The Annals of Probability
2011, Vol. 39, No. 5, 1844–1863
DOI: 10.1214/11-AOP652

SCHRAMM'S PROOF OF WATTS' FORMULA

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Gérard Watts predicted a formula for the probability in percolation that there is both a left–right and an up–down crossing, which was later proved by Julien Dubédat. Here we present a simpler proof due to Oded Schramm, which builds on Cardy's formula in a conceptually appealing way: the triple derivative of Cardy's formula is the sum of two multi-arm densities. The relative sizes of the two terms are computed with Girsanov conditioning. The triple integral of one of the terms is equivalent to Watts' formula. For the relevant calculations, we present and annotate Schramm's original (and remarkably elegant) Mathematica code.

1. Watts' formula. When Langlands, Pichet, Pouliot and Saint-Aubin (1992) were doing computer simulations to test the conformal invariance of percolation, there were several different events whose probability they measured. The first event that they studied was the probability that there is a percolation crossing connecting two disjoint boundary segments. Using conformal field theory, Cardy (1992) derived his now-famous formula for this crossing probability, and the formula was later proved rigorously by Smirnov (2001) for site percolation on the hexagonal lattice. The next event that Langlands et al. tested was the probability that there is both a percolation crossing connecting the two boundary segments and a percolation crossing connecting the complementary boundary segments (see Figure 1). This probability also appeared to be conformally invariant, but finding a formula for it was harder, and it was not until several years after Cardy's formula that Watts (1996) proposed his formula for the probability of this double crossing. Watts considered the derivation of the formula unsatisfactory, even by the standards of physics, but it matched the data of Langlands et al. very well, which lent credibility to the formula. Watts' formula was proved rigorously by Dubédat (2006a).

To express Cardy's formula and Watts' formula for the two types of crossing events, since the scaling limit of percolation is conformally invariant, it is enough to give these probabilities for one canonical domain, and this is usually taken to be the upper half-plane. There are four points on the boundary of the domain (the real line). Label them in increasing order x_1, x_2, x_3 and x_4. Cardy's formula is then the probability that there is a percolation crossing from the interval $[x_1, x_2]$ to the interval $[x_3, x_4]$. Again by conformal invariance, we may map the upper half-plane

Received March 2010; revised February 2011.
MSC2010 subject classifications. Primary 60J67; secondary 82B43.
Key words and phrases. Schramm–Loewner evolution (SLE), percolation.

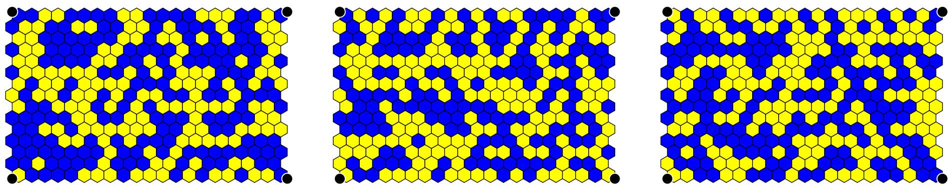


FIG. 1. *In the left panel, there is no left–right crossing in blue hexagons. In the second panel there is a blue left–right crossing, but no blue up–down crossing. In the third panel, there are both blue left–right and blue up–down crossings. Cardy's formula gives the probability of a left–right crossing in a domain, while Watts' formula gives the probability that there is both a left–right crossing and an up–down crossing.*

to itself so that $x_1 \to 0$, $x_3 \to 1$ and $x_4 \to \infty$. The remaining point x_2 gets mapped to

$$s = \operatorname{cr}(x_1, x_2, x_3, x_4) := \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}, \tag{1.1}$$

which is a point in $(0, 1)$ known as the cross-ratio. Both Cardy's formula and Watts' formula are expressed in terms of the cross-ratio. Cardy's formula for the probability of a percolation crossing is

$$\operatorname{cardy}(s) := \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} s^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; s\right), \tag{1.2}$$

where Γ is the gamma function, and ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $a, b, c \in \mathbb{C}$ are parameters, $c \notin -\mathbb{N}$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$), and $(\ell)_n$ denotes $\ell(\ell+1)\cdots(\ell+n-1)$. This series converges for $z \in \mathbb{C}$ when $|z| < 1$, and the hypergeometric function is defined by analytic continuation elsewhere (though it is then not always single-valued).

By comparison, Watts' formula for the probability of the two crossings is the same as Cardy's formula minus another term

$$\begin{aligned} \operatorname{watts}(s) :=& \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} s^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; s\right) \\ & - \frac{1}{\Gamma(1/3)\Gamma(2/3)} s \, {}_3F_2\left(1, 1, \frac{4}{3}; 2, \frac{5}{3}; s\right), \end{aligned} \tag{1.3}$$

where ${}_3F_2$ is the generalized hypergeometric function. The functions $\operatorname{cardy}(s)$ and $\operatorname{watts}(s)$ are shown in Figure 2. [The reader should not be intimidated by these formulae; the parts of the proof involving hypergeometric functions can be handled

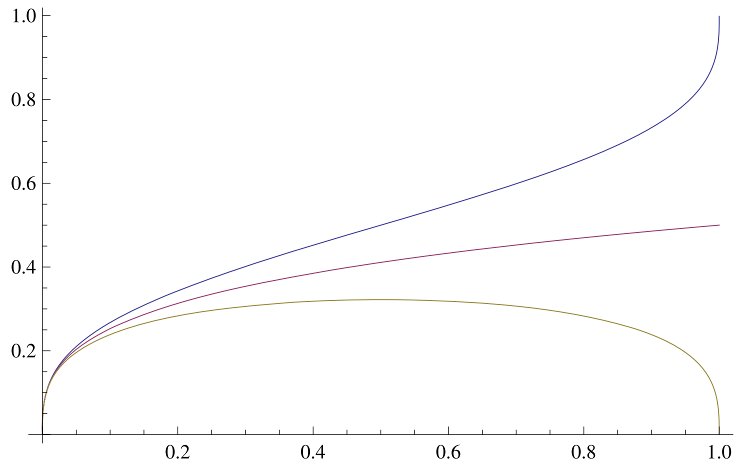


FIG. 2. *Cardy's formula* (*upper curve*), *Watts' formula* (*lower curve*), *and a tripod probability* (*defined in Section* 3) *as a function of the cross-ratio* s.

mechanically with the aid of Mathematica. See also Watts (1996) and Maier (2003) for equivalent double-integral formulations of Watts' formula.]

Schramm thought that Dubédat's paper on Watts' formula was an exciting development and started reading it as soon as it appeared in the arXiv. Schramm sometimes presented papers to interested people at Microsoft Research: for example, he presented Smirnov's proof of Cardy's formula when it came out [Smirnov (2001)], as well as Dubédat's paper on Watts' formula [Dubédat (2006a)], and later Zhan's paper on the reversibility of SLE_κ for $\kappa \leq 4$ [Zhan (2008)]. In the course of reaching his own understanding of Watts' formula, Schramm simplified Dubédat's proof, with the help of a Mathematica notebook, and it was this version that he presented at Microsoft on May 17, 2004. This proof did not come up again until an August 2008 Centre de Recherches Mathématiques (CRM) meeting on SLE in Montréal, after a talk by Jacob Simmons on his work with Kleban and Ziff on "Watts' formula and logarithmic conformal field theory" [Simmons, Kleban and Ziff (2007)]. Schramm mentioned that he had an easier proof of Watts' formula, which he recalled after just a few minutes. The people who saw his version of the proof thought it was very elegant and strongly encouraged him to write it up. The next day Oded wrote down an outline of the proof, but he tragically died a few weeks later. There is interest in seeing a written version of Schramm's version of the proof, so we present it here.

2. Outline of proof. This is a slightly edited version of the proof outline that Oded wrote down at the CRM. Steps 1 and 2 are the same as in Dubédat's proof, but with step 3 the proofs diverge. We will expand on these steps of the outline (with slightly modified notation) in subsequent sections.

- Reduce to the problem of calculating the probability that there is a crossing up–down which also connects to the right.
- Further reduce to the following problem. In the upper half-plane, say, mark points $-\infty < y_1 < x_0 < y_2 < y_3 = \infty$. Let γ be the SLE_6 interface started from x_0. Let $\tau := \inf\{t \geq 0 : \gamma_t \in \mathbb{R} \setminus [y_1, y_2]$ and $\sigma := \sup\{t < \tau : \gamma_t \in \mathbb{R}\}$. Calculate $\mathbf{P}[\gamma_\tau \in [y_2, y_3], \gamma_\sigma \in [x_0, y_2]]$.
- Let $\sigma_1 := \sup\{t < \tau : \gamma_t \in [y_1, x_0]\}$ and $\sigma_2 := \sup\{t < \tau : \gamma_t \in [x_0, y_2]\}$. Now calculate the probability density of the event $\gamma_{\sigma_1} = z_1, \gamma_{\sigma_2} = z_2, \gamma_\tau = z_3$ as $h(z_1, x_0, z_2, z_3) := \partial_{z_1} \partial_{z_2} \partial_{z_3} \mathrm{Cardy}(z_1, x_0, z_2, z_3)$.
- Now, h [times certain derivatives] is a martingale for the corresponding diffusion. Consider the Doob-transform (h-transform) of the diffusion with this h. This corresponds to conditioning on this probability zero event. For the Doob-transform, calculate the probability that $\sigma_2 > \sigma_1$. This comes out to be a hypergeometric function g. Finally,

$$\mathrm{Watts}(y_1, x_0, y_2, y_3) = \int_{[y_1, x_0]} dz_1 \int_{[x_0, y_2]} dz_2 \int_{[y_2, y_3]} gh \, dz_3,$$

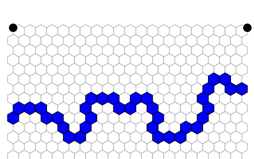
(or more precisely, the three-arm probability), and use integration by parts.

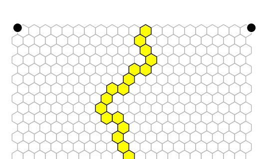
3. Reduction to tripod probabilities. The initial reduction, which is step 1 of the proof, has been derived by multiple people independently. The first place that it appeared in print appears to be in Dubédat's (2004) paper, where it is credited to Werner, who, in turn, is sure that it must have been known earlier. In the interest of keeping the exposition self-contained, we explain this reduction.

It is an elementary fact that exactly one of the following two events occurs:

(1) there is a horizontal blue crossing in the rectangle (i.e., a path of blue hexagons connecting the left and right edges of the rectangle), which we denote by H_b;

(2) there is a vertical yellow crossing, which we denote by V_y.



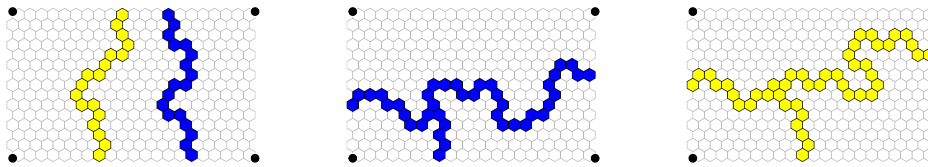


If there is a horizontal crossing, then by considering the region beneath it, using the above fact, either it connects to the bottom edge (forming a T shape) or else there is another crossing beneath it of the opposite color. Since there are finitely many hexagons, there must be a bottom-most crossing, which then necessarily forms a T shape. Thus exactly one of the following three events occurs:

(1) there is no horizontal crossing of either color (denoted by N);

(2) there is a blue T (denoted T_b);

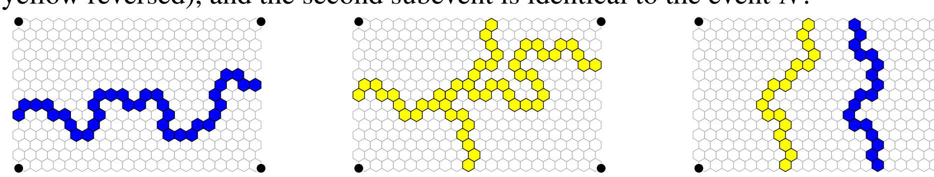
(3) there is a yellow T (denoted T_y).



Of course the latter two events have equal probability, so we have

$$\Pr[N] + 2\Pr[T_b] = 1.$$

Recall again that there is either a blue horizontal crossing or a yellow vertical crossing but not both. We can decompose the yellow vertical crossing event into two subevents according to whether or not there is also a yellow horizontal crossing. The first subevent is, of course, the event we are interested in (with blue and yellow reversed), and the second subevent is identical to the event N.



Thus we have

$$\Pr[H_b] + \Pr[H_y \wedge V_y] + \Pr[N] = 1.$$

Combining these equations, we see that

$$\Pr[H_b \wedge V_b] = 2\Pr[T_b] - \Pr[H_b].$$

In the limit of large grids with cross ratio s, the third term is given by Cardy's formula, $\mathrm{cardy}(s)$, and we seek to show that the left-hand side is given by Watts' formula, $\mathrm{watts}(s)$. Let us give another name for what we expect to be the limit of the second term. Define $\mathrm{tripod}(s)$ to satisfy

$$\mathrm{watts}(s) = 2\,\mathrm{tripod}(s) - \mathrm{cardy}(s),$$

that is [substituting (1.2) and (1.3)],

$$\begin{aligned}\mathrm{tripod}(s) &= \frac{\mathrm{watts}(s) + \mathrm{cardy}(s)}{2}\\ &= \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} s^{1/3}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; s\right)\\ &\quad - \frac{1}{2\Gamma(1/3)\Gamma(2/3)} s\,{}_3F_2\left(1, 1, \frac{4}{3}; 2, \frac{5}{3}; s\right).\end{aligned}$$

Then in light of Cardy's formula, proving Watts' formula is equivalent to showing that $\Pr[T_b]$ is given by $\mathrm{tripod}(s)$ in the fine mesh limit.

4. Comparison with SLE_6.

4.1. *Discrete derivatives of the tripod probability.* Consider percolation on the upper half-plane triangular lattice, and let $P_T[x_1, x_2, x_3, x_4]$ be the probability of a blue tripod connecting the intervals (x_1, x_2) and (x_2, x_3) and (x_3, x_4) when the four (here discrete) locations are $x_1 < x_2 < x_3 < x_4$ (each of which is a point between two boundary hexagons; see the upper image in Figure 3).

Then $\Delta_{x_4} P_T[x_1, x_2, x_3, x_4] := P_T[x_1, x_2, x_3, x_4] - P_T[x_1, x_2, x_3, x_4 - 1]$ gives the probability that there is a crossing tripod for (x_1, x_2, x_3, x_4) but *not* for $(x_1, x_2, x_3, x_4 - 1)$. (Here we assume that the lattice spacing is 1.) Since the crossing tripod for (x_1, x_2, x_3, x_4) does not extend to a crossing tripod for $(x_1, x_2, x_3, x_4 - 1)$, there must be a path of the opposite color from the hexagon just to the left of $x_4 - 1$ to the interval between $x_2 + 1$ and $x_3 + 1$; this event is

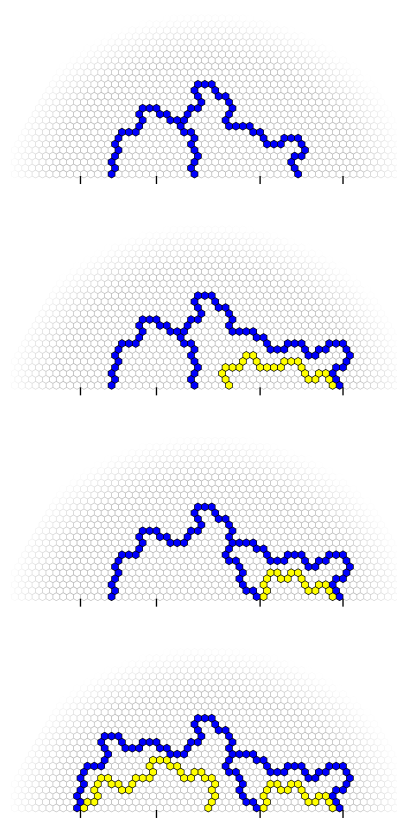


FIG. 3. *The discrete triple partial derivative of the tripod probability is the probability of a multi-arm event. The top panel illustrates the event whose probability is $P_T[x_1, x_2, x_3, x_4]$, the next panel illustrates $\Delta_{x_4} P_T[x_1, x_2, x_3, x_4]$, the third panel illustrates $\Delta_{x_3}\Delta_{x_4} P_T[x_1, x_2, x_3, x_4]$ and the bottom panel illustrates $-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4} P_T[x_1, x_2, x_3, x_4]$.*

represented by the second image in Figure 3. Similarly,

$$-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4}P_T[x_1,x_2,x_3,x_4]$$

gives the probability of a multi-arm event such as the one in the bottom image in Figure 3.

By summing these discrete differences, it is straightforward to write

$$P_T[x_1,x_2,x_3,x_4]=\sum_{c\in(x_3,x_4]}\sum_{b\in(x_2,x_3]}\sum_{a\in(x_1,x_2]}-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4}P_T[a,x_2,b,c].$$

If there is a blue tripod connecting the intervals (x_1,x_2), (x_2,x_3) and (x_3,x_4), then there is only one cluster containing such a tripod. This formula can be interpreted as partitioning the tripod event into multi-arm events of the type shown in the bottom panel of Figure 3. The triple $(a,b,c)\in(x_1,x_2]\times(x_2,x_3]\times(x_3,x_4]$ is uniquely determined by the tripod: a is (half a lattice spacing to the right of) the rightmost boundary point of the tripod cluster in the interval (x_1,x_2), b is (just right of) the rightmost point of the tripod cluster in (x_2,x_3) and c is (just right of) the leftmost point of the tripod cluster in (x_3,x_4).

4.2. *Discrete derivatives of the crossing probability.* Consider percolation on a half-plane triangular lattice, as in the previous subsection, and let $P_C[x_1,x_2,x_3,x_4]$ be the probability of at least one blue cluster spanning the intervals (x_1,x_2) and (x_3,x_4); see the upper image in Figure 4. Then $\Delta_{x_4}P_C[x_1,x_2,x_3,x_4]=P_C[x_1,x_2,x_3,x_4]-P_C[x_1,x_2,x_3,x_4-1]$ gives the probability that there is a crossing for (x_1,x_2,x_3,x_4) but *not* (x_1,x_2,x_3,x_4-1). This event is represented by the second image in Figure 4. Similarly,

$$-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4}P_C[x_1,x_2,x_3,x_4]$$

gives the probability that *one of the two* multi-arm events in the bottom image in Figure 3 occurs. The event of a crossing cluster is equivalent to the event that one of these multi-arm events occurs for *some* (necessarily unique) set of three points $(a,b,c)\in(x_1,x_2]\times(x_2,x_3]\times(x_3,x_4]$: a is (just right of) the rightmost boundary point of the crossing cluster(s) in the interval (x_1,x_2); b is (just right of) the rightmost point of the crossing cluster in (x_2,x_3) [if it exists; otherwise b is the rightmost boundary point in (x_2,x_3) of a crossing yellow cluster, as shown]; and c is (just right of) the leftmost point of the cluster(s) in (x_3,x_4).

Thus $-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4}P_C[x_1,x_2,x_3,x_4]$ decomposes into the probabilities of two multiarm events, the first of which is $-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4}P_T[x_1,x_2,x_3,x_4]$.

4.3. *Multi-arm events and the interface.* Consider the setting of Figures 3 and 4, and suppose we add an additional boundary layer of blue hexagons to the

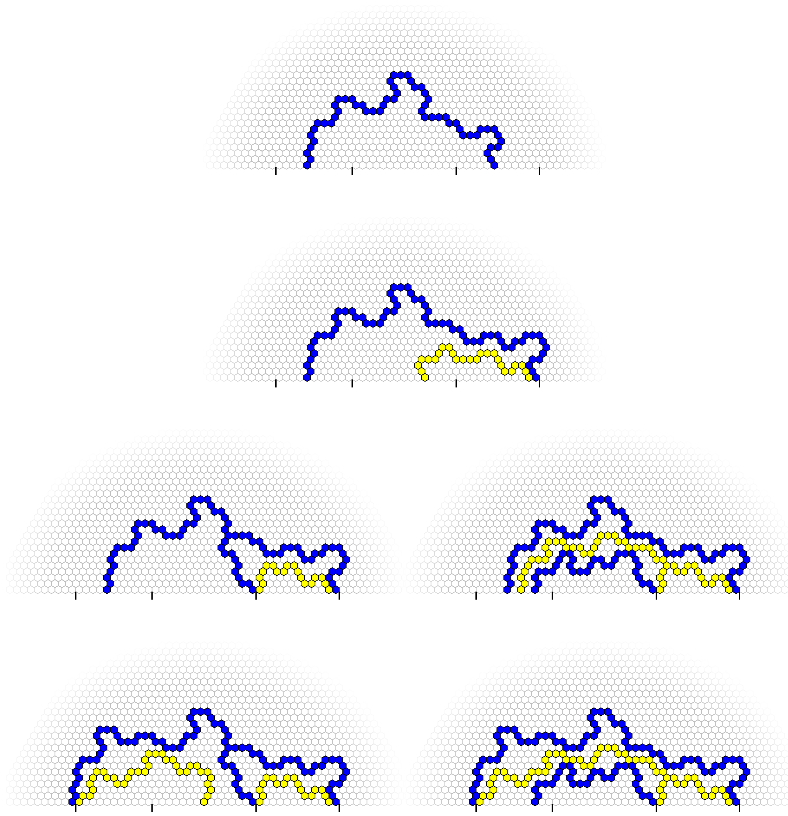


FIG. 4. *The discrete triple partial derivative of the crossing probability is sum of the probabilities of two multi-arm events. The panels illustrate the events whose probability is* $P_C[x_1, x_2, x_3, x_4]$ (*top*), $\Delta_{x_4} P_C[x_1, x_2, x_3, x_4]$ (*second*), $\Delta_{x_3}\Delta_{x_4} P_C[x_1, x_2, x_3, x_4]$ (*third row*), *and* $-\Delta_{x_1}\Delta_{x_3}\Delta_{x_4} P_C[x_1, x_2, x_3, x_4]$ (*bottom row*).

left of x_2 and yellow hexagons to the right of x_2. Then let γ_{discrete} be the discrete interface starting at x_2. (See Figure 5.)

Then the union of the two multi-arm events at the bottom of Figure 4 describes the event that that c is the first boundary point that γ_{discrete} hits outside the interval (x_1, x_3), and that a and b are the leftmost and rightmost boundary points hit by γ_{discrete} *before* c. The left figure corresponds to the case that a is hit before b, and the right figure to the case that b is hit before a.

4.4. *Continuum Watts' formula*: *A statement about SLE.* Like Cardy's formula, Watts' formula has a continuum analog, which is a statement strictly about SLE_6. Fix real numbers $x_1 < x_2 < x_3 < x_4$, and let s be their cross ratio. Consider the usual SLE_6 in the upper half-plane, where the starting point of the path is x_2. Before Smirnov proved Cardy's formula for the scaling limit of triangular

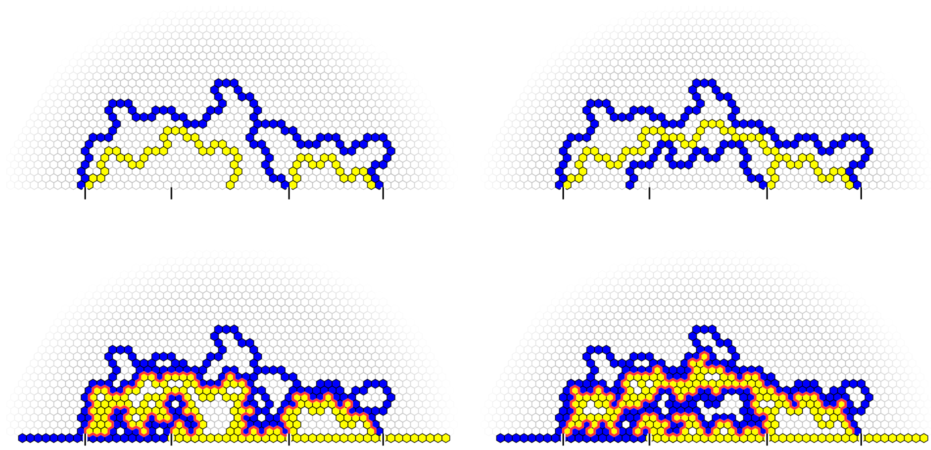


FIG. 5. *The interface interpretation of the multi-arm events.*

lattice percolation, it was already known by Schramm that $\mathrm{cardy}(s)$ represents the probability that γ hits (x_3, x_4) *before* hitting $\mathbb{R} \setminus [x_1, x_4]$. [In the discrete setting of Section 4.3, having γ_{discrete} hit (x_3, x_4) before the complement of (x_1, x_4) is equivalent to the existence of a crossing.] In light of Section 4.3, the following is the natural continuum analog of the tripod formula.

THEOREM 4.1. *Let* $\mathrm{SLEtripod}(s)$ *be the probability that both*:

(1) γ *first hits* (x_3, x_4) (*at some time* t) *before it first hits* $\mathbb{R} \setminus (x_1, x_4)$, *and*
(2) γ *hits the leftmost point of* $\mathbb{R} \cap \gamma[0, t)$ *before it hits the rightmost point*.

Then $\mathrm{SLEtripod}(s) = \mathrm{tripod}(s)$.

Theorem 4.1 is the actual statement that was proved by Dubédat, and the statement whose proof was sketched by Oded. Dubédat claimed further that Theorem 4.1 would imply the tripod formula (and hence Watts' formula) for the scaling limit of critical triangular lattice percolation if one used the (at the time unpublished) proof that SLE_6 is the scaling limit of the interface [Dubédat (2006a)]. To be fully precise, one needs slightly more than the fact that the interface scaling limit is SLE_6: it is important to know that the discrete interface is unlikely to get close to the boundary without hitting it. [Similar issues arise when using Cardy's formula to prove SLE_6 convergence; see, e.g., Camia and Newman (2007).] Rather than address this (relatively minor technical) point here, we will proceed to prove Theorem 4.1 in the manner outlined by Oded and defer this issue until Section 7.

It is convenient to have a name for the SLE versions of the multi-arm events in Figure 4. Say that a triple of distinct real numbers (a, b, c) with $a < 0 < b$ constitutes a *tripod set* for γ if for some $t > 0$ we have:

(1) $\gamma(t) = c$;
(2) $\inf(\gamma[0,t] \cap \mathbb{R}) = a$;
(3) $\sup(\gamma[0,t] \cap \mathbb{R}) = b$.

There are a.s. a countably infinite number of tripod sets, but if $x_1 < 0$ and $x_3 > 0$ is fixed, there is a.s. exactly one for which $x_1 < a < 0 < b < x_3$ and $c \notin (x_1, x_3)$. Let $\mathcal{P}_C : \mathbb{R}^3 \to \mathbb{R}$ be the probability density function for this (a, b, c). (We see in Lemma 4.2 that this density function exists.) There are also two types of tripod sets (a, b, c): those for which γ hits a first and those for which γ hits b first. Write $\mathcal{P}_C = \mathcal{P}_A + \mathcal{P}_B$, where $\mathcal{P}_A$ and $\mathcal{P}_B$ are the corresponding probability densities for a-first and b-first tripod sets.

Now, we claim the following:

LEMMA 4.2. *Using the notation above, the density functions $\mathcal{P}_C$ and $\mathcal{P}_A$ exist, and*

$$\operatorname{cardy}(s) = \int_{x_1}^{0} \int_{0}^{x_3} \int_{x_3}^{x_4} \mathcal{P}_C(a, b, c)\, dc\, db\, da$$

and

$$\operatorname{SLEtripod}(s) = \int_{x_1}^{0} \int_{0}^{x_3} \int_{x_3}^{x_4} \mathcal{P}_A(a, b, c)\, dc\, db\, da.$$

PROOF. In the event that the density functions do not exist, we abuse notation and let "$\mathcal{P}_C(a, b, c)\, dc\, db\, da$" and "$\mathcal{P}_A(a, b, c)\, dc\, db\, da$" denote the relevant measures, which must exist. It is easy to see that the event that (x_3, x_4) is hit before $\mathbb{R} \setminus [x_1, x_4]$ is equivalent to the event that $(a, b, c) \in (x_1, 0) \times (0, x_3) \times (x_3, x_4)$. By definition, SLEtripod(s) is the probability of the same event intersected with the event that γ hits a first. Finally, observe that Cardy's formula of the cross-ratio of 4-points is three-times differentiable, so the density function $\mathcal{P}_C(a, b, c)$ exists and, consequently, the density function $\mathcal{P}_A(a, b, c)$ also exists. $\square$

Of course, from this, one has the immediate corollary:

COROLLARY 4.3. *Using the above notation,*

$$\partial_{x_1} \partial_{x_3} \partial_{x_4} \operatorname{cardy}(\operatorname{cr}(x_1, 0, x_3, x_4)) = \mathcal{P}_C(x_1, x_3, x_4)$$

and

$$\partial_{x_1} \partial_{x_3} \partial_{x_4} \operatorname{SLEtripod}(\operatorname{cr}(x_1, 0, x_3, x_4)) = \mathcal{P}_A(x_1, x_3, x_4).$$

If we could show further that

$$\mathcal{P}_A(x_1, x_3, x_4) = \partial_{x_1} \partial_{x_3} \partial_{x_4} \operatorname{tripod}(\operatorname{cr}(x_1, 0, x_3, x_4)), \tag{4.1}$$

then this corollary and standard integration would imply Theorem 4.1, since we know that tripod(cr(·)) = SLEtripod(cr(·)) on the bounding planes $x_1 = 0$, $x_2 = 0$ and $x_3 = x_4$. Since we already have an explicit formula for tripod, the only remaining step is to explicitly compute $\mathcal{P}_A$. Oded's approach is to compute the ratio $\mathcal{P}_A/\mathcal{P}_C$ as the conditional probability [given that (a, b, c) form a tripod set] that γ hits a before b. Since $\mathcal{P}_C$ is known, this determines $\mathcal{P}_A$.

5. Conditional probability that *a* is hit first. Schramm was very adept with using Mathematica to calculate all manner of things. He probably would have considered this last step to be routine, since it was for him straightforward to set up the right equations and then let Mathematica solve them. At this point we refer to his original Mathematica notebook from 2004, and explain the various steps in the calculation. To be consistent with Oded's notation, we now make the following substitutions:

$$v_3 = a, \qquad W = x_2, \qquad v_1 = b, \qquad v_2 = c.$$

(We assume $v_3 < W < v_1 < v_2$. Oded apparently chose this notation because under cyclic reordering it was the same as $W = v_0, v_1, v_2, v_3$.)

First we formally define the function cardy(s) as in (1.2). The Mathematica function `cardy` defined here involves an additional parameter κ, but it specializes to cardy(s) when $\kappa = 6$. This more general formula is analogous to Cardy's formula but gives the (conjectural) crossing probability for the critical Fortuin–Kasteleyn random cluster model [with $q = 4\cos^2(4\pi/\kappa)$] with alternating wired-free-wired-free boundary conditions. (See Rohde and Schramm [(2005), conjecture 9.7], for some background.) This formula was known at Microsoft in 2003, and most likely Oded copied it from another Mathematica notebook. This formula was later independently discovered [Bauer, Bernard and Kytölä (2005)] (nonrigorously) and [Dubédat (2006b)] (rigorously).

$$\mathbf{cardy[s\_] = c[\kappa]\ s^{1-\frac{4}{\kappa}}\ Hypergeometric2F1\left[\frac{-4+\kappa}{\kappa}, \frac{4}{\kappa}, 1+\frac{-4+\kappa}{\kappa}, s\right]}$$

$$s^{1-\frac{4}{\kappa}}\ \mathrm{c}[\kappa]\ \mathrm{Hypergeometric2F1}\left[\frac{-4+\kappa}{\kappa}, \frac{4}{\kappa}, 1+\frac{-4+\kappa}{\kappa}, \mathrm{s}\right]$$

Consider the evolution of chordal SLE_6 started from W and run to ∞, when at time zero there are 3 marked points at positions v_1, v_2 and v_3. We then let $W(t)$ represent the SLE_6 driving function [i.e., $W(t) = W(0) + \sqrt{6}B_t$ where B_t is a standard Brownian motion] of the Loewner evolution

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)},$$

and interpret the v_i as functions of t, evolving under the Loewner flow, that is, $v_i(t) := g_t(v_i(0))$.

If f is any function of v_1, v_2, v_3, W, we define

$$L(f) := \frac{\partial}{\partial t}\mathbb{E}[f(W(t), v_1(t), v_2(t), v_3(t))]\Big|_{t=0}. \tag{5.1}$$

This is a new function of the same four variables which can be calculated explicitly using Itô's formula as

$$L(f) = \frac{\kappa}{2}\frac{\partial^2}{\partial W^2}f + \sum_{i=1}^{3}\frac{2(\partial/\partial v_i)f}{v_i - W}.$$

This operator is defined as `L` in the Mathematica code below.

```
L[f_] := D[f, W, W] κ / 2 + Sum[D[f, v[i]] 2 / (v[i] - W), {i, 1, 3}]
```

```
cr = (W - v[3]) (v[1] - v[2]) / ((W - v[2]) (v[1] - v[3]))
```

```
(v[1] - v[2]) (W - v[3]) / ((W - v[2]) (v[1] - v[3]))
```

Similarly in the Mathematica code, `cr` is the cross-ratio [defined in (5.2)], that is,

$$\mathrm{cr} := \mathrm{cr}(v_3, W, v_1, v_2) = \frac{(W - v_3)(v_1 - v_2)}{(W - v_2)(v_1 - v_3)}. \tag{5.2}$$

In the next line, Oded performed a consistency check. Cardy's formula should be a martingale for the SLE_κ diffusion, hence $L(\mathrm{cardy}(\mathrm{cr}(\cdot))) = 0$.

```
L[cardy[cr]] /. κ → 6 // Simplify
```

```
0
```

Next, Oded computes the triple derivative `h` of Cardy's formula (which is the same as the $\mathcal{P}_C$ defined in Section 4.4).

```
h = D[cardy[cr], v[1], v[2], v[3]] /. κ → 6 // Simplify
```

```
(2 c[6] (W - v[1]) (W - v[3])
    (v[2]^2 v[3]^2 - v[1] v[2] v[3] (v[2] + v[3]) +
      v[1]^2 (v[2]^2 - v[2] v[3] + v[3]^2) +
      W^2 (v[1]^2 + v[2]^2 - v[2] v[3] + v[3]^2 - v[1] (v[2] + v[3])) -
      W (v[1]^2 (v[2] + v[3]) + v[2] v[3] (v[2] + v[3]) +
          v[1] (v[2]^2 - 6 v[2] v[3] + v[3]^2)))) /
  (27 (W - v[2])^4 (v[1] - v[3])^5 (((W - v[1]) (v[1] - v[2])
          (W - v[3]) (v[2] - v[3])) / ((W - v[2])^2 (v[1] - v[3])^2))^(5/3))
```

That is, he computes

$$h(v_3, W, v_1, v_2) := \partial_{v_1}\partial_{v_2}\partial_{v_3} \operatorname{cardy}\left(\frac{(W - v_3)(v_1 - v_2)}{(W - v_2)(v_1 - v_3)}\right).$$

The result is somewhat complicated, but we may ignore it, since it is just an intermediate result.

The next step involves conditioning on an event of zero probability, the event that (v_3, v_1, v_2) is a tripod set. We can make sense of this by introducing a triple difference of Cardy's formula and recalling the results of Section 4.4. First we introduce notation to describe some small evolving intervals. For given values $v_1(0), v_2(0), v_3(0), W(0)$, pick ε small enough so that the intervals $(v_i(0), v_i(0) + \varepsilon)$ are disjoint and do not contain $W(0)$. Write $\tilde{v}_i(0) = v_i(0) + \varepsilon$. Define $\tilde{v}_i(t)$ using the Loewner evolution, and write $\varepsilon_i(t) := \tilde{v}_i(t) - v_i(t)$. Let us write

$$\begin{aligned} &h_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(v_3, W, v_1, v_2) \\ &\quad := \Delta_{v_1}^{(\varepsilon_1)}\Delta_{v_2}^{(\varepsilon_2)}\Delta_{v_3}^{(\varepsilon_3)} \operatorname{cardy}(\operatorname{cr}(v_3, W, v_1, v_2)), \end{aligned} \tag{5.3}$$

where $\Delta_v^{(\varepsilon)}$ is the difference operator defined by

$$\Delta_v^{(\varepsilon)} f(v) = f(v + \varepsilon) - f(v).$$

Note that the $\Delta_{v_i}^{(\varepsilon_i)}$ depend on t. By Corollary 4.3, equation (5.3) at time t represents the conditional probability (given the Loewner evolution up to time t) that there is a tripod set in $[v_3(0), \tilde{v}_3(0)] \times [v_1(0), \tilde{v}_1(0)] \times [v_2(0), \tilde{v}_2(0)]$. By Girsanov's theorem, conditioning on this event induces a drift on the Brownian motion W_t driving the SLE, where the drift is

$$\kappa\, \partial_W \log h_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(v_3, W, v_1, v_2).$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial W} \log h_{\varepsilon_1,\varepsilon_2,\varepsilon_3} &= \frac{(\partial/\partial W) h_{\varepsilon_1,\varepsilon_2,\varepsilon_3}}{h_{\varepsilon_1,\varepsilon_2,\varepsilon_3}} = \frac{\iiint (\partial/\partial W) h}{\iiint h} \\ &\approx \frac{\varepsilon_1\varepsilon_2\varepsilon_3 (\partial/\partial W) h}{\varepsilon_1\varepsilon_2\varepsilon_3 h} = \partial_W \log h, \end{aligned}$$

where there triple integral is over $\prod [v_i, \tilde{v}_i]$. Thus upon taking the limit $\varepsilon \to 0$, the drift becomes

$$\operatorname{drift}(t) := \kappa\, \partial_W \log h(v_3, W, v_1, v_2). \tag{5.4}$$

The next Mathematica code explicitly computes (5.4).

```
drift = κ D[Log[h], W] /. κ → 6 // Simplify

(2 (-2 v[2]^3 v[3]^3 + 3 v[1] v[2]^2 v[3]^2 (v[2] + v[3]) +
      3 v[1]^2 v[2] v[3] (v[2]^2 - 4 v[2] v[3] + v[3]^2) +
      v[1]^3 (-2 v[2]^3 + 3 v[2]^2 v[3] + 3 v[2] v[3]^2 - 2 v[3]^3) +
      W^3 (-2 v[1]^3 - 2 v[2]^3 + 3 v[2]^2 v[3] + 3 v[2] v[3]^2 - 2 v[3]^3 +
         3 v[1]^2 (v[2] + v[3]) + 3 v[1] (v[2]^2 - 4 v[2] v[3] + v[3]^2)) +
      3 W^2 (v[1]^3 (v[2] + v[3]) + v[1]^2 (-4 v[2]^2 + 2 v[2] v[3] -
            4 v[3]^2) + v[2] v[3] (v[2]^2 - 4 v[2] v[3] + v[3]^2) +
         v[1] (v[2]^3 + 2 v[2]^2 v[3] + 2 v[2] v[3]^2 + v[3]^3)) +
      3 W (v[2]^2 v[3]^2 (v[2] + v[3]) + 2 v[1] v[2] v[3]
          (-2 v[2]^2 + v[2] v[3] - 2 v[3]^2) +
         v[1]^3 (v[2]^2 - 4 v[2] v[3] + v[3]^2) +
         v[1]^2 (v[2]^3 + 2 v[2]^2 v[3] + 2 v[2] v[3]^2 + v[3]^3)))) /
 ((W - v[1]) (W - v[2]) (W - v[3]) (v[2]^2 v[3]^2 -
      v[1] v[2] v[3] (v[2] + v[3]) +
      v[1]^2 (v[2]^2 - v[2] v[3] + v[3]^2) +
      W^2 (v[1]^2 + v[2]^2 - v[2] v[3] + v[3]^2 - v[1] (v[2] + v[3])) -
      W (v[1]^2 (v[2] + v[3]) + v[2] v[3] (v[2] + v[3]) +
         v[1] (v[2]^2 - 6 v[2] v[3] + v[3]^2))))
```

The expression above is complicated, but again it is an intermediate result that we do not need to calculate or read ourselves. The first line of the Mathematica code below defines the generator `L1` (which we will write as L_1) for the conditioned SLE_6, where the driving function W_t has the drift given above. Here L_1 is defined as in (5.1) except that the expectation is with respect to the law of W_t with the drift term (5.4). Thus

$$L_1(f) := L(f) + \mathrm{drift}(t)\, \partial_W f.$$

As before, if f is a real function of W, v_1, v_2, v_3, then $L_1(f)$ will be a function of the same four variables.

We now compute the probability in the modified diffusion that v_3 is absorbed before v_1, that is, that $W(t)$ collides with $v_3(t)$ before colliding with $v_1(t)$. This probability will be a martingale that only depends upon the cross-ratio s. Thus, in the next paragraph, we specialize and consider functions of W, v_1, v_2, v_3 that have the form $f(\mathrm{cr}(v_3, W, v_1, v_2))$ where $f : \mathbb{R} \to \mathbb{R}$ is a function of *one* variable. We would like to find a one-parameter function f for which $f(\mathrm{cr}(v_3, W, v_1, v_2))$ is a martingale with respect to this modified diffusion, so we will require that $L_1(f(\mathrm{cr}(v_3, W, v_1, v_2))) = 0$. What one-parameter functions f have this property?

Oded answers this question with some clever Mathematica work. First, he re-expresses the differential equation $L_1(f(\mathrm{cr}(v_3, W, v_1, v_2))) = 0$—which involves the four parameters W, v_1, v_2, v_3—in terms of the parameters s, v_1, v_2, v_3. He does this by setting s equal to the expression for cr given in (5.2), solving to get W in terms of the other variables and plugging this new expression for W into the expression $L_1(f(\mathrm{cr}(v_3, W, v_1, v_2)))$.

```
L1[f_] := L[f] + drift D[f, W]
```

```
L1[f[cr]] /. Solve[cr == s, W][[1]] /. κ → 6 // Simplify
```

$$\left(((-1+s)\,v[1]+v[2]-s\,v[3])^4\left(2\left(1-6s^2+4s^3\right)f'[s]+3s\left(-1+2s-2s^2+s^3\right)f''[s]\right)\right)\Big/\left(s\left(-1+2s-2s^2+s^3\right)(v[1]-v[2])^2(v[1]-v[3])^2(v[2]-v[3])^2\right)$$

This expression for $L_1(f(\mathrm{cr}(v_3, W, v_1, v_2)))$ depends on f' and f'', and equating it to zero yields a differential equation for f, the unknown one-parameter function of the cross-ratio that we seek,

$$2(1-6s^2+4s^3)f'(s)+3s(-1+2s-2s^2+s^3)f''(s)=0.$$

Oded solves this differential equation, which yields the function f up to two free parameters C_1 and C_2.

```
DSolve[% == 0, f[s], s]
```

$$\left\{\left\{f[s]\to C[2]+\left(s^{2/3}\,C[1]\left(1-3s+2s^2-(1-s)^{1/3}\left(1-s+s^2\right)\mathrm{Hypergeometric2F1}\left[\frac{2}{3},\frac{1}{3},\frac{5}{3},s\right]\right)\right)\Big/\left(3(-1+s)^{1/3}\left(1-s+s^2\right)\right)\right\}\right\}$$

Here Mathematica gives

$$f(s)=C_2+C_1 s^{2/3}\frac{1-3s+2s^2-(1-s)^{1/3}(1-s+s^2)\,{}_2F_1(2/3,1/3;5/3;s)}{3(-1+s)^{1/3}(1-s+s^2)}.$$

The conditional probability that we seek tends to 1 when $s\to 0$ and tends to 0 when $s\to 1$, and this determines C_1 and C_2: C_2 must be 1, and C_1 follows from Gauss's hypergeometric formula,

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Solving for C_1 and substituting, we find that the conditional probability that γ hits v_3 before v_1, given that (v_3, v_1, v_2) is a tripod set, is given by

$$f(s)=1-\frac{\Gamma(4/3)}{\Gamma(2/3)\Gamma(5/3)}s^{2/3}\left[\frac{-1+3s-2s^2}{(1-s)^{1/3}(1-s+s^2)}+{}_2F_1\left(\frac{2}{3},\frac{1}{3};\frac{5}{3};s\right)\right],$$

where s is the cross-ratio of $v_3, 0, v_1, v_2$.

6. Comparison of triple derivatives. Taking $\mathcal{P}_A$ and $\mathcal{P}_C$ as defined in Section 4.4, and f and h as defined in the previous section, we now have

$$h(v_3, 0, v_1, v_2) = \mathcal{P}_C(v_3, v_1, v_2)$$

and

$$f(\mathrm{cr}(v_3, 0, v_1, v_2)) = \mathcal{P}_A(v_3, v_1, v_2)/\mathcal{P}_C(v_3, v_1, v_2).$$

In principle the next step toward proving (4.1) (and hence Theorem 4.1) would be to integrate $\mathcal{P}_A = f(\mathrm{cr}(\cdot))h$ over the three variables v_1, v_2, v_3 and show that one obtains tripod(cr(·)). In Oded's original notes, he stated that this could be done using integration by parts. Fortunately (for those who lack Oded's skill at integrating) we already know (thanks to Watts) what we expect tripod to be, so we can instead differentiate tripod(cr(·)) three times (w.r.t. v_1, v_2, v_3), and check that it equals $f(\mathrm{cr}(\cdot))h$. The Mathematica code in this final section was generated by the authors of this paper, not by Schramm.

First we redefine `cardy` to have an explicit constant and define the purported tripod probability.

```
cardy[s_] = Gamma[2/3] / (Gamma[4/3] Gamma[1/3]) s^(1/3) Hypergeometric2F1[1/3, 2/3, 4/3, s];

tripod[s_] =
  cardy[s] - s / (2 Gamma[2/3] Gamma[1/3]) HypergeometricPFQ[{1, 1, 4/3}, {2, 5/3}, s];
```

Next we differentiate Cardy's formula three times.

```
dddc = D[cardy[cr], v[1], v[2], v[3]] /. Solve[cr == s, W][[1]] // FullSimplify
```

$$\frac{2\,2^{2/3}\sqrt{\pi}\,(1+(-1+s)\,s)}{27\,(-(-1+s)\,s)^{2/3}\,\mathrm{Gamma}\left[\frac{1}{6}\right]\mathrm{Gamma}\left[\frac{4}{3}\right](v[1]-v[2])\,(v[1]-v[3])\,(v[2]-v[3])}$$

This is the same as `h` defined earlier, but with the trick of eliminating the variable W and expressing the formula in terms of s. Next we triply differentiate the purported tripod probability.

```
dddt = D[tripod[cr], v[1], v[2], v[3]] /. Solve[cr == s, W][[1]] // FullSimplify
```

$$\Big(4\,(1+(-1+s)\,s)\,\mathrm{Gamma}\left[\tfrac{2}{3}\right]^2 + s^{2/3}\,\mathrm{Gamma}\left[\tfrac{1}{3}\right]\Big(-(1+s)\,\mathrm{Hypergeometric2F1}\left[\tfrac{1}{3},\tfrac{2}{3},\tfrac{5}{3},s\right] + (-1+s)\,(-1+2\,s)\,\mathrm{Hypergeometric2F1}\left[\tfrac{2}{3},\tfrac{4}{3},\tfrac{5}{3},s\right]\Big)\Big)\Big/ \Big(36\sqrt{3}\,\pi\,(-(-1+s)\,s)^{2/3}\,\mathrm{Gamma}\left[\tfrac{4}{3}\right](v[1]-v[2])\,(v[1]-v[3])\,(v[2]-v[3])\Big)$$

Notice that the triple derivative of the tripod probability is expressed in terms of two different hypergeometric functions. In order to compare this expression with the conditional probability computed in Section 5, we need to use some hypergeometric identities. We use one of Gauss's relations between "contiguous" hypergeometric functions [Erdélyi et al. (1953), Section 2.8, equation 33], to write

$$-\tfrac{1}{3}\,{}_2F_1\big(\tfrac{2}{3},\tfrac{4}{3};\tfrac{5}{3};s\big)+\tfrac{2}{3}(1-s)\,{}_2F_1\big(\tfrac{5}{3},\tfrac{4}{3};\tfrac{5}{3};s\big)-\tfrac{1}{3}\,{}_2F_1\big(\tfrac{2}{3},\tfrac{1}{3};\tfrac{5}{3};s\big)=0.$$

But ${}_2F_1(c,b;c;s)=(1-s)^{-b}$ Erdélyi et al. (1953), Section 2.8, equation 4, so

$$ {}_2F_1\big(\tfrac{2}{3},\tfrac{4}{3};\tfrac{5}{3};s\big)=2(1-s)^{-1/3}-{}_2F_1\big(\tfrac{2}{3},\tfrac{1}{3};\tfrac{5}{3};s\big). \tag{6.1}$$

```
dddt =
  dddt /. Hypergeometric2F1[2 / 3, 4 / 3, 5 / 3, s] →
     2 (1 - s) ^ (-1 / 3) - Hypergeometric2F1[1 / 3, 2 / 3, 5 / 3, s] // Simplify
```

$$\Bigg(4\,(1+(-1+s)\,s)\,\mathrm{Gamma}\Big[\frac{2}{3}\Big]^2-\frac{1}{(1-s)^{1/3}}\,2\,s^{2/3}\,\mathrm{Gamma}\Big[\frac{1}{3}\Big]\Big(-1+3\,s-2\,s^2+(1-s)^{1/3}\,(1-s+s^2)\,\mathrm{Hypergeometric2F1}\Big[\frac{1}{3},\frac{2}{3},\frac{5}{3},s\Big]\Big)\Bigg)\Big/\Big(36\sqrt{3}\,\pi\,(-(-1+s)\,s)^{2/3}\,\mathrm{Gamma}\Big[\frac{4}{3}\Big]\,(v[1]-v[2])\,(v[1]-v[3])\,(v[2]-v[3])\Big)$$

Next we compare the two expressions for the conditional probability and verify that they are the same.

```
cp = dddt / dddc;
```

$$\mathrm{cp2}=1-\frac{\mathrm{Gamma}\left[\frac{4}{3}\right]}{\mathrm{Gamma}\left[\frac{2}{3}\right]\mathrm{Gamma}\left[\frac{5}{3}\right]}\,s^{2/3}\left(\frac{-1+3\,s-2\,s^2}{(1-s)^{1/3}\,(1-s+s^2)}+\mathrm{Hypergeometric2F1}\Big[\frac{2}{3},\frac{1}{3},\frac{5}{3},s\Big]\right);$$

```
cp - cp2 // FullSimplify
```

0

Therefore the triple derivatives agree, and we have established (4.1).

7. Percolation statement. We have the established equivalence of SLEtripod(s) and tripod(s), but we still need to make the connection to percolation.

THEOREM 7.1. *Let $D\subset\mathbb{C}$ be a fixed bounded Jordan domain with marked points x_1,x_2,x_3,x_4 on its boundary. For any ε, we may consider the hexagonal lattice rescaled to have side length ε and color the faces blue and yellow according* *to site percolation. Let B be the closure of the set of blue faces, and let P^{ε} be the probability that $B \cap \overline{D}$ contains a connected component that intersects all four boundary segments* (x_1, x_2), (x_2, x_3), (x_3, x_4) *and* (x_4, x_1). *Then*

$$\lim_{\varepsilon \to 0} P^{\varepsilon} = \text{watts}(s).$$

Proving Theorem 7.1 solves the problem addressed by Watts. However, we remark that more general statements are probably possible. Any domain D with four marked boundary points has a "center" $c(D)$ with the property that a conformal map taking the domain to a rectangle (and the points to the corners) sends c to the center of the rectangle. Oded would probably have preferred to show that for any sequence D_n of simply connected *marked hexagonal domains* (domains comprised of unions of hexagons within a fixed hexagonal lattice H with four marked boundary points of cross ratios s_n converging to s), the probability of the Watts event tends to watts(s) provided that the distance from $c(D_n)$ to ∂D_n tends to ∞. (Oded's SLE convergence results are similarly general [Lawler, Schramm and Werner (2004), Schramm and Sheffield (2005, 2009)].) However, Oded's derivation of Watts' formula (like Dubédat's derivation) depends on SLE_6 convergence, and existing SLE_6 convergence statements [e.g., Camia and Newman (2007)] are not quite general enough to imply this.

PROOF OF THEOREM 7.1. As shown in Section 3, it suffices to prove the analogous statement about tripod events (x_1, x_2), (x_2, x_3), and (x_3, x_4) and the function tripod(s).

Let ε_n be a sequence of positive reals tending to zero, and define γ_n to be the random interface in D obtained from percolation on ε_n times the hexagonal lattice, between the lattice points closest to x_2 and x_4. Let a_n, b_n, c_n be the tripod set for this interface and the points x_1 and x_3, that is, c_n is the first point on $\gamma_n \cap \partial D_n$ outside the boundary segment (x_1, x_3), and the interface γ_n up to point c_n last hits the boundary intervals (x_1, x_2) and (x_2, x_3) at a_n and b_n, respectively. From the work of Camia and Newman (2007), we can couple the γ_n and γ in such a way that $\gamma_n \to \gamma$ almost surely in the uniform topology (in which two curves are close if they can be parameterized in such a way that they are close at all times). By the compactness of ∂D (and the corresponding compactness—in the topology of convergence in law—of the space of measures on ∂D) it is not hard to see that there must be a subsequence of the n values and a coupling of the γ_n with γ in which the entire quadruple $(\gamma_n, a_n, b_n, c_n)$ converges almost surely to some limit. If we could show further that this limit must be (γ, a, b, c) almost surely, this would imply the theorem, since uniform topology convergence would imply that if γ hits a before b then γ_n hits a_n before b_n for large enough n almost surely. However, it is not clear a priori that this limit is (γ, a, b, c) almost surely (even though the γ_n converge to γ), since while γ touches the boundary at a, b and c, it could be that γ_n comes close to the boundary at these points without touching it.

To obtain a contradiction, let us suppose that there is a uniformly positive probability (i.e., bounded away from 0 as $n \to \infty$) that, say, the limit of the a_n is not a. (The argument for the b_n and the c_n is essentially the same.) Then there must be an open interval (α_1, α_2) of the boundary and an open subinterval $(\beta_1, \beta_2) \subset (\alpha_1, \alpha_2)$ of the boundary (with $\beta_1 \neq \alpha_1$ and $\beta_2 \neq \alpha_2$) such that there is a uniformly positive probability that a lies (β_1, β_2) but the limit of the a_n does not lie in that (α_1, α_2). Now we can expand the Jordan domain D to a larger Jordan domain $\tilde{D}$ that includes a neighborhood of (β_1, β_2), but where the boundary of $\tilde{D}$ agrees with boundary of D outside of (α_1, α_2). Let $\tilde{\gamma}_n$ denote the discrete interfaces in this expanded domain. We can couple the $\tilde{\gamma}_n$ with the γ_n in such a way that the two agree whenever $\tilde{\gamma}_n$ does not leave D (by using the same percolation to define both). But now we have a coupling of the $\tilde{\gamma}_n$ sequence with the property that there is a positive probability that the limit of the $\tilde{\gamma}_n$ is a path that hits the boundary of $\tilde{D} \setminus D$ without entering $\tilde{D} \setminus D$. This implies that if the $\tilde{\gamma}_n$ converge in law, they must converge to a random path that with positive probability hits the boundary of $\tilde{D} \setminus D$ without entering $\tilde{D} \setminus D$. By the Camia–Newman theorem, applied to the domain $\tilde{D}$, the $\tilde{\gamma}_n$ converge in law to chordal SLE_6 in $\tilde{D}$, and on the event that SLE_6 hits $\partial(\tilde{D} \setminus D)$, it will a.s. enter $\tilde{D} \setminus D$, a contradiction. $\square$

Acknowledgments. We thank Nike Sun, Kalle Kytölä and Stas Smirnov for their comments on an earlier version of this article.

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