $d \in [2\alpha, 3\alpha)$. As one removes Dynkin's restriction of bounding away from the diagonal, a singularity arises from "local double points" (i.e., $\mu_s \times \mu_t$ where $t = s$) of the process; cf. Adler and Lewin (1992). The true self-intersection local time should not be concerned with such local double points, and thus a heuristic approach to renormalization is naturally observed in the construction. It should be noted that though this is the method used in Adler and Lewin (1991, 1992), a quite different method for renormalization was developed by Rosen (1992). Both methods are legitimate renormalizations, and lead to existence in equivalent dimensions, but for this paper, due to the natural occurrence of

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the term involving local double points, the initial of the two methods will be employed. Moreover, the real beauty of this constructive proof of existence, as seen in Adler and Lewin (1991, 1992), is that the aforementioned approximating process is "Tanaka-like" in form. Thus the limit gives a (quite simple) "Tanaka-like" representation for the renormalized SILT.

Quite often, as in the case of Skoulakis and Adler (2001), interaction occurs between particles within the system. Thus, a major drawback in each of the previous superdiffusions is the requirement of independent spatial motion. Existence as a weak limit of a branching particle system, and uniqueness as the solution to a martingale problem, of the superprocess with dependent spatial motion (SDSM), as a measure-valued Markov process with state space $M(\hat{\mathbb{R}})$,was shown by Wang (1998). It was later shown by Dawson, Li and Wang (2001) to exist uniquely as a process in $M(\mathbb{R})$, and was then extended by Ren, Song and Wang (2009) to $M(\mathbb{R}^d)$. Skoulakis and Adler (2001) suggested a different model incorporating dependent spatial motion by replacing the space–time white noise of Wang's SDSM with a Brownian flow of homeomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^d$, which was referred to as a Superprocess over a Stochastic Flow (SSF).

As of yet, very little work has been done with regard to the self-intersection local time for superprocesses with dependent spatial motion. Of important note is the work of He (2009), in which the existence of the SILT for a superprocess with dependent spatial motion, similar to the model of Wang, but discontinuous, is shown to exist in one dimension as a probabilistic limit. Though this was known to be true, since the local time of the superprocess with dependent spatial motion was known to exist in one dimension [cf. Dawson, Li and Wang (2001)], He was able to give a similar "Tanaka-like" representation for the SILT. This paper will investigate the existence and further properties of a generalized SILT for the d-dimensional SSF, where the generalization refers to the shift of the support of the Dirac measure away from the origin, to a point $u \in \mathbb{R}^d$. Note that if X_t is a Markov process, then $Y_t \triangleq X_t + u$ is a second, dependent Markov Process. The generalized SILT at u can be realized as the intersection local time of the Markov processes X_t and Y_t.

2. Preliminary definitions. The SSF is constructed as the weak limit of an $\mathbb{R}^d$ branching particle system. Much of the work that will follow involves using properties of the branching particle system, and thus we will briefly review this construction. This section follows very closely to the work of Skoulakis and Adler (2001), and the reader is referenced to this work for further questions. We will let $\overline{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Delta\}$ denote the one-point (Alexandroff) compactification of $\mathbb{R}^d$, where Δ denotes the "cemetery." We extend measurable functions $\phi \in \mathcal{B}(\mathbb{R}^d)$ to $\mathcal{B}(\overline{\mathbb{R}}^d)$ by setting $\phi(\Delta) = 0$.

Let $\mathbb{N} = \{1, 2, \ldots\}$ and set

$$I \triangleq \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N) : N \geq 0, \alpha_0 \in \mathbb{N}, \alpha_i \in \{0, 1\}, 1 \leq i \leq N\},$$

and for any $\alpha = (\alpha_0, \dots, \alpha_N) \in I$, let $|\alpha| = N$ and $\alpha - i = (\alpha_0, \dots, \alpha_{|\alpha|-i})$. In addition, we will write $\alpha \sim_n t$ exactly when $t \in [\frac{|\alpha|}{n}, \frac{|\alpha|+1}{n})$. Let $M(n)$ be the number of particles alive at time zero, where the spatial position of each particle is written as $(x_1^n, x_2^n, \dots, x_{M(n)}^n)$, and define the initial (atomic) measure by

$$\mu_0^{(n)} \triangleq \sum_{i=1}^{M(n)} \delta_{x_i^n}.$$

For each $n \in \mathbb{N}$, $\{B^{\alpha,n} : \alpha_0 \le M(n), |\alpha| = 0\}$ is defined to be a family of independent $\mathbb{R}^d$ Brownian motions, stopped at time $t = n^{-1}$, with $B_0^{\alpha,n} = x_{\alpha_0}^n$. A recursive definition then gives a tree: for each $k \in \mathbb{N}$, let $\{B^{\alpha,n} : \alpha_0 \le M(n), |\alpha| = k\}$ be a collection of $\mathbb{R}^d$ valued Brownian motions, stopped at time $t = (|\alpha|+1)n^{-1}$, and conditionally independent given the σ-field generated by $\{B^{\alpha,n} : \alpha_0 \le M(n), |\alpha| < k\}$ and for which

$$B_t^{\alpha,n} = B_t^{\alpha-1,n}, \qquad t \le |\alpha| n^{-1}.$$

In regards to branching, for $n \in \mathbb{N}$ let $\{N^{\alpha,n} : \alpha_0 \le M(n)\}$ be a family of i.i.d. copies of N_n, where N_n is an $\mathbb{N}$-valued random variable such that

$$\mathbb{P}(N_n = k) = \begin{cases} \frac{1}{2}, & k = 2, \\ \frac{1}{2}, & k = 0. \end{cases}$$

Note that it is implicit in the above that the branching is assumed to be binary, and that for each $n \in \mathbb{N}$,

$$\mathbb{E}N_n = 1,$$
$$\mathbb{E}N_n^2 - (\mathbb{E}N_n)^2 = 1$$

and

$$\mathbb{E}N_n^q = 2^{q-1}, \qquad q \in \mathbb{N}.$$

Moreover, it is assumed that the families $\{B^{\alpha,n} : \alpha_0 \le M(n)\}$ and $\{N^{\alpha,n} : \alpha_0 \le M(n)\}$ are independent.

The final component is that of the stochastic flow. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \to M(d,m)$, where $M(d,m)$ is the space of $d \times m$ matrices, $m \in \mathbb{N}$, satisfying the following:

(i) the global Lipschitz condition

$$|b(x) - b(y)| + |c(x) - c(y)| \le C|x - y|$$

for any $x, y \in \mathbb{R}^d$;

(ii) the linear growth condition,

$$|b(x)| + |c(x)| \le C(1 + |x|)$$

for any $x \in \mathbb{R}^d$;

(iii) for all $i = 1, 2, \dots, d$, $j = 1, 2, \dots, m$, b_i and c_{ij} are bounded with bounded and continuous first and second partial derivatives.

Assume that $t \mapsto F^n_{s,t}(x)$ is the solution of the stochastic differential equation

$$dY_t = c(Y_t)\, dW^n_t, \qquad Y_s = x,$$

for all $t \geq s$ and $x \in \mathbb{R}^d$, where W^n is a $\mathbb{R}^m$-valued Brownian motion, independent of the families $\{B^{\alpha,n}\}$ and $\{N^{\alpha,n}\}$. This defines a unique Brownian flow of homeomorphisms from $\mathbb{R}^d \to \mathbb{R}^d$ [Skoulakis and Adler (2001)].

Set $a_n = n^{-1}$ and $k_n = kn^{-1}$. Then the tree of Brownian motions over the flow is given by the family of processes $Y^{\alpha,n}$, defined in the following way: let $\alpha \sim_n k_n$ for some $k \in \mathbb{N}$. Over the time interval $[0, k_n + a_n]$, $Y^{\alpha,n}$ is defined to be the solution of the d-dimensional stochastic differential equation,

$$\begin{aligned} dY_t &= b(Y_t)\, dB^{\alpha,n}_t + c(Y_t)\, dW^n_t, \\ Y_0 &= x^n_{\alpha_0}. \end{aligned}$$

Note that existence and strong uniqueness of the aforementioned solution is ensured due to the assumed conditions on b and c. Now set $Y^{\alpha,n}_t = Y^{\alpha,n}_{k_n+a_n}$ for $t > k_n + a_n$ and note that due to construction,

$$Y^{\alpha,n}_t = Y^{\alpha-1,n}_t$$

for $0 \leq t \leq k_n$, $k \in \mathbb{N}$.

We now define the stopping times $\tau^{\alpha,n}$ as follows: for each $\alpha \in I$, let

$$\tau^{\alpha,n} = \begin{cases} 0, & \text{if } \alpha_0 > K_n, \\ \min\left\{\dfrac{i+1}{n} : 0 \leq i \leq |\alpha|, N^{\alpha|_i,n} = 0\right\}, & \text{if not } \varnothing \text{ and } \alpha_0 \leq M(n), \\ \dfrac{1+|\alpha|}{n}, & \text{otherwise.} \end{cases}$$

The stopped tree of processes, with branching, is the family of processes $X^{\alpha,n}$ defined by

$$X^{\alpha,n}_t = \begin{cases} Y^{\alpha,n}_t, & t < \tau^{\alpha,n}, \\ \Delta, & t \geq \tau^{\alpha,n}. \end{cases}$$

The measure-valued process for the finite system of particles is

$$\mu^{(n)}_t(U) = \frac{\#\{\alpha \sim_n t : X^{\alpha,n}_t \in U\}}{n}$$

for $U \in \mathcal{B}(\mathbb{R}^d)$, where for a topological space E, $\mathcal{B}(E)$ denotes the σ-field of Borel measurable sets in E.

We define the corresponding filtration $\mathcal{F}^n$ by

$$\mathcal{F}_t^n \triangleq \sigma(B^{\alpha,n}, N^{\alpha,n} : |\alpha| < k) \vee \sigma(W_s^n : s \le t) \vee \sigma(B_s^{\alpha,n} : s \le t, |\alpha| = k)$$

for $t \in [k_n, k_n + a_n)$, $k = 0, 1, \ldots$.

Let $C^k(E)$ be the space of continuous functions on E having continuous partial derivatives up to order k, and for $\phi \in C^k(R^d)$ let

$$\partial^k_{i_1 i_2 \cdots i_k} \phi(x) = \left(\frac{\partial^k \phi}{\partial x_{i_1}\, \partial x_{i_2} \cdots \partial x_{i_k}} \right)(x).$$

For $\phi \in C^2(\mathbb{R}^d)$ define the second-order operators L and Λ by

$$(L\phi)(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x, x)\, \partial^2_{ij} \phi(x) \tag{1}$$

and

$$(\Lambda\phi)(x, y) = \sum_{i,j=1}^{d} \sigma_{ij}(x, y)\, \partial_i \phi(x)\, \partial_j \phi(y),$$

where

$$a_{ij}(x, y) = \delta_{ij} b_i(x) b_j(y) + \sigma_{ij}(x, y)$$

and

$$\sigma_{ij}(x, y) = \sum_{\ell=1}^{m} c_{i\ell}(x) c_{j\ell}(y),$$

$x, y \in \mathbb{R}^d$, $i, j = 1, \ldots, d$.

Furthermore, for each $n \in \mathbb{N}$, $\phi \in C^2(\mathbb{R}^{n \times d})$ define the second-order operator L^n by

$$(L^n\phi)(x) = \frac{1}{2} \sum_{p,q=1}^{n} \sum_{i,j=1}^{d} a^{pq}_{ij}(x)\, \partial_{p_i}\, \partial_{q_j} \phi(x), \tag{2}$$

where

$$a^{pq}_{ij}(x) = \delta_{pq} \delta_{ij} b_i(x_p) b_j(x_q) + \sigma_{ij}(x_p, x_q),$$

$x = (x_1, \ldots, x_n)$, $x_p \in \mathbb{R}^d$, $p = 1, \ldots, n$, and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

For any operator A on a Banach space $\mathcal{B}$, such that $A\phi = \lim_{t\to 0} t^{-1}\{T_t\phi - \phi\}$ for some semigroup T_t, we will denote by $\mathcal{D}(A) \subset \mathcal{B}$ the domain of A. That is,

$$\mathcal{D}(A) = \Big\{\phi \in \mathcal{B} : \lim_{t\to 0} t^{-1}\{T_t\phi - \phi\} \text{ exists}\Big\},$$

where the limit is in the strong sense.

ASSUMPTION 2.1. For the remainder of this paper, the assumption will be made that L is uniformly elliptic.

For each $k \in \mathbb{N}$ we will denote by $C_0^k(\mathbb{R}^d)$ the subspace of functions in $C^k(\mathbb{R}^d)$ which vanish at infinity.

For any topological space E, let $M_F(E)$ denote the space of finite Borel measures on E, $C_E[0,\infty)$ the space of continuous paths in E and for any $\ell \in \mathbb{N}$, $C_K^\ell(\mathbb{R}^d)$ the subspace of $C^\ell(\mathbb{R}^d)$ for which the elements have compact support.

Endow $D_{M_F(\mathbb{R}^d)}[0,\infty)$ with the topology of weak convergence, that is, $\mu^{(n)} \in D_{M_F(\mathbb{R}^d)}[0,\infty)$ converges to $\mu \in D_{M_F(\mathbb{R}^d)}[0,\infty)$ provided $\lim_{n\to\infty}\langle \phi, \mu^{(n)}\rangle = \langle \phi, \mu\rangle$ for any $\phi \in C_b(\mathbb{R}^d)$, and let $\Rightarrow$ denote weak convergence. In addition, for any $\mu \in M_F(E)$ and $\ell \in \mathbb{N}$, denote by μ^ℓ the product measure $\mu \times \mu \times \cdot \times \mu \in M_F(\mathbb{R}^{\ell\times d})$. Under these assumptions, and Assumption 2.1 upon L, we arrive at the following theorem.

THEOREM 2.2. *Let $\mu^{(n)}$ be defined as above with $\mu_0^{(n)} \Rightarrow \mu_0$, then $\mu^{(n)} \Rightarrow \mu$, where $\mu \in C_{M_F(\mathbb{R}^d)}[0,\infty)$ is the unique solution of the following martingale problem*:

For all $\phi \in C_K^2(\mathbb{R}^d)$,

$$Z_t(\phi) = \langle \phi, \mu_t\rangle - \langle \phi, \mu_0\rangle - \int_0^t ds\langle L\phi, \mu_s\rangle \tag{3}$$

is a continuous square integrable $\{\mathcal{F}_t^\mu\}$-martingale such that $Z_0(\phi) = 0$ and has quadratic variation process

$$\langle Z(\phi)\rangle_t = \int_0^t ds(\langle \phi^2, \mu_s\rangle + \langle \Lambda\phi, \mu_s^2\rangle). \tag{4}$$

PROOF. See Theorem 2.2.1 in Skoulakis and Adler (2001). □

ASSUMPTION 2.3. For the remainder of this work, it will be assumed that $\mu_0 \in M_F(\mathbb{R}^d)$ has compact support and satisfies

$$\mu_0(dx) \le m(x)\,dx$$

for some bounded $m \in L^1(\mathbb{R}^d)$.

3. Some needed lemmata. Some needed technical lemmata, will be presented, where due to the significantly large number of calculations required, the proof is deferred to the Appendix.

As in most existence proofs for self-intersection local time of a superprocess, higher moments of the superprocess are required; cf. Adler and Lewin (1991), Dynkin (1988). Through finding the first and second moments of the branching process, and passing to the limit as $n \to \infty$ [Skoulakis and Adler (2001)] found the first and second moments for the SSF. A variation of this method is employed to find higher moments of the SSF.

LEMMA 3.1. *If L^n is defined as* (2), *then L^n is the generator of the diffusion which describes the joint motion of n particles in the aforementioned branching particle system.*

PROOF. See the Appendix of Skoulakis and Adler (2001). □

LEMMA 3.2. *For each $n \in \mathbb{N}$, there exists a transition function q_t^n for the Markov process $Y_t = (Y_t^1, \dots, Y_t^n)$. Furthermore, $\{Q_t^n : t \geq 0\}$, defined by*

$$Q_t^n \phi(x) = \int_{\mathbb{R}^d} \phi(y) q_t^n(x, y)$$

is a strongly continuous contraction semigroup on $C_0(\mathbb{R}^d)$.

PROOF. Since it is assumed that Assumption 2.1 holds for L, it follows that for each $n \in \mathbb{N}$, Assumption 2.1 also holds for L^n. Theorem 5.11 in Dynkin (1965) then completes the proof. □

We denote by $C^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$, by $C_K^\infty(\mathbb{R}^d)$, the subspace of $C^\infty(\mathbb{R}^d)$ of which the elements have compact support, by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on $C_K^\infty(\mathbb{R}^d)$, and by $D^\alpha u$ the αth-weak partial derivative of u. Note that a differentiable function will have a weak derivative that agrees with the functions derivative, and thus we will at times use a slight abuse in notation and write the weak derivative as $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$.

We denote by S_d the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$, and the dual to S_d, the space of tempered distributions on R^d, by S_d'. For any two functions $\phi : E_1 \to \mathbb{R}$, $\psi : E_2 \to \mathbb{R}$ denote by $\phi \otimes \psi$ the concatenation of ϕ and ψ. That is, $\phi \otimes \psi : E_1 \times E_2 \to \mathbb{R}$ is the map defined by $(x_1, x_2) \mapsto \phi(x_1)\psi(x_2)$.

LEMMA 3.3. *Let $\phi \in S_{\ell \times d}$, then there exists $\{\phi_n : n \in \mathbb{N}\}$ such that*:

(i) $\phi_n = \sum_{k=1}^n \phi_k^1 \otimes \phi_k^2 \otimes \cdots \otimes \phi_k^\ell$, *for some* $\phi_k^1, \dots, \phi_k^\ell \in C_K^\infty(\mathbb{R}^d)$;
(ii) ϕ_n *converges to* ϕ *in* $S_{\ell \times d}$ *as* $n \to \infty$.

PROOF. Taylor's theorem implies the above holds for any $\phi \in C_K^\infty(\mathbb{R}^d)$; cf. Rudin (1976, 1987). From Theorem 7.10 of Rudin (1973) there exist $\{\phi_n : n \in \mathbb{N}\} \subset C_K^\infty(\mathbb{R}^{\ell\times d})$ such that ϕ_n converges to ϕ in S_d, and the result thus follows. □

Given $\phi \in \mathcal{B}(\mathbb{R}^{(n+1)\times d})$, $n \in \mathbb{N}$, define the projection π_1 by

$$(\pi_1 Q_t^n \phi)(x_1, \dots, x_{n+1}) = Q_t^n \phi_{x_1}(x_2, \dots, x_{n+1}),$$

where $\phi_x(y_1, \dots, y_n) = \phi(x, y_1, \dots, y_n)$.

Given $m \in \mathbb{N}$, $i = 1, 2, \dots, m-1$, $j = 1, 2, \dots, m$, $i \neq j$ and any function $\phi : \mathbb{R}^{m\times d} \to \mathbb{R}$, define $(\Phi_{ij}\phi) : \mathbb{R}^{(m-1)\times d} \to \mathbb{R}$ by

$$(\Phi_{ij}\phi)(x_1, \dots, x_{m-1}) = \begin{cases} \phi(x_1, \dots, x_i, \dots, x_{j-1}, x_i, x_j, \dots, x_{m-1}), & i < j, j \neq m, \\ \phi(x_1, \dots, x_i, \dots, x_{m-1}, x_i), & i < j, j = m, \\ \phi(x_1, \dots, x_{j-1}, x_i, x_j, \dots, x_i, \dots, x_{m-1}), & i > j, j \neq 1, \\ \phi(x_i, x_1, \dots, x_i, \dots, x_{m-1}), & i > j, j = 1. \end{cases}$$

Furthermore, for $\mathbf{x}_m = (x_1, x_2, \dots, x_m)$, $x_p \in \{ij, 0\}$, $p = 1, 2, \dots, m$, let

$$\zeta(x_p) = \begin{cases} \Phi_{ij}, & x_p = ij, \\ \pi_1, & x_p = 0, \end{cases}$$

and

$$\ell(x_p) = \begin{cases} \ell(x_{p-1}) + 1, & x_{p-1} = ij, \\ \ell(x_{p-1}) - 1, & x_{p-1} = 0, \end{cases}$$

$\ell(x_0) = \ell \in \mathbb{N}$. Let $\mathbf{s}_m = (s_1, s_2, \dots, s_m)$, $s_p \in [0, \infty)$, $p = 1, 2, \dots, m$, and denote

$$\Gamma_{\ell;\mathbf{s}_m}^{\mathbf{x}_{m-1}} \triangleq Q_{s_1}^{\ell} \zeta(x_1) Q_{s_2-s_1}^{\ell(x_1)} \zeta(x_2) Q_{s_3-s_2}^{\ell(x_2)} \cdots \zeta(x_{m-2}) Q_{s_{m-1}-s_{m-2}}^{\ell(x_{m-2})} \pi_1 Q_{s_m-s_{m-1}}^{\ell(x_{m-1})}. \tag{5}$$

The next lemma comes from Skoulakis and Adler (2001), though it should be noted that in the aforementioned paper the result is shown for near critical branching (as opposed to critical branching in this paper). This is of little concern though, as a modification of the original proof (making for a much simpler proof) gives the critical branching case.

LEMMA 3.4. *Let $\phi, \phi_1, \phi_2 \in C_K^2(\mathbb{R}^d)$ and $t > 0$, then*

$$\text{(i)} \quad \mathbb{E}\mu_t(\phi) = \langle Q_t \phi, \mu_0 \rangle$$

and

$$\text{(ii)} \quad \mathbb{E}\mu_{t_1}(\phi_1)\mu_{t_2}(\phi_2) = \langle Q_{t_1}^2(\pi_1 Q_{t_2-t_1}(\phi_1 \otimes \phi_2)), \mu_0^2 \rangle + \int_0^{t_1} ds \langle Q_s \Phi_{12} Q_{t_1-s}^2(\pi_1 Q_{t_2-t_1}(\phi_1 \otimes \phi_2)), \mu_0 \rangle$$

with the convention that $Q_0^n \phi = \phi$, $n \in \mathbb{N}$.

PROOF. See Skoulakis and Adler (2001), Proposition 3.2.1 [with $(1+\gamma_n/n)$ and $e^{-\lambda\gamma r_n}$ both replaced by 1]. $\square$

Before our moment calculations, some needed definitions and lemmata will be presented. In what follows (S,d) will refer to a metric space, in which it is assumed S is separable, and ρ will denote the Prohorov metric on $M_F(S)$.

LEMMA 3.5. *If $\{\mu^{(n)}:n\geq 0\}\subset M_F(\mathbb{R}^d)$ satisfies $\mu^{(n)}\Rightarrow\mu\in M_F(\mathbb{R}^d)$ then*

$$(\mu^{(n)})^\ell\Rightarrow\mu^\ell$$

for all $\ell\in\mathbb{N}$.

PROOF. Define

$$M=\left\{\phi=\bigotimes_{k=1}^{\ell}\phi_k:\ell\geq 1,\phi_k\in C_K(\mathbb{R}^d)\cup\{1\},k=1,2,\dots,\ell\right\}.$$

From Ethier and Kurtz (1986) Chapter 3, Proposition 4.4, for any $\nu,\nu^{(n)}\in M_F(\mathbb{R}^d)$, $n=1,2,\dots,$ such that $\lim_{n\to\infty}\langle\phi,\nu^{(n)}\rangle=\langle\phi,\nu\rangle$ for all $\phi\in C_K(\mathbb{R}^d)$, it follows that $\nu^{(n)}\Rightarrow\nu$. For any $\ell\in\mathbb{N}$, since $\mu^{(n)}\Rightarrow\mu$, $\lim_{n\to\infty}\langle\phi,(\mu^{(n)})^\ell\rangle=\langle\phi,\mu^\ell\rangle$, for any $\phi=\bigotimes_{k=1}^{\ell}\phi_k$ with $\phi_k\in C_K(\mathbb{R}^d)$ or $\phi_k\in\{1\}$, $k=1,\dots,\ell$. Thus, for any $\phi=\bigotimes_{k=1}^{\ell}\phi_k\in M$, $\lim_{n\to\infty}\langle\phi,(\mu^{(n)})^\ell\rangle=\langle\phi,\mu^\ell\rangle$, which implies, by Ethier and Kurtz (1986), Chapter 3, Proposition 4.6, $(\mu^{(n)})^\ell\Rightarrow\mu^\ell$. $\square$

We denote by $D_S[0,\infty)$ the Skorohod space on S, that is, the space of all càdlàg mappings from $[0,\infty)$ to S. Note that under the assumption that S is separable, $D_S[0,\infty)$ with the metric defined by Ethier and Kurtz (1986), Chapter 3, (5.2), is a separable metric space. Moreover, if (S,d) is complete, $D_S[0,\infty)$ is complete; cf. Ethier and Kurtz (1986), Theorem 5.6, Chapter 3. For $\phi\in C_b(S)$ define the metric $\|\phi\|_{bL}=\|\phi\|_\infty\vee\sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{d(x,y)}$. The next two lemmata are essential in the moment proofs for the superprocess.

LEMMA 3.6. *For $k,\ell\in\mathbb{N}$, let $\psi:\mathbb{R}_+^k\times\mathbb{R}^\ell\to\mathbb{R}$ in $C_b(\mathbb{R}_+^k\times\mathbb{R}^\ell)$ satisfy $\sup_{s\in\mathbb{R}_+^k}\|\psi(s,\cdot)\|_{bL}<\infty$, and let μ_0 be an a.s. finite measure having compact support with $\mu_0^{(n)}\Rightarrow\mu_0$. Then*

$$\lim_{n\to\infty}\Bigg|\frac{1}{n^k}\sum_{\substack{r_1,\dots,r_k=0\\ r_1<\cdots<r_k}}^{[nt]-1}\left\langle\psi\left(\frac{r}{n},\cdot\right),(\mu_0^{(n)})^\ell\right\rangle$$

$$-\int_0^t ds_k\int_0^{s_k}ds_{k-1}\cdots\int_0^{s_2}ds_1\langle\psi(s,\cdot),\mu_0^\ell\rangle\Bigg|=0,$$

where $r=(r_1,\dots,r_k)$ and $s=(s_1,\dots,s_k)$.

PROOF. Indeed,

$$\begin{aligned}&\left|\frac{1}{n^k}\sum_{\substack{r_1,\dots,r_k=0\\r_1<\cdots<r_k}}^{[nt]-1}\left\langle\psi\left(\frac{r}{n},\cdot\right),(\mu_0^{(n)})^{\ell}\right\rangle-\int_0^t ds_k\int_0^{s_k}ds_{k-1}\cdots\int_0^{s_2}ds_1\langle\psi(s,\cdot),\mu_0^{\ell}\rangle\right|\\&\quad\le\frac{1}{n^k}\sum_{\substack{r_1,\dots,r_k=0\\r_1<\cdots<r_k}}^{[nt]-1}\left|\left\langle\psi\left(\frac{r}{n},\cdot\right),(\mu_0^{(n)})^{\ell}\right\rangle-\left\langle\psi\left(\frac{r}{n},\cdot\right),\mu_0^{\ell}\right\rangle\right|\\&\qquad+\left\langle\left|\frac{1}{n^k}\sum_{\substack{r_1,\dots,r_k=0\\r_1<\cdots<r_k}}^{[nt]-1}\psi\left(\frac{r}{n},\cdot\right)-\int_0^t ds_k\int_0^{s_k}ds_{k-1}\cdots\int_0^{s_2}ds_1\psi(s,\cdot)\right|,\mu_0^{\ell}\right\rangle.\end{aligned}$$

By assumption $\sup_s\|\psi(s,\cdot)\|_{bL}<\infty$, and thus from Ethier and Kurtz (1986) the first of the above terms converges to zero. Since ψ is continuous and bounded, and μ_0^{ℓ} is finite with compact support, it follows that the second term is also convergent toward zero. □

LEMMA 3.7. *For any $\phi_i\in C_K^{\infty}(\mathbb{R}^d)$, $i=1,2,\dots,\ell$, $\ell\in\mathbb{N}$, $0<t<\infty$,*

$$\lim_{n\to\infty}\mathbb{E}\langle\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_\ell,(\mu_t^{(n)})^{\ell}\rangle=\mathbb{E}\langle\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_\ell,\mu_t^{\ell}\rangle.$$

PROOF. Let $\mu^{(n)}=\{\mu_t^{(n)}:t\ge 0\}$ be a branching process as defined above, let μ be a weak limit point of $\mu^{(n)}$, and let $\{n_k\}$ be the subsequence along which $\mu^{(n_k)}\Rightarrow\mu$. From Ethier and Kurtz (1986), Theorem 3.1, Chapter 3, there is a Skorohod representation for $\mu,\mu^{(n_k)}$, $k\in\mathbb{N}$. That is, there exist random variables X,X_k, $k\in\mathbb{N}$, defined on the same probability space, such that $X\overset{d}{=}\mu$, $X_k\overset{d}{=}\mu^{(n_k)}$, $k\in\mathbb{N}$, and $X_k\to X$ a.s. as $k\to\infty$.

For $X\in D_{M_F(\mathbb{R}^d)}[0,\infty)$, define $\mathbb{P}X(\phi_i)^{-1}$ to be the distribution of $X(\phi_i)\in D_{\mathbb{R}}[0,\infty)$ then, by dominated convergence

$$\begin{aligned}&\lim_{k\to\infty}\sup_{\|\psi\|_{bL}=1}\left|\left\langle\psi,\prod_{i=1}^{\ell}\mathbb{P}X_k(\phi_i)^{-1}\right\rangle-\left\langle\psi,\prod_{i=1}^{\ell}\mathbb{P}X(\phi_i)^{-1}\right\rangle\right|\\&\quad=\lim_{k\to\infty}\sup_{\|\psi\|_{bL}=1}|\mathbb{E}\psi(X_k(\phi_1),\dots,X_k(\phi_\ell))-\mathbb{E}\psi(X(\phi_1),\dots,X(\phi_\ell))|\\&\quad=0.\end{aligned}$$

It then follows from Ethier and Kurtz (1986) that

$$\lim_{k\to\infty}\rho\left(\prod_{i=1}^{\ell}\mathbb{P}X_k(\phi_i)^{-1},\prod_{i=1}^{\ell}\mathbb{P}X(\phi_i)^{-1}\right)=0$$

or equivalently,

$$(X_k(\phi_1), \dots, X_k(\phi_\ell)) \Rightarrow (X(\phi_1), \dots, X(\phi_\ell))$$

in $D_{\mathbb{R}^\ell}[0, \infty)$. Therefore, from Theorem 2.2,

$$\big(\mu^{(n_k)}(\phi_1), \dots, \mu^{(n_k)}(\phi_\ell)\big) \Rightarrow (\mu(\phi_1), \dots, \mu(\phi_\ell))$$

in $D_{\mathbb{R}^\ell}[0, \infty)$. Thus, from Lemma A.3.9 of Skoulakis and Adler (2001), for $i = 1, 2, \dots, \mu(\phi_i)$ is continuous. Therefore, the open mapping theorem [Ethier and Kurtz (1986), Chapter 3, Corollary 1.9] implies that

$$\big(\mu_t^{(n_k)}(\phi_1), \dots, \mu_t^{(n_k)}(\phi_\ell)\big) \Rightarrow (\mu_t(\phi_1), \dots, \mu_t(\phi_\ell))$$

in $\mathbb{R}^\ell$, which further implies that

$$\mu_t^{(n_k)}(\phi_1) \cdot \mu_t^{(n_k)}(\phi_2) \cdots \mu_t^{(n_k)}(\phi_\ell) \Rightarrow \mu_t(\phi_1) \cdot \mu_t(\phi_2) \cdots \mu_t(\phi_\ell)$$

in $\mathbb{R}$. Note that [cf. (3.1) in Skoulakis and Adler (2001)] for any $t \geq 0$, $\mathbb{E}\mu_t^{(n)}(1) = \mu_0^{(n)}(1)$, and thus $\{\mu_t^{(n)}(1) : t \geq 0\}$ is an $\mathcal{F}_t^n$-martingale. It follows from Doob's maximal inequality [Karatzas and Shreve (2000), Theorem 3.8] that for any $T \geq 0$,

$$\mathbb{E} \sup_{0 \leq t \leq T} \big[\mu_t^{(n)}(1)\big]^\ell \leq \left(\frac{\ell}{\ell - 1}\right)^\ell \mathbb{E}\big[\mu_T^{(n)}(1)\big]^\ell.$$

Since $\mu_0^{(n)} \Rightarrow \mu_0$, $\lim_{n\to\infty} \mu_0^{(n)}(1) = \mu_0(1)$, and thus, $\sup_{n\geq 1} \mu_0^{(n)}(1) < \infty$. Since $\mu_t^{(n)}(1)$ is the total mass process of the branching particle system, and is absent of influence by the stochastic flow, $[\mu_T^{(n)}(1)]^\ell$ is equivalent in distribution to a total mass process with an initial $M(n)^\ell$ particles, which implies $\mathbb{E}[\mu_T^{(n)}(1)]^\ell = [\mu_0^{(n)}(1)]^\ell$. Thus, $\sup_{n\geq 1} \mathbb{E} \sup_{0\leq t\leq T} [\mu_t^{(n)}(1)]^\ell < \infty$. Theorem 25.12 of Billingsley (1995) implies $\lim_{k\to\infty} \mathbb{E} \prod_{i=1}^\ell \mu_t^{(n_k)}(\phi_i) = \mathbb{E} \prod_{i=1}^\ell \mu_t(\phi_i)$, and thus,

$$\lim_{n\to\infty} \mathbb{E}\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, \big(\mu_t^{(n)}\big)^\ell\rangle = \mathbb{E}\langle \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_\ell, \mu_t^\ell\rangle. \qquad \square$$

In Skoulakis and Adler (2001) the first and second moment calculations are done via first finding $\mathbb{E}\langle \phi, \mu_t^{(n)}\rangle$ and $\mathbb{E}\langle \phi_1 \otimes \phi_2, \mu_{t_1}^{(n)} \mu_{t_2}^{(n)}\rangle$ then passing to the limit as $n \to \infty$. This works well when the number of cases to consider are small, but due to the rapid growth in cases to consider as the moments increase, the following method will vary slightly. The method first calculates $\mathbb{E}\langle \phi, (\mu_t^{(n)})^3\rangle$ and $\mathbb{E}\langle \psi, (\mu_t^{(n)})^4\rangle$ for $\phi \in C_K^\infty(\mathbb{R}^{3\times d})$, $\psi \in C_K^\infty(\mathbb{R}^{4\times d})$, $t \geq 0$, then passes to the limit before utilizing the Markov property to find $\mathbb{E}\langle\phi_1 \otimes \phi_2 \otimes \phi_3, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle$ and $\mathbb{E}\langle\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle$, where $\phi_i, \psi_j \in C_K^\infty(\mathbb{R}^d)$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and $0 < t_1 \le t_2 \le t_3 \le t_4$.

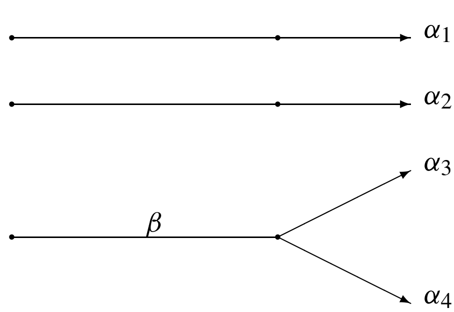
Since the calculations for the third moment are a much simpler case of the fourth, we present here only the derivation of the fourth moment. To begin, note that

$$\mathbb{E}\langle\phi, (\mu_t^{(n)})^\ell\rangle = \frac{1}{n^\ell} \sum_{\substack{\alpha_k \sim_n t \\ k=1,\dots,\ell}} \mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, \dots, Y_t^{\alpha_\ell,n})\mathbb{E}\prod_{i=1}^{\ell} 1_{\alpha_i,n}(t), \tag{6}$$

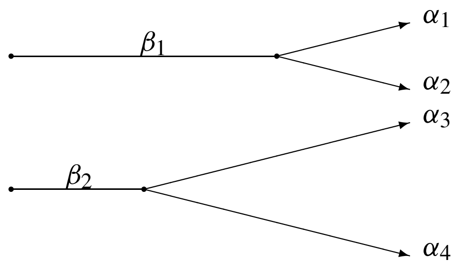
where $1_{\alpha_i,n}(t)$ is the indicator on the event that the particle α_i is alive at time t. Thus, for the fourth moment, if $\alpha_i \sim_n t$, $i = 1, 2, 3, 4$ and $N = [tn]$, we will have the following cases to consider:

(I) Each particle resides on its own tree.

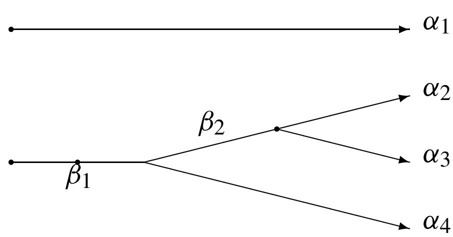
(II) Two particles reside on one tree, the other two reside on their own trees. Thus, the two particles on the common tree share a common ancestor β with $|\beta| = r$ and $r \in \{0, 1, \dots, N-1\}$.



(III) Two particles reside on one tree, the other two on a second tree. Thus, the two particles on one tree share a common ancestor β_1 with $|\beta_1| = r_1$, the two particles on the second tree have a common ancestor β_2 with $|\beta_2| = r_2$ and $r_1, r_2 \in \{0, 1, \dots, N-1\}$.

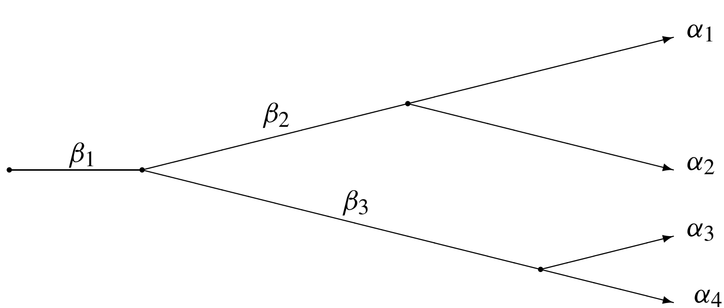


(IV) Three particles reside on one tree, the fourth on its own tree. Thus, two of the three particles share a common ancestor β_2 with $|\beta_2| = r_2$, and all three share a common ancestor β_1 with $|\beta_1| = r_1$, such that $r_1 \in \{0, 1, \ldots, r_2 - 1\}$ and $r_2 \in \{1, \ldots, N - 1\}$.

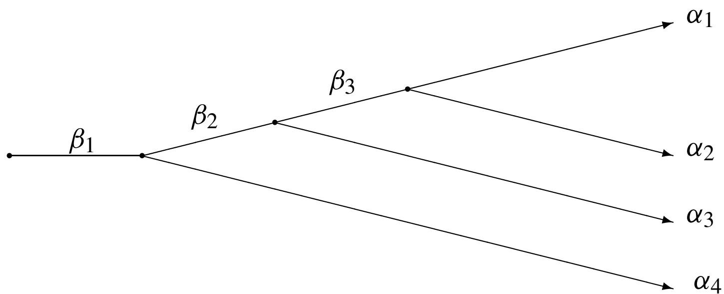


(V) All four particles reside on one tree. This gives the following two sub-cases:

(A) Two of the particles share a common ancestor β_3 with $|\beta_3| = r_3$, the other two share a common ancestor β_2 also with $|\beta_2| = r_2$, all four share a common ancestor β_1 with $|\beta_1| = r_1$, β_2 and β_3 are both descendants of β_1, and $r_1 \in \{0, 1, \ldots, (r_2 - 1) \wedge (r_3 - 1)\}$ and $r_2, r_3 \in \{1, \ldots, N - 1\}$.



(B) Two of the particles share a common ancestor β_3, another particle shares a common ancestor β_2 with β_3, all four particles share a common ancestor β_1 and β_1 is an ancestor of β_2 which is an ancestor of β_3, with $|\beta_1| = r_1$, $|\beta_2| = r_2$, $|\beta_3| = r_3$ and $r_1 \in \{0, 1, \ldots, r_2 - 1\}$, $r_2 \in \{1, \ldots, r_3 - 1\}$, $r_3 \in \{2, \ldots, N - 1\}$.



Taking into consideration the possible resulting diagrams, and defining $r(n) \in [0, r]$ by $r(n) = \frac{r}{n}$, the following lemma can be shown.

LEMMA 3.8. *Given $\phi \in C_K^2(\mathbb{R}^{3\times d})$ and $\psi \in C_K^2(\mathbb{R}^{4\times d})$, for all $n \in \mathbb{N}$, $t > 0$, it follows that*

$$\begin{aligned}\mathbb{E}\langle \phi, (\mu_t^{(n)})^3\rangle &= \langle Q_t^3\phi, (\mu_0^{(n)})^3\rangle + \frac{1}{n}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\langle \Gamma_{2;(r(n),t)}^{(ij)}\phi, (\mu_0^{(n)})^2\rangle \\ &\quad + \frac{1}{n^2}\sum_{\substack{r_1,r_2=0\\ r_1<r_2}}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\langle \Gamma_{1;(r_1(n),r_2(n),t)}^{(12,ij)}, \mu_0^{(n)}\rangle + o(1)\end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathbb{E}\langle \psi, (\mu_t^{(n)})^4\rangle &= \langle Q_t^4\psi, (\mu_0^{(n)})^4\rangle + \frac{1}{n}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\langle \Gamma_{3;(r(n),t)}^{(ij)}\psi, (\mu_0^{(n)})^3\rangle \\ &\quad + \frac{1}{n^2}\sum_{\substack{r_1,r_2=0\\ r_1<r_2}}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\langle \Gamma_{2;(r_1(n),r_2(n),t)}^{(i_2j_2,i_1j_1)}\psi, (\mu_0^{(n)})^2\rangle \\ &\quad + \frac{1}{n^3}\sum_{\substack{r_k=0\\ k=1,2,3\\ r_1<r_2<r_3}}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\langle \Gamma_{1;(r_1(n),r_2(n),r_3(n),t)}^{(12,i_2j_2,i_1j_1)}\psi, \mu_0^{(n)}\rangle \\ &\quad + o(1),\end{aligned} \tag{8}$$

where $\Gamma_{\cdot\cdot}^{\cdot}$ is defined as in (5).

PROOF. See Appendix A. $\square$

Having now a formula for both the third and fourth moments of the branching process, with the exception of a some small technicalities to mention, the moment formulae for the superprocess will follow almost immediately from Lemmas 3.6, 3.7 and 3.8.

THEOREM 3.9. *Let $\phi \in C_K^\infty(\mathbb{R}^{3\times d})$ and $\psi \in C_K^\infty(\mathbb{R}^{4\times d})$ be respectfully defined by*

$$\phi = \phi_1 \otimes \phi_2 \otimes \phi_3 \quad \textit{and} \quad \psi = \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4$$

for $\phi_i, \psi_j \in C_K^\infty(\mathbb{R}^d)$, $i = 1,2,3,\ j = 1,2,3,4$. Then for all $0 \le t_1 \le t_2 \le t_3 \le t_4 < \infty$,

$$\begin{aligned}
&\mathbb{E}\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle \\
&\quad = \langle \Gamma^{(0,0,0)}_{4;(t_1,t_2,t_3,t_4)}\psi, \mu_0^4\rangle + \sum_{\substack{i,j=1\\ i\ne j}}^{4} \int_0^{t_1} ds \langle \Gamma^{(ij,0,0,0)}_{3;(s,t_1,t_2,t_3,t_4)}\psi, \mu_0^3\rangle \\
&\qquad + \sum_{\substack{i,j=1\\ i\ne j}}^{3} \int_{t_1}^{t_2} ds \langle \Gamma^{(0,ij,0,0)}_{3;(t_1,s,t_2,t_3,t_4)}\psi, \mu_0^3\rangle + \int_{t_2}^{t_3} ds \langle \Gamma^{(0,0,12,0)}_{3;(t_1,t_2,s,t_3,t_4)}\psi, \mu_0^3\rangle \\
&\qquad + \sum_{\substack{i_1,j_1=1\\ i_1\ne j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\ne j_2}}^{3} \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \langle \Gamma^{(i_2j_2,i_1j_1,0,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3,t_4)}\psi, \mu_0^2\rangle \\
&\qquad + \sum_{\substack{i_1,j_1=1\\ i_1\ne j_1}}^{3} \sum_{\substack{i_2,j_2=1\\ i_2\ne j_2}}^{3} \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \langle \Gamma^{(i_2j_2,0,i_1j_1,0,0)}_{2;(s_1,t_1,s_2,t_2,t_3,t_4)}\psi, \mu_0^2\rangle \\
&\qquad + \sum_{\substack{i,j=1\\ i\ne j}}^{3} \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 \langle \Gamma^{(0,12,ij,0,0)}_{2;(t_1,s_1,s_2,t_2,t_3,t_4)}\psi, \mu_0^2\rangle \\
&\qquad + \sum_{\substack{i,j=1\\ i\ne j}}^{3} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \langle \Gamma^{(ij,0,0,12,0)}_{2;(s_1,t_1,t_2,s_2,t_3,t_4)}\psi, \mu_0^2\rangle \\
&\qquad + \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \langle \Gamma^{(0,12,0,12,0)}_{2;(t_1,s_1,t_2,s_2,t_3,t_4)}, \mu_0^2\rangle \\
&\qquad + \sum_{\substack{i_1,j_1=1\\ i_1\ne j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\ne j_2}}^{3} \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \langle \Gamma^{(12,i_2j_2,i_1j_1,0,0,0)}_{1;(s_1,s_2,s_3,t_1,t_2,t_3,t_4)}\psi, \mu_0\rangle
\end{aligned} \tag{9}$$

$$+\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\langle\Gamma^{(12,i_2j_2,0,i_1j_1,0,0)}_{1;(s_1,s_2,t_1,s_3,t_2,t_3,t_4)}\psi,\mu_0\rangle$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2}ds_3\int_{t_1}^{s_3}ds_2\int_0^{t_1}ds_1\langle\Gamma^{(12,0,12,ij,0,0)}_{1;(s_1,t_1,s_2,s_3,t_2,t_3,t_4)}\psi,\mu_0\rangle$$

$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\langle\Gamma^{(12,ij,0,0,12,0)}_{1;(s_1,s_2,t_1,t_2,s_3,t_3,t_4)}\psi,\mu_0\rangle$$

$$+\int_{t_2}^{t_3}ds_3\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1\langle\Gamma^{12,0,12,0,12,0)}_{1;(s_1,t_1,s_2,t_2,s_3,t_3,t_4)}\mu_0\rangle$$

and

$$\begin{aligned}\mathbb{E}\langle\phi,\mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle &= \langle\Gamma^{(0,0)}_{3;(t_1,t_2,t_3)}\phi,\mu_0^3\rangle+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_0^{t_1}ds\langle\Gamma^{(ij,0,0)}_{2;(s,t_1,t_2,t_3)}\phi,\mu_0^2\rangle\\ &\quad+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\langle\Gamma^{(12,ij,0,0)}_{1;(s_1,s_2,t_1,t_2,t_3)}\phi,\mu_0\rangle\\ &\quad+\int_{t_1}^{t_2}ds\langle\Gamma^{(0,12,0)}_{2;(t_1,s,t_2,t_3)}\phi,\mu_0^2\rangle\\ &\quad+\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1\langle\Gamma^{(12,0,12,0)}_{1;(s_1,t_1,s_2,t_2,t_3)}\phi,\mu_0\rangle.\end{aligned}\tag{10}$$

PROOF. See Appendix B. □

The purpose of the above moment formulae is due to the need for L^2 bounds, the verification of which makes up the most essential part of this paper. For the remainder, any arbitrary constant value, dependent only upon $0\le T$, will be denoted by $C=C(T)$.

LEMMA 3.10. *Let* $\phi\in S_d$, $d\le 3$, *and define for* $x=(x_1,x_2,x_3)\in\mathbb{R}^{3\times d}$, $y=(y_1,y_2,y_3,y_4)\in\mathbb{R}^{4\times d}$,

$$\psi(x)\triangleq\phi(x_1-x_3)\phi(x_2-x_3)$$

and

$$\varphi(y)\triangleq\phi(y_1-y_3)\phi(y_2-y_4).$$

Suppose that $\mu = \{\mu_t : t \geq 0\}$ *is a superprocess over a stochastic flow such that* $\mu_0 \in M_F(\mathbb{R}^d)$ *satisfies Assumption* 2.3. *Then, for any* $0 \leq t_1 \leq t_2 \leq t_3 \leq T < \infty$,

$$\text{(i)} \quad \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, \mathbb{E}\langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle \leq C\|\phi\|_{L^1}^2$$

and

$$\text{(ii)} \quad \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, \mathbb{E}\langle \varphi, \mu_{t_1}\mu_{t_2}\mu_{t_3}^2\rangle \leq C\|\phi\|_{L^1}^2.$$

PROOF. See Appendix D. □

We now proceed with the establishing existence of the GSILT.

4. Existence of generalized self-intersection local time. Generalized self-intersection local time (GSILT) at $u \in \mathbb{R}^d$, over $B \subset \mathcal{B}(\mathbb{R}^2)$, is defined formally as

$$\mathcal{L}(u; B) \triangleq \int_B dt\, ds \langle \delta_u, \mu_s \mu_t\rangle,$$

where $\delta_u(x)$ is the Dirac point-mass measure at u.

Note that in the above, and throughout the remainder of this paper, if $\varphi : \mathbb{R}^d \to \mathbb{R}$, the convention

$$\langle \varphi, \mu_s\mu_t\rangle = \int \mu_s(dx)\mu_t(dy)\varphi(x - y)$$

is made.

Since $\mu_s\mu_t = \mu_t\mu_s$, it makes sense to restrict GSILT either above or below the diagonal, and so we set

$$\mathcal{L}(u, T) = \mathcal{L}\big(u; \{(s, t) : 0 \leq s \leq t \leq T\}\big)$$

for fixed $T \in [0, \infty)$.

The above definition is clearly formal, and thus to make sense of this a limiting process will be constructed. For fixed $\lambda > 0$, define

$$G^{\lambda,u}(x) = \int_0^\infty dt\, e^{-\lambda t} q_t(u, x),$$

then, since $G^{\lambda,u}$ is the resolvent to L at λ, $(\lambda - L)G^{\lambda,u} = \delta_u$ and $\|G^{\lambda,u}\|_{L^1} \leq \lambda^{-1}$.

From Dynkin (1965), Theorem 0.5, it can be seen that $G^{\lambda,u}(x)$ is not smooth (take, e.g., $x = u$), and thus it is desired to estimate $G^{\lambda,u}$ by a class of smooth functions.

Since $G^{\lambda,u} \in L^1(\mathbb{R}^d)$, for any $\phi \in C_K^\infty(\mathbb{R}^d)$

$$\langle \phi, G^{\lambda,u}\rangle \triangleq \int dx\, G^{\lambda,u}(x)\phi(x) < \infty,$$

which implies $G^{\lambda,u}$ can be regarded as the element of S'_d which sends $\phi \in S_d$ to $\langle\phi, G^{\lambda,u}\rangle$. Thus, Lieb and Loss (2001), Theorem 7.10, implies the existence of a family $\{G_\varepsilon^{\lambda,u} : \varepsilon > 0\} \subset C_K^\infty$ such that $G_\varepsilon^{\lambda,u} \to G^{\lambda,u}$ as $\varepsilon \to 0$, in S'_d.

From Hörmander (1985), L is a continuous operator on S'_d, and it is concluded that

$$\lim_{\varepsilon\to 0}(\lambda - L)G_\varepsilon^{\lambda,u} = \delta_u,$$

where convergence is in the sense of distributions, and so a limiting process is defined by

$$\gamma_\varepsilon^\lambda(u, T) \triangleq \int_0^T dt \int_0^t ds \langle(\lambda - L)G_\varepsilon^{\lambda,u}, \mu_s\mu_t\rangle,$$

$\lambda > 0$, $\varepsilon > 0$, $0 \le T < \infty$.

The goal now is to make sense of the operator L appearing in the integrand.

4.1. *An Itô formula.* As in the independent case, the derivation of the evolution equation is accomplished through the construction, and careful application, of an appropriate Itô formula. This construction will mimic that of Adler and Lewin (1991), which begins with application of Itô's lemma to the nonanticipative functional f, given by

$$f(t, x) = x \int_0^t ds\, \mu_s(\psi),$$

where $\psi \in C_K^2(\mathbb{R}^d)$, and x is a $\mathbb{R}$-valued random variable. Note that from the SPDE (3), if $\phi \in C_K^\infty(\mathbb{R}^d)$, then $\mu_t(\phi)$ is a continuous semi-martingale with decomposition

$$\mu_t(\phi) = \mu_0(\phi) + Z_t(\phi) + V_t(\phi),$$

where

$$V_t(\phi) \triangleq \int_0^t ds\, \mu_s(L\phi).$$

THEOREM 4.1. *If $\phi \in S_d$ then $\mu_t(\phi)$ is an a.s. continuous semimartingale.*

PROOF. See Appendix D. □

Through some careful work (outlined in the Appendix), we arrive at the following.

LEMMA 4.2. *Given $\Psi \in S_{2d}$,*

$$\int_0^T dt \int_0^t ds\langle L_2\Psi, \mu_s\mu_t\rangle = \int_0^T dt\langle\Psi, \mu_t\mu_T\rangle - \int_0^T dt\langle\Psi, \mu_t\mu_t\rangle - \int_0^T \int_{\mathbb{R}^d} Z(dt, dy) \int_0^t ds\langle\Psi(\cdot, y), \mu_s\rangle,$$

where

$$(L_2\Psi)(x,y) \triangleq \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(y)\,\partial_{2_i}\,\partial_{2_j}\Psi(x,y),$$

and $Z(dt,dy)$ is the corresponding martingale measure.

PROOF. See Appendix E. □

4.2. *Existence.* Using lemma 4.2 with $G_\varepsilon^{\lambda,u}$ in place of Ψ, we now have

$$\begin{aligned}\gamma_\varepsilon^\lambda(u,T) = \lambda\int_0^T dt\int_0^s ds\langle G_\varepsilon^{\lambda,u},\mu_s\mu_t\rangle \\ -\int_0^T dt\langle G_\varepsilon^{\lambda,u},\mu_t\mu_T\rangle + \int_0^T dt\langle G_\varepsilon^{\lambda,u},\mu_t\mu_t\rangle \\ +\int_0^T\int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds\langle G_\varepsilon^{\lambda,u}(\cdot - y),\mu_s\rangle.\end{aligned}$$

As in Rosen (1992) and Adler and Lewin (1991, 1992), the issue of "local double points" must be addressed, that is, the set of points lying on the diagonal in $\mathbb{R}^2$, which will be (falsely) counted as points of self-intersection when $u=0$, and will lead to singularities in dimensions greater than one. Due to this we follow the idea first proposed by Adler and Lewin, and renormalize our GSILT via subtraction of the term involving "local double points." It is easy enough to see that the term involving the "local double points" is given by $\int_0^T dt\langle G_\varepsilon^{\lambda,u},\mu_t\mu_t\rangle$, and thus we define our renormalized limiting process to generalized self-intersection local time at $u\in\mathbb{R}^d$, over the set $\{(s,t):0\le s<t\le T\}$ by

$$\begin{aligned}\mathcal{L}_\varepsilon^\lambda(u,T) &= \gamma_\varepsilon^\lambda(u,T) - \int_0^T dt\langle G_\varepsilon^{\lambda,u},\mu_t\mu_t\rangle \\ &= \lambda\int_0^T dt\int_0^t ds\langle G_\varepsilon^{\lambda,u},\mu_s\mu_t\rangle - \int_0^T dt\langle G_\varepsilon^{\lambda,u},\mu_t\mu_T\rangle \\ &\quad + \int_0^T\int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds\langle G_\varepsilon^{\lambda,u}(\cdot - y),\mu_s\rangle.\end{aligned}$$

Using Lemma 3.10 existence follows almost immediately.

THEOREM 4.3. *Suppose that $\mu=\{\mu_t:t\ge 0\}$ is a d-dimensional superprocess over a stochastic flow such that $\mu_0\in M_F(\mathbb{R}^d)$, $d\le 3$, satisfies Assumption 2.3. Fix $T\in[0,\infty)$ and define $\mathcal{L}_\varepsilon^\lambda(u,T)$ as above, then for $0\le s<t\le T$,*

$$L^2 - \lim_{\varepsilon\to 0}\mathcal{L}_\varepsilon^\lambda(u,T) = \mathcal{L}^\lambda(u,T),$$

uniformly in $u \in \mathbb{R}^d$, *where* $\mathcal{L}^\lambda(u,T)$ *is defined by*

$$\mathcal{L}^\lambda(u,T) = \lambda \int_0^T dt \int_0^t ds \langle G^{\lambda,u}, \mu_s\mu_t\rangle - \int_0^T dt \langle G^{\lambda,u}, \mu_t\mu_T\rangle + \int_0^T \int_{\mathbb{R}^d} Z(dt,dy) \int_0^t ds \langle G^{\lambda,u}(\cdot - y), \mu_s\rangle.$$

For each $\lambda > 0$, $\mathcal{L}^\lambda(u,T)$ *is referred to as the self-intersection local time at* u, *up to time* T, *for a superprocess over a stochastic flow.*

PROOF. See Appendix F. □

It should be noted that, as with the the SILT of Adler and Lewin (1991), for $d > 3$ the GSILT can be shown to blow up to infinity. It remains an open question if renormalization processes, such as those of Rosen (1992), exist for dimensions $d > 3$.

APPENDIX A: PROOF OF LEMMA 3.8

We now proceed with the moment calculations. Much of what follows will be a consequence of the Markov property, and the reader is referred to Skoulakis and Adler (2001) for a similar calculation for the first and second moments. Note that if $t \geq 0$ and $r \in \mathbb{N}$, we define $N \in \mathbb{N}$ and $r(n) \in [0,r]$ by $N = [nt]$ and $r(n) = \frac{r}{n}$. Recall,

$$\mathbb{E}\langle\phi, (\mu_t^{(n)})^4\rangle = \frac{1}{n^4} \sum_{\substack{\alpha_j \sim_n t \\ j=1,2,3,4}} \mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) \mathbb{E}\prod_{i=1}^4 1_{\alpha_i,n}(t). \tag{11}$$

If $\alpha_1(0), \alpha_2(0), \alpha_3(0)$, and $\alpha_4(0)$ are given, case (I) gives

$$\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) = Q_t^4\phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)})$$

and $\mathbb{E}\prod_{i=1}^4 1_{\alpha_i,n}(t) = (\frac{1}{2})^{4N}$. For any $\alpha_1(0)$, $\alpha_2(0)$, $\alpha_3(0)$, $\alpha_4(0)$, there are 2^{4N} corresponding $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which result from binary branching over N steps. We thus arrive at the following contribution from case (I):

$$\begin{aligned}
&\frac{1}{n^4} \sum_{\substack{\alpha_k(0)=1, k=1,2,3,4 \\ \alpha_\ell(0)\neq\alpha_k(0), \ell\neq k}} Q_t^4\phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}) \\
&\quad = \frac{1}{n^4} \sum_{\alpha_k(0)=1, k=1,2,3,4} Q_t^4\phi(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}) \\
&\qquad - \frac{1}{n^4} \sum_{\substack{\alpha_k(0)=1 \\ k=1,2,3}} \sum_{\substack{i,j=1 \\ i\neq j}}^4 (\Phi_{ij} Q_t^4\phi)(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)})
\end{aligned}$$

$$- \frac{1}{n^4} \sum_{\substack{\alpha_k(0)=1 \\ j=1,2}} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} (\Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi)\big(x_{\alpha_1(0)}, x_{\alpha_2(0)}\big)$$

$$- \frac{1}{n^4} \sum_{\alpha_1(0)=1} \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} (\Phi_{12} \Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi)\big(x_{\alpha_1(0)}\big),$$

which by the definition of $\mu^{(n)}$,

$$\begin{aligned}
&\frac{1}{n^4} \sum_{\substack{\alpha_k(0)=1, k=1,2,3,4 \\ \alpha_\ell(0) \neq \alpha_k(0), \ell \neq k}} Q_t^4 \phi\big(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}\big) \\
&\quad = \big\langle Q^4 \phi, \big(\mu_0^{(n)}\big)^4 \big\rangle - \frac{1}{n} \big\langle \Phi_{ij} Q_t^4 \phi, \big(\mu_0^{(n)}\big)^3 \big\rangle \\
&\qquad - \frac{1}{n^2} \big\langle \Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi, \big(\mu_0^{(n)}\big)^2 \big\rangle - \frac{1}{n^3} \big\langle \Phi_{12} \Phi_{i_2 j_2} \Phi_{i_1 j_1} Q_t^4 \phi, \mu_0^{(n)} \big\rangle.
\end{aligned}$$

From Lemma 3.5 all but the first term on the right-hand side will vanish as $n \to \infty$, and thus

$$\begin{aligned}
&\frac{1}{n^4} \sum_{\substack{\alpha_k(0)=1, k=1,2,3,4 \\ \alpha_\ell(0) \neq \alpha_k(0), \ell \neq k}} Q_t^4 \phi\big(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\alpha_3(0)}, x_{\alpha_4(0)}\big) \\
&\qquad = \big\langle Q_t^4 \phi, \big(\mu_0^{(n)}\big)^4 \big\rangle + o(1).
\end{aligned} \tag{12}$$

For case (II), given $\alpha_1(0), \alpha_2(0), \beta(0)$ and r, proceeding as before,

$$\begin{aligned}
&\mathbb{E}\phi\big(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}\big) \\
&\qquad = \frac{1}{12} \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \big(Q_{r(n)}^3 \Phi_{ij} Q_{t-r(n)}^4 \phi\big)\big(x_{\alpha_1(0)}, x_{\alpha_2(0)}, x_{\beta(0)}\big),
\end{aligned}$$

and if for any distinct $i, j \in \{1, 2, 3, 4\}$ we define i', j' to be the exhaustive elements of $\{1, 2, 3, 4\} \setminus \{i, j\}$,

$$\begin{aligned}
\mathbb{E} \prod_{i=1}^{4} 1_{\alpha_i,n}(t) &= (\mathbb{E} 1_{\alpha_{i'},n}(t))(\mathbb{E} 1_{\alpha_{j'},n}(t)) \mathbb{E}\mathbb{E}\big[1_{\alpha_i,n}(t) 1_{\alpha_j,n}(t) | \mathcal{F}_{r(n)}^n\big] \\
&= 2^{-(4N-r-1)}.
\end{aligned}$$

For any $\alpha_1(0)$, $\alpha_2(0)$, $\beta_1(0)$ and r, there are 2^{4N-r-1} corresponding tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which result from binary branching over N steps and $2 \cdot \binom{4}{2}$ possible arrangements for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We thus arrive at the following contribution from case (II):

$$\frac{1}{n^4}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{\alpha_1(0),\alpha_2(0),\beta(0)=1\\ \alpha_1(0)\neq\alpha_2(0),\alpha_\ell(0)\neq\beta(0)\\ \ell=1,2}}\big(\Gamma^{(ij)}_{3;(r(n),t)}\phi\big)\big(x_{\alpha_1(0)},x_{\alpha_2(0)},x_{\beta(0)}\big)$$

$$=\frac{1}{n}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\big\langle\Gamma^{(ij)}_{3;(r(n),t)}\phi,\big(\mu_0^{(n)}\big)^3\big\rangle$$

$$-\frac{1}{n^2}\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\big\langle\Phi_{i_2j_2}\Gamma^{(i_1j_1)}_{3;(r(n),t)}\phi,\big(\mu_0^{(n)}\big)^2\big\rangle$$

$$-\frac{1}{n^3}\sum_{r=0}^{N-1}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\big\langle\Phi_{12}\Phi_{i_2j_2}\Gamma^{(i_1j_1)}_{3;(r(n),t)}\phi,\big(\mu_0^{(n)}\big)\big\rangle.$$

Again from Lemma 3.6, all but the first term on the right-hand side will vanish as $n\to\infty$ and thus,

$$\begin{aligned}&\frac{1}{n^4}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\sum_{\substack{\alpha_1(0),\alpha_2(0),\beta(0)=1\\ \alpha_1(0)\neq\alpha_2(0),\alpha_\ell(0)\neq\beta(0)\\ \ell=1,2}}\big(\Gamma^{(ij)}_{3;(r(n),t)}\phi\big)\big(x_{\alpha_1(0)},x_{\alpha_2(0)},x_{\beta(0)}\big)\\ &\quad=\frac{1}{n}\sum_{r=0}^{N-1}\sum_{\substack{i,j=1\\ i\neq j}}^{4}\big\langle\Gamma^{(ij)}_{3;(r(n),t)}\phi,\big(\mu_0^{(n)}\big)^3\big\rangle+o(1).\end{aligned}\tag{13}$$

Cases (III) and (IV) will now be considered together. For case (III), given $\beta_1(0)$, $\beta_2(0)$, r_1 and r_2,

$$\begin{aligned}&\mathbb{E}\phi\big(Y_t^{\alpha_1,n},Y_t^{\alpha_2,n},Y_t^{\alpha_3,n},Y_t^{\alpha_4,n}\big)\\ &\quad=\frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3}\big(\Gamma^{(i_2j_2,i_1j_1)}_{2;(r_1(n),r_2(n),t)}\phi\big)\big(x_{\beta_1(0)},x_{\beta_2(0)}\big)\end{aligned}$$

and

$$\mathbb{E}\prod_{i=1}^{4}1_{\alpha_i,n}(t)=2^{-(4N-r_1-r_2-2)}.$$

For case (IV), given $\alpha(0)$, $\beta_1(0)$, r_1 and r_2,

$$\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n}) = \frac{1}{48} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2=i_1 \text{ or } j_2=i_1}}^{3} \big(\Gamma^{(i_2 j_2, i_1 j_1)}_{2;(r_1(n),r_2(n),t)}\phi\big)\big(x_{\alpha(0)}, x_{\beta_1(0)}\big)$$

and

$$\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) = 2^{-(4N-r_1-r_2-2)}.$$

Given two initial ancestors, there are $2^{4N-r_1-r_2-2}$ possible trees, and a possible $2\cdot\binom{4}{2}$ arrangements for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ upon each tree (requiring $r_1 < r_2$) that result in case (III). Furthermore, there are $2^{4N-r_1-r_2-2}$ possible trees, and a possible $2\cdot\binom{2}{1}\cdot\binom{3}{2}\cdot\binom{4}{3}$ arrangements for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ upon each tree that result in case (IV). It follows that the contribution coming from the sum of case (III) and case (IV) is given by

$$\begin{aligned}
&\frac{1}{n^4} \sum_{\substack{r_1,r_2=0\\ r_1<r_2}}^{N-1} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \sum_{\substack{\alpha(0)=1\\ \beta(0)=1\\ \alpha(0)\neq\beta(0)}} \big(\Gamma^{(i_2 j_2, i_1 j_1)}_{2;(r_1(n),r_2(n),t)}\phi\big)\big(x_{\alpha(0)}, x_{\beta(0)}\big)\\
&\quad = \frac{1}{n^2} \sum_{\substack{r_1,r_2=0\\ r_1<r_2}}^{N-1} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \big\langle\Gamma^{(i_2 j_2, i_1 j_1)}_{2;(r_1(n),r_2(n),t)}\phi, \big(\mu_0^{(n)}\big)^2\big\rangle\\
&\qquad - \frac{1}{n^3} \sum_{\substack{r_1,r_2=0\\ r_1<r_2}}^{N-1} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \big\langle\Phi_{12}\Gamma^{(i_2 j_2, i_1 j_1)}_{2;(r_1(n),r_2(n),t)}\phi, \mu_0^{(n)}\big\rangle.
\end{aligned}$$

Thus, again from Lemma 3.6, the second term vanishes as $n \to \infty$, and we have the contribution

$$\frac{1}{n^2} \sum_{\substack{r_1=0\\ r_2=0\\ r_1<r_2}}^{N-1} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \big\langle\big\langle\Phi_{12}\Gamma^{(i_2 j_2, i_1 j_1)}_{2;(r_1(n),r_2(n),t)}\phi, \mu_0^{(n)}\big\rangle, \big(\mu_0^{(n)}\big)^2\big\rangle + o(1). \tag{14}$$

Considering subcase (V)(A), given r_1, r_2, r_3 and $\beta_1(0)$,

$$\begin{aligned}
&\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n})\\
&\quad = \frac{1}{6}\sum_{\substack{i,j=1\\ i\neq j}}^{4} \mathbb{E}\mathbb{E}\big[\big(\Phi_{ij} Q^4_{t-r_3(n)}\phi\big)\big(Y^{\alpha_1,n}_{r_3(n)}, Y^{\alpha_2,n}_{r_3(n)}, Y^{\beta_3,n}_{r_3(n)}\big)|\mathcal{F}^n_{r_2(n)}\big]\\
&\quad = \frac{1}{12}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2,j_2\neq i_1}}^{3} \big(\Gamma^{(12,i_2j_2,i_1j_1)}_{1;(r_1(n),r_2(n),r_3(n),t)}\phi\big)\big(x_{\beta_1(0)}\big).
\end{aligned}$$

Furthermore,

$$\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) = 2^{-(4N-r_3-r_2-r_1-3)}.$$

For subcase (V)(B), given r_1, r_2, r_3 and $\beta_1(0)$,

$$\begin{aligned}
&\mathbb{E}\phi(Y_t^{\alpha_1,n}, Y_t^{\alpha_2,n}, Y_t^{\alpha_3,n}, Y_t^{\alpha_4,n})\\
&\quad = \frac{1}{12}\sum_{\substack{i,j=1\\ i\neq j}}^{4} \mathbb{E}\mathbb{E}\big[\big(\Phi_{ij} Q^4_{t-r_3(n)}\phi\big)\big(Y^{\alpha_1,n}_{r_3(n)}, Y^{\alpha_2,n}_{r_3(n)}, Y^{\beta_3,n}_{r_3(n)}\big)|\mathcal{F}^n_{r_2(n)}\big]\\
&\quad = \frac{1}{48}\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2\\ i_2=i_1 \text{ or } j_2=i_1}}^{3} \big(\Gamma^{(12,i_2j_2,i_1j_1)}_{1;(r_1(n),r_2(n),r_3(n),t)}\phi\big)\big(x_{\beta_1(0)}\big)
\end{aligned}$$

and

$$\mathbb{E}\prod_{i=1}^{4} 1_{\alpha_i,n}(t) = 2^{-(4N-r_1-r_2-r_3-3)}.$$

Given one initial ancestor there are $2^{4N-r_1-r_2-r_3-3}$ possible trees, and a possible $\binom{1}{1}\cdot\binom{2}{1}\cdot\binom{4}{2}$ arrangements for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ upon each tree (requiring $r_2 < r_3$) that result in case (V)(A). Furthermore, there are $2^{4N-r_1-r_2-r_3-3}$ possible trees, and a possible $\binom{1}{1}\cdot\binom{2}{1}\cdot\binom{3}{2}\cdot\binom{4}{3}$ arrangements for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ upon each tree that result in case (V)(B). It follows that the contribution coming from the sum of subcase (V)(A) and subcase (V)(B) is given by

$$\frac{1}{n^3}\sum_{\substack{r_1,r_2,r_3=0\\ r_1<r_2<r_3}}^{N-1} \sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4} \sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3} \langle \Gamma^{(12,i_2j_2,i_1j_1)}_{1;(r_1(n),r_2(n),r_3(n),t)}\phi, \mu_0^{(n)}\rangle. \tag{15}$$

Therefore, from (12), (13), (14) and (15), the lemma is shown.

APPENDIX B: PROOF OF THEOREM 3.9

Again, due to similarity and escalating difficulty, we forgo the proof of the third moment in favor of the fourth moment. We first prove a needed lemma.

LEMMA B.1. *Given $\phi_k, \psi_j \in C_K^\infty(\mathbb{R}^d)$, $k = 1, 2, 3$, $j = 1, 2, 3, 4$, let $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$ and $\psi = \psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4$. For any $t \geq 0$, the following hold*:

$$\begin{aligned}\mathbb{E}\langle \phi, \mu_t^3 \rangle = \langle Q_t^3 \phi, \mu_0^3 \rangle &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_0^t ds \langle \Gamma_{2;(s,t)}^{(ij)} \phi, \mu_0^2 \rangle \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \int_0^t ds_2 \int_0^{s_2} ds_1 \langle \Gamma_{1;(s_1,s_2,t)}^{(12,ij)} \phi, \mu_0 \rangle \end{aligned} \tag{16}$$

and

$$\begin{aligned}\mathbb{E}\langle \psi, \mu_t^4 \rangle = \langle Q_t^4 \psi, \mu_0^4 \rangle &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \int_0^t ds \langle \Gamma_{3;(s,t)}^{(ij)} \psi, \mu_0^3 \rangle \\ &+ \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \int_0^t ds_2 \int_0^{s_2} ds_1 \langle \Gamma_{2;(s_1,s_2,t)}^{(i_2 j_2, i_1 j_1)} \psi, \mu_0^2 \rangle \\ &+ \sum_{\substack{i_1,j_1=1 \\ i_1 \neq j_1}}^{4} \sum_{\substack{i_2,j_2=1 \\ i_2 \neq j_2}}^{3} \int_0^t ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \langle \Gamma_{1;(s_1,s_2,s_3,t)}^{(12, i_2 j_2, i_1 j_1)} \psi, \mu_0 \rangle. \end{aligned} \tag{17}$$

PROOF. To begin, note that Lemma 3.5 implies $(\mu_0^{(n)})^\ell \Rightarrow \mu_0^\ell$ for any $\ell \in \mathbb{N}$, and thus the first term of the right-hand sides of (7) and (8) converge, respectively, to the first term of the right-hand sides of (16) and (17) as $n \to \infty$. Since Q_t^k is a strongly continuous contraction semigroup for $k \in \mathbb{N}$ (Lemma 3.2), for any $\phi \in C_b(\mathbb{R}^d)$ which satisfies $\|\phi\|_{bL} = 1$, $\|Q_t^k \phi\|_\infty \leq 1$, and $\sup_{x \neq y} \frac{|Q_t^k \phi(x) - Q_t^k \phi(y)|}{|x-y|} \leq 1$. Thus, for any $k \in N$, $\|\phi\|_{bL} = 1$ implies $\|Q_t^k \phi\|_{bL} \leq 1$. From Lemma 3.6, the remaining terms on the right-hand sides of (7) and (8) converge, respectively, to the remaining terms of the right-hand sides of (16) and (17) as $n \to \infty$. It remains to show that the left-hand sides of (7) and (8) converge, respectively, to the left-hand sides of (16) and (17), but this follows immediately from Lemma 3.7 □

The proof of the main theorem can now be shown.

PROOF OF THEOREM 3.9. Using the Markov property and Lemma 3.4, it follows that

$$\begin{aligned}
&\mathbb{E}\langle\psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle \\
&\quad= \mathbb{E}\mu_{t_1}(\psi_1)\mu_{t_2}(\psi_2)\mu_{t_3}^2(\psi_3 \otimes Q_{t_4-t_3}\psi_4) \\
&\quad= \mathbb{E}\mu_{t_1}(\psi_1)\mu_{t_2}^3\bigl(\psi_2 \otimes Q_{t_3-t_2}^2(\psi_3 \otimes Q_{t_4-t_3}\psi_4)\bigr) \\
&\qquad + \int_0^{t_3-t_2} ds\, \mu_{t_1}(\psi_1) \\
&\qquad\qquad \times \mu_{t_2}^2\bigl(\psi_2 \otimes Q_s\Phi_{12}Q_{t_3-t_2-s}^2(\psi_3 \otimes Q_{t_4-t_3}\psi_4)\bigr) \\
&\quad= \mathbb{E}\mu_{t_1}(\psi_1)\mu_{t_2}^3\bigl(\pi_1 Q_{t_3-t_2}^2 \pi_1 Q_{t_4-t_3}(\psi_2 \otimes \psi_3 \otimes \psi_4)\bigr) \\
&\qquad + \int_{t_2}^{t_3} ds\, \mathbb{E}\mu_{t_1}(\psi_1) \\
&\qquad\qquad \times \mu_{t_2}^2\bigl(\pi_1 Q_{s-t_2}\Phi_{12}Q_{t_3-s}^2 \pi_1 Q_{t_4-t_3}(\psi_2 \otimes \psi_3 \otimes \psi_4)\bigr) \\
&\quad= \mathbb{E}\mu_{t_1}^4\bigl(\pi_1\Gamma_{3;(t_2-t_1,t_3-t_1,t_4-t_1)}^{(0,0)}\psi\bigr) \\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds\, \mathbb{E}\mu_{t_1}^3\bigl(\pi_1\Gamma_{2;(s-t_1,t_2-t_1,t_3-t_1,t_4-t_1)}^{(ij,0,0)}\psi\bigr) \\
&\qquad + \int_{t_2}^{t_3} ds\, \mathbb{E}\mu_{t_1}^3\bigl(\pi_1\Gamma_{2;(t_2-t_1,s-t_1,t_3-t_1,t_4-t_1)}^{(0,12,0)}\psi\bigr) \\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1\, \mathbb{E}\mu_{t_1}^2\bigl(\pi_1\Gamma_{1;(s_1-t_1,s_2-t_1,t_2-t_1,t_3-t_1,t_4-t_1)}^{(12,ij,0,0)}\psi\bigr) \\
&\qquad + \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\, \mathbb{E}\mu_{t_1}^2\bigl(\pi_1\Gamma_{1;(s_1-t_1,t_2-t_1,s_2-t_1,t_3-t_1,t_4-t_1)}^{(12,0,12,0)}\psi\bigr).
\end{aligned}$$

To make sense of the remainder of the proof, each of the above five terms will now be considered separately.

From (17),

$$\begin{aligned}
&\mathbb{E}\mu_{t_1}^4\bigl(\pi_1\Gamma_{3;(t_2-t_1,t_3-t_1,t_4-t_1)}^{(0,0)}\psi\bigr) \\
&\quad= \mu_0^4\bigl(\Gamma_{4;(t_1,t_2,t_3,t_4)}^{(0,0,0)}\psi\bigr) + \sum_{\substack{i,j=1\\ i\neq j}}^{4} \int_0^{t_1} ds\, \mu_0^3\bigl(\Gamma_{3;(s,t_1,t_2,t_3,t_4)}^{(ij,0,0,0)}\psi\bigr)
\end{aligned} \tag{18}$$

$$+\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\,\mu_0^2\big(\Gamma_{2;(s_1,s_2,t_1,t_2,t_3,t_4)}^{(i_2j_2,i_1j_1,0,0,0)}\psi\big)$$

$$+\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{4}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_0^{t_1}ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\,\mu_0\big(\Gamma_{1;(s_1,s_2,s_3,t_1,t_2,t_3,t_4)}^{(12,i_2j_2,i_1j_1,0,0,0)}\psi\big).$$

From (16),

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2}ds\,\mathbb{E}\mu_{t_1}^3\big(\pi_1\Gamma_{2;(s-t_1,t_2-t_1,t_3-t_1,t_4-t_1)}^{(ij,0,0)}\psi\big)\\
&\quad=\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2}ds\,\mu_0^3\big(\Gamma_{3;(t_1,s,t_2,t_3,t_4)}^{(0,ij,0,0)}\psi\big)\\
&\qquad+\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1\,\mu_0^2\big(\Gamma_{2;(s_1,t_1,s_2,t_3,t_4)}^{(i_2j_2,0,i_1j_1,0,0)}\psi\big)\\
&\qquad+\sum_{\substack{i_1,j_1=1\\ i_1\neq j_1}}^{3}\sum_{\substack{i_2,j_2=1\\ i_2\neq j_2}}^{3}\int_{t_1}^{t_2}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\,\mu_0\big(\Gamma_{1;(s_1,s_2,t_1,s_3,t_2,t_3,t_4)}^{(12,i_2j_2,0,i_1j_1,0,0)}\psi\big).
\end{aligned}\tag{19}$$

Again from (16),

$$\begin{aligned}
&\int_{t_2}^{t_3}ds\,\mathbb{E}\mu_{t_1}^3\big(\pi_1\Gamma_{2;(t_2-t_1,s-t_1,t_3-t_1,t_4-t_1)}^{(0,12,0)}\psi\big)\\
&\quad=\int_{t_2}^{t_3}ds\,\mu_0^3\big(\Gamma_{3;(t_1,t_2,s,t_3,t_4)}^{(0,0,12,0)}\psi\big)\\
&\qquad+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3}ds_2\int_0^{t_1}ds_1\,\mu_0^2\big(\Gamma_{2;(s_1,t_1,t_2,s_2,t_3,t_4)}^{(ij,0,0,12,0)}\psi\big)\\
&\qquad+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_2}^{t_3}ds_3\int_0^{t_1}ds_2\int_0^{s_2}ds_1\,\mu_0\big(\Gamma_{1;(s_1,s_2,t_1,t_2,s_3,t_3,t_4)}^{(12,ij,0,0,12,0)}\psi\big).
\end{aligned}\tag{20}$$

From Lemma 3.4,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1\, \mathbb{E}\mu_{t_1}^2\big(\pi_1\Gamma_{1;(s_1-t_1,s_2-t_1,t_2-t_1,t_3-t_1,t_4-t_1)}^{(12,ij,0,0)}\psi\big)$$

$$(21)\qquad = \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1\, \mu_0^2\big(\Gamma_{2;(t_1,s_1,s_2,t_2,t_3,t_4)}^{(0,12,ij,0,0)}\psi\big)$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{3}\int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_{0}^{t_1} ds_1\, \mu_0\big(\Gamma_{1;(s_1,t_1,s_2,t_2,s_3,t_3,t_4)}^{(12,0,12,ij,0,0)}\psi\big)$$

and

$$\int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\, \mathbb{E}\mu_{t_1}^2\big(\pi_1\Gamma_{1;(s_1-t_1,t_2-t_1,s_2-t_1,t_3-t_1,t_4-t_1)}^{(12,0,12,0)}\psi\big)$$

$$(22)\qquad = \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1\, \mu_0^2\big(\Gamma_{2;(t_1,s_1,t_2,s_2,t_3,t_4)}^{(0,12,0,12,0)}\psi\big)$$

$$+ \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_{0}^{t_1} ds_1\, \mu_0\big(\Gamma_{1;(s_1,t_1,s_2,t_2,s_3,t_3,t_4)}^{(12,0,12,0,12,0)}\psi\big).$$

Combining (18), (19), (20), (21) and (22), the desired formula follows. □

APPENDIX C: PROOF OF LEMMA 3.10

We begin with some needed corollaries (of Theorem 3.9) and lemmata.

COROLLARY C.1. *For $i, j = 1, 2, 3, 4$, let $\phi_j^i \in C_K^\infty(\mathbb{R}^d)$ and define $\phi_i \in C_K^\infty(\mathbb{R}^{i\times d})$ by*

$$\phi_i = \phi_1^i \otimes \cdots \otimes \phi_i^i.$$

Then if $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T < \infty$,

$$\mathbb{E}\langle \phi_1, \mu_{t_1}\rangle \le C(T)\|\phi_1\|_\infty,$$

$$\mathbb{E}\langle \phi_2, \mu_{t_1}\mu_{t_2}\rangle \le C(T)\|\phi_2\|_\infty,$$

$$\mathbb{E}\langle \phi_3, \mu_{t_1}\mu_{t_2}\mu_{t_3}\rangle \le C(T)\|\phi_3\|_\infty$$

and

$$\mathbb{E}\langle \phi_4, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle \le C(T)\|\phi_4\|_\infty.$$

PROOF. Since $\int dy q_t^k(x, y) = 1$ for any $k \in \mathbb{N}$ and all $x \in \mathbb{R}^{k\times d}$, and since μ_0 is a finite measure having compact support, this follows immediately from Theorem 3.9. □

COROLLARY C.2. *Equations* (10) *and* (9) *continue to hold for* $\phi \in S_{3\times d}$ *and* $\psi \in S_{4\times d}$.

PROOF. From Lemma 3.3 there exist $\{\phi_n \triangleq \sum_{k=1}^n \phi_k^1 \otimes \phi_k^2 \otimes \phi_k^3 : k \in \mathbb{N}\}$ and $\{\psi_n \triangleq \sum_{k=1}^n \psi_k^1 \otimes \psi_k^2 \otimes \psi_k^3 \otimes \psi_k^4 : k \in \mathbb{N}\}$ such that $\phi_k^j, \psi_m^i \in C_K^\infty(\mathbb{R}^d)$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$, $k, m \in \mathbb{N}$, and $\lim_{n\to\infty} \phi_n = \phi$, $\lim_{n\to\infty} \psi_n = \psi$, where the convergence is uniform. For any $n, m \in N$, from equation (9) and Corollary C.1, it follows that

$$\mathbb{E}\langle |\psi_n - \psi_m|, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle \le C(T)\|\psi_n - \psi_m\|_\infty.$$

Thus $\langle \psi_n, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_1}\rangle$ is Cauchy in the complete space $L^1(\mathbb{P})$, and hence convergent. Uniform convergence of ψ_n implies

$$\lim_{n\to\infty} \langle \psi_n, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle = \langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle, \qquad \text{a.s.}$$

Since the L^1 and a.s. limits must agree when they both exist,

$$\lim_{n\to\infty} \mathbb{E}\langle \psi_n, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle = \langle \psi, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_4}\rangle.$$

Considering now the right-hand sides of equations (9) and (10), by uniform convergence, and since μ_0 is finite with compact support, the desired convergence is shown. □

For ease in reading, we introduce the notation

$$\Xi_{\ell;\mathbf{s}_m}^{\mathbf{x}_{m-1}} \triangleq Q_{s_1}^{\ell}\zeta(x_1)Q_{s_2-s_1}^{\ell(x_1)}\zeta(x_2)Q_{s_3-s_2}^{\ell(x_2)}\cdots\zeta(x_{m-2})Q_{s_{m-1}-s_{m-2}}^3\pi_1 Q_{s_m-s_{m-1}}^2, \tag{23}$$

where $\mathbf{x}_m, \mathbf{s}_m, \ell, \ell(x)$ and ζ are defined as in (5), with the convention that $x_{m-1} = 0$ and $\ell(m-1) = 2$.

PROOF OF LEMMA 3.10. Throughout this proof, the norm on L^p will be denoted by $\|\cdot\|_p$. From the moment equation (9) and the preceding corollary, it follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\langle \varphi, \mu_{t_1}\mu_{t_2}\mu_{t_3}^2\rangle \le C \sum_{k=1}^{14} J_k(t_1, t_2, t_3),$$

where the definition of each J_k is implicit in equation (9).

To begin, note that from Dynkin (1965), Theorem 0.5, for $n \in \mathbb{N}$, $x, y \in \mathbb{R}^{n\times d}$,

$$q_t^n(x, y) \le C p_{\iota t}^n(x, y),$$

where C and ι are constants, and $p_\cdot^n = \prod_{i=1}^n p_\cdot$, where $p_\cdot$ is the Brownian transition function on $\mathbb{R}^d$. It thus follows that

$$\begin{aligned} &\Xi_{4;(t_1-s,t_2-s,t_3-s)}^{(0,0)}\varphi(x) \\ &\quad\le C\int da\, p_{\iota(t_1-s)}(x_1, a_1)p_{\iota(t_2-s)}(x_2, a_2) \\ &\qquad\times p_{\iota(t_3-s)}(x_3, a_3)p_{\iota(t_3-s)}(x_4, a_4)\varphi(a) \end{aligned} \tag{24}$$

for all $x \in \mathbb{R}^{4\times d}$, $s \in [0, t_1]$, $a = (a_1, a_2, a_3, a_4)$.

Using inequality (24), it follows that

$$\begin{aligned}
&\langle \Xi^{(0,0)}_{4;(t_1,t_2,t_3)}\varphi, \mu_0^4\rangle \\
&\quad\le C\int \mu_0^4(dx)\int da\, p_{\iota t_3}(x_4, a_4)\prod_{i=1}^{3} p_{\iota t_i}(x_i, a_i)\varphi(a_1, a_2, a_3, a_4) \\
&\quad\le C\int \mu_0(dx_3)\mu_0(dx_4)\int da_3\, da_4\, p_{\iota t_3}(x_3, a_3)p_{\iota t_3}(x_4, a_4)\int da_1\,\phi(a_1 - a_3) \\
&\qquad\times\int da_2\,\phi(a_2 - a_4)\int \mu_0(dx_1)p_{\iota t_1}(x_1, a_1)\int \mu_0(dx_2)p_{\iota t_2}(x_2, a_2) \\
&\quad\le C\|\phi\|_1^2,
\end{aligned}$$

and thus, since $d \le 3$,

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \langle \Xi^{(0,0)}_{4;(t_1,t_2,t_3)}\varphi, \mu_0^4\rangle \le C(T)\|\phi\|_1^2.$$

Let $\{i', j'\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, $i' < j'$, then again from (24),

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\ne j}}^{4}\int_0^{t_1} ds \langle \Xi^{(ij,0,0)}_{3;(s,t_1,t_2,t_3)}\varphi, \mu_0^3\rangle \\
&\quad\le C\sum_{\substack{i,j=1\\ i<j}}^{4}\int_0^{t_1} ds\int \mu_0^3(dx)\int dy\, p_{\iota s}(x_1, y)\int da_1\, da_2\, da_3\, da_4\, p_{\iota(t_i - s)}(y, a_i) \\
&\qquad\qquad\times p_{\iota(t_{j\wedge 3}-s)}(y, a_j)p_{\iota t_{i'}}(x_2, a_{i'})p_{\iota t_{j'\wedge 3}}(x_3, a_{j'})\phi(a_1 - a_3)\phi(a_2 - a_4)
\end{aligned}$$

and so, with a some applications of the Kolmogorv–Chapman equation to the above expression,

$$\sum_{\substack{i,j=1\\ i\ne j}}^{4}\int_0^{t_1} ds \langle \Xi^{(ij,0,0)}_{3;(s,t_1,t_2,t_3)}\varphi, \mu_0^3\rangle \le C(T)\|\phi\|_1^2 + C\int_0^{t_1} ds (t_3 - s)^{-d/2}.$$

Since $d \le 3$, it follows that

$$\sum_{\substack{i,j=1\\ i\ne j}}^{4}\int_0^{t_3} dt_2\int_0^{t_2} dt_1\int_0^{t_1} ds \langle \Xi^{(ij,0,0)}_{3;(s,t_1,t_2,t_3)}\varphi, \mu_0^3\rangle \le C(T)\|\phi\|_1^2.$$

This next case becomes quite a bit more complicated, so we explain with more detail. Consider

$$\sum_{\substack{i,j=1\\i\neq j}}^{4}\sum_{\substack{n,m=1\\n\neq m}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\langle\Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)}\varphi,\mu_0^2\rangle,$$

wherein the presence of both Φ_{nm} and Φ_{ij} greatly increase the number of cases. In bounding, we again may assume, with the addition of a multiplicative constant to the bound, that $i<j$. Note first that when $m+n\neq 6-i$ it will follow that either

$$\begin{aligned}
&\Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)}\varphi(x_1,x_2)\\
&\quad\le C\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)\\
&\qquad\times\int dw_1\,dw_2\,dw_3\,dw_4\,p_{\iota(t_i-s_2)}(z,w_i)\\
&\qquad\quad\times p_{\iota(t_{j\wedge3}-s_2)}(z,w_j)p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota t_{j'\wedge3}}(x_2,w_{j'})\varphi(w)
\end{aligned}$$

or

$$\begin{aligned}
&\Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)}\varphi(x_1,x_2)\\
&\quad\le C\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota(s_2-s_1)}(y,z)\\
&\qquad\times\int dw_1\,dw_2\,dw_3\,dw_4\,p_{\iota(t_i-s_2)}(z,w_i)\\
&\qquad\quad\times p_{\iota(t_{j\wedge3}-s_2)}(z,w_j)p_{\iota(t_{j'\wedge3}-s_1)}(y,w_{j'})p_{\iota t_{i'}}(x_2,w_{i'})\varphi(w),
\end{aligned}$$

where again $\{i',j'\}=\{1,2,3,4\}\setminus\{i,j\}$, with $i'<j'$. In the case that $m+n=6-i$, we have the bound

$$\begin{aligned}
&\Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)}\varphi(x_1,x_2)\\
&\quad\le C\int dy\,p_{\iota s_1}(x_1,y)\int dz\,p_{\iota s_2}(x_2,z)\\
&\qquad\times\int dw_1\,dw_2\,dw_3\,dw_4\,p_{\iota(t_i-s_2)}(z,w_i)\\
&\qquad\quad\times p_{\iota(t_{j\wedge3}-s_2)}(z,w_j)p_{\iota(t_{i'}-s_1)}(y,w_{i'})p_{\iota(t_{j'\wedge3}-s_1)}(y,w_{j'})\varphi(w).
\end{aligned}$$

It thus follows that

$$\sum_{\substack{i,j=1\\i\neq j}}^{4}\sum_{\substack{n,m=1\\n\neq m}}^{3}\int_0^{t_1}ds_2\int_0^{s_2}ds_1\langle\Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)}\varphi,\mu_0^2\rangle\le C(A_1+A_2+A_3),$$

where

$$
\begin{aligned}
A_1 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3} & \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx_1) \int dy\, p_{\iota s_1}(x_1, y) \\
& \times \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
& \times \int dw_i\, dw_j\, dw_{i'}\, p_{\iota(t_{i'}-s_1)}(y, w_{i'}) p_{\iota(t_i-s_2)}(z, w_i) \\
& \qquad \times p_{\iota(t_{j\wedge 3}-s_2)}(z, w_j) \int dw_{j'}\, \phi(w_1 - w_3)\phi(w_2 - w_4) \\
& \times \int \mu_0(dx_2) p_{\iota t_{j'\wedge 3}}(x_2, w_{j'}),
\end{aligned}
$$

$$
\begin{aligned}
A_2 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3} & \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx_1) \int dy\, p_{\iota s_1}(x_1, y) \\
& \times \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
& \times \int dw_i\, dw_j\, dw_{j'}\, p_{\iota(t_{j'\wedge 3}-s_1)}(y, w_{j'}) p_{\iota(t_i-s_2)}(z, w_i) \\
& \qquad \times p_{\iota(t_{j\wedge 3}-s_2)}(z, w_j) \int dw_{i'}\, \phi(w_1 - w_3)\phi(w_2 - w_4) \\
& \times \int \mu_0(dx_2) p_{\iota t_{i'}}(x_2, w_{i'})
\end{aligned}
$$

and

$$
\begin{aligned}
A_3 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m= i}}^{3} & \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \\
& \times \int dy\, p_{\iota s_1}(x_1, y) \int dz\, p_{\iota s_2}(x_2, z) \\
& \times \int dw_1\, dw_2\, dw_3\, dw_4\, p_{\iota(t_{i'}-s_1)}(y, w_{i'}) \\
& \qquad \times p_{\iota(t_{j'\wedge 3}-s_1)}(y, w_{j'}) p_{\iota(t_i-s_2)}(z, w_i) \\
& \qquad \times p_{\iota(t_{j\wedge 3}-s_2)}(z, w_j) \varphi(w_1, w_2, w_3, w_4).
\end{aligned}
$$

For the first of the above three terms, the process is as follows. Bound $\mu_0(dx_2)$ by $\|m\|_\infty\, dx_2$, integrate $p_{\iota t_{j'\wedge 3}}(x_2, w_j')$ with respect to dx_2, then ϕ with respect to $dw_{j'}$. In doing so, we may then integrate out one of the remaining w_i, w_j or w'_j. In what remains, if $(i,j) \neq (1,2)$ there will be the term $p_{\iota(t_3-s_2)}(z, w_\cdot)$, or if $(i,j) = (1,2)$, the term $p_{\iota(t_3-s_1)}(y, w_3)$. In either case, bound the respective term by $C(t_3 - s_\cdot)^{-d/2}$. This allows for the integration of the second ϕ.

For the second of the two above terms, bound $\mu_0(dx_2)$ by $\|m\|_\infty \, dx_2$, integrate $p_{\iota t_{i'\wedge 3}}(x_2, w'_i)$ with respect to dx_2, then ϕ with respect to $dw_{i'}$. In doing so, we may then integrate out one of the remaining w_i, w_j or w'_j. In what remains, if $(i,j) \in \{(1,2),(1,4),(2,3)\}$ there will be the term $p_{\iota(t_3-s_1)}(y, w_{j'})$, otherwise there will exist the term $p_{\iota(t_3-s_2)}(z, w_\cdot)$. In either case, bound the respective term by $C(t_3 - s_\cdot)^{-d/2}$. This allows for the integration of the second ϕ.

For the third and final term, if $(i,j) \notin \{(1,2),(3,4)\}$, there will exist the terms $p_{\iota(t_3-s_2)}(z, w_j)$ and $p_{\iota(t_3-s_1)}$, which are bounded, respectively, by $C(t_3 - s_2)^{-d/2}$ and $C(t_3 - s_1)^{-d/2}$. When $(i,j) = (1,2)$ we bound $p_{\iota(t_3-s_1)}(y, w_3)$ and $p_{\iota(t_2-s_2)}(z, w_2)$, respectively, by $C(t_3 - s_1)^{-d/2}$ and $C(t_2 - s_2)^{-d/2}$. Finally, when $(i,j) = (3,4)$, bound the terms $p_{\iota(t_3-s_2)}(z, w_3)$ and $p_{\iota(t_2-s_1)}(y, w_2)$, respectively, by $C(t_3 - s_2)^{-d/2}$ and $C(t_2 - s_1)^{-d/2}$. This allows for the desired integration of $\phi(w_1 - w_3)\phi(w_2 - w_4)$.

Combining the above, and since $d \le 3$, we arrive at the bound

$$\sum_{\substack{i,j=1\\ i\neq j}}^{4} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_1 \int_0^{s_1} ds_2 \langle \Xi^{(nm,ij,0,0)}_{2;(s_1,s_2,t_1,t_2,t_3)} \varphi, \mu_0^2 \rangle \le C(T)\|\phi\|_1^2.$$

Considering the next case, note first the similarities in the respective corresponding particle pictures of this and the previous case. This case can be seen as a modification of the previous case in which the two original particles were both born from a common ancestor. Thus, arguing as before, we arrive at the bound

$$\sum_{\substack{i,j=1\\ i\neq j}}^{4} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \langle \Xi^{(12,nm,ij,0,0)}_{1;(s_1,s_2,s_3,t_1,t_2,t_3)} \varphi, \mu_0 \rangle \le C(B_1 + B_2 + B_3),$$

where for the B_k we have

$$\begin{aligned} B_1 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3} & \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy \, p_{\iota s_1}(x,y) \\ & \times \int dz \, p_{\iota(s_2-s_1)}(y,z) \int dw \, p_{\iota(s_3-s_2)}(z,w) \\ & \times \int dv_1 \, dv_2 \, dv_3 \, dv_4 \, p_{\iota(t_{j'\wedge 3}-s_1)}(y, v_{j'}) p_{\iota(t_{i'}-s_2)}(z, v_{i'}) \\ & \qquad \times p_{\iota(t_i-s_3)}(w, v_i) p_{\iota(t_{j\wedge 3}-s_3)}(w, v_j) \\ & \qquad \times \phi(v_1 - v_3)\phi(v_2 - v_4), \end{aligned}$$

$$B_2 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m\neq i}}^{3} \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$

$$\times \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w)$$

$$\times \int dv_1\, dv_2\, dv_3\, dv_4\, p_{\iota(t_{i'}-s_1)}(y, v_{i'}) p_{\iota(t_{j'\wedge 3}-s_2)}(z, v_{j'})$$

$$\times\, p_{\iota(t_i-s_3)}(w, v_i) p_{\iota(t_{j\wedge 3}-s_3)}(w, v_j)$$

$$\times\, \phi(v_1 - v_3)\phi(v_2 - v_4)$$

and

$$B_3 = \sum_{\substack{i,j,i',j'=1\\ i'\neq i, j'\neq j\\ i<j, i'<j'}}^{4} \sum_{\substack{n,m=1\\ n<m\\ 6-n-m=i}}^{3} \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$

$$\times \int dz_1\, dz_2\, p_{\iota(s_2-s_1)}(y, z_1) p_{\iota(s_3-s_1)}(y, z_2)$$

$$\times \int dv_1\, dv_2\, dv_3\, dv_4\, p_{\iota(t_i-s_3)}(z_2, v_i) p_{\iota(t_{j\wedge 3}-s_3)}(z_2, v_j)$$

$$\times\, p_{\iota(t_{i'}-s_2)}(z_1, v_{i'}) p_{\iota(t_{j'\wedge 3}-s_2)}(z_1, v_{j'})$$

$$\times\, \phi(v_1 - v_3)\phi(v_2 - v_4).$$

Thus,

$$\begin{aligned} &B_1 + B_2 + B_3 \\ &\quad \le C\|\phi\|_1^2 \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \\ &\qquad \times \int_0^{s_2} ds_1 [(t_3 - s_2)^{-d/2}(t_2 - s_2)^{-d/2} + (t_3 - s_2)^{-d/2}(t_3 - s_3)^{-d/2} \\ &\qquad\qquad + (t_2 - s_2)^{-d/2}(t_3 - s_3)^{-d/2} + (t_3 - s_2)^{-d/2}(t_2 - s_3)^{-d/2}], \end{aligned}$$

where the above bounds are obtained similarly to the previous case. And so, since $d \le 3$,

$$\begin{aligned} &\sum_{\substack{i,j=1\\ i\neq j}}^{4} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \langle \Xi_{1;(s_3,s_2,s_1,t_1,t_2,t_3)}^{(12,nm,ij,0,0)} \varphi, \mu_0 \rangle \\ &\qquad \le C(T)\|\phi\|_1^2. \end{aligned}$$

This takes care of four of the fourteen J_k, and we consider now the next three integrals which are dependent upon the expression

$$
\begin{aligned}
&\Xi^{(0,ij,0)}_{3;(t_1-s_1,s_2-s_1,t_2-s_1,t_3-s_1)}\varphi(x) \\
&\quad\le C\int db\, p_{\iota(s_2-s_1)}(x_3,b) \\
&\qquad\times\int da_1\,da_2\,da_3\,da_4\, p_{\iota(t_1-s_1)}(x_1,a_1)p_{\iota(t_{4-i}-s_1)}(x_2,a_{7-i-j}) \\
&\qquad\quad\times p_{\iota(t_{i+1}-s_2)}(b,a_{i+1})p_{\iota(t_3-s_2)}(b,a_{j+1}) \\
&\qquad\quad\times\phi(a_1-a_3)\phi(a_2-a_4)
\end{aligned}
\tag{25}
$$

for all $x\in\mathbb{R}^{3\times d}$, $0\le s_1\le t_1$, $t_1\le s_2\le t_2$ and $i,j=1,2,3$, $i\ne j$. In the above $z^{ij}_{\cdot}$ refers to the particular arrangement of z_1,z_2 given the pair (i,j). Applying (25) now gives

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\ne j}}^{3}\int_{t_1}^{t_2}ds\langle\Xi^{(0,ij,0)}_{3;(t_1,s,t_2,t_3)}\varphi,\mu_0^3\rangle \\
&\quad\le C\sum_{\substack{i,j=1\\ i<j}}^{3}\int_{t_1}^{t_2}ds\int\mu_0(dx_2)\mu_0(dx_3)\int dy\, p_{\iota s}(x_3,y) \\
&\qquad\times\int dz_{7-i-j}\,dz_{i+1}\,p_{\iota t_{4-i}}(x_2,z_{7-i-j})p_{\iota(t_{i+1}-s)}(y,z_{i+1}) \\
&\qquad\times\int dz_{j+1}\,p_{\iota(t_3-s)}(y,z_{j+1})\int dz_1\,\phi(z_1-z_3)\phi(z_2-z_4) \\
&\qquad\times\int\mu_0(dx_1)p_{\iota t_1}(x_1,z_1)
\end{aligned}
$$

and, by integrating out terms and using known bounds,

$$
\sum_{\substack{i,j=1\\ i\ne j}}^{3}\int_{t_1}^{t_2}ds\langle\Xi^{(0,ij,0)}_{3;(t_1,s,t_2,t_3)}\varphi,\mu_0^3\rangle\le C\|\phi\|_1^2\int_{t_1}^{t_2}ds[1+(t_3-s)^{-d/2}].
$$

And so, since $d\le 3$,

$$
\sum_{\substack{i,j=1\\ i\ne j}}^{3}\int_0^{t_3}dt_2\int_0^{t_2}dt_1\int_{t_1}^{t_2}ds\langle\Xi^{(0,ij,0)}_{3;(t_1,s,t_2,t_3)}\varphi,\mu_0^3\rangle\le C(T)\|\phi\|_1^2.
$$

Again from (25),

$$
\sum_{\substack{i,j=1\\ i\ne j}}^{3}\sum_{\substack{n,m=1\\ n\ne m}}^{3}\int_{t_1}^{t_2}ds_2\int_0^{t_1}ds_1\langle\Xi^{(nm,0,ij,0)}_{2;(s_1,t_1,s_2,t_2,t_3)}\varphi,\mu_0^2\rangle\le C(C_1+C_2+C_3),
$$

where for the C_k we have

$$
\begin{aligned}
C_1 = \sum_{\substack{i,j=1\\ i<j}}^{3} \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1) \int dy\, p_{\iota s_1}(x_1, y) \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
\times \int dw_1\, dw_{i+1}\, dw_{j+1}\, p_{\iota(t_1-s_1)}(y, w_1) \\
\times p_{\iota(t_{i+1}-s_2)}(z, w_{i+1}) p_{\iota(t_3-s_2)}(z, w_{j+1}) \\
\times \int dw_{7-i-j}\, \phi(w_1 - w_3)\phi(w_2 - w_4) \\
\times \int \mu_0(dx_2) p_{\iota(t_{(7-i-j)\wedge 3})}(x_2, w_{7-i-j}) \\
\le C\|\phi\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 (t_3 - s_2)^{-d/2},
\end{aligned}
$$

$$
\begin{aligned}
C_2 = \sum_{\substack{i,j=1\\ i<j}}^{3} \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1) \int dy\, p_{\iota s_1}(x_1, y) \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
\times \int dw_2\, dw_3\, dw_4\, p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y, w_{7-i-j}) p_{\iota(t_{i+1}-s_2)}(z, w_{i+1}) \\
\times p_{\iota(t_3-s_2)}(z, w_{j+1})\phi(w_2 - w_4) \int dw_1\, \phi(w_1 - w_3) \\
\times \int \mu_0(dx_2) p_{\iota t_1}(x_2, w_1) \\
\le C\|\phi\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 [(t_3 - s_1)^{-d/2} + (t_3 - s_2)^{-d/2}]
\end{aligned}
$$

and

$$
\begin{aligned}
C_3 = C \sum_{\substack{i,j=1\\ i<j}}^{3} \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1)\mu_0(dx_2) \\
\times \int dy\, p_{\iota s_1}(x_1, y) \int dz\, p_{\iota s_2}(x_2, z) \\
\times \int dw_1\, dw_2\, dw_3\, dw_4\, p_{\iota(t_1-s_1)}(y, w_1) p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y, w_{7-i-j}) \\
\times p_{\iota(t_{i+1}-s_2)}(z, w_{i+1}) p_{\iota(t_3-s_2)}(z, w_{j+1}) \phi(w_1 - w_3)\phi(w_2 - w_4) \\
\le C\|\phi\|_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 [(t_3 - s_1)^{-d/2}(t_3 - s_2)^{-d/2} + (t_3 - s_2)^{-d/2}].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&C_1 + C_2 + C_3 \\
&\quad \le C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 [(t_3 - s_1)^{-d/2} + (t_3 - s_1)^{-d/2}(t_3 - s_2)^{-d/2} \\
&\qquad\qquad + (t_3 - s_2)^{-d/2}].
\end{aligned}
$$

Since $d \le 3$,

$$
\sum_{\substack{i,j=1\\ i\neq j}}^{3} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \langle \Xi_{2;(s_1,t_1,s_2,t_2,t_3)}^{(nm,0,ij,0)} \varphi, \mu_0^2 \rangle \le C(T)\|\phi\|_1^2.
$$

After one final application of (25),

$$
\sum_{\substack{i,j=1\\ i\neq j}}^{3} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \langle \Xi_{1;(s_1,s_2,t_1,s_3,t_2,t_3)}^{(12,nm,0,ij,0)} \varphi, \mu_0 \rangle \le C(D_1 + D_2 + D_3),
$$

where for the D_k we have

$$
\begin{aligned}
D_1 &= C \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \\
&\qquad \times \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) \\
&\qquad \times \int dv_1\, dv_2\, dv_3\, dv_4\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_{(7-i-j)\wedge 3}-s_1)}(y, v_{7-i-j}) \\
&\qquad\quad \times p_{\iota(t_{i+1}-s_3)}(w, v_{i+1}) p_{\iota(t_3-s_3)}(w, v_{j+1}) \phi(v_1 - v_3)\phi(v_2 - v_4) \\
&\le C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 [(t_3 - s_1)^{-d/2}(t_3 - s_3)^{-d/2} \\
&\qquad\qquad + (t_2 - s_1)^{-d/2}(t_3 - s_3)^{-d/2}], \\
D_2 &= C \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\qquad \times \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) \\
&\qquad \times \int dv_1\, dv_2\, dv_3\, dv_4\, p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(z, v_{7-i-j}) p_{\iota(t_1-s_1)}(y, v_1) \\
&\qquad\quad \times p_{\iota(t_{i+1}-s_3)}(w, v_{i+1}) p_{\iota(t_3-s_3)}(w, v_{j+1}) \phi(v_1 - v_3)\phi(v_2 - v_4)
\end{aligned}
$$

$$\leq C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 [(t_3 - s_2)^{-d/2}(t_3 - s_3)^{-d/2} + (t_2 - s_2)^{-d/2}(t_3 - s_3)^{-d/2}]$$

and

$$\begin{aligned} D_3 = C \sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\ \times \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_1)}(y,w) \\ \times \int dw_1\, dw_2\, dw_3\, dw_4\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(z, v_{7-i-j}) \\ \times p_{\iota(t_{i+1}-s_3)}(w, v_{i+1}) p_{\iota(t_3-s_3)}(w, v_{j+1}) \phi(v_1 - v_3)\phi(v_2 - v_4) \\ \leq C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 [(t_3 - s_2)^{-d/2}(t_3 - s_3)^{-d/2} \\ + (t_2 - s_2)^{-d/2}(t_3 - s_3)^{-d/2}]. \end{aligned}$$

Thus,

$$\begin{aligned} D_1 + D_2 + D_3 \\ \leq C\|\phi\|_1^2 \sum_{k=1}^{2} \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \\ \times \int_0^{s_2} ds_1 \big[(t_3 - s_3)^{-d/2}\big((t_3 - s_k)^{-d/2} + (t_2 - s_k)^{-d/2}\big)\big]. \end{aligned}$$

Therefore, since $d \leq 3$,

$$\begin{aligned} \sum_{\substack{i,j=1\\ i\neq j}}^{3} \sum_{\substack{n,m=1\\ n\neq m}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \langle \Xi^{(12,nm,0,ij,0)}_{1;(s_1,s_2,t_1,s_3,t_2,t_3)}, \mu_0\rangle \\ \leq C(T)\|\phi\|_1^2. \end{aligned}$$

Thus seven of the fourteen J_k are now shown to have the desired bound, we continue with three more of the J_k.

$$\begin{aligned} \Xi^{(0,0,12)}_{3;(t_1-s_1,t_2-s_1,s_2-s_1,t_3-s_1)}\varphi(x) \\ \leq C \int db\, p_{\iota(s_2-s_1)}(x_3, b) \\ \times \int da_1\, da_2\, da_3\, da_4\, p_{\iota(t_1-s_1)}(x_1, a_1) p_{\iota(t_2-s_1)}(x_2, a_2) \\ \times p_{\iota(t_3-s_2)}(b, a_3) p_{\iota(t_3-s_2)}(b, a_4) \varphi(a_1, a_2, a_3, a_4) \end{aligned} \tag{26}$$

for all $x \in \mathbb{R}^{3\times d}$, $0 \le s_1 \le t_1, t_2 \le s_2 \le t_3$. Now, from inequality (26),

$$\begin{aligned}
&\int_{t_2}^{t_3} ds \langle \Xi^{(0,0,12)}_{3;(t_1,t_2,s,t_3)} \varphi, \mu_0^3 \rangle \\
&\quad \le C \int_{t_2}^{t_3} ds \int \mu_0(dx_3) \int dy\, p_{\iota s}(x_3, y) \\
&\qquad \times \int dz_3\, dz_4\, p_{\iota(t_3-s)}(y, z_3) p_{\iota(t_3-s)}(y, z_4) \int dz_1\, \phi(z_1 - z_3) \\
&\qquad \times \int dz_2\, \phi(z_2 - z_4) \int \mu_0(dx_1) p_{\iota t_1}(x_1, z_1) \int \mu_0(dx_2) p_{\iota t_2}(x_2, z_2) \\
&\quad \le C(T) \|\phi\|_1^2.
\end{aligned}$$

It thus follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds \langle \Xi^{(0,0,12)}_{3;(t_1,t_2,s,t_3)} \varphi, \mu_0^3 \rangle \le C(T) \|\phi\|_1^2.$$

Again from (26), we have that

$$\sum_{\substack{i,j=1 \\ i \ne j}}^{3} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \langle \Xi^{(ij,0,0,12)}_{2;(s_1,t_1,t_2,s_2,t_3)} \varphi, \mu_0^2 \rangle \le C(E_1 + E_2),$$

where for E_1 and E_2 we have

$$\begin{aligned}
E_1 &= \sum_{k=1}^{2} \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \int \mu_0(dx_1) \int dy\, p_{\iota s_1}(x_1, y) \int dz\, p_{\iota(s_2-s_1)}(y, z) \\
&\qquad \times \int dw_{3-k}\, dw_{5-k}\, p_{\iota(t_{3-k}-s_1)}(y, w_{3-k}) p_{\iota(t_3-s_2)}(z, w_{5-k}) \\
&\qquad \times \int dw_{k+2}\, p_{\iota(t_3-s_2)}(z, w_{k+2}) \int dw_k\, \phi(w_1 - w_3) \phi(w_2 - w_4) \\
&\qquad \times \int \mu_0(dx_2) p_{\iota t_k}(x_2, w_k) \\
&\le C \|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 (t_3 - s_1)^{-d/2}
\end{aligned}$$

and

$$\begin{aligned}
E_2 &= \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 (t_3 - s_2)^{-d/2} \\
&\qquad \times \int \mu_0(dx_1) \mu_0(dx_2) \int dw_1\, dw_3\, \phi(w_1 - w_3) \\
&\qquad \times \int dy\, p_{\iota s_1}(x_1, y) p_{\iota(t_1-s_1)}(y, w_1) \int dz\, p_{\iota s_2}(x_2, z) p_{\iota(t_3-s_2)}(z, w_3)
\end{aligned}$$

$$\times \int dw_4\, p_{\iota(t_3-s_2)}(z,w_4) \int dw_2\, \phi(w_2-w_4)$$

$$\le C\|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_2)^{-d/2}.$$

Since $d \le 3$, it follows that

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_2 \int_0^{t_1} ds_1 \langle \Xi^{(ij,0,0,12)}_{2;(s_1,t_1,t_2,s_2,t_3)} \varphi, \mu_0^2\rangle \le C(T)\|\phi\|_1^2.$$

With one final application of (26), we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \langle \Xi^{(12,ij,0,0,12)}_{1;(s_1,s_2,t_1,t_2,s_3,t_3)} \varphi, \mu_0\rangle \le C(F_1+F_2+F_3),$$

where, for F_1 and F_2, we have

$$F_1 = \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2-s_1)^{-d/2} \int \mu_0(dx) \int dz\, p_{\iota s_2}(x,z)$$

$$\times \int dw\, p_{\iota(s_3-s_2)}(z,w) \int dv_1\, dv_3\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_3-s_3)}(w,v_3)\phi(v_1-v_3)$$

$$\times \int dv_4\, p_{\iota(t_3-s_3)}(w,v_4) \int dv_2\, \phi(v_2-v_4)$$

$$\le C\|\phi\|_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2-s_1)^{-d/2} \int \mu_0(dx) \int dz\, p_{\iota s_2}(x,z)$$

$$\times \int dv_1\, dv_3\, p_{\iota(t_1-s_2)}(z,v_1) p_{\iota(t_3-s_2)}(z,v_3)\phi(v_1-v_3)$$

and

$$F_2 = \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y)$$

$$\times \int dw\, p_{\iota(s_3-s_1)}(y,w) \int dv_1\, dv_3\, p_{\iota(t_1-s_1)}(y,v_1) p_{\iota(t_3-s_3)}(w,v_3)\phi(v_1-v_3)$$

$$\times \int dv_4\, p_{\iota(t_3-s_3)}(w,v_4) \int dv_2\, \phi(v_2-v_4)$$

$$\le C\|\phi\|_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 (t_2-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_2}(x,y)$$

$$\times \int dv_1\, dv_3\, p_{\iota(t_1-s_2)}(y,v_1) p_{\iota(t_3-s_1)}(y,v_3)\phi(v_1-v_3).$$

Therefore, since $d \le 3$,

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_3 \int_0^{t_1} ds_2 \int_0^{s_2} ds_1 \langle \Xi^{(12,ij,0,0,12)}_{1;(s_1,s_2,t_1,t_2,s_3,t_3)} \varphi, \mu_0\rangle$$
$$\le C(T)\|\phi\|_1^2.$$

As a total count of the original fourteen J_k, the desired bound has now been shown for ten. We continue now with

$$\begin{aligned}
&\Xi^{(0,12,ij,0)}_{2;(t_1-s_1,s_2-s_1,s_3-s_1,t_2-s_1,t_3-s_1)}\varphi(x)\\
&\quad\le C\int db_1\, db_2\, p_{\iota(s_2-s_1)}(x_2,b_1)p_{\iota(s_3-s_2)}(b_1,b_2)\\
&\qquad\times\int da_1\, da_2\, da_3\, da_4\, p_{\iota(t_1-s_1)}(x_1,a_1)p_{\iota(t_{(7-i-j)\wedge 3}-s_2)}(b_1,a_{7-i-j})\\
&\qquad\quad\times p_{\iota(t_{i+1}-s_3)}(b_2,a_{i+1})p_{\iota(t_3-s_3)}(b_2,a_{j+1})\varphi(a)
\end{aligned}\tag{27}$$

for any $x \in \mathbb{R}^{2\times d}$, $0 \le s_1 \le t_1 \le s_2 \le s_3 \le t_2$, and $i, j = 1, 2, 3$, $i < j$. Applying inequality (27) gives

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 \langle \Xi^{(0,12,ij,0)}_{2;(t_1,s_1,s_2,t_2,t_3)}\varphi, \mu_0^2\rangle\\
&\quad\le C\|\phi\|_1^2 \sum_{k=1}^{2} \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 \int \mu_0(dx_2) \int dy\, p_{\iota s_k}(x_2,y)\\
&\qquad\times \int dw_2\, dw_4\, p_{\iota(t_2-s_k)}(y,w_2)p_{\iota(t_3-s_k)}(y,w_4)\phi(w_2-w_4)\\
&\quad\le C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 [(t_3-s_1)^{-d/2} + (t_3-s_2)^{-d/2}].
\end{aligned}$$

Therefore, since $d \le 3$, we have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_2 \int_{t_1}^{s_2} ds_1 \langle \Xi^{(0,12,ij,0)}_{2;(t_1,s_1,s_2,t_2,t_3)}\varphi, \mu_0^2\rangle \le C(T)\|\phi\|_1^2.$$

With a second and final application of (27), it follows that

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 \langle \Xi^{(12,0,12,ij,0)}_{1;(s_1,t_1,s_2,s_3,t_2,t_3)}\varphi, \mu_0\rangle \le C(G_1+G_2+G_3),$$

where for G_1, G_2 and G_3 we have

$$
\begin{aligned}
G_1 &= \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\quad \times \int dv_3 \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) p_{\iota(t_3-s_3)}(w,v_3) \\
&\quad \times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1)\phi(v_1-v_3) \int dv_2\, p_{\iota(s_3-t_1)}(w,v_2) \\
&\quad \times \int dv_4\, \phi(v_2-v_4) \\
&\le C\|\phi\|_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_2)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\quad \times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, p_{\iota(t_3-s_1)}(y,v_3)\phi(v_1-v_3), \\
G_2 &= \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\quad \times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, \phi(v_1-v_3) \\
&\quad \times \int dz\, p_{\iota(s_2-s_1)}(y,z) p_{\iota(t_3-s_2)}(z,v_3) \\
&\quad \times \int dw\, p_{\iota(s_3-s_2)}(z,w) \int dv_2\, p_{\iota(t_2-s_3)}(w,v_2) \int dv_4\, \phi(v_2-v_4) \\
&\le C\|\phi\|_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\quad \times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1) \int dv_3\, p_{\iota(t_3-s_1)}(y,v_3)\phi(v_1-v_3)
\end{aligned}
$$

and

$$
\begin{aligned}
G_3 &= \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_3)^{-d/2} \int \mu_0(dx) \int dy\, p_{\iota s_1}(x,y) \\
&\quad \times \int dv_3 \int dz\, p_{\iota(s_2-s_1)}(y,z) \int dw\, p_{\iota(s_3-s_2)}(z,w) p_{\iota(t_3-s_3)}(w,v_3) \\
&\quad \times \int dv_1\, p_{\iota(t_1-s_1)}(y,v_1)\phi(v_1-v_3) \int dv_2\, p_{\iota(t_3-s_2)}(z,v_2) \\
&\quad \times \int dv_4\, \phi(v_2-v_4) \\
&\le C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_1 (t_3-s_1)^{-d/2} \\
&\qquad \times \big((t_3-s_2)^{-d/2} + (t_3-s_3)^{-d/2}\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&G_1 + G_2 + G_2 \\
&\quad \le C\|\phi\|_1^2 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \\
&\qquad \times \int_0^{t_1} ds_1 (t_3 - s_1)^{-d/2} \big((t_3 - s_2)^{-d/2} + (t_3 - s_3)^{-d/2} \big).
\end{aligned}
$$

And so, since $d \le 3$,

$$
\sum_{\substack{i,j=1 \\ i \ne j}}^{3} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_1}^{t_2} ds_3 \int_{t_1}^{s_3} ds_2 \int_0^{t_1} ds_3 \big\langle \Xi^{(12,0,12,ij,0)}_{1;(s_1,t_1,s_2,s_3,t_2,t_3)}, \mu_0 \big\rangle \le C(T)\|\phi\|_1^2.
$$

It thus remains to show the desired bound on two of the fourteen original J_k. As in the previous steps, the bounds will result from the following simpler bound:

$$
\begin{aligned}
&\Xi^{(0,12,0,12)}_{2;(t_1-s_1,s_2-s_1,t_2-s_1,s_3-s_1,t_3-s_1)} \varphi(x) \\
&\quad \le C \int db_1\, p_{\iota(s_2-s_1)}(x_2, b_1) \int db_2\, p_{\iota(s_3-s_2)}(b_1, b_2) \\
&\qquad \times \int da_1\, da_2\, da_3\, da_4\, p_{\iota(t_1-s_1)}(x_1, a_1) p_{\iota(t_2-s_2)}(b_1, a_2) p_{\iota(t_3-s_3)}(b_2, a_3) \\
&\qquad\quad \times p_{\iota(t_3-s_3)}(b_2, a_4) \varphi(a)
\end{aligned}
\tag{28}
$$

for any $x \in \mathbb{R}^{2\times d}$, $0 \le s_1 \le t_1 \le s_2 \le t_2 \le s_3 \le t_3$. Using inequality (28), it follows that

$$
\begin{aligned}
&\int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \big\langle \Xi^{(0,12,0,12)}_{2;(t_1,s_1,t_2,s_2,t_3)} \varphi, \mu_0^2 \big\rangle \\
&\quad \le C\|\phi\|_1 \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 \int \mu_0(dx_2) \\
&\qquad \times \int dy\, p_{\iota s_1}(x_2, y) \int dw_2\, p_{\iota(t_2-s_1)}(y, w_2) \\
&\qquad \times \int dw_4\, p_{\iota(t_3-s_1)}(y, w_4) \phi(w_2 - w_4) \\
&\quad \le C\|\phi\|_1^2 \int_{t_2}^{t_3} ds_2 \int_{t_1}^{t_2} ds_1 (t_3 - s_1)^{-d/2}.
\end{aligned}
$$

And so, since $d \le 3$,

$$
\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_1 \int_{t_1}^{t_2} ds_2 \big\langle \Xi^{(0,12,0,12)}_{2;(t_1,s_1,t_2,s_2,t_3)} \varphi, \mu_0^2 \big\rangle \le C(T)\|\phi\|_1^2.
$$

Finally, once again by (28),

$$\int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \langle \Xi^{(12,0,12,0,12)}_{1;(s_1,t_1,s_2,t_2,s_3,t_3)} \varphi, \mu_0 \rangle$$
$$\leq C\|\phi\|_1^2 \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 (t_2 - s_1)^{-d/2}(s_3 - s_2)^{-d/2}.$$

It thus follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \int_{t_2}^{t_3} ds_3 \int_{t_1}^{t_2} ds_2 \int_0^{t_1} ds_1 \langle \Xi^{(12,0,12,0,12)}_{1;(s_1,t_1,s_2,t_2,s_3,t_3)} \varphi, \mu_0 \rangle \leq C(T)\|\phi\|_1^2.$$

Therefore, from the bounds established above for each J_k, $k = 1, \dots, 14$, it follows that

$$\int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \, \mathbb{E}\langle \varphi, \mu_{t_1}\mu_{t_2}\mu_{t_3}^2 \rangle \leq C(T)\|\phi\|_{L^1}^2. \qquad \square$$

APPENDIX D: PROOF OF THEOREM 4.1

From Doob's maximal inequality for martingales and Theorem 2.2 we have that for $\phi \in C_K^\infty(\mathbb{R}^d)$, $0 \leq T < \infty$,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} \mu_t(\phi)\Big)^2 \leq 2\mu_0(\phi)^2 + 8\mathbb{E}Z_T(\phi)^2 + 2\mathbb{E}\Big(\int_0^T ds\, \mu_s(L\phi)\Big)^2.$$

For the second term, from Lemma 3.4 and Hölder's inequality,

$$\begin{aligned}
\mathbb{E}Z_T(\phi)^2 &= \int_0^T ds\, \mathbb{E}\mu_s(\phi^2) + \int_0^T ds\, \mathbb{E}\mu_s^2(\Lambda\phi) \\
&= \int_0^T ds\, \mu_0(Q_s\phi^2) + \int_0^T ds\, \mu_0^2(Q_s^2\Lambda\phi) \\
&\quad + \int_0^T ds_1 \int_0^{s_1} ds_2\, \mu_0(Q_{s_2}\Phi_{12}Q^2_{s_1-s_2}\Lambda\phi) \\
&\leq \|m\|_\infty \int_0^T ds\, \|Q_s\phi^2\|_{L^1} + \|m\|_\infty^2 \int_0^T ds\, \|Q_s^2\Lambda\phi\|_{L^1} \\
&\quad + \|m\|_\infty \int_0^T ds_1 \int_0^{s_1} ds_2\, \|\Phi_{12}Q^2_{s_1-s_2}\Lambda\phi\|_{L^1} \\
&\leq C(T)\|\phi\|_{L^2}^2 + C(T)\sum_{i,j=1}^d \|\partial_i\phi\|_{L^1}\|\partial_j\phi\|_{L^1} \\
&\quad + C(T)\sum_{i,j=1}^d \|\partial_i\phi\|_{L^2}\|\partial_j\phi\|_{L^2},
\end{aligned}$$

where in the above $\{S_t : t \geq 0\}$ is the Brownian transition semigroup. With regards to the third term above,

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T ds\, \mu_s(L\phi)\right)^2 \\
&\quad \leq \int_0^T ds_1 \int_0^T ds_2 (\mathbb{E}\mu_{s_1}(L\phi)^2 \mathbb{E}\mu_{s_2}(L\phi)^2)^{1/2} \\
&\quad \leq T^2 \sup_{0\leq s\leq T} \left(\mu_0^2(Q_s^2(L\phi \otimes L\phi)) + \int_0^s dr\, \mu_0(Q_r \Phi_{12} Q_{s-r}^2(L\phi \otimes L\phi))\right) \\
&\quad \leq T^2 \left(C\|L\phi \otimes L\phi\|_{L^1} + \sup_{0\leq s\leq T} \int_0^s dr \int dy (S_{\iota(s-r)} L\phi)(y)^2\right) \\
&\quad \leq C(T) \sum_{i,j,p,q=1}^d (\|\partial_i\, \partial_j \phi\|_{L^1} \|\partial_p\, \partial_q \phi\|_{L^1} + \|\partial_i\, \partial_j \phi\|_{L^2} \|\partial_p\, \partial_q \phi\|_{L^2}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq T} \mu_t(\phi)\Big)^2 \\
&\qquad \leq C(T)\|\phi\|_{L^2}^2 + \sum_{i,j=1}^d (\|\partial_i \phi\|_{L^1} \|\partial_j \phi\|_{L^1} + \|\partial_i \phi\|_{L^2} \|\partial_j \phi\|_{L^2}) \\
&\qquad\quad + \sum_{i,j,p,q=1}^d (\|\partial_i\, \partial_j \phi\|_{L^1} \|\partial_p\, \partial_q \phi\|_{L^1} + \|\partial_i\, \partial_j \phi\|_{L^2} \|\partial_p\, \partial_q \phi\|_{L^2}).
\end{aligned}
\tag{29}
$$

If $\phi \in S_d$, from Rudin (1973), Theorem 7.10, there exists a Cauchy sequence $\{\phi_n\} \subset C_K^\infty(\mathbb{R}^d)$ converging to ϕ in S_d. Thus, from (29), Chebyshev's inequality and a subsequence argument from the Borel–Cantelli lemma [cf. Theorem 4.2.3 of Chung (1974)], there is a subsequence $\{\phi_{n_k}\}$ such that $\mu_t(\phi_{n_k})$ converges uniformly in $t \in [0, T]$ to $\mu_t(\phi)$ with probability one. Therefore, $\mu_t(\phi)$ is an a.s. continuous semimartingale for S_d.

APPENDIX E: PROOF OF LEMMA 4.2

Fix $T \geq 0$, and set $\phi \in C_K^\infty(\mathbb{R}^d)$, then from Itô's lemma [cf. Ikeda and Watanabe (1981)],

$$
\begin{aligned}
\int_0^T dt \langle \psi \otimes \phi, \mu_t \mu_T\rangle &= \int_0^T dt \langle \psi \otimes \phi, \mu_t \mu_t\rangle + \int_0^T dt \int_0^t ds \langle \psi \otimes L\phi, \mu_s \mu_t\rangle \\
&\quad + \int_0^T dZ_t(\phi) \int_0^t ds \langle \psi, \mu_s\rangle.
\end{aligned}
$$

LEMMA E.1. *For any $\phi, \psi \in C_K^\infty(\mathbb{R}^d)$,*

$$\int_0^T dZ_t(\phi) \int_0^t ds\, \mu_s(\psi) = \int_0^T \int_{\mathbb{R}^d} Z(dt, dx) \int_0^t ds\, \phi(x)\mu_s(\psi).$$

PROOF. Let $0 \le t \le T$. It follows from (3) and Corollary C.1 that

$$\begin{aligned}
&\mathbb{E}\biggl(\int_0^t dZ_s(\phi) \int_0^s dv\, \mu_v(\psi)\biggr)^2 \\
&\quad = \mathbb{E}\int_0^t d\langle Z(\phi).\rangle_s \biggl(\int_0^s dv\, \mu_v(\psi)\biggr)^2 \\
&\quad = \int_0^t ds \int_0^s dv_1 \int_0^s dv_2\, \mathbb{E}\mu_s(\phi^2)\mu_{v_1}(\psi)\mu_{v_2}(\psi) \\
&\qquad + \int_0^t ds \int_0^s dv_1 \int_0^s dv_2\, \mathbb{E}\mu_s(\Lambda\phi)^2\mu_{v_1}(\psi)\mu_{v_2}(\psi) \\
&\quad \le C(T)(\|\phi\|_\infty^2 \|\psi\|_\infty^2 + \|\Lambda\phi\|_\infty^2 \|\psi\|_\infty^2).
\end{aligned}$$

By assumption on Λ and since $\phi, \psi \in C_K^\infty(\mathbb{R}^d)$, $\|\Lambda\phi\|_\infty < \infty$ and $\|\Lambda\psi\|_\infty < \infty$, which, since $\|\psi\|_\infty, \|\phi\|_\infty < \infty$, implies by the definition of the stochastic integral [cf. Karatzas and Shreve (2000), Chapter 3] that

$$\int_0^t dZ_s(\phi) \int_0^s dv\, \mu_v(\psi) \in L^2(\mathbb{P}).$$

In addition, it is clear from Lemma C.1 that $\int_0^s dv\, \mu_v(\psi) \in L^2(\mathbb{P})$, and thus, again from the definition of the stochastic integral, $\int_0^s dv\, \mu_v(\psi)$ can be approximated in $L^2(\mathbb{P})$ by simple functions of the form

$$\sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s),$$

where $\bigcup_i A_i^{(n)} = \Omega$, $A_i^{(n)} \cap A_j^{(n)} = \varnothing$ if $i \ne j$, $\bigcup_i (t_i^{(n)}, t_{i+1}^{(n)}] = [0, \infty)$, and $(t_i^{(n)}, t_{i+1}^{(n)}] \cap (t_k^{(n)}, t_{k+1}^{(n)}] = \varnothing$ if $i \ne k$. It follows that an L^2 approximation to $\int_0^t dZ_s(\phi) \int_0^s dv\, \mu_v(\psi)$ is given by

$$\begin{aligned}
&\int_0^t dZ_s(\phi) \sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s) \\
&\qquad = \sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(0,t]}(t_i) \bigl(Z_{t_{i+1}}(\phi) - Z_{t_i}(\phi)\bigr).
\end{aligned}$$

Clearly $f(s,\phi(x)) = \phi(x)\int_0^s dv\, \mu_v(\psi)$ is also in $L^2(\mathbb{P})$, and so there exist simple functions of the form

$$\sum_i^n \sum_{A_i} c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(t_i^{(n)}, t_{i+1}^{(n)}]}(s)\phi(x),$$

converging to $f(s,\phi(x))$ in $L^2(\mathbb{P})$. From Walsh's construction of the stochastic integral with respect to a martingale measure [Walsh (1986)], an L^2 approximation to $\int_0^t \int Z(ds,dx)\phi(x)\int_0^s dv\, \mu_v(\psi)$ is then given by

$$\begin{aligned}\sum_i^n \sum_{A_i} \int c_{A_i} 1_{A_i^{(n)}}(\omega) 1_{(0,t]}(t_i)\phi(x)(Z_{t_{i+1}} - Z_{t_i})(dx)\\ = \sum_i^n \sum_{A_i} c_{A_i} 1_{A_j^{(n)}}(\omega) 1_{(0,t]}(t_i)\big(Z_{t_{i+1}}(\phi) - Z_{t_i}(\phi)\big).\end{aligned}$$

Since any two L^2 limits of a sequence must agree, it follows that

$$\int_0^T dZ_t(\phi)\int_0^t ds\, \mu_s(\psi) = \int_0^T \int_{\mathbb{R}^d} Z(dt,dx)\int_0^t ds\, \phi(x)\mu_s(\psi). \qquad \square$$

Immediately, we arrive at the corollary:

COROLLARY E.2.

$$\begin{aligned}\int_0^T dt \int_0^t ds\, \mu_s(\psi)\mu_t(L\phi) &= \int_0^T dt\, \mu_t(\psi)\mu_T(\phi) - \int_0^T dt\, \mu_t(\psi)\mu_t(\phi)\\ &\quad - \int_0^T \int_{\mathbb{R}^d} Z(dt,dx)\int_0^t ds\, \phi(x)\mu_s(\psi)\end{aligned} \tag{30}$$

for any $\psi, \phi \in C_K^\infty(\mathbb{R}^d)$.

We can now prove the desired lemma.

PROOF. Assume that $\Psi \in S_{2d}$, then from Lemma 3.3 we can choose $\{\Psi_n; n \in \mathbb{N}\}$ such that $\Psi_n(x,y) = \sum_{k=1}^n (\psi_k \otimes \phi_k)(x,y)$, for some $\{\psi_k : k \in \mathbb{N}\}, \{\phi_k : k \in \mathbb{N}\} \subset C_K^\infty(\mathbb{R}^d)$, and Ψ_n converges to Ψ in S_{2d} as $n \to \infty$. It is clear from (30) that

$$\begin{aligned}\int_0^T dt \int_0^t ds \langle L_2\Psi_n, \mu_s\mu_t\rangle &= \int_0^T dt \langle \Psi_n, \mu_t\mu_T\rangle - \int_0^T dt \langle \Psi_n, \mu_t\mu_t\rangle\\ &\quad - \int_0^T \int_{\mathbb{R}^d} Z(dt,dy)\int_0^t ds \langle \Psi_n(\cdot,y), \mu_s\rangle.\end{aligned} \tag{31}$$

From Corollary C.1,

$$\mathbb{E}\left\{\int_0^T dt\langle\Psi_n-\Psi_m,\mu_t\mu_T\rangle\right\}^2$$
$$=\int_0^T dt\int_0^T ds\,\mathbb{E}\langle(\Psi_n-\Psi_m)\otimes(\Psi_n-\Psi_m),\mu_t\mu_T\mu_s\mu_T\rangle$$
$$\le C(T)\|\Psi_n-\Psi_m\|_\infty^2$$

and

$$\mathbb{E}\left\{\int_0^T dt\langle\Psi_n-\Psi_m,\mu_t\mu_t\rangle\right\}^2$$
$$=\int_0^T dt\int_0^T ds\,\mathbb{E}\langle(\Psi_n-\Psi_m)\otimes(\Psi_n-\Psi_m),\mu_t\mu_t\mu_s\mu_s\rangle$$
$$\le C(T)\|\Psi_n-\Psi_m\|_\infty^2,$$

since Ψ_n converges in S_d to Ψ, $\lim_{n\to\infty}\|\Psi_n-\Psi\|_\infty$ and both of the above two terms are L^2 convergent.

For any $t,s\ge 0$, since $\mu_\cdot\in C_{M_F(\mathbb{R}^d)}[0,\infty)$, and $\Psi_n\to\Psi$ uniformly, $\langle\Psi_n,\mu_s\mu_t\rangle\to\langle\Psi,\mu_s\mu_t\rangle$ a.s. Since the L^2 limit must agree with the a.s. limit, $L^2-\lim_{n\to\infty}\langle\Psi_n,\mu_s\mu_t\rangle=\langle\Psi,\mu_s\mu_t\rangle$. Thus, $L^2-\lim_{n\to\infty}\int_0^T dt\langle\Psi_n,\mu_t\mu_T\rangle=\int_0^T dt\langle\Psi,\mu_t\mu_T\rangle$, and $L^2-\lim_{n\to\infty}\int_0^T dt\langle\Psi_n,\mu_t\mu_t\rangle=\int_0^T dt\langle\Psi,\mu_t\mu_t\rangle$.

Consider next the stochastic integral term and the term involving the generator L. From Lemma C.1 it follows that

$$\mathbb{E}\left\{\int_0^T dt\int_0^t ds\langle L_2\Psi_n-L_2\Psi_m,\mu_s\mu_t\rangle\right\}^2$$
$$\le C(T)\|L_2(\Psi_n-\Psi_m)\|_\infty^2$$
$$\le C(T)\sum_{i,j,p,q=1}^d\|\partial_{2_i}\partial_{2_j}(\Psi_n-\Psi_m)\|_\infty\|\partial_{2_p}\partial_{2_q}(\Psi_n-\Psi_m)\|_\infty$$

and

$$\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\Psi_n(\cdot,y)-\Psi_m(\cdot,y),\mu_s\rangle\right\}^2$$
$$\le C(H_1+H_2),$$

where H_1 and H_2 satisfy

$$H_1=\int_0^T dt\int_0^t ds_1\int_0^t ds_2\,\mathbb{E}\langle(\Psi_n-\Psi_m)(x-z)(\Psi_n-\Psi_m)(y-z),$$
$$\mu_{s_1}(dx)\mu_{s_2}(dy)\mu_t(dz)\rangle$$

$$\le C\int_0^T dt\int_0^t ds_2\int_0^{s_2} ds_1\,\mathbb{E}\langle(\Psi_n-\Psi_m)(x-z)\cdot(\Psi_n-\Psi_m)(y-z),\\ \mu_{s_1}(dx)\mu_{s_2}(dy)\mu_t(dz)\rangle$$

and

$$H_2=\sum_{i,j=1}^d\int_0^T dt\int_0^t ds_2\int_0^t ds_1\,\mathbb{E}\langle\partial_i(\Psi_n-\Psi_m)\otimes\partial_j(\Psi_n-\Psi_m),\mu_{s_1}\mu_t\mu_{s_2}\mu_t\rangle$$

$$\le C\sum_{i,j=1}^d\int_0^T dt\int_0^t ds_2\int_0^{s_2} ds_1\,\mathbb{E}\langle\partial_i(\Psi_n-\Psi_m)\,\partial_j(\Psi_n-\Psi_m),\mu_{s_1}\mu_t\mu_{s_2}\mu_t\rangle.$$

Thus,

$$H_1+H_2\le C(T)\|\Psi_n-\Psi_m\|_\infty^2\\ +C(T)\sum_{i,j=1}^d\|\partial_{2_i}(\Psi_n-\Psi_m)\|_\infty\|\partial_{2_j}(\Psi_n-\Psi_m)\|_\infty.$$

Lemma 3.3 implies Ψ_n converges in the Schwartz space $S_{2\times d}$, and thus $\{\partial_{2_i}\Psi_n:n\in\mathbb{N}\}$ and $\{\partial_{2_i}\partial_{2_j}\Psi_n:n\in\mathbb{N}\}$, for all $i,j=1,2,\dots,d$, are uniformly Cauchy sequences.

For any $t,s\ge 0$, since $\mu_\cdot\in C_{M_F(\mathbb{R}^d)}[0,\infty)$, and $D^\alpha\Psi_n\to D^\alpha\Psi$ uniformly for any multiindex α, $\langle D^\alpha\Psi_n,\mu_s\mu_t\rangle\to\langle D^\alpha\Psi,\mu_s\mu_t\rangle$ a.s. Since the L^2 limit must agree with the a.s. limit, $L^2-\lim_{n\to\infty}\langle D^\alpha\Psi_n,\mu_s\mu_t\rangle=\langle D^\alpha\Psi,\mu_s\mu_t\rangle$, and so $L^2-\lim_{n\to\infty}\int_0^T dt\int_0^t ds\langle L_2\Psi_n,\mu_s\mu_t\rangle=\int_0^T dt\int_0^t ds\langle L_2\Psi,\mu_s\mu_t\rangle$.

Finally, $\{\int_0^T\int Z(dt,dy)\int_0^t ds\langle(\Psi_n-\Psi_m)(x,y),\mu_s(dx)\rangle\}$ is a Cauchy sequence in L^2. Now, for each $y\in\mathbb{R}^d$, and $t\in[0,T]$, $\langle\Psi_n(\cdot,y),\mu_t\rangle$ is Cauchy in L^2, and so there exists an a.s convergent subsequence $\langle\Psi_{n_k}(\cdot,y),\mu_t\rangle$. Since μ_t is almost surely finite and $\Psi_n\to\Psi$ uniformly, $\langle\Psi_{n_k}(\cdot,y),\mu_t\rangle\to\langle\Psi(\cdot,y),\mu_t\rangle$ a.s. as $k\to\infty$. Furthermore, both $\langle\Psi_n(\cdot,y),\mu_t\rangle$ and $\langle\Psi(\cdot,y),\mu_t\rangle$ are uniformly continuous in $y\in\mathbb{R}^d$ and $t\in[0,T]$, and so

$$\lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\Psi_{n_k}(\cdot,y),\mu_s\rangle\\ =\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\Psi(\cdot,y),\mu_s\rangle\qquad\text{a.s.}$$

Since the L^2 limit must agree with the a.s. limit,

$$L^2-\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\Psi_n(\cdot,y),\mu_s\rangle\\ =\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\Psi(\cdot,y),\mu_s\rangle.\qquad\square$$

APPENDIX F: PROOF OF THEOREM 4.3

Let $\{G_\varepsilon\} \subset C^\infty(\mathbb{R}^d)$ be any sequence such that G_ε and $\partial_i G_\varepsilon$ converge, respectively, in L^1 to $G^{\lambda,u}$ and $\partial_i G^{\lambda,u}$, and for $\varepsilon_1, \varepsilon_2 > 0$, $x \in \mathbb{R}^d$, define

$$\phi_{\varepsilon_1,\varepsilon_2}(x) = G_{\varepsilon_1}(x) - G_{\varepsilon_2}(x).$$

Then for the two nonstochastic integral terms, it is clear that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T dt \int_0^t ds \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_s \mu_t \rangle\right]^2 \\ &\quad \le C \int_0^T dt\, \mathbb{E}\left[\int_0^t ds \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_s \mu_t \rangle\right]^2 \\ &\quad = C \int_0^T dt \int_0^t ds_1 \int_0^t ds_2\, \mathbb{E}\langle \phi_{\varepsilon_1,\varepsilon_2} \otimes \phi_{\varepsilon_1,\varepsilon_2}, \mu_{s_1}\mu_t \mu_{s_1}\mu_t \rangle \\ &\quad \le C \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\langle \varphi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_3} \rangle \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\mathbb{E}\left[\int_0^T dt \langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_t \mu_T \rangle\right]^2 \\ &\quad = \int_0^T dt_1 \int_0^T dt_2\, \mathbb{E}\langle \phi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_T \mu_{t_2}\mu_T \rangle \\ &\quad \le C \int_0^T dt_2 \int_0^{t_2} dt_1\, \mathbb{E}\langle \varphi_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_T \mu_T \rangle, \end{aligned} \tag{33}$$

where $\varphi_{\varepsilon_1,\varepsilon_2}(x_1, x_2, x_3, x_4) \triangleq \phi_{\varepsilon_1,\varepsilon_2}(x_1 - x_3)\phi_{\varepsilon_1,\varepsilon_2}(x_2 - x_4)$.

For the stochastic integral term, we have

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} Z(dt, dy) \int_0^t ds \langle \phi_{\varepsilon_1,\varepsilon_2}(\cdot - y), \mu_s \rangle\right]^2 \\ &\quad = \int_0^T dt\, \mathbb{E}\left\langle \left[\int_0^t ds \langle \phi_{\varepsilon_1,\varepsilon_2}(\cdot - \cdot\cdot), \mu_s(\cdot)\rangle\right]^2, \mu_t \right\rangle \\ &\qquad + \int_0^T dt \left\langle \Lambda\left[\int_0^t ds \langle \phi_{\varepsilon_1,\varepsilon_2}(\cdot - \cdot\cdot), \mu_s(\cdot)\rangle\right], \mu_t \mu_t \right\rangle \\ &\quad \le C \int_0^T dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \Bigg[\mathbb{E}\langle \hat{\varphi}_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_{t_3} \rangle \\ &\qquad\qquad + \sum_{p,q=1}^{d} \mathbb{E}\langle \varphi^{pq}_{\varepsilon_1,\varepsilon_2}, \mu_{t_1}\mu_{t_2}\mu_{t_3}\mu_{t_3} \rangle \Bigg], \end{aligned} \tag{34}$$

where

$$\varphi^{pq}_{\varepsilon_1,\varepsilon_2}(x_1,x_2,x_3,x_4) \triangleq \partial_p\phi_{\varepsilon_1,\varepsilon_2}(x_1-x_3)\,\partial_q\phi_{\varepsilon_1,\varepsilon_2}(x_2-x_4),$$

for each $p,q=1,2,\dots,d$, and $(x_1,x_2,x_3,x_4)\in\mathbb{R}^{4\times d}$, and

$$\hat{\varphi}_{\varepsilon_1,\varepsilon_2}(x_1,x_2,x_3) \triangleq \varphi_{\varepsilon_1,\varepsilon_2}(x_1,x_2,x_3,x_3),$$

for each $(x_1,x_2,x_3)\in\mathbb{R}^{3\times d}$. Thus, from (32), (33), (34) and Lemma 3.10, it follows that

$$\mathbb{E}\bigg[\int_0^T dt\int_0^t ds\langle\phi_{\varepsilon_1,\varepsilon_2},\mu_s\mu_t\rangle\bigg]^2 \le C(T)\|\phi_{\varepsilon_1,\varepsilon_2}\|_1^2,$$

$$\mathbb{E}\bigg[\int_0^T dt\langle\phi_{\varepsilon_1,\varepsilon_2},\mu_t\mu_T\rangle\bigg]^2 \le C(T)\|\phi_{\varepsilon_1,\varepsilon_2}\|_1^2$$

and

$$\mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}^d}Z(dt,dy)\int_0^t ds\langle\phi_{\varepsilon_1,\varepsilon_2}(\cdot-y),\mu_s\rangle\bigg]^2$$
$$\le C(T)\|\phi_{\varepsilon_1,\varepsilon_2}\|_1^2 + C(T)\sum_{p,q=1}^d\|\partial_p\phi_{\varepsilon_1,\varepsilon_2}\|_1\|\partial_q\phi_{\varepsilon_1,\varepsilon_2}\|_1.$$

Since G_ε and $\partial_i G_\varepsilon$ converge, respectively, to $G^{\lambda,u}$ and $\partial_i G^{\lambda,u}$ in L^1, $i=1,\dots,d$, we have that $\lim_{\varepsilon_1,\varepsilon_2\to 0}\|\phi_{\varepsilon_1,\varepsilon_2}\|_1=0$ and $\lim_{\varepsilon_1,\varepsilon_2\to 0}\|\partial_i\phi_{\varepsilon_1,\varepsilon_2}\|_1=0$.

Since the choice of the $\{G_\varepsilon\}$ is arbitrary, it may be assumed that $G_\varepsilon = G^{\lambda,u}_\varepsilon$ for each $\varepsilon>0$, and the result follows.

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REFERENCES

ADLER, R. J. and LEWIN, M. (1991). An evolution equation for the intersection local times of superprocesses. In *Stochastic Analysis*: *Proceeding of the Durahm Symposium on Stochastic Analysis*. Cambridge Univ. Press, Cambridge. MR1166405

ADLER, R. J. and LEWIN, M. (1992). Local time and Tanaka formulae for super Brownian motion and super stable processes. *Stochastic Process*. *Appl*. **41** 45–67. MR1162718

BILLINGSLEY, P. (1995). *Probability and Measure*. Wiley, New York. MR1324786

CHUNG, K. L. (1974). *A Course In Probability Theory*. Academic Press, New York.

DAWSON, D. A. (1977). The critical measure diffusion process. *Z. Wahrsch. Verw. Gebiete* **40** 125–145. MR0478374

DAWSON, D. A. (1993). Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI*—1991. *Lecture Notes in Math*. **1541**. Springer, Berlin. MR1242575

DAWSON, D. A., LI, Z. and WANG, H. (2001). Superprocesses with dependent spatial motion and general branching densities. *Electron*. *J*. *Probab*. **6** 1–33. MR1873302

DYNKIN, E. B. (1965). *Markov Processes*, *Vols I*, *II*. Academic Press, New York. MR193671

DYNKIN, E. B. (1988). Representation for functionals of superprocesses by multiple stochastic integrals, with application to self-intersection local times. *Astérisque* **157–158** 147–171. MR0976217

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes*, *Characterization and Convergence*. Wiley, New York. MR0838085

HE, H. (2009). Discontinuous superprocesses with dependent spatial motion. *Stochastic Process*. *Appl*. **119** 130–166. MR2485022

HÖRMANDER, L. (1985). *The Analysis of Linear Partial Differential Operators*, *Vol*. *III*. Springer, Berlin. MR0781536

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. North-Holland, Amsterdam. MR0637061

KARATZAS, I. and SHREVE, S. E. (2000). *Brownian Motion and Stochastic Calculus*. Springer, New York. MR0917065

LIEB, E. H. and LOSS, M. (2001). *Analysis*, 2nd ed. *Graduate Studies in Mathematics* **14**. Amer. Math. Soc., Providence, RI. MR1817225

MYTNIK, L. and VILLA, J. (2007). Self-intersection local time of (α, d, β)-superprocesses. *Ann*. *Inst*. *H*. *Poincaré Probab*. *Stat*. **43** 481–507. MR2329513

REN, Y., SONG, R. and WANG, H. (2009). A class of stochastic partial differential equations for interacting superprocesses on a bounded domain. *Osaka J*. *Math*. **46** 373–401. MR2549592

ROSEN, J. (1992). Renormalization and limit theorems for self-intersections of superprocesses. *Ann*. *Probab*. **20** 1341–1368. MR1175265

RUDIN, W. (1973). *Functional Analysis*. McGraw-Hill, New York. MR0365062

RUDIN, W. (1976). *Principles of Mathematical Analysis*. McGraw-Hill, New York. MR0385023

RUDIN, W. (1987). *Real and Complex Analysis*. McGraw-Hill, New York. MR0924157

SKOULAKIS, G. and ADLER, R. J. (2001). Superprocesses over a stochastic flow. *Ann*. *Appl*. *Probab*. **11** 488–543. MR1843056

WALSH, J. (1986). An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour*, *XIV*—1984. *Lecture Notes in Math*. **1180** 265–437. Springer, Berlin. MR0876085

WANG, H. (1998). A class of measure valued branching diffusions in a random medium. *Stoch*. *Anal*. *Appl*. **16** 753–768. MR1632574

WATANABE, S. (1968). A limit theorem of branching processes and continuous state branching processes. *J*. *Math*. *Kyoto Univ*. **8** 141–167. MR0237008

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