

SPANNING FORESTS AND THE VECTOR BUNDLE LAPLACIAN¹

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Dedicated to the memory of Oded Schramm

The classical matrix-tree theorem relates the determinant of the combinatorial Laplacian on a graph to the number of spanning trees. We generalize this result to Laplacians on one- and two-dimensional vector bundles, giving a combinatorial interpretation of their determinants in terms of so-called cycle rooted spanning forests (CRSFs). We construct natural measures on CRSFs for which the edges form a determinantal process.

This theory gives a natural generalization of the spanning tree process adapted to graphs embedded on surfaces. We give a number of other applications, for example, we compute the probability that a loop-erased random walk on a planar graph between two vertices on the outer boundary passes left of two given faces. This probability cannot be computed using the standard Laplacian alone.

1. Introduction. The classical matrix-tree theorem, which is usually attributed to either Kirchhoff [15] or Brooks et al. [2], states that the product of the nonzero eigenvalues of the combinatorial Laplacian on a connected graph is equal to the number of rooted spanning trees of that graph. This theorem has extensions to graphs with weighted edges (resistor networks) and more generally to Markov chains. Forman [5] extended the theorem in a more interesting direction, to the setting of a line bundle on a graph. In this paper, we reprove his result and extend the theorem further to two-dimensional vector bundles on graphs (with $SL_2\mathbb{C}$ connection). Our main theorem gives a combinatorial interpretation of the determinant of the Laplacian as a sum over cycle-rooted spanning forests (CRSFs). These are simply collections of edges each of whose connected components has as many vertices as edges (and therefore, each component is a tree plus an edge: a unicycle). Here the weight of a configuration is the product of 2 minus the trace of the holonomy around each of the cycles.

By varying the connection on the vector bundle, one constructs in this way many natural measures on CRSFs. If the monodromy of the underlying connection is unitary, then these measures are determinantal processes for the edges (or q -determinantal in the case of an $SL_2\mathbb{C}$ -connection).

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We give here a number of applications of these results. Further applications can be found in the papers [6, 9] and [13].

1.1. *Spanning trees.* One of the main applications is to the study of spanning trees, in particular spanning trees on planar graphs or graphs on surfaces. By a judicious choice of vector bundle connection, the natural probability measure on CRSFs can be made to model the uniform spanning tree measure conditioned on having certain “boundary connections.” As simple examples we compute the probability that, on a finite planar graph, the branch of the uniform spanning tree connecting two boundary points z_1 and z_2 passes left of a given face or a given two faces. These probabilities are given in terms of the Green’s function for the standard Laplacian on the graph.

1.2. *θ -functions.* Suppose \mathcal{G} is a graph embedded on an annulus. On such a graph, a θ -function is a multi-valued harmonic function whose “analytic continuation” around the annulus is a constant times the original function. In other terminology (defined below), it is a harmonic section of a flat line bundle. The constant is called the *multiplier* of the θ function. We prove that if k is the largest integer such that one can simultaneously embed k pairwise disjoint cycles in \mathcal{G} , each winding once around the annulus, then \mathcal{G} has at most $2k - 1$ θ -functions. These θ -functions have distinct positive real multipliers; these multipliers are closed under inverses (if λ is a multiplier then so is λ^{-1}). For generic conductances, \mathcal{G} will have exactly $2k - 1$ θ -functions; we conjecture that this is always the case. The multipliers have a probabilistic meaning for CRSFs: see below.

1.3. *Graphs on surfaces.* For a graph embedded on a surface (in such a way that complementary components are contractible or peripheral annuli), a natural probability model is the uniform random CRSF whose cycles are topologically nontrivial (not null-homotopic). Such CRSFs are called *incompressible*. The cycles in a CRSF give a finite lamination of the surface, that is, a finite collection of disjoint simple closed curves. The Laplacian determinant on a flat line bundle on the surface counts CRSFs weighted by a function of the monodromy of the connection. By considering the Laplacian determinant as a function on the representation variety (consisting of flat connections modulo conjugacy), one can extract the terms for each possible topological type of finite lamination. In particular, one can study the uniform measure on incompressible CRSFs. For example, the number of components for a uniform incompressible CRSF on a graph on an annulus is distributed as a sum of a finite number of independent Bernoulli random variables, with biases given by $\frac{\lambda}{\lambda+1}$, where λ runs over the θ -function multipliers defined above. See Corollary 1 below.

Given a Riemann surface one can take a sequence of finer and finer graphs adapted to the metric so that the potential theory on the graph and on the Riemann surface agree in the limit. In this case, one can show [9] that the uniform

incompressible CRSF has a scaling limit, whose distribution only depends on the conformal structure of the underlying surface. This is similar to the theorem of Lawler, Schramm and Werner on conformal invariance of the uniform spanning tree [16]. We compute here for the annulus and the square torus the exact distribution of the number and homology class of the cycles of an incompressible CRSF in the scaling limit.

1.4. *Monotone lattice paths.* The papers [7] and [14] studied CRSFs on a north/east-directed $m \times n$ grid on a torus. The distribution of the number of cycles was shown to have a highly nontrivial structure as a function of m, n . Here the line bundle Laplacian gives quantitative information about the model which was not available in [7]. In particular, we obtain an exact expression for the generating function for the number and homology type of the cycles. For the $n \times n$ torus, for example, we show that the number of cycles tends to a Gaussian as $n \rightarrow \infty$ with expectation $\sqrt{\frac{n}{4\pi}}$ and variance $\sqrt{\frac{n}{4\pi}}(1 - \frac{1}{\sqrt{2}})$.

2. Background. The uniform probability measure on spanning trees on a graph, called the UST measure, has for the past 20 years been a remarkably successful and rich area of study in probability theory. Pemantle [19] showed that the unique path between two points in a uniform spanning tree has the same distribution as the loop-erased random walk between those two points.² Wilson [20] extended this to give a simple method of sampling a uniform spanning tree in any graph. Burton and Pemantle [3] proved that the edges of the uniform spanning tree form a determinantal process (see definition below). This allows computation of multi-edge probabilities in terms of the Green’s function.

The UST on \mathbb{Z}^2 received particular attention due to the conformal invariance properties of its scaling limit. Pemantle showed that almost surely the UST in \mathbb{Z}^2 has one component. In [10], we proved that the expected length of the LERW in \mathbb{Z}^2 , and therefore a branch of the UST of diameter n was of order $n^{5/4}$, a result predicted earlier by conformal field theory [17]. In [12], we showed that the scaling limit of the “winding field” (describing how the branches of the UST wind around faces) was a Gaussian free field. Further conformally invariant properties were proved in [11]. In [16], Lawler, Schramm and Werner proved that the \mathbb{Z}^2 -LERW converges to SLE₂, and the peano curve winding around the \mathbb{Z}^2 -UST converges to SLE₈.

As a result of these works, we have a decent understanding of the scaling limit of the UST on \mathbb{Z}^2 . However, some important questions remain. For example, how are different points in the UST connected: given a set of points in the plane, what

²The loop-erased random walk (LERW) between a and b is defined as follows: draw the trace a simple random walk from a stopped when it reaches b , and then erase from the trace all loops in chronological order. What remains is a simple path from a to b .

is the topology of their tree convex hull, that is, the union of the branches of the UST connecting them in pairs? Can one compute various connection probabilities, for example, the probability that the LERW from $(0, 0)$ passes through the points v_1, v_2, \dots, v_n in order? What is the distribution of the “bush size,” the finite component of the tree obtained by removing a single random edge? While many of these can be answered in principle using SLE techniques, in practice one must solve a hard PDE.

Many of these questions can be answered with the bundle Laplacian. Some of these are illustrated below. We will discuss how these results can be used to compute various connection probabilities for the UST in [13], and discuss the connection with integrable systems in [6].

3. Vector bundles on graphs.

3.1. *Definitions.* Let \mathcal{G} be a finite graph. Given a fixed vector space V , a V -bundle, or simply a *vector bundle on \mathcal{G}* is the choice of a vector space V_v isomorphic to V for every vertex v of \mathcal{G} . A vector bundle can be identified with the vector space $V_{\mathcal{G}} := \bigoplus_v V_v \cong V^{|\mathcal{G}|}$. A *section* of a vector bundle is an element of $V_{\mathcal{G}}$.

A *connection* Φ on a V -bundle is the choice for each edge $e = vv'$ of \mathcal{G} of an isomorphism $\phi_{vv'}$ between the corresponding vector spaces $\phi_{vv'} : V_v \rightarrow V_{v'}$, with the property that $\phi_{vv'} = \phi_{v'v}^{-1}$. This isomorphism is called the *parallel transport* of vectors in V_v to vectors in $V_{v'}$. Two connections Φ, Φ' are said to be *gauge equivalent* if there is for each vertex an isomorphism $\psi_v : V_v \rightarrow V_v$ such that the diagram

$$\begin{array}{ccc} V_v & \xrightarrow{\phi_{vv'}} & V_{v'} \\ \downarrow \psi_v & & \downarrow \psi_{v'} \\ V_v & \xrightarrow{\phi'_{vv'}} & V_{v'} \end{array}$$

commutes. In other words Φ' is just a base change of Φ . Given an oriented cycle γ in \mathcal{G} starting at v , the *monodromy* of the connection is the element of $\text{End}(V_v)$ which is the product of the parallel transports around γ . Monodromies starting at different vertices on γ are conjugate, as are monodromies of gauge-equivalent connections.

A *line bundle* is a V -bundle where $V \cong \mathbb{C}$, the one-dimensional complex vector space. In this case, if we choose a basis for each \mathbb{C} then the parallel transport is just multiplication by an element of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The monodromy of a cycle is in \mathbb{C}^* and does not depend on the starting vertex of the cycle (or gauge).

3.2. *The Laplacian.* The Laplacian Δ on a V -bundle with connection Φ is the linear operator $\Delta : V_{\mathcal{G}} \rightarrow V_{\mathcal{G}}$ defined by

$$\Delta f(v) = \sum_{v' \sim v} f(v) - \phi_{v'v} f(v'),$$

where the sum is over neighbors v' of v .

If we assign to each edge a positive real weight (a conductance) $c_{vv'} = c_{v'v}$, the weighted Laplacian is defined by

$$\Delta f(v) = \sum_{v'} c_{vv'}(f(v) - \phi_{v'v}f(v')).$$

Note that if the vector bundle is *trivial*, in the sense that $\phi_{vv'}$ is the identity for all edges, this is the classical notion of graph Laplacian (or more precisely, the direct sum of $\dim V$ copies of the Laplacian).

Here is an example. Let $\mathcal{G} = K_3$ with vertices $\{v_1, v_2, v_3\}$. Let Φ be the line bundle connection with $\phi_{v_i v_j} = z_{ij} \in \mathbb{C}^*$. Then in the natural basis, Δ has matrix

$$(1) \quad \Delta = \begin{pmatrix} 2 & -z_{12} & -z_{13} \\ -z_{12}^{-1} & 2 & -z_{23} \\ -z_{13}^{-1} & -z_{23}^{-1} & 2 \end{pmatrix}.$$

3.3. *Edge bundle.* One can extend the definition of a vector bundle to the edges of \mathcal{G} . In this case, there is a vector space $V_e \cong V$ for each edge e as well as each vertex. One defines connection isomorphisms $\phi_{ve} = \phi_{ev}^{-1}$ for a vertex v and edge e containing that vertex, in such a way that if $e = vv'$ then $\phi_{vv'} = \phi_{ev'} \circ \phi_{ve}$, where $\phi_{vv'}$ is the connection on the vertex bundle.

The vertex/edge bundle can be identified with $V^{|\mathcal{G}|+|E|} = V_{\mathcal{G}} \oplus V_E$, where V_E is the direct sum of the edge vector spaces.

A 1-form (or cochain) is a function on oriented edges which is antisymmetric under changing orientation. If we fix an orientation for each edge, a 1-form is a section of the edge bundle, that is, an element of $V^{|E|}$. We denote by $\Lambda^1(\mathcal{G}, \Phi)$ the space of 1-forms and $\Lambda^0(\mathcal{G}, \Phi)$ the space of 0-forms, that is, sections of the vertex bundle.

We define a map $d : \Lambda^0(\mathcal{G}, \Phi) \rightarrow \Lambda^1(\mathcal{G}, \Phi)$ by $df(e) = \phi_{ye}f(y) - \phi_{xe}f(x)$ where $e = xy$ is an oriented edge from vertex x to vertex y . We also define an operator $d^* : \Lambda^1 \rightarrow \Lambda^0$ as follows:

$$d^* \omega(v) = \sum_{e=v'v} \phi_{ev} \omega(e),$$

where the sum is over edges containing v and oriented toward v . Despite the notation, this operator d^* is not a standard adjoint of d unless ϕ_{ev} and ϕ_{ve} are adjoints themselves, that is, if parallel transports are unitary operators (see below).

The Laplacian Δ on Λ^0 can then be defined as the operator $\Delta = d^*d$:

$$\begin{aligned} d^*df(v) &= \sum_{e=v'v} \phi_{ev} df(e) \\ &= \sum_{e=v'v} \phi_{ev}(\phi_{ve}f(v) - \phi_{v'e}f(v')) \\ &= \sum_{v'} f(v) - \phi_{v'v}f(v') \\ &= \Delta f(v). \end{aligned}$$

We can see from the example (1) above on K_3 that Δ is not necessarily self-adjoint. However, if $\phi_{vv'}$ is unitary: $\phi_{vv'}^{-1} = \phi_{v'v}^*$, then d^* will be the adjoint of d for the standard Hermitian inner products on $V^{|\mathcal{G}|}$ and $V^{|E|}$, and so in this case Δ is a Hermitian, positive semidefinite operator. In particular on a line bundle if $|\phi_{vv'}| = 1$ for all edges $e = vv'$, then Δ is Hermitian and positive semidefinite.

4. Spanning forests associated to a line bundle.

4.1. *Cycle-rooted spanning forests (CRSFs).* For a line bundle (\mathcal{G}, Φ) on a finite graph \mathcal{G} , we have a combinatorial interpretation of the determinant of the Laplacian in terms of cycle-rooted spanning forests. A *cycle-rooted spanning forest* (CRSF) is a subset S of the edges of \mathcal{G} , spanning all vertices (in the sense that every vertex is the endpoint of some edge) and with the property that each connected component of S has as many vertices as edges (and so has a unique cycle). See Figure 1.

An *oriented CRSF* is a CRSF in which we orient each edge in such a way that cycles are oriented coherently and we orient the edges not in a cycle toward the cycle. An oriented CRSF is the same as a *nonzero vector field* on \mathcal{G} , which is the choice of a single outgoing edge from each vertex.

A *cycle-rooted tree* (CRT), also called unicycle, is a component of a CRSF (oriented or not).

4.2. *The matrix-CRSF theorem.* The following theorem is due to Forman [5].

THEOREM 1 [5]. *For a line bundle on a connected finite graph,*

$$\det \Delta = \sum_{\text{CRSFs}} \prod_{\text{cycles}} (2 - w - 1/w),$$

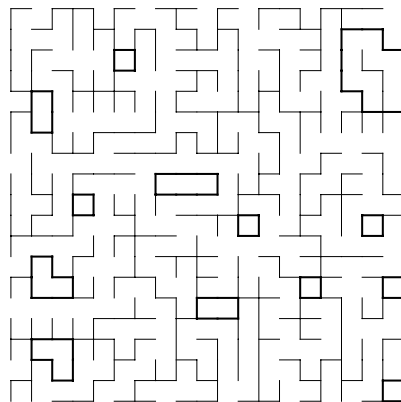


FIG. 1. A CRSF on a square grid.

where the sum is over all unoriented CRSFs C , the product is over the cycles of C , and $w, 1/w$ are the monodromies of the two orientations of the cycle.

Recall that for a line bundle the monodromy of an oriented cycle does not depend on the starting vertex. Note that we could as well have written $\det \Delta = \sum_{\text{OCRSFs}} \prod_{\text{cycles}} (1 - w)$ where now the sum is over oriented CRSFs (OCRSFs).

Forman uses an explicit expansion of the determinant as a sum over the symmetric group, and a careful rearrangement of the terms. We give a different proof using Cauchy–Binet formula which allows us to bypass this step.

PROOF OF THEOREM 1. We use the Cauchy–Binet formula, $\det \Delta = \det d^*d = \sum_B \det(B^*B)$, where B is a maximal minor of d (i.e., if d is $n \times m$ then B runs over all $n \times n$ submatrices of d obtained by choosing n columns).

Columns of B index vertices and rows of B index edges of \mathcal{G} . Since B is square, it corresponds to a selection of n edges (where n is the number of vertices of \mathcal{G}).

If the set of edges in B contains a component C with no cycles, say of size $|C| = k$, this component necessarily has $k - 1$ edges. So B maps a k -dimensional subspace of the vertex space to a $k - 1$ -dimensional subspace of the edge space and so $\det B = 0$. Therefore to have a nonzero contribution to the sum every component of B has at least one cycle, and, since the total number of vertices equals the total number of edges, every component must have exactly one cycle. We have proved that the sum is over CRSFs.

If B has only one component which is a CRT, orient its edges so that all edges are oriented toward the unique cycle, and along the cycle choose one of the two coherent orientations. Write

$$\det B = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) B_{1\sigma(1)} \cdots B_{n\sigma(n)}.$$

The only nonzero terms in this expansion are those in which vertex i is adjacent to edge $\sigma(i)$, that is, the two terms which are consistent with one of these two orientations. Thus

$$\begin{aligned} \det B &= \text{sgn}(\sigma_1) \left(\prod_{j \in \text{bushes}} -\phi_{v_j e_j} \prod_{i \in \text{cycle}} -\phi_{v_i e_i} \right) + \text{sgn}(\sigma_2) \\ (2) \quad &\times \left(\prod_{j \in \text{bushes}} -\phi_{v_j e_j} \prod_{i \in \text{cycle}} -\phi_{v_{i+1} e_i} \right) \\ &= \text{sgn}(\sigma_1) (-1)^n \prod_{j \in \text{bushes}} \phi_{v_j e_j} \left(\prod_{i \in \text{cycle}} \phi_{v_i e_i} - \prod_{i \in \text{cycle}} \phi_{v_{i+1} e_i} \right), \end{aligned}$$

where the products over j are over edges not in the cycle (called bush edges), and those over i are for the two cyclic orientations (indices are taken cyclically). If the cycle has odd length, $\text{sgn}(\sigma_1) = \text{sgn}(\sigma_2)$ and the sign change in front of the second

product in (2) is due to switching the sign on the odd number of edges of the cycle. If the cycle has even length, the sign change is due to $\text{sgn}(\sigma_1) = -\text{sgn}(\sigma_2)$ (a cyclic permutation of even length has signature -1).

We similarly have

$$\det B^* = \text{sgn}(\sigma_1)(-1)^n \prod_{j \in \text{bushes}} \phi_{e_j v_j} \left(\prod_{i \in \text{cycle}} \phi_{e_i v_i} - \prod_{i \in \text{cycle}} \phi_{e_i v_{i+1}} \right),$$

and multiplying these two and using $\phi_{ev} \phi_{ve} = 1$, we get

$$\det B^* B = (A_1 - A_2) \left(\frac{1}{A_1} - \frac{1}{A_2} \right) = 2 - \frac{A_1}{A_2} - \frac{A_2}{A_1},$$

where $\frac{A_1}{A_2} = \prod_{i \in \text{cycle}} \phi_{v_i v_{i+1}}$ which is the monodromy of the cycle.

If each component of B is a CRT, then $\det B^* B$ is the product over components of the above contributions. \square

5. Measures on CRSFs. In the case of a line bundle Φ with unitary connection, Theorem 1 provides a definition of a natural probability measure μ_Φ on CRSFs. A CRSF has probability equal to $\frac{1}{Z} \prod (2 - w - \frac{1}{w})$ where the product is over its cycles. The constant of proportionality Z , or partition function, is then just $\det \Delta$. Note that as long as the (unitary) bundle is not trivializable (gauge equivalent to the trivial bundle) then some cycle has nontrivial monodromy and so $\det \Delta > 0$.

The goal of this section is to show that the measure is determinantal for the edges.

5.1. *Harmonic sections.* The Laplacian is the composition:

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d^*} \Lambda^0.$$

If Δ has full rank, then $\Lambda^1(\mathcal{G}, \Phi)$ splits as

$$\Lambda^1 = \text{Im}(d) \oplus \text{Ker}(d^*).$$

A section $f \in \Lambda^0$ is *harmonic* if $\Delta f = 0$. If Δ does not have full rank, there are nontrivial harmonic sections and the above is no longer a direct sum. A 1-form $\omega \in \Lambda^1$ is harmonic if it is in the intersection $\text{Im}(d) \cap \text{Ker}(d^*)$. Such a form is both *exact* ($\omega = df$ for some f) and *co-closed* ($d^* \omega = 0$).

PROPOSITION 1. *When Δ has full rank, the projection along $\text{Ker}(d^*)$ from Λ^1 to the space of exact 1-forms $\text{Im}(d)$ is given by the operator $P = dg d^*$ where $g = \Delta^{-1}$ is the Green's function for Δ on Λ^0 .*

PROOF. Note that $P = dg d^*$ is zero on co-closed forms and the identity on exact forms: $P(df) = dg d^* df = dg \Delta f = df$. \square

This projection operator P plays a role below.

5.2. *Determinantal processes.* Let μ be a probability measure on $\Omega = \{0, 1\}^n$. For a set of indices $S \subset \{1, 2, \dots, n\}$, define the set

$$E_S = \{(x_1, \dots, x_n) \in \Omega \mid x_s = 1 \ \forall s \in S\}.$$

We say μ is a *determinantal* probability measure (see, e.g., [8]) if there is an $n \times n$ matrix K with the property that for any set of indices $S \subset \{1, 2, \dots, n\}$, we have

$$\mu(E_S) = \det[(K_{i,j})_{i,j \in S}].$$

That is, principal minors of K determine the probability of events of type E_S .

The matrix K is called the *kernel* of the measure. A simple example of a determinantal process is the product measure, in which case K is a diagonal matrix with diagonal entries in $[0, 1]$. Another well-known example of a determinantal process is the uniform spanning tree measure on a finite graph [3]. Here $\{1, 2, \dots, n\}$ index the edges of a connected graph. The kernel K is the so-called *transfer-current matrix*, defined by $K = dGd^*$, where G is the Green's function, see [3].

A simple inclusion-exclusion argument shows that individual point probabilities are also given by determinants:

PROPOSITION 2. *Let $X = (x_1, x_2, \dots, x_n) \in \Omega$. Then*

$$\mu(X) = (-1)^{n-|X|} \det(\text{diag}(X) - K),$$

where $\text{diag}(X)$ is the diagonal matrix whose diagonal entries are $1 - x_i$, and $|X| = \sum_{i=1}^n x_i$.

5.3. *A determinantal process on CRSFs.* We will need the following well-known lemma.

LEMMA 1. *If $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix with A, D square and D invertible then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C).$$

PROOF. This follows from

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}. \quad \square$$

Recall the definition of the projection P from 1-forms to exact 1-forms (Proposition 1).

THEOREM 2. *For a line bundle Φ with unitary connection, with respect to the measure μ_Φ the edges of the CRSF form a determinantal process with kernel $P = dgd^*$.*

PROOF. Let \mathcal{G} be a graph with n vertices and m edges. Let $\{e_1, \dots, e_n\} \subset E$ be the edges of a CRSF γ . Write the matrix for d and d^* so that the first n edges are e_1, \dots, e_n . Then $d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ where d_1 is $n \times n$ and d_2 is $(m - n) \times n$.

From Proposition 2, the probability of γ is $\Pr(\gamma) = (-1)^{m-n} \det(X - dg d^*)$ which by Lemma 1 with $A = X, B = d, D = \Delta$ and $C = d^*$ can be written

$$\Pr(\gamma) = (-1)^{m-n} \frac{\det \begin{pmatrix} 0 & 0 & d_1 \\ 0 & I_{m-n} & d_2 \\ d_1^* & d_2^* & \Delta \end{pmatrix}}{\det \Delta} = \frac{\det d_1^* d_1}{\det \Delta}.$$

By Theorem 1 this is exactly the probability of γ . \square

Among other things, this theorem along with Proposition 2 allows us to do exact sampling from the measure μ_Φ , as follows (see [8]). Pick an edge e_1 ; it is present with probability $P(e_1, e_1)$. Take another edge e_2 ; if e_1 is present, e_2 will be present with probability

$$\Pr(e_2|e_1) = \frac{1}{P(e_1, e_1)} \det \begin{pmatrix} P(e_1, e_1) & P(e_1, e_2) \\ P(e_2, e_1) & P(e_2, e_2) \end{pmatrix}.$$

If e_1 is not present, e_2 will be present with probability

$$\Pr(e_2|-e_1) = \frac{-1}{1 - P(e_1, e_1)} \det \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} P(e_1, e_1) & P(e_1, e_2) \\ P(e_2, e_1) & P(e_2, e_2) \end{pmatrix} \right)$$

and so on.

5.4. *Small monodromies.* An important limit, or actually set of limits of these determinantal processes, is when the monodromies tend to 1. Even though $\det \Delta$ tends to zero, the probability measures μ_Φ may converge. We get determinantal processes supported on CRSFs on the graph with trivial line bundle.

THEOREM 3. *Choose an orientation for each edge and suppose the parallel transport is $\phi_{e_j} = e^{itc_j}$ for the j th edge in the direction of its orientation. Fix the c_j and let $t \rightarrow 0$. In the limit $t \rightarrow 0$ the determinantal process μ_{Φ_t} tends to a determinantal process $\mu_0 = \mu_0(\{c_j\})$ supported on CRSFs with a single component, that is, CRTs. The probability of a CRT is proportional to $(\sum c_j)^2$, where the sum is over oriented edges in the unique cycle.*

PROOF. The monodromy of a loop is $w = e^{it \sum c_j}$. We have $2 - w - \frac{1}{w} = 2 - 2 \cos(t \sum c_j) = t^2 (\sum c_j)^2 + O(t^3)$. In the limit $t \rightarrow 0$, the partition function is $Z = O(t^2)$ so only CRSFs with one component contribute. \square

6. An application to $\mathcal{G} \times \mathbb{Z}_n$. Let \mathcal{G} be a finite graph with m vertices and \mathbb{Z}_n the n -cycle. Let H_n be the product graph whose vertices are $\mathcal{G} \times \mathbb{Z}_n$ and edges connect (x, j) to $(x, j + 1 \bmod n)$ and (x, j) to (x', j) when x, x' are neighbors in \mathcal{G} .

We compare the uniform spanning tree on H_n and the uniform cycle-rooted tree whose cycle winds around \mathbb{Z}_n . The *minimum cut set* of a spanning tree is the set of edges not in the tree such that when added to the tree make a cycle winding nontrivially around \mathbb{Z}_n .

Fix $z \in \mathbb{C}$ with $|z| = 1$. On the product graph H_n let $\phi_{vv'} = 1$ except when $v = (x, n - 1)$ and $v' = (x, 0)$ in which case $\phi_{vv'} = z$. Then cycles have trivial monodromy unless they wind nontrivially around the \mathbb{Z}_n direction. So a CRSF has nonzero probability only if all of its cycles wind nontrivially around \mathbb{Z}_n (possibly many times).

Because of the product structure the eigenvectors of the Laplacian on H_n are products of eigenvectors of the standard Laplacian on \mathcal{G} and the line bundle Laplacian on \mathbb{Z}_n . Thus, the eigenvalues of the Laplacian on H_n are sums of eigenvalues of $\Delta_{\mathcal{G}}$ (the standard Laplacian) and $\Delta_{\mathbb{Z}_n}$ (the line bundle Laplacian). The eigenvalues of $\Delta_{\mathbb{Z}_n}$ are $2 - \zeta - \zeta^{-1}$ where ζ ranges over the roots of $\zeta^n = z$ (the corresponding eigenvectors are exponential functions). The Laplacian determinant of H_n is then

$$\begin{aligned}
 \det \Delta &= \prod_{\zeta^n=z} \det(\Delta_{\mathcal{G}} + 2 - \zeta - \zeta^{-1}) = \prod_{\zeta^n=z} \prod_{\lambda} (\lambda + 2 - \zeta - \zeta^{-1}) \\
 (3) \quad &= (2 - z - z^{-1}) \prod_{\zeta^n=z} \prod_{\lambda \neq 0} (\lambda + 2 - \zeta - \zeta^{-1}),
 \end{aligned}$$

where λ runs over the eigenvalues of the Laplacian on \mathcal{G} .

Compare this to $\det' \Delta_0$, the product of nonzero eigenvalues of the standard Laplacian Δ_0 on H_n :

$$\begin{aligned}
 \det' \Delta_0 &= \prod_{(\zeta, \lambda) \neq (1, 0)} \lambda + 2 - \zeta - \zeta^{-1} \\
 (4) \quad &= \prod_{\zeta^n=1, \zeta \neq 1} 2 - \zeta - \zeta^{-1} \prod_{\zeta^n=1, \lambda \neq 0} \lambda + 2 - \zeta - \zeta^{-1} \\
 &= n^2 \prod_{\zeta^n=1} \prod_{\lambda \neq 0} \lambda + 2 - \zeta - \zeta^{-1}.
 \end{aligned}$$

THEOREM 4. *As $n \rightarrow \infty$ a μ_{ϕ} -random CRSF has one component with probability tending to 1, that is, is a CRT. Its unique cycle winds around \mathbb{Z}_n once with probability tending to 1. The ratio R_n of the number of such CRTs and number of spanning trees satisfies $\lim_{n \rightarrow \infty} \frac{R_n}{n} = \frac{1}{m}$. The expected length $\mathbb{E}(L_n)$ of the cycle*

in such a random CRT of H_n satisfies

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(L_n)}{n\mathbb{E}(S_n)} = \frac{1}{m},$$

where $\mathbb{E}(S_n)$ is the expected size of the minimum cut set of a random spanning tree of H_n .

PROOF. The first two statements can be seen as follows. Take n large and consider the part of the configuration in a piece $\mathcal{G} \times [i, i + N]$ for large N . We claim that the number of spanning tree configurations restricted to this subgraph, as a function of N , has a strictly larger exponential growth rate than that of the number of spanning forest configurations in this subgraph with two or more components (“strands”) each of which intersects both ends $\mathcal{G} \times \{i\}$ and $\mathcal{G} \times \{i + N\}$. The growth rate of spanning trees can be computed as the leading eigenvalue of nonnegative finite matrix T , the transfer matrix.³ It is easily seen that the transfer matrix T is primitive, that is, has the property that some power is strictly positive. On the other hand, the transfer matrix for spanning forest configurations with two or more strands is a matrix which can be obtained from T by setting some of its positive entries to 0 (those entries whose components are all connected to the left). This strictly decreases its leading eigenvalue. This proves the claim and the first two statements of the theorem.

If we divide (3) by $2 - z - z^{-1}$ and let z tend to 1 the expression counts CRSFs with one component, that is, CRTs, with a weight k^2 if they wind k times around $\mathbb{Z}/n\mathbb{Z}$, because

$$\lim_{z \rightarrow 1} \frac{2 - z^k - z^{-k}}{2 - z - z^{-1}} = k^2.$$

As n gets large CRTs which wind more than once around have probability tending to zero. Line (4) is the number of rooted trees; dividing by nm , the number of locations for the root, gives the number of trees, which by (3) and the above remarks is $\frac{n}{m}$ times the number of CRTs winding once around, plus errors tending to zero as $n \rightarrow \infty$. This proves the third statement.

The following two sets A and B are in bijection: A is the set of CRTs whose cycle winds once around \mathbb{Z}_n , along with the choice of an edge on this cycle. B is the set of spanning trees with a choice of a complementary edge in the minimum cut set. The bijection consists in adding the edge to the tree. Therefore, the expected length of the unique cycle in a random CRT is

$$\frac{|A|}{|\text{CRTs}|} = \frac{|B|}{|\text{CRTs}|} = \frac{|B|}{|\text{trees}|} \frac{|\text{trees}|}{|\text{CRTs}|}.$$

³The states of the transfer matrix are all possible forests (edge subsets) of $\mathcal{G} \times \{i\}$ occurring in a spanning tree of $\mathcal{G} \times \mathbb{Z}$ along with a partition of the components in this forest, which describes which components of this forest are connected to each other to the left, that is, in the part of the tree in $\mathcal{G} \times (-\infty, i - 1]$.

By the above, $\frac{|\text{trees}|}{|\text{CRTs}|} \rightarrow \frac{n}{m}$. \square

7. Weighted and/or directed graphs.

7.1. *Edge weights.* Putting weights on the edges is a minor generalization; let $c: E \rightarrow \mathbb{R}_{>0}$ be a conductance associated to each edge, with $c_{vv'} = c_{v'v}$. We then define

$$\Delta f(v) = \sum_{v'} c_{vv'}(f(v) - \phi_{v'v}f(v')).$$

In other words, letting C be the diagonal matrix, indexed by the edges, whose diagonal entries are the conductances, we have

$$\Delta = d^*Cd.$$

THEOREM 5. $\det \Delta = \sum_{\text{CRSFs}} \prod_e C(e) \prod (2 - w - \frac{1}{w})$ where the first product is over all edges of the configuration and the second is over cycles, and w is the monodromy of the cycle as before.

This follows directly from the proof of Theorem 1 above.

7.2. *Directed graphs.* We can also have weights which depend on direction, so that each edge has two weights, one associated to each direction. This makes \mathcal{G} into a Markov chain, in which the transition probabilities are proportional to the edge weights.

In this case the Laplacian has the same form

$$\Delta f(v) = \sum_{v'} c_{vv'}(f(v) - \phi_{v'v}f(v')),$$

except that $c_{vv'} \neq c_{v'v}$ in general. The operator $d: \Lambda^0 \rightarrow \Lambda^1$ can be defined by

$$df(e) = \phi_{ve}f(v) - \phi_{v'e}f(v')$$

as in the unweighted case, where $e = v'v$, and d^* by

$$d^*(\omega)(v) = \sum_{e=v'v} c_{vv'}\phi_{ev}\omega(e),$$

where the sum is over edges $e = v'v$ containing v .

In this case, each oriented CRSF has a different weight. Forman's theorem in this case is the following.

THEOREM 6 [5].

$$\det \Delta = \sum_{\text{OCRSFs}} \prod_{e \in \text{bushes}} c(e) \prod_{\text{cycles } \gamma} C(\gamma)(1 - w(\gamma)),$$

where the sum is over oriented CRSFs, the first product is over the edges in the bushes (i.e., not in the cycles), oriented toward the cycle, and the second product is over oriented cycles, $C(\gamma)$ is the product of semiconductances along γ and w is the monodromy of γ .

An application is in Section 13 below.

8. Laplacian with Dirichlet boundary. Let \mathcal{G} be a graph with line bundle and connection Φ and S a subset of its vertices, which play the role of boundary vertices. We can define a Laplacian $\Delta_{\mathcal{G},S}$ with Dirichlet boundary conditions at S as follows: for $f \in V^{\mathcal{G} \setminus S}$ and $v \in \mathcal{G} \setminus S$,

$$\Delta_{\mathcal{G},S} f(v) = (\deg v) f(v) - \sum_{v' \in \mathcal{G} \setminus S, v' \sim v} \phi_{v'v} f(v').$$

This is of course just the Laplacian $\Delta_{\mathcal{G}}$ restricted to the subspace $V^{\mathcal{G} \setminus S}$ of functions which are zero on S and projected back to this subspace. As a matrix, it is just a submatrix of $\Delta_{\mathcal{G}}$.

An *essential CRSF* of (\mathcal{G}, S) is a set of edges in which each component is either a tree containing a unique vertex in S or a cycle-rooted tree containing no vertices in S .

THEOREM 7. *det $\Delta_{\mathcal{G},S}$ is the weighted sum of essential CRSFs, where each configuration has weight $\prod_{\text{CRTs}} (2 - w - w^{-1})$, where w is the monodromy of the cycle.*

This generalizes the matrix-tree theorem⁴ (when Φ is the identity and $|S| = 1$) and the matrix-CRSF theorem (when $|S| = 0$).

PROOF OF THEOREM 7. The proof follows the same lines as the proof of Theorem 1. Now d is an $m \times n$ matrix with $n = |\mathcal{G}| - |S|$. A maximal minor B of d corresponds to a choice of n edges. Each component of B is either a CRT, in which case its weight is computed as before, or a tree connecting some vertices of $\mathcal{G} \setminus S$ with a single vertex of S . In this case, the determinant of B (we mean, that part of B coming from this component) has a single nonzero term in its expansion, and the corresponding term in B^* is its inverse. So each tree component counts 1. □

Again when $|w| = 1$ there is a determinantal measure associated with Δ .

⁴One version of the classical matrix-tree theorem is that the determinant of the matrix obtained by removing a single row and column v from Δ is equal to the number of spanning trees rooted at v .

9. Bundles with $SL_2\mathbb{C}$ connections. We can extend many of the above results to the case of a \mathbb{C}^2 -bundle with $SL_2\mathbb{C}$ -connection.

9.1. *Q-determinants.* Given $A \in GL_2(\mathbb{C})$ define $\tilde{A} = (\det A)A^{-1}$, the adjugate of A . That is, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\tilde{A} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Note that

$$(5) \quad A + \tilde{A} = \text{Tr}(A)I$$

is a scalar.

Let M be a matrix with entries in $GL_2(\mathbb{C})$. M is said to be *self-dual* if $M_{ij} = \tilde{M}_{ji}$. In particular, diagonal entries must be scalar matrices.

For a matrix M with noncommuting entries, there are many possible ways one might define its determinant: however these all involve the expansion

$$(6) \quad \det M = \sum_{\sigma \in S_n} \text{sgn}(\sigma) [M_{1\sigma(1)} \cdots M_{n\sigma(n)}]_{\sigma},$$

where $[M_{1\sigma(1)} \cdots M_{n\sigma(n)}]_{\sigma}$ denotes the product of the terms $M_{i\sigma(i)}$ after they have been rearranged in some specified order depending on σ . Each of these “order functions” determines a different possible determinant.

In the current case, there is one very natural condition to put on these orders. If σ is written as a product of disjoint cycles, then in the corresponding order the terms in each cycle of σ should appear consecutively. Moreover, if σ, σ' are two permutations with the same cycles, except that some of the cycles are reversed, then the corresponding order should have the cycles appear in the same relative order.

The advantage of this for self-dual matrices is that the product of entries along a cycle is the adjugate of the product along the reversed cycle; so that the sum of these is a scalar by (5). Thus by grouping permutations into sets with the same cycles up to reversals, one arrives at a product of scalar matrices. The sum (6) with appropriately rearranged products then yields a 2×2 scalar matrix $q \cdot I$, and the number q is defined to be the determinant.

The Q -determinant of self-dual matrix M [18] is defined in precisely this way. We define

$$\text{Qdet}(M) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{\text{cycles}} \frac{1}{2} \text{tr}(w),$$

where the sum is over the symmetric group, each permutation σ is written as a product of disjoint cycles, and $\text{tr}(w)$ is the trace of the product of the matrix entries in that cycle. If we group together terms above with the same cycles—up to the order of traversal of each cycle—then the contribution from each of these terms is identical: reversing the orientation of a cycle does not change its trace. So we can write

$$\text{Qdet}(M) = \sum (-1)^{c+n} \prod_{i=1}^c \widehat{\text{tr}}(w_i),$$

where the sum is over cycle decompositions of the indices, c is the number of cycles, and w_i is the monodromy (in one direction or the other) of each cycle. Here $\widehat{\text{tr}}$ is equal to the trace for cycles of length at least 3; cycles of length 1 or 2 are their own reversals so we define $\widehat{\text{tr}}(w) = \frac{1}{2} \text{tr}(w)$ for these cycles.

As an example, let $A = aI$, $C = cI$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. Then

$$\text{Qdet} \begin{pmatrix} A & B \\ \tilde{B} & C \end{pmatrix} = \widehat{\text{tr}}(A)\widehat{\text{tr}}(C) - \widehat{\text{tr}}(B\tilde{B}) = ac - (b_1b_4 - b_2b_3).$$

Note that if M is a self-dual $n \times n$ matrix then ZM , considered as a $2n \times 2n$ matrix is antisymmetric, where Z is the matrix with diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and zeros elsewhere. The following theorem allows us to compute Q -determinants explicitly.

THEOREM 8 [18]. *Let M be an $n \times n$ self-dual matrix with entries in $\text{GL}_2(\mathbb{C})$ and M' the associated $2n \times 2n$ matrix, obtained by replacing each entry with the 2×2 block of its entries. Then $\text{Qdet}(M) = \text{Pf}(ZM')$, the Pfaffian of the antisymmetric matrix ZM' .*

9.2. Laplacian determinant. For a \mathbb{C}^2 -bundle on a graph \mathcal{G} with $\text{SL}_2\mathbb{C}$ connection Δ is a self-dual operator (in the above sense).

THEOREM 9. *For a \mathbb{C}^2 -bundle on a connected graph \mathcal{G} with $\text{SL}_2\mathbb{C}$ connection,*

$$\text{Qdet} \Delta = \sum_{\text{CRSFs}} \prod_{\text{cycles}} 2 - \text{tr}(w),$$

where $\text{tr}(w)$ is the trace of the monodromy of the cycle.

PROOF. The proof generalizes the proof of Theorem 1. We use as before the Cauchy–Binet formula which holds for general Q -determinants (Theorem 10 below).

Suppose \mathcal{G} has n vertices. Write $\text{Qdet} \Delta = \text{Qdet}(d^*d) = \sum_B \text{Qdet} B^*B$. Here B runs over choices of n edges of \mathcal{G} . If B corresponds to a set of edges which has a component with no cycle, that is, a component with $k - 1$ edges and k vertices, the corresponding matrix B is singular (in fact has a block form with nonsquare blocks) and so $\text{Qdet} B = \sqrt{\det B} = 0$.

Thus, as before, each component of B must have exactly one cycle. We now work directly with B^*B . Note that this is the Laplacian on the subgraph defined by B . Suppose that B has only one component; the general case is similar. A nonzero term in the expansion of the determinant corresponds to a decomposition of σ into cycles; the only such terms which are nonzero are when adjacent vertices are paired, each vertex is fixed, or the vertices on the unique cycle are advanced or retreated in the direction of that cycle and the other vertices are fixed.

Consider the case when some vertices are paired; if i, j are paired in B^*B then B maps i and j to the same edge e_{ij} , so this term does not contribute. If no terms are paired, the expansion of the Q -determinant for B^*B therefore has exactly four terms. It has two terms in which σ fixes the bush vertices and advances the cycle vertices around the cycle in one of the two orientations. It also has two terms in which all vertices are fixed: the vertices in the cycle advance to the edges in one of the two directions and then retreat back to their original positions. The signs work out as before. \square

THEOREM 10 (Noncommutative Cauchy–Binet formula).

$$Q\det(M_1 M_2) = \sum_{B_1} Q\det(B_1 B_2),$$

where B_1 runs over maximal minors of M_1 , and B_2 is the corresponding minor of M_2 .

PROOF (Sketch). Suppose M_1 is $n \times m$ with $m > n$. The standard proof only relies on the multilinearity of the determinant. Write columns of $M_1 M_2$ as linear combinations of columns of M_1 with coefficients in M_2 (acting on the right). Use multilinearity to write this as a sum of m^n Q -determinants; the Q -determinant of one of these is zero if two of the same columns are used. Group the remaining m choose n nonzero terms into products $Q\det B_1 Q\det B_2$. \square

10. Graphs on surfaces.

10.1. *Definitions.* Let Σ be an oriented surface, possibly with boundary, and \mathcal{G} a graph embedded on Σ in such a way that complementary components are contractible or peripheral annuli (i.e., an annular neighborhood of a boundary component). We call the pair (\mathcal{G}, Σ) a *surface graph*.

For each boundary component there is a “peripheral” cycle on \mathcal{G} , consisting of those vertices and edges bounding the same complementary component as the boundary component.

An *finite lamination* or *simple closed curve system* on Σ is (the isotopy class of) a finite set of pairwise disjoint simple closed curves on Σ each of which has nontrivial homotopy type (i.e., no curve bounds a disk). We allow two or more curves in the lamination to be isotopic.

10.2. *Bundles on surface graphs.* Given a surface graph (\mathcal{G}, Σ) a vector bundle on \mathcal{G} with connection Φ is *flat* if it has trivial monodromy around any face (contractible complementary component of Σ ; peripheral cycles do not bound faces). In this case, given a closed loop starting at a base point v , the monodromy around a loop only depends on the homotopy class of the (pointed) loop in $\pi_1(\Sigma)$, and

so the monodromy determines a representation of $\pi_1(\Sigma)$ into $\text{End}(V)$. This representation depends on the base point for π_1 ; choosing a different base point will conjugate the representation.

Conversely, let $\rho \in \text{Hom}(\pi_1(\Sigma), \text{End}(V))$ be a representation of $\pi_1(\Sigma)$ into $\text{End}(V)$; for a fixed base point there is a unique flat line bundle, up to gauge equivalence, with monodromy ρ . It is easy to construct: start with trivial parallel transport on the edges of a spanning tree of \mathcal{G} . Now on each additional edge, adding it to the tree makes a unique cycle; define the parallel transport along this edge to be the image of the homotopy class of this cycle under ρ . Thus, we have a correspondence between flat bundles modulo gauge equivalence and a space $X = \text{Hom}(\pi_1(\Sigma), \text{End}(V))/\text{End}(V)$, the space of homomorphisms of $\pi_1(\Sigma)$ into $\text{End}(V)$ modulo conjugation. This space X has the structure of an algebraic variety: it can be represented using variables for the matrix entries of a set of generators of $\pi_1(\Sigma)$, modulo an ideal corresponding to the relations in $\pi_1(\Sigma)$ and the conjugation relations. The variety X is called the *representation variety* of $\pi_1(\Sigma)$ in $\text{End}(V)$.

When dealing with flat bundles it is natural to restrict our attention to CRSFs each of whose cycles is topologically nontrivial, since other CRSFs have zero weight in $\det \Delta$. We call these *incompressible CRSFs*. The cycles in an incompressible CRSF form a finite lamination.

We now restrict to $V = \mathbb{C}^2$ and $\text{SU}_2\mathbb{C}$ -connections. For a flat bundle, $\det \Delta$ is a regular function on the representation variety $X = \text{Hom}(\pi_1(\Sigma), \text{SU}_2\mathbb{C})/\text{SU}_2\mathbb{C}$ (i.e., it is a polynomial function of the matrix entries). Remarkably, by varying the representation it is possible to extract from $\det \Delta$ those terms for incompressible CRSFs whose cycles have any given set of homotopy types. The following theorem is due to Fock and Goncharov [4]:

THEOREM 11 ([4], Theorem 12.3). *If Σ has nonempty boundary, the products*

$$\prod_{\text{cycles } \gamma} \text{Tr}(\gamma)$$

over all finite laminations form a basis for the vector space of regular functions on the representation variety $\text{Hom}(\pi_1(\Sigma), \text{SU}_2\mathbb{C})/\text{SU}_2\mathbb{C}$.

In our case, $\det \Delta$ is a regular function on X ; hence it can be written

$$\det \Delta = \sum_L c_L \prod_{\text{cycles}} \text{Tr}(w),$$

where $c_L \in \mathbb{Z}$, the sum is over finite laminations L , the product is over curves γ in L and w is the monodromy of γ . By an integral change of basis (in fact an upper triangular integer matrix, if we order isotopy types of finite laminations by inclusion) we can write this as

$$\det \Delta = \sum_L c'_L \prod_{\text{cycles}} (2 - \text{Tr}(w)),$$

and therefore by Theorem 9, c'_L is the desired weighted sum of CRSFs with lamination type L .

The actual extraction of a coefficient can be done as follows. There is a natural measure μ on X , essentially just the product of k copies of Haar measure on $SU_2\mathbb{C}$, where k is the rank of the free group $\pi_1(\Sigma)$. The regular functions discussed above form a basis for the Hilbert space $L^2(X, \mu)$ (not an orthonormal basis but obtained from an orthonormal basis by an upper-triangular linear transformation). Thus, one can obtain c_L above by an integration of $\det \Delta$ against a particular function over X .

10.3. Examples.

10.3.1. *Annulus.* The simplest example of a surface graph, after a graph on a disk, is a graph on an annulus.

Suppose (\mathcal{G}, Σ) is a surface graph on an annulus. Since $\pi_1(\Sigma) = \mathbb{Z}$ is Abelian, it will suffice to work with a line bundle (a \mathbb{Z} subgroup of $SL_2\mathbb{C}$ will have the same traces as a \mathbb{Z} subgroup of diagonal matrices in $SL_2\mathbb{C}$, which is equivalent to taking a line bundle connection). Let $z \in \mathbb{C}^*$ be the monodromy of a flat line bundle.

Then $P(z) = \det \Delta$ is a Laurent polynomial in z . It is reciprocal: $P(z) = P(1/z)$ by Theorem 1.

THEOREM 12. *$P(z)$ is a reciprocal polynomial with real and positive roots and a double root at $z = 1$.*

We emphasize here that the roots of P are unrelated to the eigenvalues of Δ . Rather they are special values of the monodromy z for which Δ is singular. We conjecture that the roots are distinct.

PROOF OF THEOREM 12. When $z = 1$, the line bundle is trivializable and Δ is the usual graph Laplacian. Thus, $P(1) = 0$. However, there exists a CRSF on \mathcal{G} with a single component winding once around Σ : just take any cycle of this type and complete it to a CRT. So in Theorem 1, the coefficient of $(2 - z - 1/z)^1$ is nonzero, and $z = 1$ is a double root (i.e., not of higher order).

We prove reality by a deformation argument. At each real root $r > 1$ of P , Δ has a kernel W_r . We claim that $\dim W_r = 1$. If it were of larger dimension, let $f_1, f_2 \in W_r$ be independent. These are real and harmonic for Δ , and so lift to actual harmonic functions on the embedded graph $(\tilde{\mathcal{G}}, \tilde{\Sigma})$ on the universal cover $\tilde{\Sigma}$ of Σ . These functions have the property that $f(x + 1) = rf(x)$ where “+1” represents the deck transformation, that is, on a path winding once around the annulus the values of f_1, f_2 are multiplied by r .

Let v be a vertex on the boundary of Σ (i.e., on a peripheral cycle) and f_3 be a linear combination of f_1, f_2 which is zero at v . Then f_3 is a harmonic function on $\tilde{\mathcal{G}}$ which is zero on a biinfinite sequence of boundary points. We claim that this is impossible unless f_3 is identically zero. By the mean-value principle, from each

zero of a (not identically zero) harmonic function on an infinite graph there are two paths to ∞ , one on which the function is increasing (and eventually positive) and another on which the function is decreasing (and eventually negative). Let α be a choice of increasing and eventually positive infinite path from a fixed lift \tilde{v} of v , and β be a choice of decreasing and eventually negative infinite path from the same point \tilde{v} . The union of the lifts of α starting at all lifts of v is a nonnegative connected set of vertices containing all lifts of v , and containing a strictly positive path from $-\infty$ to ∞ ; this contradicts the existence of β .

This completes the proof of the claim that f_3 must be identically zero, and thus the claim that $\dim W_r = 1$.

The roots of P vary continuously with the conductances. We can find a set of conductances for which all roots are real, see the next paragraph. As we vary the conductances, if two real roots r_1, r_2 collide and become a complex conjugate pair, the sum $W_{r_1} \oplus W_{r_2}$ at the point of collision must be a two-dimensional subspace of the kernel of Δ , which we showed above cannot happen. Thus, the roots must remain real. From Theorem 1, $P(z_1)$ cannot have negative real roots so all roots are real and positive.

We now find specific conductances for which all roots are real. Take a set of maximal cardinality of disjoint cycles, each winding around Σ , and complete it to a CRSF. Let C_1, \dots, C_k be the cycles in order from innermost to outermost. Add $k - 1$ more edges to join these CRTs up into a connected set U (a ‘‘chain of loops,’’ with bushes), so that C_i is connected to C_{i+1} with a path γ_i . Put a large conductance R on every edge of U except for one edge of every C_i , and one edge of every path γ_i ; these edges of U have weight 1. Put a small conductance ε on the remaining edges of \mathcal{G} not in U .

When ε is small and R is large, $\det \Delta$ is to leading order a power of R times the determinant of the graph of Figure 2. This is because with high probability all edges with conductance R , none of the edges of conductance ε , and some of the edges of conductance 1 will be present in a random CRSF. Contracting all edges of conductance R and removing those of conductance ε , we are left with the graph of Figure 2 with all edges of conductance 1. For this graph, it is easy to verify that the roots r_i of $\det \Delta$ are real and distinct (see the example after Corollary 1). The actual Laplacian is a small perturbation of this so its determinant also has real distinct roots. \square

COROLLARY 1. *The number of cycles in a uniform random incompressible CRSF on an annulus is distributed as 1 plus a sum of k independent Bernoulli*

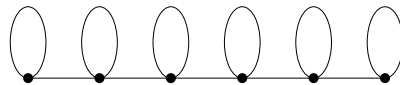


FIG. 2. A chain of loops. The roots of $\det \Delta$ are real and distinct.

random variables, where $k + 1$ is the maximal number of nonintersecting incompressible cycles one can simultaneously draw on \mathcal{G} .

PROOF. Let $w = 2 - z - 1/z$. Then $Q(w) = P(z)$ has only real negative roots:

$$Q(w) = w \prod_{i=1}^k (w + \lambda_i).$$

By Theorem 1, $Q(w)/Q(1)$ is the probability generating function for the number of loops. It is also the probability generating function for a sum of independent Bernoullis, with the i th Bernoulli being biased as $(\frac{1}{1+\lambda_i}, \frac{\lambda_i}{1+\lambda_i})$. \square

As an example, let us compute, for a rectangular cylinder, $\det \Delta$ and the corresponding distribution of cycles in a uniform random incompressible CRSF. Let \mathcal{G}_m be a line graph of length m and $H_{m,n} = \mathcal{G}_m \times \mathbb{Z}_n$. The eigenvalues of $\Delta_{\mathcal{G}_m}$ are $2 + 2 \cos \frac{k\pi}{m}$ for $k = 1, 2, \dots, m$ [the corresponding eigenvectors are $f_k(x) = \cos \frac{\pi k(x+1/2)}{m}$ for $x = 0, 1, 2, \dots, m - 1$]. By (3), we have

$$\begin{aligned} \det \Delta &= (2 - z - z^{-1}) \prod \left(\text{Ch}_n(\lambda + 2) - z - \frac{1}{z} \right) \\ &= w \prod_{\lambda} w + \text{Ch}_n(\lambda + 2) - 2, \end{aligned}$$

where $w = 2 - z - \frac{1}{z}$ and Ch_n is defined by $\text{Ch}_n(\alpha + \frac{1}{\alpha}) = \alpha^n + \alpha^{-n}$ (a variant of the Chebyshev polynomial). For nonnegative λ and large n , $\text{Ch}_n(\lambda + 2)$ is large unless λ is near 0, so the relevant roots are $\lambda_j = 2 + 2 \cos \frac{\pi(m-j)}{m}$ for small j , $j = 1, 2, \dots$. We have $2 + \lambda_j = 2 + \frac{\pi^2 j^2}{m^2} + O(\frac{j}{m})^4 = \alpha_j + 1/\alpha_j$ where $\alpha_j = 1 + \frac{\pi j}{m} + O(\frac{j}{m})^2$. Thus, in the limit $m, n \rightarrow \infty$ with $m/n \rightarrow \tau$ the roots satisfy $\text{Ch}_n(\lambda_j + 2) = 2 \cosh \frac{\pi j}{\tau} + o(1)$.

The limit probability generating function for the number of cycles is then

$$Q(w)/Q(1) = w \prod_{j=1}^{\infty} \left(\frac{w + 2 \cosh(\pi j/\tau) - 2}{2 \cosh(\pi j/\tau) - 1} \right).$$

For example, for a square annulus ($m = n$) the probability of a single cycle is approximately 95%.

10.3.2. *Torus example.* We compute the Laplacian determinant for a flat line bundle on a $n \times n$ grid on a torus, that is, for the graph $\mathbb{Z}^2/n\mathbb{Z}^2$. In this case, the determinant is

$$\det \Delta = \prod_{\zeta^n = z_1} \prod_{\xi^n = z_2} 4 - \zeta - \frac{1}{\zeta} - \xi - \frac{1}{\xi}.$$

This product was evaluated in [1], yielding $\det \Delta = C_n P_n(z_1, z_2)$ where C_n is a constant tending to ∞ and P_n tends to P , where

$$(7) \quad P(z_1, z_2) = \sum_{j,k \in \mathbb{Z}} e^{-(\pi/2)(j^2+k^2)} z_1^j z_2^k (-1)^{j+k+jk}.$$

We wish to rewrite this as

$$P(z_1, z_2) = \sum_{(j,k) \text{ primitive}} \sum_{m \geq 1} C_{jkm} (2 - z_1^j z_2^k - z_1^{-j} z_2^{-k})^m,$$

where the first sum is over primitive vectors (j, k) in \mathbb{Z}^2 , one per direction [i.e., only one of (j, k) and $(-j, -k)$ appears]. Thus, C_{jkm} will be the relative probability that a CRSF has m cycles in homology class (j, k) . To this end, pick a primitive vector (j, k) and consider the terms in (7) with monomials $z_1^{mj} z_2^{mk}$ for $m \in \mathbb{Z}$. Letting $u = z_1^j z_2^k$ and $q = e^{-(\pi/2)(j^2+k^2)}$ the sum of these terms is

$$\sum_{m \in \mathbb{Z}} q^{m^2} (-u)^m,$$

which can be rewritten as a power series in $2 - u - 1/u$ as⁵

$$\sum_{\ell \in \mathbb{Z}} q^{\ell^2} (-1)^\ell + \sum_{m=1}^{\infty} \left(\sum_{\ell=1}^{\infty} \frac{\ell}{m} \binom{m+\ell-1}{2m-1} (-1)^{\ell+m} q^{\ell^2} \right) (2 - u - 1/u)^m.$$

Thus, we find

$$C_{jkm} = \sum_{\ell=1}^{\infty} \frac{\ell}{m} \binom{m+\ell-1}{2m-1} (-1)^{\ell+m} e^{-(\pi/2)(j^2+k^2)\ell^2}.$$

For example, the probability of a single cycle in homology class $(1, 0)$ is

$$\frac{C_{101}}{\sum C_{jkm}} \approx 41\%.$$

10.3.3. *Pants example.* We consider here the case when Σ is a 3-holed sphere. This is the simplest case where the noncommutativity of the connection plays a role. We consider a flat $SL_2\mathbb{C}$ -connection on (\mathcal{G}, Σ) . Let a, b and c be the monodromies around the three holes, which satisfy $abc = 1$. Since $\pi_1(\Sigma)$ is the free group on two generators, there are no other relations. Note that a simple closed curve on Σ has only one of three possibly homotopy types: it must be homotopic

⁵We use the fact that

$$2 - z^\ell - z^{-\ell} = \sum_{m \geq 1} \frac{\ell}{m} \binom{m+\ell-1}{2m-1} (-1)^{m+1} (2 - z - z^{-1})^m.$$

to one of the three boundary curves. A finite lamination up to homotopy is therefore described by a triple $(i, j, k) \neq (0, 0, 0)$ of nonnegative integers, where there are i curves homotopic to a , j homotopic to b , and k homotopic to c . Theorem 9 gives the following.

THEOREM 13. *Let $X = 2 - \text{Tr}(a)$, $Y = 2 - \text{Tr}(b)$, $Z = 2 - \text{Tr}(c) = 2 - \text{Tr}(ab)$. Then $\det \Delta = P(X, Y, Z) = \sum_{\text{CRSFs}} X^i Y^j Z^k$ where (i, j, k) is the number of cycles with homotopy type a, b, c , respectively.*

Note in particular that P is a polynomial with nonnegative coefficients. What polynomials occur? Is there an analog of Theorem 12 in this setting? We do not know. However, it is not difficult to show that the set of exponents (i, j, k) of the nonzero monomials of $P(X, Y, Z)$ form a convex set in \mathbb{Z}^3 : let ρ_{ab} be the length in terms of the number of vertices of the shortest path from face a (the face with monodromy a) to face b (the face with monodromy b). Similarly define ρ_{ac} and ρ_{bc} . Then we can simultaneously embed i, j, k cycles of types a, b, c if and only if

$$(8) \quad \begin{aligned} i + j &\leq \rho_{ab}, \\ i + k &\leq \rho_{ac}, \\ j + k &\leq \rho_{bc}. \end{aligned}$$

This is easily proved by induction on $\rho_{ab}, \rho_{ac}, \rho_{bc}$. So the Newton polyhedron of P is defined by $i, j, k \geq 0$, $(i, j, k) \neq (0, 0, 0)$, and (8).

11. Planar loop-erased random walk. In this section, we use the vector-bundle Laplacian to study the loop-erased walk on a planar graph. In Section 11.1, we compute the probability that the LERW between two boundary points passes left or right of a given face. In Section 11.2, we compute this probability for two faces.

11.1. *LERW around a hole.* Let \mathcal{G} be a finite planar graph. Let z_1, z_2 vertices on the outer boundary and f a bounded face. We compute the probability that the LERW from z_1 to z_2 goes left of f . This can be computed quite simply using duality: given a UST on a planar graph \mathcal{G} , the duals of the edges not in T form a UST of the planar dual graph. In the dual graph, our question is equivalent to asking whether the LERW from the face f first reaches the outer face along the edge between z_1 and z_2 or outside this edge. This is just the harmonic measure of the path from z_1 to z_2 as seen from f .

Here we will compute this in another way, using line bundle technology. It illustrates a general method which will be useful in the next section.

Take a line bundle with monodromy b around f and trivial around all other faces. This can be achieved by putting in a zipper of edges (dual of a simple path

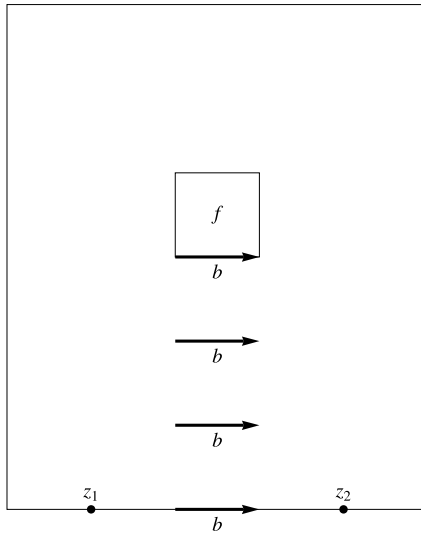


FIG. 3. Defining the connection for the LERW around a hole.

in the dual graph from f to the boundary) each with parallel transport b as in Figure 3. Suppose without loss of generality that the zipper ends between z_1 and z_2 as in the figure.

Put in an extra edge from z_1 to z_2 with parallel transport $\phi_{z_1 z_2} = a$. Then by Theorem 1,

$$\begin{aligned}
 \det \Delta &= p_1 + p_2 \left(2 - a - \frac{1}{a} \right) + p_3 \left(2 - \frac{a}{b} - \frac{b}{a} \right) \\
 (9) \qquad &= p_1 + 2p_2 + 2p_3 - a \left(p_2 + \frac{p_3}{b} \right) - \frac{1}{a} (p_2 + bp_3).
 \end{aligned}$$

Here $p_2 = p_2(b)$ is the weighted sum of CRSFs which contain a cycle containing edge a and going left of f (when oriented so that a is the edge before z_1); $p_3 = p_3(b)$ is the weighted sum of CRSFs containing a cycle through a going “right” of f .

We wish to compute $p_2(b = 1)$, $p_3(b = 1)$. These can be extracted from the coefficients of a and $\frac{1}{a}$ in $\det \Delta$.

Letting z_1, z_2 be the first two vertices, we have

$$\Delta = \Delta(b) + \begin{pmatrix} 1 & -a & \mathbf{0} \\ -a^{-1} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\Delta(b)$ is the Laplacian without the extra edge from z_1 to z_2 (and $\mathbf{0}$ represents a vector or matrix of zeros). As a consequence,

$$\det \Delta = \det \Delta(b) \det \left(I + \begin{pmatrix} 1 & -a \\ -1/a & 1 \end{pmatrix} \begin{pmatrix} G(z_1, z_1) & G(z_1, z_2) \\ G(z_2, z_1) & G(z_2, z_2) \end{pmatrix} \right),$$

where $G = \Delta(b)^{-1}$.

This last determinant is $1 + G(z_1, z_1) + G(z_2, z_2) - aG(z_2, z_1) - \frac{G(z_1, z_2)}{a}$. Comparing coefficients with (9) yields

$$p_2 = \frac{b^2 G(z_2, z_1) - G(z_1, z_2)}{b^2 - 1} \det \Delta(b)$$

and

$$p_3 = \frac{b(G(z_1, z_2) - G(z_2, z_1))}{b^2 - 1} \det \Delta(b).$$

The desired quantity is

$$(10) \quad \frac{p_2}{p_2 + p_3} = \frac{b^2 G(z_2, z_1) - G(z_1, z_2)}{(b - 1)(bG(z_2, z_1) + G(z_1, z_2))}$$

in the limit $b \rightarrow 1$.

We need to compute $G(v, v')$ when b is near 1.

LEMMA 2. *When $b = 1 + \varepsilon$, $G(v, v') = \frac{\kappa}{\det \Delta(b)}(1 + \varepsilon X(v, v') + O(\varepsilon^2))$ where κ is the number of spanning trees of \mathcal{G} and $X(v, v')$ is the expected signed number of crossings of the zipper on a simple random walk from v to v' .*

PROOF. For a matrix M , let M_A^B denote $(-1)^{i_A+i_B}$ times the determinant of M upon removing rows A and columns B . Here i_A is the sum of the indices of elements of A and likewise for i_B . (We can assume without loss of generality that all relevant vertices have even index so that we can ignore the signs.) Recall that for the standard Laplacian Δ_0 , we have for any v, v' that $(\Delta_0)_v^{v'} = \kappa$, the number of spanning trees of \mathcal{G} . We also have $(\Delta_0)_{ab}^{cd} = G(a, c) - G(a, d) - G(b, c) + G(b, d)$ where G is the Green's function.⁶

Letting $Z = \Delta(b) - \Delta_0$, a matrix supported on the zipper edges. We have

$$\begin{aligned} &G(v, v') \det \Delta(b) \\ &= \Delta(b)_v^{v'} = (\Delta_0 + Z)_v^{v'} = (\Delta_0)_v^{v'} + \sum_{x,y} Z_x^y (\Delta_0)_{v',x}^{v,y} + \dots \\ (11) \quad &= \kappa + \sum_{\vec{u}\vec{w}} (b - 1) (\Delta_0)_{v',u}^{v,w} + \left(\frac{1}{b} - 1\right) (\Delta_0)_{v',w}^{v,u} + O(\varepsilon^2) \end{aligned}$$

⁶More generally,

$$(\Delta_0)_{a_1, \dots, a_k}^{b_1, \dots, b_k} = [t] \kappa \det(t + G(a_i, b_j))_{1 \leq i, j \leq k},$$

where $[t]P(t)$ denotes the coefficient of t^1 in $P(t)$. This can be proved by induction on k and expanding the determinant on the left-hand side along row a_k .

$$= \kappa \left(1 + \varepsilon \sum_{\vec{u}\vec{w}} G_0(v', u) + G_0(v, w) - G_0(v', w) - G_0(v, u) + O(\varepsilon^2) \right).$$

Here the sum is over the zipper edges. For an edge $\vec{u}\vec{w}$ the quantity $G_0(v', u) + G_0(v, w) - G_0(v', w) - G_0(v, u)$ is the transfer current, equivalently, the expected signed number of crossings of uw of the simple random walk started at v' and stopped at v . \square

By the lemma, when $b = 1 + \varepsilon$ the probability (10) becomes

$$\frac{p_2}{p_2 + p_3} = 1 + \frac{X(z_2, z_1) - X(z_1, z_2)}{2} + O(\varepsilon) = 1 - X(z_1, z_2) + O(\varepsilon)$$

since $X(z_1, z_2) = -X(z_2, z_1)$.

COROLLARY 2. *The probability that the LERW from z_1 to z_2 goes left of f is the total amount of current flowing left of f when 1 unit of current enters at z_1 and exits at z_2 .*

11.2. LERW around two holes. Let f_1, f_2 be two faces in a finite planar graph \mathcal{G} and z_1, z_2 two points on the boundary. Consider the LERW from z_1 to z_2 . We compute the probability that it goes left of both faces. Although this can be computed in principle using Theorem 11 and the comments thereafter, in practice that method involves an integration of $\det \Delta$ over the representation variety X ; in the case of a large graph one might not have access to an explicit expression for $\det \Delta$. Here, we do the computation using only a bundle which is close to the trivial bundle; the determinant as a function on X can be expanded as a power series around the trivial bundle case and we only need the first two terms in the expansion, which we can write down explicitly in terms of the standard green's function.

We use an $SL_2\mathbb{C}$ -bundle. Choose a bundle with monodromy A around f_1 and B around f_2 , and trivial monodromy around the other faces. This can be achieved by putting trivial parallel transport on all edges except for two zippers, one from each of f_1, f_2 to the boundary, as in Figure 4. Let Z_A, Z_B denote these two zippers.

Add a new edge connecting z_1 to z_2 with parallel transport C . Suppose that A, B are close to the identity. Then a CRSF will have with high probability only a loop containing C . The first-order correction to this consists in CRSFs having two loops, one of them containing C and the other of monodromy A, B or AB (or their inverses). The possible loops in a CRSF with one or two loops, one of which contains C , have one of the seven monodromy types (or their inverses):

$$A, B, AB, C, CB^{-1}A^{-1}, C(BA)^n A^{-1}(BA)^{-n}, C(AB)^n B^{-1}(AB)^{-n},$$

where $n \in \mathbb{Z}$.

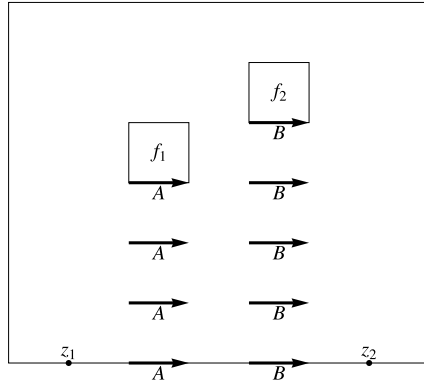


FIG. 4. Defining the connection for the LERW around two holes.

Let

$$A = \begin{pmatrix} 1 + xu & y\sqrt{u} \\ z\sqrt{u} & 1 + wu \end{pmatrix},$$

$$B = \begin{pmatrix} 1 + au & b\sqrt{u} \\ c\sqrt{u} & 1 + du \end{pmatrix},$$

$$C = \begin{pmatrix} e & 1 \\ -1 & 0 \end{pmatrix}$$

be the representations in $SL_2\mathbb{C}$, where u is small and $x, y, z, w, a, b, c, d, e$ are variables. The equations $\det A = \det B = 1$ yield $x + w - yz = o(u)$ and $a + d - bc = o(u)$.

Computing the traces of the above seven classes, we have

$$2 - \text{Tr}(A) = -(x + w)u + o(u),$$

$$2 - \text{Tr}(B) = -(a + d)u + o(u),$$

$$2 - \text{Tr}(AB) = -(a + d + w + x + cy + bz)u + o(u),$$

$$(12) \quad 2 - \text{Tr}(C) = 2 - e,$$

$$2 - \text{Tr}(C(AB)^n B^{-1}(AB)^{-n}) = 2 - e(1 + du + o(u)),$$

$$2 - \text{Tr}(C(BA)^n A^{-1}(BA)^{-n}) = 2 - e(1 + wu + o(u)),$$

$$2 - \text{Tr}(CB^{-1}A^{-1}) = 2 - e(1 + (d + w + bz)u).$$

The relevant quantities for us are the last four and the products of any one the first three of these with the last four.

We let N_{ll} be the weight sum of configurations in which the LERW from z_1 to z_2 goes left of both faces, and having no other loops (i.e., CRSFs with one loop containing the extra edge, with the property that this loop surrounds both f_1, f_2).

Similarly, define N_{lr}, N_{rl}, N_{rr} . Let N_1, N_2, N_3 be, respectively, the weighted sum of those configurations having one loop containing C and one loop surrounding, respectively, hole f_1 , hole f_2 , and both holes.

If we extract the coefficient of e in $-\text{Qdet } \Delta$, from (12) the result will be of the form

$$N_{ll} + N_{lr}(1 + du) + N_{rl}(1 + wu) + N_{rr}(1 + (d + w + bz)u) - N_1(w + x)u - N_2(a + d)u - N_3(a + d + w + x + cy + bz)u + O(u^2).$$

Recall that these variables are subject to the constraints $x + w - yz = o(u)$ and $a + d - bc = o(u)$. The coefficients of the N_* are linearly independent given these constraints. Thus given an explicit expression for $\text{Qdet } \Delta$ to first order in u , it is a simple matter to extract these various coefficients N_* . The desired probabilities are

$$p_{ll} = \frac{N_{ll}}{N_{ll} + N_{lr} + N_{rl} + N_{rr}}$$

and so on.

Let us show how to compute the coefficient of e in $\text{Qdet } \Delta$, to first order in u .

Indexing the vertices so that z_1, z_2 are the first two vertices, we have

$$\Delta(A, B, C) = \Delta(A, B) + \begin{pmatrix} I & -C & \mathbf{0} \\ -C^{-1} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\Delta(A, B)$ is the Laplacian without the extra edge from z_1 to z_2 (and $\mathbf{0}$ represents a vector or matrix of zeros). As a consequence,

$$\begin{aligned} \det \Delta(A, B, C) &= \det \Delta(A, B) \det \left(I + \begin{pmatrix} 1 & 0 & -e & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -e & 0 & 1 \end{pmatrix} \right. \\ &\quad \left. \times \begin{pmatrix} G(z_1^{(1)}, z_1^{(1)}) & \cdots & G(z_1^{(1)}, z_2^{(2)}) \\ G(z_2^{(2)}, z_1^{(1)}) & \cdots & G(z_2^{(2)}, z_2^{(2)}) \end{pmatrix} \right), \end{aligned}$$

where $G = \Delta(A, B)^{-1}$. Note that G is self-dual, since it is the inverse of a self-dual matrix. Thus, we have

$$\begin{aligned} \text{Qdet } \Delta(A, B, C) &= \text{Qdet } \Delta(A, B) \\ &\quad \times \text{Qdet} \left(I + \begin{pmatrix} I & -C \\ -C^{-1} & I \end{pmatrix} \begin{pmatrix} \mathbf{G}(z_1, z_1) & \mathbf{G}(z_1, z_2) \\ \tilde{\mathbf{G}}(z_1, z_2) & \mathbf{G}(z_2, z_2) \end{pmatrix} \right), \end{aligned}$$

where $\mathbf{G}(z_1, z_1)$ and $\mathbf{G}(z_2, z_2)$ are diagonal matrices. The right-hand side is $\text{Qdet } \Delta(A, B)$ times

$$1 + G(z_1^{(1)}, z_1^{(1)}) - G(z_1^{(1)}, z_2^{(2)}) + G(z_1^{(2)}, z_2^{(1)}) + G(z_2^{(1)}, z_2^{(1)}) - eG(z_1^{(2)}, z_2^{(2)}).$$

Thus, the coefficient of e in $-\text{Qdet } \Delta(A, B, C)$ is

$$G(z_1^{(2)}, z_2^{(2)}) \text{Qdet } \Delta(A, B)$$

to first order in u .

It is possible to give an exact expression for $G(z_1^{(2)}, z_2^{(2)}) \text{Qdet } \Delta(A, B)$ as a sum over Z_A and Z_B of products of standard Green's functions. Since A, B are close to the identity, we can write

$$\Delta(A, B) = \begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_0 \end{pmatrix} + Z,$$

where Z is a perturbation supported on the zippers. As a consequence, both

$$G(z_1^{(2)}, z_2^{(2)}) \det \Delta(A, B) = \det \Delta(A, B)_{z_2^{(2)}, z_1^{(2)}}$$

and $\text{Qdet } \Delta(A, B)$ itself have expansions in powers of u . The ratio of these is the desired quantity.

For the present purposes, we need the expansion of $G(z_1^{(2)}, z_2^{(2)}) \det \Delta(A, B)$ to first order (which is the term of order u^2) and $\sqrt{\det \Delta(A, B)}$ to first order (which is the term of order u). Unfortunately these computations, although elementary, can be quite long, and we have not been able to get an explicit answer for a main case of interest which is the upper half plane.

12. Green's function and random walks. We give here a probabilistic interpretation of the Green's function of the line bundle Laplacian in the case where each edge has parallel transport either 1 or z . Let \mathcal{G} be a graph, E_Z a collection of directed edges, and consider the line bundle with parallel transport z on E_Z and 1 elsewhere. Let G be the inverse of the line bundle Laplacian.

Let v, v' be vertices of \mathcal{G} . Let us first ask: what is the distribution of the signed number of crossings of E_Z on the SRW from v to v' ?

Let P be the transition matrix of the simple random walk on \mathcal{G} (ignoring the line bundle) with absorbing state v' . Thus, $P_{v_2 v_1}$ is the probability of going to v_2 from state v_1 (as usual the indices are reversed so that composition respects the natural path order). Multiply the entry $P_{v_2 v_1}$ by z or $1/z$ if $v_1 v_2$ is in E_Z or $-E_Z$, respectively. With v' as the vertex with last index P has the form $P = \begin{pmatrix} N & 0 \\ A & 1 \end{pmatrix}$. Then the probability generating function of the number of crossings is

$$\sum_{k=0}^{\infty} (AN^k)_{v'v} = (A(I - N)^{-1})_{v'v}.$$

PROPOSITION 3. *The probability generating function of the signed number of crossings of E_Z on a simple random walk from v to v' is*

$$(A(I - N)^{-1})_{v'v} = \frac{G(v, v')}{G(v', v')}.$$

PROOF. Note that $((I - P)D)_{v_1v_2} = \Delta_{v_1v_2}$ for $v_2 \neq v'$, where D is the diagonal matrix of vertex degrees. As a consequence

$$\begin{aligned} (A(I - N)^{-1})_{v'v} &= \sum_{v_2} A_{v'v_2} (I - N)_{v_2v}^{-1} \\ &= \sum_{v_2} A_{v'v_2} \frac{\text{cof}(I - N)_{vv_2}^{v_2}}{\det(I - N)} \\ &= \sum_{v_2} A_{v'v_2} \frac{\text{cof}(\Delta D^{-1})_{vv_2}^{v_2v'}}{\text{cof}(\Delta D^{-1})_{v'v}^{v'}} \\ &= \sum_{v_2} A_{v'v_2} \frac{\text{cof} \Delta_{vv'}^{v_2v'} (\text{deg}(v') \text{deg}(v_2) / \det D)}{(\det \Delta) G(v', v') \text{deg}(v') / \det D} \\ &= \frac{1}{(\det \Delta) G(v', v')} \sum_{v_2} \Delta_{v'v_2} \text{cof} \Delta_{vv'}^{v_2v'} \\ &= \frac{G(v, v')}{G(v', v')}. \quad \square \end{aligned}$$

The quantity $1 - \frac{1}{G(v', v') \text{deg} v'}$ also has a probabilistic interpretation. Choosing a gauge such that $\phi_{v'v} = 1$ for all v adjacent to v' , and using

$$\sum_{v_2} \Delta_{v, v_2} G(v_2, v') = \delta_{v, v'}$$

we have (setting $v = v'$)

$$\text{deg} v' G(v', v') - \sum_{v_2 \sim v'} G(v_2, v') = 1$$

giving

$$1 - \frac{1}{G(v', v') \text{deg} v'} = \sum_{v_2 \sim v'} \frac{1}{\text{deg} v'} \frac{G(v_2, v')}{G(v', v')}.$$

Thus, we have the following proposition.

PROPOSITION 4. $1 - \frac{1}{G(v', v') \text{deg} v'}$ is the probability generating function of the signed number of crossings of E_Z on a SRW started at v' and stopped on its first return to v' .

Now let $z = 1 + \varepsilon$ with small ε and

$$(13) \quad \frac{G(v, v')}{G(v', v')} = 1 + X_1\varepsilon + X_2\varepsilon^2 + \dots,$$

so that X_1 is the expected signed number of crossings of E_Z on a SRW from v to v' . Similarly let

$$(14) \quad 1 - \frac{1}{G(v', v') \deg v'} = 1 + Y_2\varepsilon^2 + Y_3\varepsilon^3 + \dots,$$

so that $2Y_2$ is the expectation of the square of the signed number of crossings of E_Z of the SRW from v' until its first return to v' (note that Y_1 is zero and $Y_3 = -Y_2$ by symmetry).

PROPOSITION 5. *The determinant of the line bundle Laplacian Δ above satisfies*

$$(15) \quad \det \Delta = \kappa \deg v' (-Y_2\varepsilon^2 + Y_2\varepsilon^3 + O(\varepsilon^4)),$$

where $2Y_2$ is the expected square of the signed number of crossings of E_Z of a SRW from v' until its first return to v' . In particular $Y_2 \deg v'$ does not depend on v' .

PROOF. From Lemma 2, we have

$$G(v, v') \det \Delta = \kappa (1 + X_1\varepsilon + O(\varepsilon^2)).$$

However by (13) and (14), we have

$$G(v, v') \det \Delta = \frac{1}{\deg v'} \frac{1 + X_1\varepsilon + X_2\varepsilon^2 + O(\varepsilon^3)}{-Y_2\varepsilon^2 + Y_2\varepsilon^3 + O(\varepsilon^4)} \det \Delta,$$

so we must have $\det \Delta = \kappa \deg v' (-Y_2\varepsilon^2 + Y_2\varepsilon^3 + O(\varepsilon^4))$. \square

We can get a full expansion of $\det \Delta$ as a power series in z around $z = 1$ as follows. We have

$$\frac{\partial}{\partial z} \log \det \Delta = \sum_{uw \in E_Z} G_u^w - \frac{1}{z^2} G_w^u.$$

This follows from differentiating the entries in Δ . Here using Propositions 4 and 3, the quantities on the right-hand side have explicit probabilistic interpretations. Integrating both sides from $z = 1$ to $z = z_0$, gives the desired expansion.

13. Monotone lattice paths. Let $\Gamma_{m,n}$ be the $m \times n$ grid on a torus [the nearest neighbor grid $\{0, 1, \dots, m\} \times \{0, 1, \dots, n\}$ with opposite sides identified: $(m, j) \sim (0, j)$ and $(i, n) \sim (i, 0)$]. Choose for each vertex a north- or east-going edge, independently and with probability $p = 1/2$. The resulting configuration of edges makes a directed CRSF, and we wish to determine the distribution of the number and homology type of the cycles. This problem was studied in [7] who showed among other things that when m/n is close to a rational with small denominator the number of cycles is (in the limit $m, n \rightarrow \infty$) tending to a Gaussian with expectation on the order of \sqrt{n} .

Here we show how to compute explicitly for each m, n the probability generating function of the total homology class of the cycles (which determines both the number of cycles and their direction).

We make $\Gamma_{m,n}$ into a directed graph, directing all edges northward or eastward. Let Φ be a flat line bundle on $\Gamma_{m,n}$ with monodromy z and w for loops in homology class $(1, 0)$ and $(0, 1)$, respectively. Φ can be obtained by putting parallel transports 1 on all edges except edges $(m - 1, i)(0, i)$ which get parallel transport z and edges $(i, n - 1)(i, 0)$ which get parallel transport w .

By Theorem 6, the determinant of the line bundle Laplacian is

$$(16) \quad F_{m,n}(z, w) = \det \Delta = \sum_{j,k \geq 0} C_{j,k} (1 - z^p w^q)^\ell,$$

where $(j, k) = (p\ell, q\ell)$ and p, q are coprime, and $C_{j,k}$ is the number of CRSFs with total homology class (j, k) , that is, with ℓ cycles of homology (p, q) .

This determinant can be explicitly computed using a standard Fourier diagonalization of Δ .

PROPOSITION 6.

$$(17) \quad F_{m,n}(z, w) = \prod_{u^m=z} \prod_{v^n=w} 2 - u - v.$$

PROOF. If $u^m = z$ and $v^n = w$, then the function $f(x, y) = u^x v^y$ is an eigenvector of Δ with eigenvalue $2 - u - v$. These eigenvectors are independent and span \mathbb{C}^{mn} . \square

From (17), we can extract the coefficients $C_{m,n}$. Let us consider for simplicity the case $m = n$. In [7], it was shown that with probability tending to 1 as $n \rightarrow \infty$, all cycles will have homology class $(1, 1)$. Thus, from (16),

$$(18) \quad F_{n,n}(z, w) = \sum_{j \geq 0} C_{j,j} (1 - zw)^j + \sum_{j \neq k \geq 0} C_{j,k} (1 - z^p w^q)^\ell,$$

where the second sum is negligible, in the sense that the sum of its coefficients is $o(\sum C_{j,j})$. We can thus ignore the second sum, let $z = 1$ and expand $F_{n,n}(1, w)$

around $w = 1$. We have

$$F_{n,n}(z, w) = \prod_{u^n=z, v^n=w} 2 - u - v = \prod_{u^n=z} (2 - u)^n - w.$$

Letting $Y = 1 - w$ and $F_{n,n}(1, w) = H_n(1, Y)$, we have

$$H_n(1, Y) = \prod_{j=0}^{n-1} (2 - e^{2\pi i j/n})^n - (1 - Y)$$

and

$$\frac{H_n(1, Y)}{H_n(1, 1)} = \prod_{j=-\lfloor n/2 \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{(2 - e^{2\pi i j/n})^n - (1 - Y)}{(2 - e^{2\pi i j/n})^n}.$$

The product $(2 - e^{2\pi i j/n})^n$ is large in absolute value unless $|j|$ is small, that is, when $|j|$ is $O(n^{1/2})$, so the dependence on Y comes from terms with $|j|$ small. That is, the product can be written as

$$\prod_{|j| < n^{1/2+\epsilon}} (\dots) \prod_{|j| > n^{1/2+\epsilon}} (\dots)$$

and the second product is $1 + o(e^{-n^\epsilon})$. When $|\theta|$ is small,

$$(2 - e^{i\theta})^n = (1 - i\theta + \theta^2/2 + \dots)^n = e^{-in\theta + n\theta^2 + \dots}.$$

Plugging in $\theta = 2\pi j/n$ yields

$$\frac{H_n(1, Y)}{H_n(1, 1)} (1 + o(e^{-n^\epsilon})) = \prod_{j=-\infty}^{\infty} \frac{e^{(2\pi j)^2/n} - 1 + Y}{e^{(2\pi j)^2/n}},$$

where we have extended the range of j without loss of precision. Thus up to exponentially small errors,

$$\frac{H_n(1, Y)}{H_n(1, 1)} = Y \prod_{j \neq 0} (1 - e^{-(2\pi j)^2/n} + Y e^{-(2\pi j)^2/n})$$

is the probability generating function for the number $N_{(1,1)}$ of $(1, 1)$ -cycles. We see that the number of $(1, 1)$ cycles is a sum of independent Bernoulli random variables with the j th having bias $p_j = e^{-(2\pi j)^2/n}$. The expectation of $N_{(1,1)}$ is then (using $t = 2\pi j/\sqrt{n}$)

$$\sum p_j = \sum_j e^{-(2\pi j)^2/n} \approx \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\frac{n}{4\pi}}.$$

The variance is

$$\sum p_j (1 - p_j) \approx \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{\infty} e^{-t^2} (1 - e^{-t^2}) dt = \sqrt{\frac{n}{4\pi}} \left(1 - \frac{1}{\sqrt{2}}\right).$$

Since the variance tends to ∞ , the central limit theorem implies that distribution tends to a Gaussian as $n \rightarrow \infty$.

A similar computation holds when $m = pk$, $n = qk$ and $k \rightarrow \infty$.

14. Open questions.

1. Consider a graph embedded on a Riemann surface, such that edges are geodesic segments. Now use a naturally associated connection, for example, the Levi-Civita connection on the tangent bundle. What are the probabilistic consequences of choosing such a connection?
2. Is there any combinatorial meaning to the vector bundle Laplacian for higher-rank bundles?
3. What information does the line bundle Laplacian give about the three-dimensional monotone lattice path problem, generalizing the results of Section 13?

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