A PROBABILISTIC APPROACH TO DIRICHLET PROBLEMS OF SEMILINEAR ELLIPTIC PDEs WITH SINGULAR COEFFICIENTS

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In this paper, we prove that there exists a unique solution to the Dirichlet boundary value problem for a general class of semilinear second order elliptic partial differential equations. Our approach is probabilistic. The theory of Dirichlet processes and backward stochastic differential equations play a crucial role.

1. Introduction. In this paper, we will use probabilistic methods to solve the Dirichlet boundary value problem for the semilinear second order elliptic PDE of the following form:

$$
(1.1) \qquad
\begin{cases}
\mathcal{A}u(x) = -f(x, u(x), \nabla u(x)), & \forall x \in D, \\
u(x)|_{\partial D} = \varphi, & \forall x \in \partial D,
\end{cases}
$$

where D is a bounded domain in R^d. The operator $\mathcal{A}$ is given by

$$
(1.2) \quad \mathcal{A}u = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} - \text{``div}(\hat{b}u)\text{''} + q(x)u,
$$

where $a = (a_{i,j}(x))_{1 \le i, j \le d} : D \to R^{d \times d}$ $(d > 2)$ is a measurable, symmetric matrix-valued function satisfying a uniform elliptic condition, $b = (b_1, b_2, \ldots, b_d)$, $\hat{b} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_d) : D \to R^d$ and $q : D \to R$ are merely measurable functions belonging to some L^p spaces, and $f(\cdot, \cdot, \cdot)$ is a nonlinear function. The operator $\mathcal{A}$ is rigorously determined by the following quadratic form:

$$
Q(u, v) = (-\mathcal{A}u, v)_{L^2(R^d)}
$$

$$
(1.3) \qquad = \frac{1}{2} \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx - \sum_{i=1}^{d} \int_{R^d} b_i(x) \frac{\partial u}{\partial x_i} v(x) \, dx
$$

$$
- \sum_{i=1}^{d} \int_D \hat{b}_i(x) u \frac{\partial v}{\partial x_i} \, dx - \int_D q(x) u(x) v(x) \, dx.
$$

We refer readers to [14, 18] and [24] for details of the operator $\mathcal{A}$.

Probabilistic approaches to boundary value problems of second order differential operators have been adopted by many people. The earlier work went back as early as 1944 in [15]. See the books [1, 7] and references therein. If $f = 0$ (i.e., the linear case), and moreover $\hat{b} = 0$, the solution u to problem (1.1) can be solved by a Feynman–Kac formula

$$u(x) = E_x\left[\exp\left(\int_0^{\tau_D} q(X(s))\,ds\right)\varphi(X(\tau_D))\right] \qquad \text{for } x \in D,$$

where $X(t), t \geq 0$ is the diffusion process associated with the infinitesimal generator

$$(1.4) \qquad L_1 = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i},$$

τ_D is the first exit time of the diffusion process $X(t), t \geq 0$ from the domain D. Very general results are obtained in the paper [6] for this case. When $\hat{b} \neq 0$, "div($\hat{b}\cdot$)" in (1.2) is just a formal writing because the divergence does not really exist for the merely measurable vector field $\hat{b}$. It should be interpreted in the distributional sense. It is exactly due to the nondifferentiability of $\hat{b}$, all the previous known probabilistic methods in solving the elliptic boundary value problems such as those in [1, 6, 15] and [13] could not be applied. We stress that the lower order term div($\hat{b}\cdot$) cannot be handled by Girsanov transform or Feynman–Kac transform either. In a recent work [5], we show that the term $\hat{b}$ in fact can be tackled by the time-reversal of Girsanov transform from the first exit time τ_D from D by the symmetric diffusion X^0 associated with $L_0 = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$, the symmetric part of $\mathcal{A}$. The solution to equation (1.1) (when $f = 0$) is given by

$$
\begin{aligned}
u(x) = E_x^0\Bigg[\varphi(X^0(\tau_D))\exp\Bigg\{ &\int_0^{\tau_D}\langle(a^{-1}b)(X^0(s)),dM^0(s)\rangle \\
&+ \left(\int_0^{\tau_D}\langle(a^{-1}\hat{b})(X^0(s)),dM^0(s)\rangle\right)\circ r_{\tau_D} \\
&- \frac{1}{2}\int_0^{\tau_D}(b-\hat{b})a^{-1}(b-\hat{b})^*(X^0(s))\,ds \\
&+ \int_0^{\tau_D}q(X^0(s))\,ds\Bigg\}\Bigg],
\end{aligned}
$$

(1.5)

where $M^0(s)$ is the martingale part of the diffusion X^0, r_t denotes the reverse operator, and $\langle\cdot,\cdot\rangle$ stands for the inner product in R^d.

Nonlinear elliptic PDEs [i.e., $f \neq 0$ in (1.1)] are generally very hard to solve. One can not expect explicit expressions for the solutions. However, in recent years backward stochastic differential equations (BSDEs) have been used effectively to solve certain nonlinear PDEs. The general approach is to represent the solution of

the nonlinear equation (1.1) as the solution of certain BSDEs associated with the diffusion process generated by the linear operator $\mathcal{A}$. But so far, only the cases where $\hat{b} = 0$ and b being bounded were considered. The main difficulty for treating the general operator $\mathcal{A}$ in (1.2) with $\hat{b} \neq 0$, $q \neq 0$ is that there are no associated diffusion processes anymore. The mentioned methods used so far in the literature ceased to work. Our approach is to transform the problem (1.1) to a similar problem for which the operator $\mathcal{A}$ does not have the "bad" term $\hat{b}$. See below for detailed description.

There exist many papers on BSDEs and their applications to nonlinear PDEs. We mention some related earlier results. The first result on probabilistic interpretation for solutions of semilinear parabolic PDE's was obtained by Peng in [19] and subsequently in [21]. In [8], Darling and Pardoux obtained a viscosity solution to the Dirichlet problem for a class of semilinear elliptic PDEs (through BSDEs with random terminal time) for which the linear operator $\mathcal{A}$ is of the form

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},$$

where $a_{ij} \in C_b^2(D)$ and $b \in C_b^1(D)$. BSDEs associated with Dirichlet processes and weak solutions of semi-linear parabolic PDEs were considered by Lejay in [16] where the linear operator $\mathcal{A}$ is assumed to be

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i},$$

for bounded coefficients a and b. BSDEs associated with symmetric Markov processes and weak solutions of semi-linear parabolic PDEs were studied by Bally, Pardoux and Stoica in [2] where the linear operator $\mathcal{A}$ is assumed to be symmetric with respect to some measure m. BSDEs and solutions of semi-linear parabolic PDEs were also considered by Rozkosz in [23] for the linear operator $\mathcal{A}$ of the form

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial}{\partial x_j} \right).$$

Now we describe the contents of this paper in more details. Our strategy is to transform the problem (1.1) by a kind of h-transform to a problem of a similar kind, but with an operator $\mathcal{A}$ that does not have the "bad" term $\hat{b}$. The first step will be to solve (1.1) assuming $\hat{b} = 0$. In Section 2, we introduce the Feller diffusion process $(\Omega, \mathcal{F}, \mathcal{F}_t, X(t), P_x, x \in R^d)$ whose infinitesimal generator is given by

$$(1.6) \qquad L_1 = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}.$$

In general, $X(t)$, $t \geq 0$ is not a semimartingale. But it has a nice martingale part $M(t)$, $t \geq 0$. In this section, we prove a martingale representation theorem for the martingale part $M(t)$, which is crucial for the study of BSDEs in subsequent sections. In Section 3, we solve a class of BSDEs associated with the martingale part $M(t)$, $t \geq 0$:

$$(1.7) \qquad Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) \, ds - \int_t^T \langle Z(s), dM(s) \rangle.$$

The random coefficient $f(t, y, z, \omega)$ satisfies a certain monotonicity condition which is particularly fulfilled in the situation we are interested. The BSDEs with deterministic terminal time were solved first and then the BSDEs with random terminal time were studied. In Section 4, we consider the Dirichelt problem for the second order differential operator

$$(1.8) \qquad L_2 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} + q(x),$$

where $b_i \in L^p$ for some $p > d$ and $q \in L^\beta$ for some $\beta > \frac{d}{2}$. We first solve the linear problem with a given function F

$$(1.9) \qquad \begin{cases} L_2 u = F, & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases}$$

and then the nonlinear problem

$$(1.10) \qquad \begin{cases} L_2 u = -g(x, u(x), \nabla u(x)), & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases}$$

with the help of BSDEs. Finally, in Section 5, we study the Dirichlet problem

$$(1.11) \qquad \begin{cases} \mathcal{A}u(x) = -f(x, u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D, \end{cases}$$

where $\mathcal{A}$ is a general second order differential operator given in (1.2). We apply a transform we introduced in [5] to transform the above problem to a problem like (1.10) and then a reverse transformation will solve the final problem.

2. Preliminaries. Let $\mathcal{A}$ be an elliptic operator of the following general form:

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} - \text{``div}(\hat{b} \cdot)\text{''} + q(x),$$

where $a = (a_{ij}(x)) : D \to R^{d \times d}$ $(d > 2)$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$(2.1) \qquad \lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \qquad \forall \xi \in R^d \text{ and } x \in D$$

for some constant $\lambda, \Lambda > 0$, $b = (b_1, \ldots, b_d)$, $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_d) : D \to R^d$ and $q : D \to R$ are measurable functions which could be singular and such that

$$|b|^2 \in L^p(D), \qquad |\hat{b}|^2 \in L^p(D) \quad \text{and} \quad q \in L^p(D),$$

for some $p > \frac{d}{2}$. Here D is a bounded domain in R^d whose boundary is regular, that is, for every $x \in \partial D$, $P(\tau_D^x = 0) = 1$, where τ_D^x is the first exit time of a standard Brownian motion started at x from the domain D. Let $f : R^d \times R \times R^d \to R$ be a measurable nonlinear function. Consider the following nonlinear Dirichlet boundary value problem:

$$(2.2) \qquad \begin{cases} \mathcal{A}u(x) = -f(x, u(x), \nabla u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases}$$

Let $W^{1,2}(D)$ denote the usual Sobolev space of order one:

$$W^{1,2}(D) = \{u \in L^2(D) : \nabla u \in L^2(D; R^d)\}.$$

DEFINITION 2.1. We say that $u \in W^{1,2}(D)$ is a continuous, weak solution of (2.2) if:

(i) for any $\phi \in W_0^{1,2}(D)$,

$$\frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} \phi \, dx$$

$$- \sum_{i=1}^d \int_D \hat{b}_i(x) u \frac{\partial \phi}{\partial x_i} \, dx - \int_D q(x) u(x) \phi \, dx = \int_D f(x, u, \nabla u) \phi \, dx,$$

(ii) $u \in C(\bar{D})$,
(iii) $\lim_{y \to x} u(y) = \varphi(x)$, $\forall x \in \partial D$.

Next we introduce two diffusion processes which will be used later.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X(t), P_x, x \in R^d)$ be the Feller diffusion process whose infinitesimal generator is given by

$$(2.3) \qquad L_1 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i},$$

where $\mathcal{F}_t$ is the completed, minimal admissible filtration generated by $X(s)$, $s \geq 0$. The associated nonsymmetric, semi-Dirichlet form with L_1 is defined by

$$(2.4) \quad \begin{aligned} Q_1(u, v) &= (-L_1 u, v)_{L^2(R^d)} \\ &= \frac{1}{2} \sum_{i,j=1}^d \int_{R^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx - \sum_{i=1}^d \int_{R^d} b_i(x) \frac{\partial u}{\partial x_i} v(x) \, dx. \end{aligned}$$

The process $X(t)$, $t \geq 0$ is not a semimartingale in general. However, it is known (see, e.g., [6, 10, 12] and [17]) that the following Fukushima's decomposition holds:

$$(2.5) \qquad X(t) = x + M(t) + N(t) \qquad P_x\text{-a.s.,}$$

where $M(t)$ is a continuous square integrable martingale with sharp bracket being given by

$$(2.6) \qquad \langle\langle M^i, M^j \rangle\rangle_t = \int_0^t a_{i,j}(X(s)) \, ds,$$

and $N(t)$ is a continuous process of zero quadratic variation. Later we also write $X_x(t)$, $M_x(t)$ to emphasize the dependence on the initial value x. Let $\mathcal{M}$ denote the space of square integrable martingales w.r.t. the filtration $\mathcal{F}_t$, $t \geq 0$. The following result is a martingale representation theorem whose proof is a modification of that of Theorem A.3.20 in [12]. It will play an important role in our study of the backward stochastic differential equations associated with the martingale part M.

THEOREM 2.1. *For any $L \in \mathcal{M}$, there exist predictable processes $H_i(t)$, $i = 1, \ldots, d$ such that*

$$(2.7) \qquad L_t = \sum_{i=1}^d \int_0^t H_i(s) \, dM^i(s).$$

PROOF. It is sufficient to prove (2.7) for $0 \leq t \leq T$, where T is an arbitrary, but fixed constant T. Recall that $\mathcal{M}$ is a Hilbert space w.r.t. the inner product $(K_1, K_2)_{\mathcal{M}} = E[\langle\langle K_1, K_2 \rangle\rangle_T]$, where $\langle\langle K_1, K_2 \rangle\rangle$ denotes the sharp bracket of K_1 and K_2. Let $\mathcal{M}_1$ denote the subspace of square integrable martingales of the form (2.7). Let R_α, $\alpha > 0$ be the resolvent operators of the diffusion process $X(t)$, $t \geq 0$. Fix any $g \in C_b(R^d)$, we know that $R_\alpha g \in D(L_1)$ and $L_1 R_\alpha g = \alpha R_\alpha g - g$. Moreover, it follows from [12] and [17] that

$$R_\alpha g(X(t)) - R_\alpha g(X(0)) = \int_0^t \langle \nabla R_\alpha g(X(s)), dM(s) \rangle$$
$$+ \int_0^t (\alpha R_\alpha g - g)(X(s)) \, ds.$$

Hence,

$$J_t := \int_0^t e^{-\alpha s} \langle \nabla R_\alpha g(X(s)), dM(s) \rangle$$
$$= e^{-\alpha t} R_\alpha g(X(t)) - R_\alpha g(X(0)) + \int_0^t e^{-\alpha s} g(X(s)) \, ds$$

is a bounded martingale that belongs to $\mathcal{M}_1$. The theorem will be proved if we can show that $\mathcal{M}_1^\perp = \{0\}$. Take $K \in \mathcal{M}_1^\perp$. Since $\mathcal{M}_1$ is stable under stopping, by

Lemma 2 in Chapter IV in [22], we deduce $\langle\langle K, L \rangle\rangle = 0$ for all $L \in \mathcal{M}_1$. In particular, $\langle\langle K, J \rangle\rangle = 0$. From here, we can follow the same proof of Theorem A.3.20 in [12] to conclude $K = 0$. $\square$

We will denote by $(\Omega, \mathcal{F}^0, \mathcal{F}_t^0, X^0(t), P_x^0, x \in R^d)$ the diffusion process generated by

$$(2.8) \qquad L_0 = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j}\right).$$

The corresponding Fukushima's decomposition is written as

$$X^0(t) = x + M^0(t) + N^0(t), \qquad t \geq 0.$$

For $v \in W^{1,2}(R^d)$, the Fukushima's decomposition for the Dirichlet process $v(X^0(t))$ reads as

$$(2.9) \qquad v(X^0(t)) = v(X^0(0)) + M^v(t) + N^v(t),$$

where $M^v(t) = \int_0^t \nabla v(X^0(s)) \cdot dM^0(s)$, $N^v(t)$ is a continuous process of zero energy (the zero energy part). See [3, 4, 12] for details of symmetric Markov processes.

3. Backward SDEs with singular coefficients. Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ be the probability space carrying the diffusion process $X(t)$ described in Section 2. Recall $M(t), t \geq 0$ is the martingale part of X. In this section, we will study backward stochastic differential equations (BSDEs) with singular coefficients associated with the martingale part $M(t)$.

3.1. *BSDEs with deterministic terminal times.* Let $f(s, y, z, \omega) : [0, T] \times R \times R^d \times \Omega \to R$ be a given progressively measurable function. For simplicity, we omit the random parameter ω. Assume that f is continuous in y and satisfies:

(A.1) $(y_1 - y_2)(f(s, y_1, z) - f(s, y_2, z)) \leq -d_1(s)|y_1 - y_2|^2$,
(A.2) $|f(s, y, z_1) - f(s, y, z_2)| \leq d_2|z_1 - z_2|$,
(A.3) $|f(s, y, z)| \leq |f(s, 0, z)| + K(s)(1 + |y|)$,

where $d_1(\cdot)$, $K(s)$ are a progressively measurable stochastic process and d_2 is a constant. Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Let λ be the constant defined in (2.1).

THEOREM 3.1.   *Assume* $E[e^{-\int_0^T 2d_1(s)\,ds}|\xi|^2] < \infty$, $E[\int_0^T K(s)\,ds] < \infty$ *and*

$$E\left[\int_0^T e^{-\int_0^s 2d_1(u)\,du}|f(s, 0, 0)|^2\,ds\right] < \infty.$$

Then, there exists a unique ($\mathcal{F}_t$-adapted) solution (Y, Z) to the following BSDE:

$$(3.1) \qquad Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\,ds - \int_t^T \langle Z(s), dM(s)\rangle,$$

where $Z(s) = (Z_1(s), \ldots, Z_d(s))$.

PROOF.   We first prove the uniqueness. Set $d(s) = -2d_1(s)$. Suppose $(Y^1(t), Z^1(t))$ and $(Y^2(t), Z^2(t))$ are two solutions to equation (3.1). Then

(3.2)
$$
\begin{aligned}
d(|Y^1(t) &- Y^2(t)|^2) \\
&= -2(Y^1(t) - Y^2(t))(f(t, Y^1(t), Z^1(t)) - f(t, Y^2(t), Z^2(t))) \, dt \\
&\quad + 2(Y^1(t) - Y^2(t))\langle Z^1(t) - Z^2(t), dM(t)\rangle \\
&\quad + \langle a(X(t))(Z^1(t) - Z^2(t)), Z^1(t) - Z^2(t)\rangle \, dt.
\end{aligned}
$$

By the chain rule, using the assumptions (A.1), (A.2) and Young's inequality, we get

$$
\begin{aligned}
|Y^1(t) &- Y^2(t)|^2 e^{\int_0^t d(s)\,ds} \\
&+ \int_t^T e^{\int_0^s d(u)\,du} \langle a(X(s))(Z^1(s) - Z^2(s)), Z^1(s) - Z^2(s)\rangle \, ds \\
&= -\int_t^T e^{\int_0^s d(u)\,du} |Y^1(s) - Y^2(s)|^2 d(s) \, ds \\
&\quad + 2 \int_t^T e^{\int_0^s d(u)\,du} (Y^1(s) - Y^2(s)) \\
&\qquad\qquad \times (f(s, Y^1(s), Z^1(s)) - f(s, Y^2(s), Z^2(s))) \, ds \\
&\quad - 2 \int_t^T e^{\int_0^s d(u)\,du} (Y^1(s) - Y^2(s)) \langle Z^1(s) - Z^2(s), dM(s)\rangle
\end{aligned}
$$

(3.3)
$$
\begin{aligned}
&\leq -\int_t^T e^{\int_0^s d(u)\,du} |Y^1(s) - Y^2(s)|^2 d(s) \, ds \\
&\quad - 2 \int_t^T e^{\int_0^s d(u)\,du} |Y^1(s) - Y^2(s)|^2 d_1(s) \, ds \\
&\quad + 2 \int_t^T e^{\int_0^s d(u)\,du} d_2 |Y^1(s) - Y^2(s)||Z^1(s) - Z^2(s)| \, ds \\
&\quad - 2 \int_t^T e^{\int_0^s d(u)\,du} (Y^1(s) - Y^2(s)) \langle Z^1(s) - Z^2(s), dM(s)\rangle \\
&\leq C_\lambda \int_t^T e^{\int_0^s d(u)\,du} |Y^1(s) - Y^2(s)|^2 \, ds \\
&\quad + \frac{1}{2} \int_t^T e^{\int_0^s d(u)\,du} \langle a(X(s))(Z^1(s) - Z^2(s)), (Z^1(s) - Z^2(s))\rangle \, ds \\
&\quad - 2 \int_t^T e^{\int_0^s d(u)\,du} (Y^1(s) - Y^2(s)) \langle Z^1(s) - Z^2(s), dM(s)\rangle.
\end{aligned}
$$

Take expectation in above inequality to get

$$E\big[|Y^1(t) - Y^2(t)|^2 e^{\int_0^t d(s)\,ds}\big] \le C_\lambda \int_t^T E\big[e^{\int_0^s d(u)\,du}|Y^1(s) - Y^2(s)|^2\big]\,ds.$$

By Gronwall's inequality, we conclude $Y^1(t) = Y^2(t)$ and hence $Z^1(t) = Z^2(t)$ by (3.3).

Next, we prove the existence. Take an even, nonnegative function $\phi \in C_0^\infty(R)$ with $\int_R \phi(x)\,dx = 1$. Define

$$f_n(t, y, z) = \int_R f(t, x, z)\phi_n(y - x)\,dx,$$

where $\phi_n(x) = n\phi(nx)$. Since f is continuous in y, it follows that $f_n(t, y, z) \to f(t, y, z)$ as $n \to \infty$. Furthermore, it is easy to see that for every $n \ge 1$,

(3.4) $$|f_n(t, y_1, z) - f_n(t, y_2, z)| \le C_n|y_1 - y_2|, \qquad y_1, y_2 \in R,$$

for some constant C_n. Consider the following BSDE:

(3.5) $$Y_n(t) = \xi + \int_t^T f_n(s, Y_n(s), Z_n(s))\,ds - \int_t^T \langle Z_n(s), dM(s)\rangle.$$

In view of (3.4) and the assumptions (A.2), (A.3), it is known (e.g., [20]) that the above equation admits a unique solution (Y_n, Z_n). Our aim now is to show that there exists a convergent subsequence (Y_{n_k}, Z_{n_k}). To this end, we need some estimates. Applying Itô's formula, in view of assumptions (A.1)–(A.3) it follows that

$$|Y_n(t)|^2 e^{\int_0^t d(s)\,ds} + \int_t^T e^{\int_0^s d(u)\,du}\langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds$$

$$= |\xi|^2 e^{\int_0^T d(s)\,ds} - \int_t^T e^{\int_0^s d(u)\,du}Y_n^2(s)d(s)\,ds$$

$$+ 2\int_t^T e^{\int_0^s d(u)\,du}Y_n(s)f_n(s, Y_n(s), Z_n(s))\,ds$$

$$- 2\int_t^T e^{\int_0^s d(u)\,du}Y_n(s)\langle Z_n(s), dM(s)\rangle$$

$$\le |\xi|^2 e^{\int_0^T d(s)\,ds} - \int_t^T e^{\int_0^s d(u)\,du}Y_n^2(s)d(s)\,ds$$

$$- 2\int_t^T e^{\int_0^s d(u)\,du}d_1(s)Y_n^2(s)\,ds + 2C\int_t^T e^{\int_0^s d(u)\,du}|Y_n(s)||Z_n(s)|\,ds$$

$$+ 2\int_t^T e^{\int_0^s d(u)\,du}|Y_n(s)|f(s, 0, 0)\,ds$$

(3.6) $$- 2\int_t^T e^{\int_0^s d(u)\,du}Y_n(s)\langle Z_n(s), dM(s)\rangle$$

$$\leq |\xi|^2 e^{\int_0^T d(s)\,ds} + C_\lambda \int_t^T e^{\int_0^s d(u)\,du} Y_n^2(s)\,ds$$

$$+ \frac{1}{2} \int_t^T e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds$$

$$+ \int_t^T e^{\int_0^s d(u)\,du} Y_n^2(s)\,ds + \int_t^T e^{\int_0^s d(u)\,du} |f(s,0,0)|^2\,ds$$

$$- 2 \int_t^T e^{\int_0^s d(u)\,du} Y_n(s)\langle Z_n(s), dM(s)\rangle.$$

Take expectation in (3.6) to obtain

$$E\big[|Y_n(t)|^2 e^{\int_0^t d(s)\,ds}\big] + \frac{1}{2} E\bigg[\int_t^T e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\bigg]$$

$$(3.7) \qquad \leq E\big[|\xi|^2 e^{\int_0^T d(s)\,ds}\big] + C_\lambda \int_t^T E\big[e^{\int_0^s d(u)\,du} Y_n^2(s)\big]\,ds$$

$$+ E\bigg[\int_t^T e^{\int_0^s d(u)\,du} |f(s,0,0)|^2\,ds\bigg].$$

Gronwall's inequality yields

$$(3.8) \qquad \sup_n \sup_{0\leq t\leq T} E\big[|Y_n(t)|^2 e^{\int_0^t d(s)\,ds}\big]$$

$$\leq C\bigg\{E\big[|\xi|^2 e^{\int_0^T d(s)\,ds}\big] + E\bigg[\int_0^T e^{\int_0^s d(u)\,du} |f(s,0,0)|^2\,ds\bigg]\bigg\}$$

and also

$$(3.9) \qquad \sup_n E\bigg[\int_0^T e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\bigg] < \infty.$$

Moreover, (3.6)–(3.9) further imply that there exists some constant C such that

$$E\bigg[\sup_{0\leq t\leq T} Y_n^2(t) e^{\int_0^t d(s)\,ds}\bigg]$$

$$\leq C + CE\bigg[\sup_{0\leq t\leq T} \int_0^t e^{\int_0^s d(u)\,du} Y_n(s)\langle Z_n(s), dM(s)\rangle\bigg]$$

$$\leq C + CE\bigg[\bigg(\int_0^T e^{2\int_0^s d(u)\,du} Y_n^2(s)\langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\bigg)^{1/2}\bigg]$$

$$\leq C + CE\bigg[\sup_{0\leq s\leq T} \big(e^{(1/2)\int_0^s d(u)\,du} |Y_n(s)|\big)$$

$$(3.10) \qquad \times \bigg(\int_0^T e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\bigg)^{1/2}\bigg]$$

$$\leq C + \frac{1}{2} E\Big[\sup_{0 \leq s \leq T} \big(e^{\int_0^s d(u)\,du} Y_n^2(s) \big) \Big]$$

$$+ C_1 E\Big[\int_0^T e^{\int_0^s d(u)\,du} \langle a(X(s)) Z_n(s), Z_n(s) \rangle \, ds \Big].$$

In view of (3.9), this yields

$$(3.11) \qquad \qquad \sup_n E\Big[\sup_{0 \leq t \leq T} Y_n^2(t) e^{\int_0^t d(s)\,ds} \Big] < \infty.$$

By (3.9) and (3.11), we can extract a subsequence n_k such that $Y_{n_k}(t) e^{(1/2) \int_0^t d(s)\,ds}$ converges to some $\hat{Y}(t)$ in $L^2(\Omega, L^\infty[0, T])$ equipped with the weak star topology and $Z_{n_k}(t) e^{(1/2) \int_0^t d(s)\,ds}$ converges weakly to some $\hat{Z}(t)$ in $L^2(\Omega_T; R)$, where $\Omega_T = [0, T] \times \Omega$. Observe that

$$Y_{n_k}(t) e^{(1/2) \int_0^t d(s)\,ds}$$

$$= e^{(1/2) \int_0^T d(s)\,ds} \xi + \int_t^T e^{(1/2) \int_0^s d(u)\,du} f_{n_k}(s, Y_{n_k}(s), Z_{n_k}(s)) \, ds$$

$$(3.12) \qquad - \frac{1}{2} \int_t^T e^{(1/2) \int_0^s d(u)\,du} Y_{n_k}(s) d(s) \, ds$$

$$- \int_t^T e^{(1/2) \int_0^s d(u)\,du} \langle Z_{n_k}(s), dM(s) \rangle.$$

Letting $k \to \infty$ in (3.12), using the monotonicity of f, following the same arguments as that in the proof of Proposition 2.3 in Darling and Pardoux in [8], we can show that the limit $(\hat{Y}, \hat{Z})$ satisfies

$$\hat{Y}(t) = e^{(1/2) \int_0^T d(s)\,ds} \xi$$

$$(3.13) \qquad + \int_t^T e^{(1/2) \int_0^s d(u)\,du} f\big(s, e^{-(1/2) \int_0^s d(u)\,du} \hat{Y}(s), e^{-(1/2) \int_0^s d(u)\,du} \hat{Z}(s) \big) \, ds$$

$$- \frac{1}{2} \int_t^T \hat{Y}(s) d(s) \, ds - \int_t^T \langle \hat{Z}(s), dM(s) \rangle.$$

Set

$$Y(t) = e^{-(1/2) \int_0^t d(u)\,du} \hat{Y}(t), \qquad Z(t) = e^{-(1/2) \int_0^t d(u)\,du} \hat{Z}(t).$$

An application of Itô's formula yields that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)) \, ds - \int_t^T \langle Z(s), dM(s) \rangle,$$

namely, (Y, Z) is a solution to the backward equation (3.1). The proof is complete.

$\square$

3.2. *BSDEs with random terminal times.* Let $f(t, y, z)$ satisfy (A.1)–(A.3) in Section 3.1. In this subsection, set $d(s) = -2d_1(s) + \delta d_2^2$. The following result provides existence and uniqueness for BSDEs with random terminal time. Let τ be a stopping time. Suppose ξ is $\mathcal{F}_\tau$-measurable.

THEOREM 3.2. *Assume $E[e^{\int_0^\tau d(s)\,ds}|\xi|^2] < \infty$, $E[\int_0^\tau K(s)\,ds] < \infty$ and*

$$(3.14) \qquad E\left[\int_0^\tau e^{\int_0^s d(u)\,du}|f(s,0,0)|^2\,ds\right] < \infty,$$

for some $\delta > \frac{1}{\lambda}$, where λ is the constant appeared in (2.1). Then, there exists a unique solution (Y, Z) to the BSDE

$$(3.15) \qquad Y(t) = \xi + \int_{\tau \wedge t}^\tau f(s, Y(s), Z(s))\,ds - \int_{\tau \wedge t}^\tau \langle Z(s), dM(s)\rangle.$$

Furthermore, the solution (Y, Z) satisfies

$$(3.16) \quad E\left[\int_0^\tau e^{\int_0^s d(u)\,du} Y^2(s)\,ds\right] < \infty, \qquad E\left[\int_0^\tau e^{\int_0^s d(u)\,du}|Z(s)|^2\,ds\right] < \infty,$$

and

$$(3.17) \qquad E\left[\sup_{0 \le s \le \tau}\{e^{\int_0^s d(u)\,du} Y^2(s)\}\right] < \infty.$$

PROOF. After the preparation of Theorem 3.1, the proof of this theorem is similar to that of Theorem 3.4 in [8], where $d_1(s)$, d_2 were both assumed to be constants. We only give a sketch highlighting the differences. For every $n \ge 1$, from Theorem 3.1 we know that the following BSDE has a unique solution (Y_n, Z_n) on $0 \le t \le n$:

$$(3.18) \quad Y_n(t) = E[\xi|\mathcal{F}_n] + \int_{\tau \wedge t}^{\tau \wedge n} f(s, Y_n(s), Z_n(s))\,ds - \int_{\tau \wedge t}^{\tau \wedge n} \langle Z_n(s), dM(s)\rangle.$$

Extend the definition of (Y_n, Z_n) to all $t \ge 0$ by setting

$$Y_n(t) = E[\xi|\mathcal{F}_n], \qquad Z_n(t) = 0 \qquad \text{for } t \ge n.$$

Then the extended (Y_n, Z_n) satisfies a bsde similar to (3.18) with f replaced by $\chi_{\{s \le n \wedge \tau\}} f(s, y, z)$. Let $n \ge m$. By Itô's formula, we have

$$|Y_n(t \wedge \tau) - Y_m(t \wedge \tau)|^2 e^{\int_0^{t \wedge \tau} d(s)\,ds}$$

$$+ \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u)\,du}\langle a(X(s))(Z_n(s) - Z_m(s)\chi_{\{s \le m \wedge \tau\}}),$$

$$Z_n(s) - Z_m(s)\chi_{\{s \le m \wedge \tau\}}\rangle\,ds$$

$$(3.19) \quad = e^{\int_0^{n \wedge \tau} d(s)\,ds}(E[\xi|\mathcal{F}_n] - E[\xi|\mathcal{F}_m])^2$$

$$- \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} |Y_n(s \wedge \tau) - Y_m(s \wedge \tau)|^2 d(s)\, ds$$

$$+ 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)$$

$$\times \big(f\big(s, Y_n(s \wedge \tau), Z_n(s \wedge \tau)\big)$$

$$- f\big(s, Y_m(s \wedge \tau), Z_m(s \wedge \tau)\big)\big)\, ds$$

$$+ 2 \int_{m \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)$$

$$\times f\big(s, Y_m(s \wedge \tau), Z_m(s \wedge \tau)\big)\, ds$$

$$- 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)\langle Z_n(s \wedge \tau), dM(s)\rangle$$

$$+ 2 \int_{t \wedge \tau}^{m \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)\langle Z_m(s \wedge \tau), dM(s)\rangle.$$

Choose δ_1, δ_2 such that $\frac{1}{\lambda} < \delta_1 < \delta$ and $0 < \delta_2 < \delta - \delta_1$. In view of the (A.1) and (A.2), we have

$$2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)$$

$$\times \big(f\big(s, Y_n(s \wedge \tau), Z_n(s \wedge \tau)\big) - f\big(s, Y_m(s \wedge \tau), Z_m(s \wedge \tau)\big)\big)\, ds$$

$$\leq -2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)^2 d_1(s)\, ds$$

(3.20)

$$+ \delta_1 d_2^2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)^2 ds$$

$$+ \frac{1}{\lambda \delta_1} \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s} d(u)\, du} \langle a(X(s))(Z_n(s) - Z_m(s)\chi_{\{s \leq m \wedge \tau\}}),$$

$$Z_n(s) - Z_m(s)\chi_{\{s \leq m \wedge \tau\}}\rangle ds.$$

On the other hand, by (A.3), it follows that

$$2 \int_{m \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big) f\big(s, Y_m(s \wedge \tau), Z_m(s \wedge \tau)\big)\, ds$$

(3.21)

$$\leq \delta_2 d_2^2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(Y_n(s \wedge \tau) - Y_m(s \wedge \tau)\big)^2 ds$$

$$+ \frac{1}{\delta_2 d_2^2} \int_{m \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\, du} \big(|f(s, 0, 0)| + K + K|E[\xi|\mathcal{F}_m]|\big)^2 ds.$$

Take expectation and utilize (3.19)–(3.21) to obtain

$$E\big[|Y_n(t \wedge \tau) - Y_m(t \wedge \tau)|^2 e^{\int_0^{t \wedge \tau} d(s)\,ds}\big]$$

$$+ \Big(1 - \frac{1}{\lambda \delta_1}\Big) E\bigg[\int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u)\,du} \langle a(X(s))(Z_n(s) - Z_m(s)\chi_{\{s \le m \wedge \tau\}}),$$

$$Z_n(s) - Z_m(s)\chi_{\{s \le m \wedge \tau\}}\rangle\,ds\bigg]$$

(3.22)

$$+ (\delta - \delta_1 - \delta_2) d_2^2 E\bigg[\int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u)\,du} (Y_n(s \wedge \tau) - Y_m(s \wedge \tau))^2\,ds\bigg]$$

$$\le E\big[e^{\int_0^{n \wedge \tau} d(s)\,ds} (E[\xi|\mathcal{F}_n] - E[\xi|\mathcal{F}_m])^2\big]$$

$$+ \frac{1}{\delta_2 d_2^2} E\bigg[\int_{m \wedge \tau}^{n \wedge \tau} e^{\int_0^{s \wedge \tau} d(u)\,du} (|f(s,0,0)| + K + K|E[\xi|\mathcal{F}_m]|)^2\,ds\bigg].$$

Since the right-hand side tends to zero as $n, m \to \infty$, we deduce that

$$\big\{\big(e^{(1/2)\int_0^{t \wedge \tau} d(s)\,ds} Y_n(t), e^{(1/2)\int_0^{t \wedge \tau} d(s)\,ds} Z_n(t)\big)\big\}$$

converges to some $(\hat{Y}, \hat{Z})$ in $M^2(0, \tau; R \times R^d)$. Furthermore, for every $t \ge 0$, $e^{(1/2)\int_0^{t \wedge \tau} d(s)\,ds} Y_n(t)$ converges in L^2. We may as well assume

$$(3.23) \qquad\qquad \hat{Y}(t) = \lim_{n \to \infty} e^{(1/2)\int_0^{t \wedge \tau} d(s)\,ds} Y_n(t)$$

for all t. Observe that for any $n \ge t \ge 0$,

$$e^{(1/2)\int_0^{t \wedge \tau} d(s)\,ds} Y_n(t)$$

$$= e^{(1/2)\int_0^{n \wedge \tau} d(s)\,ds} E[\xi|\mathcal{F}_n] + \int_{\tau \wedge t}^{n \wedge \tau} e^{(1/2)\int_0^{s \wedge \tau} d(u)\,du} f(s, Y_n(s), Z_n(s))\,ds$$

(3.24)

$$- \frac{1}{2}\int_{\tau \wedge t}^{n \wedge \tau} e^{(1/2)\int_0^{s \wedge \tau} d(u)\,du} Y_n(s)d(s)\,ds$$

$$- \int_{\tau \wedge t}^{n \wedge \tau} e^{(1/2)\int_0^{s \wedge \tau} d(u)\,du} \langle Z_n(s), dM(s)\rangle.$$

Letting $n \to \infty$ yields that

$$\hat{Y}(t) = e^{(1/2)\int_0^{\tau} d(s)\,ds} \xi + \int_{\tau \wedge t}^{\tau} e^{(1/2)\int_0^{s \wedge \tau} d(u)\,du} f\big(s, e^{-(1/2)\int_0^{s \wedge \tau} d(u)\,du}\hat{Y}(s),$$

(3.25)

$$e^{-(1/2)\int_0^{s \wedge \tau} d(u)\,du}\hat{Z}(s)\big)\,ds$$

$$- \frac{1}{2}\int_{\tau \wedge t}^{\tau} \hat{Y}(s)d(s)\,ds - \int_{\tau \wedge t}^{\tau} \langle \hat{Z}(s), dM(s)\rangle.$$

Put

$$Y(t) = e^{-(1/2)\int_0^{t \wedge \tau} d(s)\,ds} \hat{Y}(t), \qquad Z(t) = e^{-(1/2)\int_0^{t \wedge \tau} d(s)\,ds} \hat{Z}(t).$$

An application of Itô's formula and (3.25) yield that

$$(3.26) \qquad Y(t) = \xi + \int_{\tau \wedge t}^{\tau} f(s, Y(s), Z(s)) \, ds - \int_{\tau \wedge t}^{\tau} \langle Z(s), dM(s) \rangle.$$

Hence, (Y, Z) is a solution to the bsde (3.15) proving the existence. To obtain the estimates (3.16) and (3.17), we proceed to get an uniform estimate for $Y_n(s)$ and then pass to the limit. Let δ_1, δ_2 be chosen as before. Similar to the proof of (3.8), by Itô's formula, we have

$$|Y_n(t \wedge \tau)|^2 e^{\int_0^{t \wedge \tau} d(s) \, ds} + \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} \langle a(X(s)) Z_n(s), Z_n(s) \rangle \, ds$$

$$\leq |E[\xi | \mathcal{F}_n]|^2 e^{\int_0^{n \wedge \tau} d(s) \, ds} - \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} |Y_n(s)|^2 d(s) \, ds$$

$$- 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} d_1(s) Y_n^2(s) \, ds$$

$$+ 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} d_2 |Y_n(s)| |Z_n(s)| \, ds$$

$$+ 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} |Y_n(s)| |f(s, 0, 0)| \, ds$$

$$(3.27) \qquad - 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} Y_n(s) \langle Z_n(s), dM(s) \rangle$$

$$\leq |E[\xi | \mathcal{F}_n]|^2 e^{\int_0^{n \wedge \tau} d(s) \, ds} - \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} \delta d_2^2 Y_n^2(s) \, ds$$

$$+ \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} \delta_1 d_2^2 Y_n^2(s) \, ds$$

$$+ \frac{1}{\delta_1 \lambda} \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} \langle a(X(s)) Z_n(s), Z_n(s) \rangle \, ds$$

$$+ \int_{t \wedge \tau}^{n \wedge \tau} \delta_2 d_2^2 e^{\int_0^s d(u) \, du} Y_n^2(s) \, ds + \frac{1}{\delta_2 d_2^2} \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} |f(s, 0, 0)|^2 \, ds$$

$$- 2 \int_{t \wedge \tau}^{n \wedge \tau} e^{\int_0^s d(u) \, du} Y_n(s) \langle Z_n(s), dM(s) \rangle.$$

Recalling the choices of $d(s)$, δ_1 and δ_2, using Burkholder's inequality, we obtain from (3.27) that

$$E\left[\sup_{0 \leq t \leq n} |Y_n(t \wedge \tau)|^2 e^{\int_0^{t \wedge \tau} d(s) \, ds} \right]$$

$$(3.28) \qquad \leq E\left[|\xi|^2 e^{\int_0^{n \wedge \tau} d(s) \, ds} \right] + E\left[\int_0^{\tau} e^{\int_0^s d(u) \, du} \frac{1}{\delta_2 d_2^2} |f(s, 0, 0)|^2 \, ds \right]$$

$$+ 2CE\left[\left(\int_0^{n\wedge\tau} e^{2\int_0^s d(u)\,du} Y_n^2(s)\langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\right)^{1/2}\right]$$

$$\leq E\left[|\xi|^2 e^{\int_0^{n\wedge\tau} d(s)\,ds}\right] + E\left[\int_0^\tau e^{\int_0^s d(u)\,du} \frac{1}{\delta_2 d_2^2}|f(s,0,0)|^2\,ds\right]$$

$$+ \frac{1}{2}E\left[\sup_{0\leq t\leq n} |Y_n(t\wedge\tau)|^2 e^{\int_0^{t\wedge\tau} d(s)\,ds}\right]$$

$$+ C_1 E\left[\int_0^{n\wedge\tau} e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\right].$$

In view of (3.27), as the proof of (3.9), we can show that

$$(3.29) \qquad \sup_n E\left[\int_0^{n\wedge\tau} e^{\int_0^s d(u)\,du} \langle a(X(s))Z_n(s), Z_n(s)\rangle\,ds\right] < \infty.$$

(3.29) and (3.28) together with our assumptions on f and ξ imply

$$(3.30) \qquad \sup_n E\left[\sup_{0\leq t\leq n} |Y_n(t\wedge\tau)|^2 e^{\int_0^{t\wedge\tau} d(s)\,ds}\right] < \infty.$$

Applying Fatou lemma, (3.17) follows. □

3.3. *A particular case.* Let $f(x, y, z): R^d \times R \times R^d \to R$ be a Borel measurable function. Assume that f is continuous in y and satisfies:

(B.1) $(y_1 - y_2)(f(x, y_1, z) - f(x, y_2, z)) \leq -c_1(x)|y_1 - y_2|^2$, where $c_1(x)$ is a measurable function on R^d.

(B.2) $|f(x, y, z_1) - f(x, y, z_2)| \leq c_2|z_1 - z_2|$.

(B.3) $|f(x, y, z)| \leq |f(x, 0, z)| + c_3(x)(1 + |y|)$.

Let D be a bounded regular domain. Define

$$(3.31) \qquad\qquad \tau_D^x = \inf\{t \geq 0: X_x(t) \notin D\}.$$

Given $g \in C_b(R^d)$. Consider for each $x \in D$ the following BSDE:

$$
(3.32)
\begin{aligned}
Y_x(t) &= g(X_x(\tau_D^x)) + \int_{t\wedge\tau_D^x}^{\tau_D^x} f(X_x(s), Y_x(s), Z_x(s))\,ds \\
&\quad - \int_{t\wedge\tau_D^x}^{\tau_D^x} \langle Z_x(s), dM_x(s)\rangle,
\end{aligned}
$$

where $M_x(s)$ is the martingale part of $X_x(s)$. As a consequence of Theorem 3.2, we have the following theorem.

THEOREM 3.3. *Suppose $c_3 \in L^P(D)$ for $p > \frac{d}{2}$,*

$$E_x\left[\exp\left(\int_0^{\tau_D^x} \left(-2c_1(X(s)) + \delta c_2^2\right)ds\right)\right] < \infty,$$

for some $\delta > \frac{1}{\lambda}$ and

$$E_x\left[\int_0^{\tau_D^x} |f(X(s), 0, 0)|^2\, ds\right] < \infty.$$

The BSDE (3.32) *admits a unique solution* $(Y_x(t), Z_x(t))$. *Furthermore,*

(3.33)
$$\sup_{x \in \bar{D}} |Y_x(0)| < \infty.$$

4. Semilinear PDEs. As in previous sections, $(X(t), P_x)$ will denote the diffusion process defined in (2.5).

4.1. *Linear case.* Consider the second order differential operator

(4.1)
$$L_2 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} + q(x).$$

Let D be a bounded domain with regular boundary (w.r.t. the Laplace operator Δ) and $F(x)$ a measurable function satisfying

(4.2)
$$|F(x)| \leq C + C|q(x)|.$$

Take $\varphi \in C(\partial D)$ and consider the Dirichlet boundary value problem

(4.3)
$$\begin{cases} L_2 u = F, & \text{in } D, \\ u = \varphi, & \text{on } \partial D. \end{cases}$$

THEOREM 4.1. *Assume* (4.2) *and that there exists* $x_0 \in D$ *such that*

$$E_{x_0}\left[\exp\left(\int_0^{\tau_D^{x_0}} q(X(s))\, ds\right)\right] < \infty.$$

Then there is a unique, continuous weak solution u to the Dirichlet boundary value problem (4.3) *which is given by*

(4.4)
$$u(x) = E_x\left[\varphi(X(\tau_D^x)) + \int_0^{\tau_D^x} e^{\int_0^t q(X(s))\, ds} F(X(t)\, dt\right].$$

PROOF. Write

$$u_1(x) = E_x[\varphi(X(\tau_D^x))],$$

and

$$u_2(x) = E_x\left[\int_0^{\tau_D^x} e^{\int_0^t q(X(s))\, ds} F(X(t))\, dt\right].$$

We know from Theorem 4.3 in [6] that u_1 is the unique, continuous weak solution to the problem

(4.5)
$$\begin{cases} L_2 u = 0, & \text{in } D, \\ u = \varphi, & \text{on } \partial D. \end{cases}$$

So it is sufficient to show that u_2 is the unique, continuous weak solution to the following problem:

$$(4.6) \qquad \begin{cases} L_2 u = F, & \text{in } D, \\ u = 0, & \text{on } \partial D. \end{cases}$$

By Lemma 5.7 in [6] and Proposition 3.16 in [7], we know that u_2 belong to $C_0(D)$. Let $G_\beta, \beta \geq 0$ denote the resolvent operators of the generator L_2 on D with Dirichlet boundary condition, that is,

$$G_\beta f(x) = E_x \left[\int_0^{\tau_D^x} e^{-\beta t} e^{\int_0^t q(X(s))\, ds} f(X(t))\, dt \right].$$

By the Markov property, it is easy to see that

$$\beta(u_2(x) - \beta G_\beta u_2(x)) = \beta G_\beta F(x).$$

Since G_β is strong continuous, it follows that

$$\lim_{\beta \to \infty} \beta(u_2 - \beta G_\beta u_2) = F$$

in $L^2(D)$. This shows that $u_2 \in D(L_2) \subset W^{1,2}(D)$ and $L_2 u_2 = F$. The proof is complete. $\square$

4.2. Semilinear case.

Let $g(x, y, z): R^d \times R \times R^d \to R$ be a Borel measurable function that satisfies:

(C.1) $(y_1 - y_2)(g(x, y_1, z) - g(x, y_2, z)) \leq -k_1(x)|y_1 - y_2|^2$,
(C.2) $|g(x, y, z_1) - g(x, y, z_2)| \leq k_2|z_1 - z_2|$,
(C.3) $|g(x, y, z)| \leq C + C|q(x)|$,

where $k_1(x)$ is a measurable function and k_2, C are constants. Consider the semilinear Dirichlet boundary value problem

$$(4.7) \qquad \begin{cases} L_2 u = -g(x, u(x), \nabla u(x)), & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases}$$

where $\varphi \in C(\partial D)$.

THEOREM 4.2.   *Assume*

$$E_x \left[\exp\left(\int_0^{\tau_D^x} (q(X(s)) - 2k_1(X(s)) + \delta k_2^2)\, ds \right) \right] < \infty,$$

for some $\delta > \frac{1}{\lambda}$.
   The Dirichlet boundary value problem (4.7) *has a unique continuous weak solution.*

PROOF.   Set $f(x, y, z) = q(x)y + g(x, y, z)$. According to Theorem 3.3, for every $x \in D$ the following BSDE:

(4.8)
$$Y_x(t) = \phi(X_x(\tau_D^x)) + \int_{t \wedge \tau_D^x}^{\tau_D^x} f(X_x(s), Y_x(s), Z_x(s)) \, ds$$
$$- \int_{t \wedge \tau_D^x}^{\tau_D^x} \langle Z_x(s), dM_x(s) \rangle,$$

admits a unique solution $(Y_x(t), Z_x(t)), t \geq 0$. Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$. By the strong Markov property of X and the uniqueness of the BSDE (4.8), it is easy to see that

(4.9) $Y_x(t) = u_0(X_x(t)),$ $Z_x(t) = v_0(X_x(t)),$ $0 \leq t \leq \tau_D^x.$

Now consider the following problem:

(4.10)
$$\begin{cases} L_1 u = -f(x, u_0(x), v_0(x)), & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases}$$

where L_1 is defined as in Section 2. By Theorem 4.1, problem (4.10) has a unique continuous weak solution $u(x)$. As $u \in W^{1,2}(D)$, it follows from the decomposition of the Dirichlet process $u(X(t \wedge \tau_D^x))$ (see [12, 17]) that

(4.11)
$$u(X(t \wedge \tau_D^x)) = \varphi(X_x(\tau_D^x)) + \int_{t \wedge \tau_D^x}^{\tau_D^x} f(X_x(s), u_0(X(s \wedge \tau_D^x)), v_0(X(s \wedge \tau_D^x))) \, ds$$
$$- \int_{t \wedge \tau_D^x}^{\tau_D^x} \langle \nabla u(X(s \wedge \tau_D^x)), dM_x(s) \rangle$$
$$= \varphi(X_x(\tau_D^x)) + \int_{t \wedge \tau_D^x}^{\tau_D^x} f(X_x(s), Y_x(s), Z_x(s))) \, ds$$
$$- \int_{t \wedge \tau_D^x}^{\tau_D^x} \langle \nabla u(X(s \wedge \tau_D^x)), dM_x(s) \rangle.$$

Take conditional expectation both in (4.11) and (4.8) to discover
$$Y_x(t \wedge \tau_D^x) = u(X(t \wedge \tau_D^x))$$
$$= E\left[\varphi(X_x(\tau_D^x)) + \int_{t \wedge \tau_D^x}^{\tau_D^x} f(X_x(s), Y_x(s), Z_x(s)) \, ds \Big| \mathcal{F}_{t \wedge \tau_D^x} \right].$$

In particular, let $t = 0$ to obtain $u(x) = u_0(x)$. On the other hand, comparing (4.8) with (4.11) and by the uniqueness of decomposition of semimartingales, we deduce that
$$\int_{t \wedge \tau_D^x}^{\tau_D^x} \langle \nabla u(X(s \wedge \tau_D^x)), dM_x(s) \rangle = \int_{t \wedge \tau_D^x}^{\tau_D^x} \langle Z_x(s), dM_x(s) \rangle$$

for all t. By Itô's isometry, we have

$$E\left[\left(\int_0^\infty \langle(\nabla u(X(s)) - Z_x(s))\chi_{\{s<\tau_D^x\}}, dM_x(s)\rangle\right)^2\right]$$

$$= E\left[\left(\int_0^\infty \langle(\nabla u(X(s)) - v_0(X_x(s)))\chi_{\{s<\tau_D^x\}}, dM_x(s)\rangle\right)^2\right]$$

(4.12)

$$= E\left[\int_0^\infty \langle a(X_x(s))(\nabla u(X(s)) - v_0(X_x(s))),\right.$$

$$\left. \nabla u(X(s)) - v_0(X_x(s))\rangle\chi_{\{s<\tau_D^x\}}\, ds\right] = 0.$$

By Fubini theorem and the uniform ellipticity of the matrix $a(x)$, we deduce that

$$P_s^D(|\nabla u - v_0|^2) = E[|\nabla u(X(s)) - v_0(X_x(s))|^2\chi_{\{s<\tau_D^x\}}] = 0$$

a.e. in s with respect to the Lebesgue measure, where $P_s^D h(x) = E_x[h(X(t)), t < \tau_D^x]$. The strong continuity of the semigroup P_s^D, $s \geq 0$ implies that

(4.13)
$$|\nabla u - v_0|^2(x) = \lim_{s\to 0} P_s^D(|\nabla u - v_0|^2) = 0$$

a.e. Returning to problem (4.10), we see that u actually is a weak solution to the nonlinear problem:

(4.14)
$$\begin{cases} L_0 u = -f(x, u(x), \nabla u(x)), & \text{in } D, \\ u = \phi, & \text{on } \partial D. \end{cases}$$

Suppose $\bar{u}$ is another solution to the problem (4.14). By the decomposition of the Dirichlet process $\bar{u}(X_x(s))$, we find that $(\bar{u}(X_x(s)), \nabla\bar{u}(X_x(s)))$ is also a solution to the BSDE (4.8). The uniqueness of the BSDE implies that $\bar{u}(X_x(s)) = Y_x(s)$. In particular, $\bar{u}(x) = u_0(x) = Y_x(0)$. This proves the uniqueness. $\quad\square$

5. Semilinear elliptic PDEs with singular coefficients.

In this section, we study the semilinear second order elliptic PDEs of the following form:

(5.1)
$$\begin{cases} \mathcal{A}u(x) = -f(x, u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D, \end{cases}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} - \text{``div}(\hat{b}\cdot)\text{''} + q(x)$$

as in Section 2 and $\varphi \in C(\partial D)$. Consider the following conditions:

(D.1) $(y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J_1(x)|y_1 - y_2|^2$,
(D.2) $|f(x, y, z)| \leq C$,

where $J_1(x)$ is a measurable function, C is a constant. The following theorem is the main result of this section.

THEOREM 5.1. *Suppose that* (D.1), (D.2) *hold and*

$$
\begin{aligned}
E_x^0 \Bigg[\exp \Bigg\{ & \int_0^{\tau_D} \langle (a^{-1}b)(X^0(s)), dM^0(s) \rangle \\
& + \left(\int_0^{\tau_D} \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle \right) \circ r_{\tau_D} \\
(5.2) \quad & - \frac{1}{2} \int_0^{\tau_D} (b - \hat{b})a^{-1}(b - \hat{b})^*(X^0(s)) \, ds \\
& + \int_0^{\tau_D} q(X^0(s)) \, ds - 2 \int_0^{\tau_D} J_1(X^0(s)) \, ds \Bigg\} \Bigg] \\
& < \infty
\end{aligned}
$$

for some $x \in D$, *where* X^0 *is the diffusion generated by* L_0 *as in Section* 2 *and* τ_D *is the first exit time of* X^0 *from* D. *Then there exists a unique, continuous weak solution to equation* (5.1).

PROOF. Set

$$
\begin{aligned}
Z_t = \exp \Bigg\{ & \int_0^t \langle (a^{-1}b)(X^0(s)), dM^0(s) \rangle + \left(\int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle \right) \circ r_t \\
(5.3) \quad & - \frac{1}{2} \int_0^t (b - \hat{b})a^{-1}(b - \hat{b})^*(X^0(s)) \, ds + \int_0^t q(X^0(s)) \, ds \\
& \qquad\qquad\qquad\qquad\qquad\qquad - 2 \int_0^t J_1(X^0(s)) \, ds \Bigg\}.
\end{aligned}
$$

Put

$$
\hat{M}(t) = \int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle \qquad \text{for } t \geq 0.
$$

Let $R > 0$ so that $D \subset B_R := B(0, R)$. By Lemma 3.2 in [5] (see also [9]), there exits a bounded function $v \in W_0^{1,p}(B_R) \subset W_0^{1,2}(B_R)$ such that

$$
(\hat{M}(t)) \circ r_t = -\hat{M}(t) + N^v(t),
$$

where N^v is the zero energy part of the Fukushima decomposition for the Dirichlet process $v(X^0(t))$. Furthermore, v satisfies the following equation in the distributional sense:

$$
(5.4) \qquad\qquad \operatorname{div}(a\nabla v) = -2 \operatorname{div}(\hat{b}) \qquad \text{in } B_R.
$$

Note that by Sobolev embedding theorem, $v \in C(R^d)$ if we extend $v = 0$ on D^c. This implies that $\hat{M}$ and N^v are continuous additive functionals of X^0 in the strict sense (see [9, 12]), and so is $t \to (\hat{M}(t)) \circ r_t$. Thus,

$$
\left(\int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle \right) \circ r_t
$$

$$
= -\int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle + N^v(t)
$$

$$
= -\int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle + v(X^0(t)) - v(X^0(0)) - M^v(t)
$$

$$
= -\int_0^t \langle (a^{-1}\hat{b})(X^0(s)), dM^0(s) \rangle + v(X^0(t)) - v(X^0(0))
$$

$$
- \int_0^t \langle \nabla v(X^0(s)), dM^0(s) \rangle.
$$

Hence,

$$
\begin{aligned}
Z_t = {} & \frac{e^{v(X^0(t))}}{e^{v(X^0(0))}} \\
& \times \exp\Bigg(\int_0^t \langle a^{-1}(b - \hat{b} - a\nabla v)(X^0(s)), dM^0(s) \rangle - 2\int_0^t J_1(X^0(s))\, ds \\
& \qquad + \int_0^t \left(q - \frac{1}{2}(b - \hat{b} - a\nabla v)a^{-1}(b - \hat{b} - a\nabla v)^* \right)(X^0(s))\, ds \\
& \qquad\qquad + \int_0^t \left(\frac{1}{2}(\nabla v)a(\nabla v)^* - \langle b - \hat{b}, \nabla v \rangle \right)(X^0(s))\, ds \Bigg).
\end{aligned}
$$

(5.5)

Note that Z_t is well defined under P_x^0 for every $x \in D$. Set $h(x) = e^{v(x)}$. Introduce

$$
\begin{aligned}
\hat{\mathcal{A}} = {} & \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d [b_i(x) - \hat{b}_i(x) - (a\nabla v)_i(x)]\frac{\partial}{\partial x_i} \\
& - \langle b - \hat{b}, \nabla v \rangle(x) + \frac{1}{2}(\nabla v)a(\nabla v)^*(x) + q(x).
\end{aligned}
$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \hat{X}(t), \hat{P}_x, x \in R^d)$ be the diffusion process whose infinitesimal generator is given by

$$
\hat{L} = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d [b_i(x) - \hat{b}_i(x) - (a\nabla v)_i(x)]\frac{\partial}{\partial x_i}.
$$

It is known from [17] that $\hat{P}_x$ is absolutely continuous with respect to P_x^0 and

$$
\left. \frac{d\hat{P}_x}{dP_x^0} \right|_{\mathcal{F}_t} = \hat{Z}_t,
$$

where

$$\hat{Z}_t = \exp\left(\int_0^t \langle a^{-1}(b - \hat{b} - a\nabla v)(X^0(s)), dM^0(s)\rangle \right.$$

(5.6)

$$\left. - \int_0^t \left(\frac{1}{2}(b - \hat{b} - a\nabla v)a^{-1}(b - \hat{b} - a\nabla v)^* \right)(X^0(s)) \, ds \right).$$

Put

$$\hat{f}(x, y) = h(x)f(x, h^{-1}(x)y).$$

Then

$$(y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J_1(x)|y_1 - y_2|^2.$$

Consider the following nonlinear elliptic partial differential equation:

(5.7)
$$\begin{cases} \hat{\mathcal{A}}\hat{u}(x) = -\hat{f}(x, \hat{u}(x)), & \forall x \in D, \\ \hat{u}(x)|_{\partial D} = h(x)v(x), & \forall x \in \partial D. \end{cases}$$

In view of (5.5), condition (5.2) implies that

$$\hat{E}_x\left[\exp\left(-2\int_0^{\tau_D} J_1(X^0(s)) \, ds + \int_0^{\tau_D} q(X^0(s)) \, ds \right.\right.$$

(5.8)

$$\left.\left. + \int_0^{\tau_D} \left(\frac{1}{2}(\nabla v)a(\nabla v)^* - \langle b - \hat{b}, \nabla v\rangle \right)(X^0(s)) \, ds \right) \right] < \infty,$$

where $\hat{E}_x$ indicates that the expectation is taken under $\hat{P}_x$. From Theorem 4.2, it follows that equation (5.7) admits a unique weak solution $\hat{u}$. Set $u(x) = h^{-1}(x)\hat{u}(x)$. We will verify that u is a weak solution to equation (5.1).

Indeed, for $\psi \in W_0^{1,2}(D)$, since $\hat{u}(x) = h(x)u(x)$ is a weak solution to equation (5.7), it follows that

$$\frac{1}{2}\sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial[h(x)u(x)]}{\partial x_i} \frac{\partial[h^{-1}(x)\psi]}{\partial x_j} \, dx$$

$$- \sum_{i=1}^d \int_D [b_i(x) - \hat{b}_i(x) - (a\nabla v)_i(x)] \frac{\partial[h(x)u(x)]}{\partial x_i} h^{-1}(x)\psi \, dx$$

$$+ \int_D \langle b - \hat{b}, \nabla v(x)\rangle u(x)\psi(x) \, dx$$

$$- \frac{1}{2}\int_D (\nabla v)a(\nabla v)^*(x)u(x)\psi \, dx - \int_D q(x)u(x)\psi(x) \, dx$$

$$= \int_D f(x, u(x))\psi(x) \, dx.$$

Denote the terms on the left of the above equality, respectively, by T_1, T_2, T_3, T_4, T_5. Clearly,

$$T_1 = \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx - \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v}{\partial x_j} \psi \, dx$$

$$(5.9) \qquad + \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \psi}{\partial x_j} u(x) \, dx$$

$$- \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \psi u(x) \, dx.$$

Using chain rules, rearranging terms, it turns out that

$$T_2 + T_3 = -\sum_{i=1}^{d} \int_D b_i(x) \frac{\partial u(x)}{\partial x_i} \psi \, dx - \sum_{i=1}^{d} \int_D \hat{b}_i(x) \frac{\partial \psi}{\partial x_i} u(x) \, dx$$

$$(5.10) \qquad + \sum_{i=1}^{d} \int_D [\hat{b}_i(x) + (a\nabla v)_i(x)] \frac{\partial [\psi u(x)]}{\partial x_i} dx$$

$$- \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial \psi}{\partial x_i} u(x) \, dx + \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial v}{\partial x_i} u(x) \psi \, dx.$$

In view of (5.4),

$$\sum_{i=1}^{d} \int_D [\hat{b}_i(x) + (a\nabla v)_i(x)] \frac{\partial [\psi u(x)]}{\partial x_i} dx$$

$$(5.11)$$

$$= \frac{1}{2} \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial [\psi u(x)]}{\partial x_i} dx.$$

Thus,

$$T_2 + T_3 = -\sum_{i=1}^{d} \int_D b_i(x) \frac{\partial u(x)}{\partial x_i} \psi \, dx - \sum_{i=1}^{d} \int_D \hat{b}_i(x) \frac{\partial \psi}{\partial x_i} u(x) \, dx$$

$$(5.12) \qquad + \frac{1}{2} \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial [\psi u(x)]}{\partial x_i} dx$$

$$- \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial \psi}{\partial x_i} u(x) \, dx + \sum_{i=1}^{d} \int_D (a\nabla v)_i(x) \frac{\partial v}{\partial x_i} u(x) \psi \, dx.$$

After cancelations, it is now easy to see that

$$
\begin{aligned}
T_1 + T_2 + T_3 + T_4 + T_5 &= \frac{1}{2} \sum_{i,j=1}^{d} \int_D a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \\
&\quad - \sum_{i=1}^{d} \int_D b_i(x) \frac{\partial u(x)}{\partial x_i} \psi \, dx \\
&\quad - \sum_{i}^{d} \int_D \hat{b} \frac{\partial \psi}{\partial x_i} u(x) \, dx - \int_D q(x) u(x) \psi(x) \, dx \\
&= \int_D f(x, u(x)) \psi(x) \, dx.
\end{aligned}
$$

(5.13)

Since ψ is arbitrary, we conclude that u is a weak solution of equation (5.1). Suppose u is a continuous weak solution to equation (5.1). Put $\hat{u}(x) = h(x)u(x)$. Reversing the above process, we see that $\hat{u}$ is a weak solution to equation (5.7). The uniqueness of the solution of equation (5.1) follows from that of equation (5.7). □

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