THE REGULARIZING EFFECTS OF RESETTING IN A PARTICLE SYSTEM FOR THE BURGERS EQUATION

By Gautam Iyer[1] and Alexei Novikov[2]

Carnegie Mellon University and Pennsylvania State University

We study the dissipation mechanism of a stochastic particle system for the Burgers equation. The velocity field of the viscous Burgers and Navier–Stokes equations can be expressed as an expected value of a stochastic process based on noisy particle trajectories [Constantin and Iyer *Comm. Pure Appl. Math.* **3** (2008) 330–345]. In this paper we study a particle system for the viscous Burgers equations using a Monte–Carlo version of the above; we consider N copies of the above stochastic flow, each driven by independent Wiener processes, and replace the expected value with $\frac{1}{N}$ times the sum over these copies. A similar construction for the Navier–Stokes equations was studied by Mattingly and the first author of this paper [Iyer and Mattingly *Nonlinearity* **21** (2008) 2537–2553].

Surprisingly, for any finite N, the particle system for the Burgers equations shocks almost surely in finite time. In contrast to the full expected value, the empirical mean $\frac{1}{N}\sum_1^N$ does not regularize the system enough to ensure a time global solution. To avoid these shocks, we consider a resetting procedure, which at first sight should have no regularizing effect at all. However, we prove that this procedure prevents the formation of shocks for any $N \geq 2$, and consequently as $N \to \infty$ we get convergence to the solution of the viscous Burgers equation on long time intervals.

1. Introduction.

The viscous Burgers equation,

$$(1.1) \qquad \partial_t u + u\,\partial_x u - \nu\,\partial_x^2 u = 0,$$

has been studied extensively from several different points of view. Here $\nu > 0$ represents the viscosity, making the equation dissipative in nature. The inviscid Burgers equation [equation (1.1) with $\nu = 0$] is studied as the basic example of a scalar conservation law (see, e.g., [4, 5]). The Burgers equation is also linked to the KAM and Aubry–Mather theories [6, 10]. It is the simplest PDE that models the Euler and the Navier–Stokes nonlinearity. As such, it has been extensively studied as the first step in understanding the two key unresolved issues in fluid mechanics: turbulence and regularity of the Navier–Stokes equations in three dimensions. In the first category the objective is to characterize the statistical properties of turbulence [21]. In the second category the objective is to understand the regularizing

mechanism of dissipation [1, 12]. This paper falls into the latter category: we study the regularizing mechanism of a particle system for the Burgers equations, analogous to the particle system for the Navier–Stokes equations developed in [3, 9].

In [3], a class of second-order nonlinear transport equations (the Navier–Stokes and viscous Burgers in particular) were formulated as the average of a stochastic process along noisy particle trajectories. The formulation for the Navier–Stokes equations developed in [3] involves recovering the velocity u via the average of a nonlocal functional of the initial data. For the viscous Burgers' equation, however, the formulation is simpler. Explicitly, consider the stochastic flow

$$dX_t = u_t(X_t) + \sqrt{2\nu}\, dW_t \tag{1.2}$$

with initial data $X_0(a) = a$ for all $a \in \mathbb{R}$. Here W denotes a standard 1D Wiener process. If we require that the velocity u satisfies

$$u_t = \mathbf{E}[u_0 \circ (X_t^{-1})], \tag{1.3}$$

where $\mathbf{E}$ denotes the expected value with respect to the Wiener measure, then u satisfies[3] the viscous Burgers equation (1.1) and initial data u_0. We clarify that in (1.3) and subsequently, for any given time $t \geq 0$, X_t^{-1} denotes the spatial inverse of the diffeomorphism X_t. Namely, we know [16], Theorems 4.5.1, 4.6.5, that for regular drifts u, the stochastic flow X has a modification which is a stochastic flow of diffeomorphisms of $\mathbb{R}$. Replacing X with this modification if necessary, for any $t \geq 0$, we define X_t^{-1} to be the inverse of the diffeomorphism X_t. That is, for any $t \geq 0$, we have $X_t(X_t^{-1}(x)) = x$ surely for all $x \in \mathbb{R}$, and $X_t^{-1}(X_t(a)) = a$ surely for all $a \in \mathbb{R}$.

Observe that when $\nu = 0$, the system (1.2) and (1.3) is exactly the method of characteristics for the inviscid Burgers equation. Indeed trajectories of the flow X are now characteristics, and equation (1.3) states that the velocity is transported along characteristics. Thus, the $\nu > 0$ case could be viewed as a stochastic generalization of the method of characteristics: we transport the initial data along noisy characteristics, and then average with respect to the Wiener measure.

The usual Monte–Carlo method of solving (1.2) and (1.3) numerically [18, 19] is to replace the flow X_t with N different copies $X_t^{i,N}$, each driven by an independent Wiener process W_t^i, and replace the expected value in (1.3) by the empirical mean, $\frac{1}{N}\sum_{i=1}^{N}$. Explicitly, the system in question becomes

$$dX_t^{i,N} = u_t^N(X_t^{i,N})\, dt + \sqrt{2\nu}\, dW_t^i, \tag{1.4}$$

$$X_0^{i,N}(a) = a, \tag{1.5}$$

$$A_t^{i,N} = (X_t^{i,N})^{-1}, \tag{1.6}$$

$$u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ A_t^{i,N}, \tag{1.7}$$

---

[3]This is only valid for spatially periodic or decay at infinity boundary conditions.

where u_0 is the given initial data, W^i a sequence of independent Wiener processes and $\nu > 0$ the viscosity. As before, for any $t \geq 0$, $(X_t^{i,N})^{-1}$ denotes the spatial inverse of $X_t^{i,N}$. Throughout this paper, with the exception of Section 3, we impose periodic boundary conditions on the above, and assume the initial data is periodic with period 1.

For the Navier–Stokes equations, the particle system in [9] involves using a higher-dimensional Wiener process, and replacing (1.7) with the average of vorticity transport and the Biot–Savart law,

$$(1.8) \qquad\qquad \omega_t^N = \mathbf{E}[((\nabla X_t^{i,N})\omega_0) \circ A_t^{i,N}],$$

$$(1.9) \qquad\qquad u_t^N = (-\triangle)^{-1}\nabla \times \omega_t^N,$$

where $\omega_0 = \nabla \times u_0$ is the initial vorticity. In [9], the authors considered the system (1.4)–(1.6) and (1.8), (1.9), with spatially periodic boundary conditions, and proved global existence in two dimensions, local existence in three dimensions, convergence to the correct limit as $N \to \infty$ and described the asymptotic behavior for fixed N as $t \to \infty$.

Surprisingly, the techniques of [9] *fail* for the particle system for the Burgers equation [the system (1.4)–(1.7)]. Indeed, preliminary numerical simulations indicate that the system (1.4)–(1.7) shocks almost surely, in time *independent* of N. We provide a class of initial data for which we can prove (1.4)–(1.7) shocks almost surely. We, however, we are unable to analytically prove that the shock time is independent of N.

One heuristic explanation for the shock is as follows: this particle system (1.4)–(1.7) is dissipative *only* for short time ([9], Theorem 5.2). Once the system (1.4)–(1.7) stops dissipating energy, the growth from the nonlinear term should force the system to inherit properties of the *inviscid* Burgers equation, which shocks if the initial data is not monotonically nondecreasing.

We remark that the particle system for the Navier–Stokes equations [the system (1.4)–(1.6) and (1.8)–(1.9)] also dissipates energy *only* for short time ([9], Figure 1 and Theorem 5.2). However, no dissipation is required to prove 2D global existence for this system ([9], Theorem 3.5). This is because (1.8) and (1.9) are structurally similar to Euler equations, for which 2D global existence is well known [22] (see also [2, 17]). In contrast, however, the particle system (1.4)–(1.7) is structurally similar to the inviscid Burgers equation which is known to shock in finite time.

The natural approach one would expect to use "overcoming" the shocks in (1.4)–(1.7), would be to continue the system past shocks as weak solutions using an analogue of the Rankine–Hugoniot condition ([5], Section 3.4.1), and then prove that as $N \to \infty$, these weak solutions converge to the smooth solutions of the Burgers equation. This approach, however, is impossible to use as the stochastic PDE satisfied by u^N involves second-order terms, for which the classical techniques ([5], Section 3.4.1) will not work.

While the system (1.4)–(1.7) cannot be continued past shocks, the shocks can (surprisingly!) be "avoided with large probability" by resetting the Lagrangian maps. This is the main content of this paper. Namely, suppose we solve (1.4)–(1.7) for short time δ_t, and then replace the initial data with $u_{\delta_t}^N$, and restart the system (1.4)–(1.7) with this new initial data. Our main theorem shows that, if we repeat this procedure often enough, then we can avoid shocks on an arbitrarily large time interval, with probability arbitrarily close to 1.

Explicitly, consider the system

$$(1.10) \qquad dX_{k\delta_t,t}^{i,N} = u_t^N(X_{k\delta_t,t}^{i,N})\,dt + \sqrt{2\nu}\,dW_t^i,$$

$$(1.11) \qquad X_{k\delta_t,k\delta_t}^{i,N}(a) = a,$$

$$(1.12) \qquad A_{k\delta_t,t}^{i,N} = (X_{k\delta_t,t}^{i,N})^{-1},$$

$$(1.13) \qquad u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_{k\delta_t}^N \circ A_{k\delta_t,t}^{i,N},$$

where $k \in \mathbb{N}$, and t is always assumed to be in the interval $(k\delta_t, (k+1)\delta_t]$.

If δ_t is small enough, we show that solutions to this system exist on arbitrarily large time intervals, with probability arbitrarily close to one. Once existence with large probability is established, it is easy to show that as $N \to \infty$ these solutions converge to the smooth solutions of the viscous Burgers equation.

Before proceeding further, we remark that the fact that the shocks can be avoided by resetting is doubly unexpected! First, we know that the inviscid Burgers equations need to be regularized in order for them to have smooth solutions. The resetting procedure above should morally provide *no* regularization, as explained in Section 2! Second, the system (1.2) and (1.3) is Markovian; if we reset it at regular intervals (as above), then the new solution obtained will be no different from the original solution without resetting.

Fortunately, the system (1.4)–(1.7) is not Markovian, and if we reset often enough, the generic short time dissipative effect is strong enough to overcome the nonlinear growth, and with large probability prevents the formation of shocks. We observed numerically that even large resetting time δ_t (i.e., comparable to half the shock time of the inviscid Burgers system) is enough to ensure that the system (1.10)–(1.13) is globally well posed. With the techniques in this paper, however, we are only able to prove a global existence result for (1.10)–(1.13) when δ_t is small. The question for large δ_t remains open, and cannot be addressed using techniques in this paper.

Finally, we mention that our technique can be used to show global existence of the analogue of (1.10)–(1.13) for the Navier–Stokes equations in *two dimensions*. As this is already known [9], *without resetting*, we do not carry out the details here.

One interesting application would be to the three-dimensional Navier–Stokes equations. There are numerous results showing global existence of solutions to

the Navier–Stokes equations with small initial data. One new, interesting question that can be asked in this framework is the existence of solutions for arbitrary initial data, which are time global for some small (nonzero) probability. The McKean–Vlasov-type nonlinearity prevents us from asking this question for the stochastic Lagrangian formulation for the Navier–Stokes equations ([3], equations (2.3)–(2.6)). For the system (1.4)–(1.6) and (1.8), (1.9), the empirical mean $\frac{1}{N} \sum_1^N$ provides no regularization, so it is unlikely to expect small probability time global solutions. However, the repeatedly reset version of (1.4)–(1.6) and (1.8), (1.9) is free of the McKean–Vlasov nonlinearity, *and* is dissipative, making it a better candidate for small probability time global solutions. Unfortunately, there are obstructions in proving this result directly with the techniques used here, and we are working on addressing this issue.

The plan of this paper is as follows: in Section 2 we establish our notational convention, and prove our main theorem. This proof relies on a few lemmas, the proofs of which we postpone to Sections 4, 5 and 6. In Section 3, we provide an example showing that without resetting, the system (1.4)–(1.7) shocks almost surely. As mentioned earlier, once global existence is established the question of convergence as $N \to \infty$ is easily handled. We conclude the paper by studying this in Section 7.

2. The main theorem and its proof. Throughout this paper, we assume (Ω, Σ, P) is a probability space and use $\mathbf{E}$ to denote the expected value with respect to the probability measure P. Let $N \geq 2$ be a natural number (which will be fixed throughout this paper), $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration satisfying the usual conditions[4] on Ω and $W^1, \ldots, W^N$ be N independent Wiener processes adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We assume subsequently, without loss of generality, that $\nu = \frac{1}{2}$.

We use $C^k(\mathbb{T})$, to denote the space of all periodic functions on $\mathbb{R}$ (with period 1) which have k continuous derivatives. We use $L^p(\mathbb{T})$, $H^s(\mathbb{T})$ to be the Lebesgue p-space, and the Sobolev space of order s, respectively, consisting of periodic functions. When writing norms of functions in these spaces, we drop $\mathbb{T}$. For instance, we use the notation $\|u\|_{H^s}$ to denote the $H^s(\mathbb{T})$ norm of u.

We use a calligraphic script to denote the analogous spaces for processes, on random time intervals. Namely, given $t_0 \geq 0$, and a stopping time τ such that $\tau \geq t_0$ almost surely, we define

$$\mathcal{C}^k([t_0, \tau]; \mathbb{T}) = \{u \mid u \in C^0([t_0, \tau]; C^k(\mathbb{T})) \text{ a.s., and } u_{\tau \wedge t} \text{ is } \mathcal{F}_t \text{ adapted}\},$$

$$\mathcal{L}^p([t_0, \tau]; \mathbb{T}) = \{u \mid u \in C^0([t_0, \tau]; L^p(\mathbb{T})) \text{ a.s., and } u_{\tau \wedge t} \text{ is } \mathcal{F}_t \text{ adapted}\},$$

$$\mathcal{H}^s([t_0, \tau]; \mathbb{T}) = \{u \mid u \in C^0([t_0, \tau]; H^s(\mathbb{T})) \text{ a.s., and } u_{\tau \wedge t} \text{ is } \mathcal{F}_t \text{ adapted}\},$$

_____

[4]By "usual conditions" ([11], Definition 2.25) we mean that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous, and $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}_\infty$.

where we use the abbreviation "a.s." for almost surely. For convenience, if τ is any stopping time, not necessarily greater than or equal to t_0, we define $\mathcal{C}^k([t_0, \tau]; \mathbb{T}) = \mathcal{C}^k([t_0, \tau \vee t_0]; \mathbb{T})$, and similarly for $\mathcal{H}^s$, $\mathcal{L}^p$. To avoid confusion with the $C^k(\mathbb{T})$ norms, we explicitly use

$$\sup_{\omega \in \Omega} \sup_{t_0 \leq t \leq \tau} \|u_t(\omega)\|_{C^k}$$

to denote the $\mathcal{C}^k([t_0, \tau]; \mathbb{T})$ norm of u.

We clarify, $u \in \mathcal{C}^k([t_0, \tau]; \mathbb{T})$ means that there exists an event $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that $\forall \omega \in \Omega'$, $t \in [t_0, \tau(\omega)]$, $u_t(\omega) \in C^k(\mathbb{T})$ and $u_t(\omega)$ is continuous in t. Further, $\forall t \geq t_0$, $u_{\tau \wedge t}$ is $\mathcal{F}_t$-measurable. In words, $\mathcal{C}^k([t_0, \tau]; \mathbb{T})$ is the set of all processes which have a $C^k(\mathbb{T})$ valued, continuous paths modification and are defined on the random interval $[t_0, \tau]$.

Note that our spaces involve processes which are continuous in time almost surely, and we are not interested in quantifying any further regularity with respect to time. When the regularity in time needs to be quantified, the definition the analogous spaces is not as elementary (see, e.g., [13]).

Our main theorem shows, given any arbitrarily large T, we can make our resetting time δ_t small enough so that a regular solution to (1.10)–(1.13) exists up to time T with probability arbitrarily close to 1. In order to formulate our theorem precisely, we will need to define the notion of solutions to the reset system (1.10)–(1.13) with respect to a stopping time. This is our next definition.

DEFINITION 2.1. Let $t_0 \geq 0$, τ be a *spatially independent* stopping time such that $\tau \geq t_0$ almost surely, and u_{t_0} be a $C^1(\mathbb{T})$ valued $\mathcal{F}_{t_0}$-measurable random random variable. Suppose $u \in \mathcal{C}^1([t_0, \tau]; \mathbb{T})$ is a unique fixed point of the system

$$(2.1) \qquad X_{t_0,t}^{i,N}(a) = a + \int_{t_0}^{\tau \wedge t} u_s(X_{t_0,s}^{i,N}) \, ds + \int_{t_0}^{\tau \wedge t} dW_s^i,$$

$$(2.2) \qquad A_{t_0,t}^{i,N} = (X_{t_0,t}^{i,N})^{-1},$$

$$(2.3) \qquad u_t = \frac{1}{N} \sum_{i=1}^{N} u_{t_0} \circ A_{t_0,t}^{i,N}.$$

Then we define

$$S_{t_0,t}^{N,\tau} u_{t_0} = u_t.$$

For convenience, we adopt the convention that if τ is any stopping time (not necessarily satisfying $\tau \geq t_0$), we define $S_{t_0,t}^{N,\tau} u_{t_0} = S_{t_0,t}^{N,\tau \vee t_0} u_{t_0}$.

REMARK. Note that it is essential to assume τ does not depend on the spatial variable, as in this case if the drift u is spatially regular, then the process $X_{t_0,t}^{i,N}$ admits a modification which is a stochastic flow of diffeomorphisms. Hence the

spatial inverse $A_{t_0,t}^{i,N} = (X_{t_0,t}^{i,N})^{-1}$ is well defined. We will subsequently always assume our stopping times are spatially independent.

REMARK. In Lemma 2.7, we will show that $S_{t_0,\cdot}^{N,\tau}$ is well defined. Namely, if $k \geq 1$, and $u_{t_0} \in C^k(\mathbb{T})$ is $\mathcal{F}_{t_0}$-measurable, then Lemma 2.7 shows that there exists a (deterministic) time $t_1 > t_0$ such that the process u defined by $u_t = S_{t_0,t}^{N,t_1} u_{t_0}$ belongs to $\mathcal{C}^k([t_0, t_1]; \mathbb{T})$.

REMARK 2.2. Note that we can view the dependence of the operator $S_{t_0,\cdot}^{N,\tau}$ on the stopping time τ as a dependence only through the time interval of definition. Indeed, for any fixed δ_t and stopping time τ, $\mathcal{C}^1([0, \tau]; \mathbb{T})$ solutions of (2.1)–(2.3) are unique up to indistinguishability. This follows immediately from a standard argument using Gronwall's Lemma, and we omit the proof. Strong uniqueness implies that the operator $S_{t_0,\cdot}^{N,\tau}$ satisfies a compatibility condition: for $t_0 \geq 0$, consider two stopping times $\tau_1, \tau_2 \geq t_0$ such that $u_t^1 := S_{t_0,t}^{N,\tau_1} u_{t_0} \in \mathcal{C}^1([0, \tau_1]; \mathbb{T})$ and $u_2 := S_{t_0,t}^{N,\tau_2} u_{t_0} \in \mathcal{C}^1([0, \tau_2]; \mathbb{T})$, then $u^2 = u^1$ before $\tau_1 \wedge \tau_2$. That is, u^2 has a modification such that for all $t \geq t_0$, $u_{t \wedge \tau_1 \wedge \tau_2}^2 = u_{t \wedge \tau_1 \wedge \tau_2}^1$. Thus, when the time interval of definition is clear, we sometimes omit the stopping time τ as a superscript of our operator S.

For notational convenience, we omit the first superscript N for the remainder of this section. Given a (spatially independent) stopping time τ, a deterministic starting time $t_0 \geq 0$, a $C^1(\mathbb{T})$ valued $\mathcal{F}_{t_0}$-measurable initial data u_{t_0}, and a resetting time $\delta_t > 0$ small enough, we can define a $\mathcal{C}^1([t_0, \tau]; \mathbb{T})$ solution of the (stopped) system (1.10)–(1.13) iteratively by

$$(2.4) \quad \begin{cases} u_t = u_{t_0}, & \text{when } t = t_0, \\ u_t^{\delta_t} = S_{t_0+k\delta_t,t}^{\tau} u_{t_0+k\delta_t}^{\delta_t}, & \text{whenever } t \in (t_0 + k\delta_t, t_0 + (k+1)\delta_t] \\ & \text{for some } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

We are now ready to state our main theorem.

THEOREM 2.3. *Let $N > 1$, $T > 0$, $\varepsilon > 0$, $s > 6 + \frac{1}{2}$, and suppose*[5] *$u_0 \in H^s(\mathbb{T})$. Then there exists $\delta_T = \delta_T(T, \varepsilon, s, \|u_0\|_{H^s})$, independent of N, such that for all $\delta_t < \delta_T$, there exists a spatially independent stopping time τ such that $P(\tau > T) > 1 - \varepsilon$ and the process u defined by (2.4) (with $t_0 = 0$) is in the space $\mathcal{C}^6([0, \tau]; \mathbb{T})$.*

REMARK 2.4. The compatibility condition in Remark 2.2 allows us to discuss the notion of a maximal stopping time $\tau_{\max}$ for which the iterative procedure

---

[5]The theorem and proof remain unchanged if we instead assume that u_0 is a $H^s(\mathbb{T})$ valued, $\mathcal{F}_0$-measurable *bounded* random variable.

in (2.4) is well defined. Consequently, Theorem 2.3 will show that this maximal stopping time $\tau_{\max}$ is in fact at least T with probability at least $1 - \varepsilon$.

We emphasize that the operator $S_{t_0,t}$ is *not* a smoothing operator, which, as mentioned earlier, is part of the reason why Theorem 2.3 is surprising. We can see $S_{t_0,t}$ is not smoothing from the fact that (4.3), the stochastic partial differential equation (SPDE) satisfied by $u_t = S_{0,t}u_0$ is not dissipative [13]. One can immediately verify this as the diffusive term in (4.3) does not necessarily dominate the noise.

Another (perhaps more intuitive) way of understanding the regularity properties of $S_{t_0,t}$ is via time splitting. The S_{t_0} can be time split into two parts: $\bar{S}_{t_0}^1$, the nonlinear solution operator associated with the *inviscid* Burgers equations, and $\bar{S}_{t_0}^2$ the operator corresponding to resetting. By considering time split version of (1.10), one can see that $\bar{S}_{t_0}^2$ corresponds exactly to the operator

$$(2.5) \qquad \bar{S}_{t_0,t_0+\delta_t}^2 f(x) = \frac{1}{N} \sum_{j=1}^{N} f\big(x - (W_{t_0+\delta_t}^j - W_{t_0}^j)\big).$$

The operator $\bar{S}_{t_0}^1$ causes growth on the Fourier modes. It is well known that the damping provided by $\nu \partial_x^2$, for any $\nu > 0$, is enough to overcome this growth, and this gives us global existence for the viscous Burgers equations for any strictly positive viscosity. Thus if the operator $\bar{S}_{t_0}^2$ provides damping comparable to $\nu \partial_x^2$, then the usual methods can be used to prove Theorem 2.3. However, the operator $\bar{S}_{t_0}^2$ provides no damping, as can immediately be checked from (2.5): the operator norm of $\bar{S}_{t_0}^2$ is *exactly* 1 (surely) in all Sobolev and Hölder spaces. This is the main difficulty in proving Theorem 2.3.

We overcome this difficulty by considering the limit $v := \lim_{\delta_t \to 0} u^{\delta_t}$. It turns out that v satisfies a dissipative SPDE, and if the initial data is regular enough we obtain convergence in a strong norm of u^{δ_t} to v. This is the key to the proof of Theorem 2.3, and is formulated below.

LEMMA 2.5 (Key lemma). *Let[6] $N > 1$, $\beta \in \mathbb{N} \cup \{0\}$, $T_0 > t_0 \geq 0$, and τ be a (spatially independent) stopping time. Let u_{t_0} be a $C^{4+\beta}(\mathbb{T})$ valued $\mathcal{F}_{t_0}$-measurable random variable, and $u^{\delta_t} \in C^{4+\beta}([t_0, \tau]; \mathbb{T})$ be defined by (2.4). Let $v \in C^{4+\beta}([t_0, \tau]; \mathbb{T})$ be the solution of the SPDE*

$$(2.6) \qquad dv_t + v_t \, \partial_x v_t \, dt - \frac{1}{2} \partial_x^2 v_t \, dt + \frac{\partial_x v_t}{N} \sum_{j=1}^{N} dW_t^j = 0,$$

---

[6]The proof of this lemma never uses the assumption $N > 1$, and is valid even for $N = 1$. However, for $N = 1$, Lemma 2.5 is vacuously true as assumptions (2.7) and (2.8) will never be satisfied for nonconstant initial data.

with initial data $v|_{t=t_0} = u_{t_0}$, *and spatially periodic boundary conditions. Let* $\tau_0 = (\tau \vee t_0) \wedge T_0$, *and* U *be a constant such that*[7]

$$(2.7) \qquad \sup_{t_0 \leq t \leq \tau_0} \|u_t^{\delta_t}\|_{C^{4+\beta}} \leq U \qquad a.s.,$$

$$(2.8) \qquad \sup_{t_0 \leq t \leq \tau_0} \|v_t\|_{C^{4+\beta}} \leq U \qquad a.s.,$$

and let $w_t^{\delta_t} = u_{\tau \wedge t}^{\delta_t} - v_{\tau \wedge t}$. *Then there exists a constant* $C = C(\beta, U, T_0)$, *independent of* N, δ_t *and* τ, *such that*

$$(2.9) \qquad \sup_{t_0 \leq t \leq T_0} \mathbf{E}\|w_t^{\delta_t}\|_{H^\beta}^2 \leq C\delta_t^{1/2}.$$

Our main interest in this lemma will be for $\beta = 2$, as it will enable us to obtain a $\mathcal{C}^1([t_0, T_0]; \mathbb{T})$ bound on u from a $\mathcal{C}^1([t_0, T_0]; \mathbb{T})$ bound on v. A $C^1(\mathbb{T})$ bound is all that is needed to continue a solution locally, thus controlling the $C^1(\mathbb{T})$ norm of u with large probability, independent of δ_t, will prove our theorem. Since (2.6) is dissipative, uniform in time bounds of strong norms of v are readily obtained.

LEMMA 2.6. *Let* $N > 1$, $s \in \mathbb{N}$, $u_0 \in H^s(\mathbb{T})$. *There exists a process* $v \in \mathcal{H}^s([0, \infty); \mathbb{T})$ *which is a solution to the SPDE* (2.6) *with initial data* u_0 *and periodic boundary conditions. Further, there exists a constant* $V_s = V_s(s, \|u_0\|_{H^s})$ *such that*

$$(2.10) \qquad \sup_{t \geq 0} \|v_t\|_{H^s} \leq V_s$$

almost surely.

We remark that (2.10) is an almost sure bound on a strong norm of v. The reason we are able to obtain almost sure bounds is because if we "multiply by v and integrate by parts" (or more precisely, apply Itô's formula to $\|v_t\|_{L^2}^2$), we obtain an equation with no martingale part! This is carried out in detail in Section 5.

Lemmas 2.5 and 2.6 will now allow uniform in time control of a strong norm of u^{δ_t}. The only remaining ingredient is to obtain a $C^1(\mathbb{T})$ local existence result, and guarantee that inequality (2.7) is satisfied uniformly in δ_t.

---

[7]Assumptions (2.7) and (2.8) can be weakened slightly at the expense of a lengthier, more technical proof. The weakened assumptions, however, still require more than β derivatives. While replacing (2.7) and (2.8) with a condition involving only β derivatives would be of sufficient interest to warrant a more technical proof, reducing $4 + \beta$ to $4 + \beta - \varepsilon$ only obscures the heart of the matter. Since sufficient regularity on our initial data will guarantee (2.7) and (2.8) anyway, we assume they hold and avoid unnecessary technicalities.

LEMMA 2.7. *Suppose u_{t_0} is a $C^1(\mathbb{T})$ valued F_{t_0}-measurable random variable such that there exists a constant $U_1^0 \geq 0$ such that $\|u_{t_0}\|_{C^1} \leq U_1^0$ almost surely. There exists $T_0 = T_0(U_1^0) > t_0$ and a process $u^{\delta_t} \in C^1([t_0, T_0]; \mathbb{T})$ such that u^{δ_t} is a solution to (2.4) with $\tau = T_0$.*

If further for some $n \in \mathbb{N}$, u_{t_0} is a $C^n(\mathbb{T})$ valued F_{t_0}-measurable random variable, and there exists a constant $U_n^0 \geq 0$ such that $\|u_{t_0}\|_{C^n} \leq U_n^0$ almost surely, then u^{δ_t} is $C^n([0, T]; \mathbb{T})$, and further there exists a constant $U_n = U_n(U_n^0, n)$, independent of N and δ_t, such that

$$
(2.11) \qquad \sup_{t_0 \leq t \leq T_0} \|u_t^{\delta_t}\|_{C^n} \leq U_n \qquad a.s.
$$

for all $\delta_t < T_0$.

REMARK. The existence time T_0 above only depends on a the $C^1(\mathbb{T})$ norm of the initial data. However, on the existence interval, any additional regularity of the initial data is preserved.

We are now ready to prove the main theorem. (Lemmas 2.5, 2.6 and 2.7 will be proved in Sections 4, 5 and 6, resp.)

PROOF OF THEOREM 2.3. Let $\delta_T > 0$ be a small time, to be specified later, and let $\delta_t \in (0, \delta_T)$ be arbitrary. Given a stopping time τ, we define the operator $\mathscr{S}_{m\delta_t, t}^{\delta_t, \tau}$ by

$$
\mathscr{S}_{m\delta_t, t}^{\delta_t, \tau} = S_{k\delta_t, t}^{\tau} \circ S_{(k-1)\delta_t, k\delta_t}^{\tau} \circ \cdots \circ S_{(m+1)\delta_t, (m+2)\delta_t}^{\tau} \circ S_{m\delta_t, (m+1)\delta_t}^{\tau},
$$

where $k \in \mathbb{N}$ is such that $k\delta_t < t \leq (k+1)\delta_t$.

Let v_t be the solution of (2.6). By Lemma 2.6 and the Sobolev embedding theorem there is a constant V_1, such that

$$
\sup_{t \geq 0} \|v_t\|_{C^1} \leq V_1
$$

almost surely. Let $T_0 = T_0(2V_1)$ be the local existence time in Lemma 2.7; namely, for any initial data u_0 with $\|u_0\|_{C^1} \leq 2V_1$, and for any $\delta_t < T_0$, the process $\mathscr{S}_{0, \cdot}^{\delta_t, T_0} u_0$ is $C^1([0, T_0]; \mathbb{T})$. Without loss of generality we can assume that T_0 is an integer multiple of δ_t.

Note that our assumption $u_0 \in H^{13/2+}$, Lemma 2.6 and the Sobolev embedding theorem imply that assumption (2.8) is valid for $\beta = 2$ (in this case, the supremum can in fact be taken over all $t \in \mathbb{R}$). Similarly, Lemma 2.7 guarantees that assumption (2.7) is valid for $\beta = 2$ and all $\delta_t < T_0$. Thus Lemma 2.5 can be applied.

Let Ω_1 be the event $\{\|u_{T_0}^{\delta_t}\|_{C^1} \le 2V_1\}$. Then

$$P(\Omega_1) \ge P(\|u_{T_0}^{\delta_t} - v_{T_0}\|_{C^1} \le V_1)$$

$$\ge P\left(\|u_{T_0}^{\delta_t} - v_{T_0}\|_{H^2} \le \frac{V_1}{c_1}\right) \qquad \text{(Sobolev embedding)}$$

$$\ge 1 - \frac{c_1^2}{V_1^2}\mathbf{E}(\|u_{T_0}^{\delta_t} - v_{T_0}\|_{H^2}^2) \qquad \text{(Chebyshev's inequality)}$$

$$\ge 1 - \frac{C\delta_t^{1/2}}{V_1^2} \qquad \text{(Lemma 2.5)}$$

$$\ge 1 - \frac{C\delta_T^{1/2}}{V_1^2},$$

where the constant c_1 above is the constant arising in the Sobolev embedding theorem. An appropriate choice of δ_T will make $P(\Omega_1)$ arbitrarily close to 1. We clarify that while our bound on $P(\Omega_1)$ depends only on δ_T, the event Ω_1 depends on δ_t.

We define a stopping time τ_1 by

$$\tau_1(\omega) = \begin{cases} T_0, & \text{if } \omega \notin \Omega_1, \\ 2T_0, & \text{if } \omega \in \Omega_1. \end{cases}$$

Note that by Remark 2.2, we have $\mathscr{S}_{0,t}^{\delta_t,\tau_1}u_0 = \mathscr{S}_{0,t}^{\delta_t,T_0}u_0$ for all $t \in [0, T_0]$. Thus, by the semi-group property, and the fact that T_0 is an integer multiple of δ_t,

$$\mathscr{S}_{0,t}^{\delta_t,\tau_1}u_0 = \begin{cases} \mathscr{S}_{0,t}^{\delta_t,T_0}u_0, & \text{for } t \in [0, T_0], \\ \mathscr{S}_{T_0,t}^{\delta_t,\tau_1} \circ \mathscr{S}_{0,T_0}^{\delta_t,T_0}u_0, & \text{for } t \in (T_0, 2T_0], \end{cases}$$

as long as either side is defined. We claim that the right-hand side above is well defined and in $\mathcal{C}^6([0, \tau_1]; \mathbb{T})$. We see this as follows: first for $t \in [0, T_0]$, this is true by Lemma 2.7 and Remark 2.2. Now, for $\omega \notin \Omega_1$ and any $t \in [T_0, 2T_0]$, $\mathscr{S}_{T_0,t}^{\delta_t,\tau_1}$ is just the identity operator. Further for almost every $\omega \in \Omega_1$ we have $\mathscr{S}_{0,T_0}^{\delta_t,T_0}u_0(\omega) = u_{T_0}^{\delta_t}(\omega) \in C^6(\mathbb{T})$ and $\|u_{T_0}^{\delta_t}(\omega)\|_{C^1} \le 2V_1$. Thus for almost any $\omega \in \Omega_1$, and for every $t \in [T_0, 2T_0]$, $\mathscr{S}_{T_0,t}^{\delta_t,\tau_1}u_{T_0}^{\delta_t}(\omega) \in C^6(\mathbb{T})$ by Lemma 2.7.

Using Sobolev embedding, Chebyshev's inequality and Lemma 2.5 as above, we can find an event $\Omega_2 \subset \Omega_1$ such that $P(\Omega_2)$ is arbitrarily close to $P(\Omega_1)$. As before we define a stopping time τ_2 by

$$\tau_2(\omega) = \begin{cases} \tau_1(\omega), & \text{if } \omega \notin \Omega_2, \\ 3T_0, & \text{if } \omega \in \Omega_2, \end{cases}$$

and the solution $u_\cdot^{\delta_t} = \mathscr{S}_{0,\cdot}^{\delta_t,\tau_2}u_0 \in \mathcal{C}^6([0, \tau_3]; \mathbb{T})$. A finite iteration will complete the proof. $\square$

Finally we address the question of $N \to \infty$. For this purpose, we re-introduce the superscript of N to indicate the dependence on N of the process considered. Using techniques similar to [9], we show that the solution v^N of (2.6) converges to the solution of the viscous Burgers equation as $N \to \infty$.

PROPOSITION 2.8.   *Let v^N be the solution of* (2.6) *with initial data u_0, and u_t^b be the solution of the viscous Burgers equation* (1.1) *with the same initial data. If $u_0 \in H^s$, $s > \frac{3}{2}$, then for any $T > 0$, there exists a constant $C = C(T, s, \|u_0\|_{H^s})$ such that*

$$\sup_{t \in [0,T]} \mathbf{E}\|u_t^b - v_t^N\|_{L^2}^2 \leq \frac{C}{N}.$$

We prove Proposition 2.8 in Section 7. We conclude by remarking that Proposition 2.8, Lemma 2.5 and an argument similar to the proof of Theorem 2.3 will show that for small enough δ_t, as $N \to \infty$, u^{N,δ_t} converges to the same limit on an event of almost full probability.

3. Almost sure existence of shocks without resetting.

In this section we show that the system (1.4)–(1.7) develops shocks almost surely, for any N. The existence of shocks is simpler to prove if we work with monotone functions on $\mathbb{R}$, instead of periodic functions, and thus for this section only, we will work with (1.4)–(1.7) on $\mathbb{R}$ instead of on $\mathbb{T}$.

Let τ be a (spatially-independent) stopping time, and we interpret $\mathcal{C}^1([0, \tau]; \mathbb{R})$ solutions to (1.4)–(1.7), in the natural way [analogous to (2.1)–(2.3)]. The main result of this section shows that even if we stop "bad" realizations of (1.4)–(1.7), we can never continue solutions past the time $\frac{N}{\|\partial_x u_0\|_{L^\infty}}$, unless we introduce a regularizing mechanism.

PROPOSITION 3.1.   *Suppose $u_0 \in C^1(\mathbb{R})$ is a decreasing function, and let u be a $\mathcal{C}^1([0, \tau']; \mathbb{R})$ a solution of* (1.4)–(1.7) *with initial data u_0. Then, almost surely,*

$$(3.1) \qquad\qquad \tau' < \frac{N}{\|\partial_x u_0\|_{L^\infty}}.$$

REMARK 3.2.   The numerically observed shock time, in the periodic case, is independent of N, and it is of the order $\frac{1}{\|\partial_x u_0\|_{L^\infty}}$ with large probability. This indicates our bound (3.1) is far from optimal.

REMARK 3.3.   One can show[8] that as $N \to \infty$ the solution to (1.4)–(1.7) approaches the solution to (1.1) at a rate of $\frac{1}{\sqrt{N}}$. However, it is well known that the

---

[8]See, for instance, [9], Theorem 4.1, where the analogous result is proved for the Navier–Stokes equations.

solution to (1.1) is smooth for all time and no shock develops, provided the initial data is, for instance, C^1 and bounded [5].

The numerics mentioned in Remark 3.2, however, indicate that no matter how large N is, the system (1.4)–(1.7) will only be a good approximation to the true solution of (1.1) for short time, in the order of $\frac{1}{\|\partial_x u_0\|_{L^\infty}}$.

REMARK 3.4.   Monotonicity of the initial data u_0 is precisely the condition that constrained us to work on the line instead of on the torus. Specifically, the assumption $\partial_x u_0(x) < 0$ for arbitrary $x \in \mathbb{R}$ simplifies the proof of 3.1 considerably. Numerics, however, indicate that this monotonicity assumption is redundant, and (1.4)–(1.7) develops shocks for arbitrary (periodic) initial data.

PROOF OF PROPOSITION 3.1.   Assume for simplicity, and without loss of generality, that $\|\partial_x u_0\|_{L^\infty} = -\partial_x u_0(0) = 1$. Let the stopping time τ be the first time $t \le \tau'$ such that $\partial_x X_t^{1,N}(0) = 0$. Explicitly,

$$\tau = \tau' \wedge \inf\{t \mid \partial_x X_t^{1,N}(0) = 0\}.$$

We will first show that, $\tau \le N$, almost surely.

Differentiating (1.4) in space gives

$$(3.2) \qquad d(\partial_x X_t^{1,N}) = \partial_x u_t^N|_{X_t^{1,N}} \partial_x X_t^{1,N}\, dt,$$

for $t < \tau'$, almost surely. Here, our notation $\partial_x u_t^N|_{X_t^{1,N}}$ means

$$\partial_x u_t^N|_{X_t^{1,N}}(x) = \partial_x u_t^N(X_t^{1,N}(x)).$$

Differentiating equation (1.7) in space, we obtain

$$(3.3) \qquad \begin{aligned} \partial_x u_t^N|_{X_t^{1,N}} &= \frac{1}{N} \sum_{i=1}^N \partial_x (u_0 \circ A_t^{i,N})|_{X_t^{1,N}} \\ &= \frac{1}{N} \sum_{i=1}^N \partial_x u_0|_{A_t^{i,N} \circ X_t^{1,N}}\, \partial_x A_t^{i,N}|_{X_t^{1,N}}, \end{aligned}$$

for $t < \tau'$ almost surely. Since by the chain rule,

$$(3.4) \qquad \partial_x A_t^{1,N}|_{X_t^{1,N}} \partial_x X_t^{1,N} = \partial_x(A_t^{1,N} \circ X_t^{1,N}) = 1,$$

multiplying (3.3) by $\partial_x X_t^{1,N}$ gives

$$(3.5) \qquad \begin{aligned} &\partial_x u_t^N|_{X_t^{1,N}}\, \partial_x X_t^{1,N} \\ &= \frac{1}{N}\left[\partial_x u_0 + \sum_{i=2}^N \partial_x u_0|_{A_t^{i,N} \circ X_t^{1,N}} \cdot \partial_x A_t^{i,N}|_{X_t^{1,N}}\, \partial_x X_t^{1,N} \right] \end{aligned}$$

for $t < \tau'$ almost surely.

Note that for a C^1 solution of the system (1.4)–(1.7), for all i, the flow $X_t^{i,N} : \mathbb{R} \to \mathbb{R}$ is homotopic to the identity map via C^1 diffeomorphisms of $\mathbb{R}$. The same is true for the inverse inverse $A_t^{i,N}$, and thus $\partial_x A_t^{i,N}|_{X_t^{1,N}} \partial_x X_t^{1,N} > 0$. Finally, since u_0 is assumed to be decreasing, we know that $\partial_x u_0 < 0$, and thus equations (3.2) and (3.5) yield

$$\partial_t \partial_x X_t^{1,N}(0) < \frac{-1}{N}$$

for $t < \tau'$ almost surely. This (ordinary) differential inequality, along with the fact that $\partial_x X_0^{1,N} = 1$, necessitates $\tau < N$ almost surely.

Now, by definition of τ, and continuity (in time) of $\partial_x X_t^{1,N}$,

$$(3.6) \qquad \lim_{t \to \tau^-} \partial_x X_t^{1,N}(0) = 0$$

on the event $\{\tau < \tau'\}$. From (3.5) and the chain rule we have

$$\partial_x u_t^N|_{X_t^{1,N}} = \frac{1}{N}\left[\frac{\partial_x u_0}{\partial_x X_t^{1,N}} + \sum_{i=2}^{N} \partial_x u_0|_{A_t^{i,N} \circ X_t^{1,N}} \cdot \partial_x A_t^{i,N}|_{X_t^{1,N}} \right]$$

for $t < \tau'$ almost surely. Note that all the terms on the right-hand side (3.5) have the same sign. Thus if one of these terms approaches $-\infty$, then necessarily the entire right-hand side approaches $-\infty$. Equation (3.6) immediately implies the first term approaches $-\infty$ at $x = 0$ on the event $\{\tau < \tau'\}$. Hence, on this event we have

$$\lim_{t \to \tau^-} \|\partial_x u_t\|_{L^\infty} \geq - \lim_{t \to \tau^-} \partial_x u_t^N(X_t^{1,N}(0)) = \infty,$$

almost surely. Consequently, if $u \in \mathcal{C}^1([0, \tau']; \mathbb{R})$, we must have $P(\tau < \tau') = 0$. Hence $\tau' = \tau < N$ almost surely. $\square$

4. Proof of the key lemma (Lemma 2.5). In this section we prove convergence of u^{δ_t} to v as $\delta_t \to 0$. The basic idea is to show that the velocity in our reset system (1.10)–(1.13) satisfies the limiting SPDE (2.6) with a small error which is controlled as $\delta_t \to 0$.

By shifting time, we may assume without loss of generality that $t_0 = 0$. Further replacing τ with $\tau \wedge T_0$ if necessary, we may assume $\tau = \tau_0 \leq T_0$. Throughout this section, we adopt the convention that $t_0 = 0$, and N, β, T_0, τ, τ_0, u_0, u^{δ_t}, v and U are as in the statement of Lemma 2.5. We also assume the processes $X_{k\delta_t, \cdot}^i$, $A_{k\delta_t, \cdot}^i$ are all as in (1.10)–(1.13), and for notational convenience, we will omit the N and δ_t as superscripts throughout this section.

We need a few lemmas before we can prove Lemma 2.5. In our first lemma we determine an SPDE satisfied by u on the interval $(k\delta_t, (k + 1)\delta_t]$.

LEMMA 4.1.   *We define the process u^i to be the ith summand in* (1.13). *Explicitly,*

(4.1) $$u_t^i = \begin{cases} u_0, & for\ t = 0, \\ u_{k\delta_t} \circ A_{k\delta_t, t}^i, & for\ t \in (\tau \wedge k\delta_t, \tau \wedge (k+1)\delta_t]. \end{cases}$$

Then for all $i \in \{1, \dots, N\}$, the process $u^i \in \mathcal{C}^{4+\beta}([0, \tau]; \mathbb{T})$, and satisfies the SPDE

(4.2)
$$\chi_{\{\tau \geq k\delta_t\}}(u_{\tau \wedge t}^i - u_{k\delta_t}) + \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \left(u_s\, \partial_x u_s^i - \frac{1}{2} \partial_x^2 u_s^i \right) ds$$
$$+ \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x u_s^i\, dW_s^i = 0$$

on the interval $t \in [k\delta_t, (k+1)\delta_t]$. Similarly, the process $u \in \mathcal{C}^{4+\beta}([0, \tau]; \mathbb{T})$, and satisfies the SPDE

(4.3)
$$u_{\tau \wedge t} - u_{\tau \wedge k\delta_t} + \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \left(u_s \partial_x u_s - \frac{1}{2} \partial_x^2 u_s \right) ds$$
$$+ \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \frac{1}{N} \sum_{j=1}^{N} \partial_x u_s^j\, dW_s^j = 0$$

on the interval $t \in [k\delta_t, (k+1)\delta_t]$.

REMARK 4.2.   A more intuitive, though less precise, way of phrasing the SPDEs (4.2) and (4.3) would be to say for $t \in (\tau \wedge k\delta_t, \tau \wedge (k+1)\delta_t]$, u, u^i satisfy the SPDEs

$$du_t^i + u_t\, \partial_x u_t^i\, dt - \frac{1}{2} \partial_x^2 u_t^i\, dt + \partial_x u_t^i\, dW_t^i = 0 \qquad \text{for all } i \in \{1, \dots, N\},$$

$$du_t + u_t\, \partial_x u_t\, dt - \frac{1}{2} \partial_x^2 u_t\, dt + \frac{1}{N} \sum_{j=1}^{N} \partial_x u_t^j\, dW_t^j = 0$$

with initial data $u^i|_{t=k\delta_t} = u_{k\delta_t}$ and $u|_{t=k\delta_t} = u_{k\delta_t}$.

PROOF OF LEMMA 4.1.   From [3, 9] (see also [14, 20]) we know that when $\tau \equiv \infty$, the process $A_{k\delta_t, \cdot}^i$ satisfies the SPDE

$$dA_{k\delta_t, t}^i + u_t\, \partial_x A_{k\delta_t, t}^i\, dt - \frac{1}{2} \partial_x^2 A_{k\delta_t, t}^i\, dt + \partial_x A_{k\delta_t, t}^i\, dW_t^i = 0$$

on the time interval $(k\delta_t, (k+1)\delta_t]$. Writing down an integral version of this in the presence of a stopping time, equations (4.2) and (4.3) follow immediately from (2.3) and (4.1) by a direct application of Itô's formula.

To check[9] $u, u^1, \ldots, u^N \in \mathcal{C}^{4+\beta}([0, \tau]; \mathbb{T})$, note that continuity in time is immediate. Further, the spatial regularity of u has already been assumed in the statement of Lemma 2.5. For $u^1, \ldots, u^N$, note that the τ and the noise are spatially independent in (1.10), and it immediately shows that each $X^i_{k\delta_t, \cdot}$ (and hence each $A^i_{k\delta_t, \cdot}$) is as spatially regular as u, which in turn shows that each $u^i \in \mathcal{C}([0, \tau]; \mathbb{T})$. $\square$

Now we show that with a small error u satisfies the SPDE (2.6) stopped at τ, and obtain bounds on this error. Let $\mathbf{E}_{\mathcal{F}_{k\delta_t}} Y$ denote the conditional expectation of Y given $\mathcal{F}_{k\delta_t}$. Given any process f, and a stopping time τ, we define the stopped increment $\Delta^\tau_k f$ by

$$\Delta^\tau_k f = f_{\tau \wedge (k+1)\delta_t} - f_{\tau \wedge k\delta_t}.$$

For the (deterministic) process $f_t = t$, we define $\Delta^\tau_k t$ by

$$\Delta^\tau_k t = (\tau \wedge (k+1)\delta_t) - (\tau \wedge k\delta_t) = \begin{cases} \delta_t, & \text{if } \tau \geq (k+1)\delta_t, \\ \tau - k\delta_t, & \text{if } k\delta_t \leq \tau < (k+1)\delta_t, \\ 0, & \text{if } \tau < k\delta_t. \end{cases}$$

Finally, let L be the (nonlinear) operator defined by

$$Lu = u\partial_x u - \tfrac{1}{2}\partial_x^2 u.$$

LEMMA 4.3. *Suppose* (2.7) *holds for some* $\beta \in \mathbb{N} \cup \{0\}$, *and let* ε'_k *be defined by*[10]

$$(4.4) \qquad \varepsilon'_k = \Delta^\tau_k u + Lu_{k\delta_t} \Delta^\tau_k t + \partial_x u_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N \Delta^\tau_k W^j\right).$$

Then there exists a constant $C = C(\beta, U, T_0)$ *(independent of* N, k, δ_t *and* τ*) such that for all* $\delta_t \leq T_0$ *and* $k \leq \frac{T_0}{\delta_t}$ *we have*

$$(4.5) \qquad \sup_{x \in \mathbb{T}} \mathbf{E}|\partial_x^\beta \varepsilon'_k(x)|^2 \leq C\delta_t^2,$$

$$(4.6) \qquad \sup_{x \in \mathbb{T}} \mathbf{E}|\mathbf{E}_{\mathcal{F}_{k\delta_t}} \partial_x^\beta \varepsilon'_k(x)|^2 \leq C\delta_t^3.$$

REMARK. Since u and all derivatives of u are a priori uniformly bounded almost surely, the proof of this lemma is straightforward. Without this a priori bound, we would only obtain similar bounds on $\mathbf{E}\|\partial_x^\beta \varepsilon'_{k\delta_t}\|_{L^2}^2$ and $\mathbf{E}\|\mathbf{E}_{\mathcal{F}_{k\delta_t}} \partial_x^\beta \varepsilon'_{k\delta_t}\|_{L^2}^2$, which are still sufficient for Lemma 2.5.

---

[9]The spatial regularity of $u, u^1, \ldots, u^n$ follows directly from an assumption only on the initial data, and a standard iteration argument. This is contained in Section 6. However, for Lemma 4.1, an iteration argument is unnecessary because of assumption (2.7).

[10]In equation (4.4), technically, $Lu_{k\delta_t}$ is not defined when $\tau < k\delta_t$. However, in this case, $\Delta^\tau_k t = 0$, so the value of $Lu_{k\delta_t}$ does not matter. We use this convention subsequently without further mention.

PROOF OF LEMMA 4.3.   We assume throughout this section that C is a constant only depending on U and T_0 which could change from line to line. Note first that assumption (2.7) and equation (2.1) immediately imply that for any $i \in \{1, \ldots, N\}$,

$$\sup_{0 \le t \le \tau_0} \|\partial_x X_t^i\|_{C^{3+\beta}} \le C \quad \text{and} \quad \sup_{0 \le t \le \tau_0} \|\partial_x A_t^i\|_{C^{3+\beta}} \le C$$

almost surely. Now equation (2.3) immediately yields the same bound for u, independent of N. Thus, making U larger if necessary, we may assume without loss of generality that (2.7) holds for all the processes $u, u^1, u^2, \ldots, u^n$.

For any $k \in \mathbb{N} \cup \{0\}$, $t \in (k\delta_t, (k+1)\delta_t]$, $n \le \beta + 2$, differentiating (4.3) n times gives

$$\partial_x^n u_{\tau \wedge t} - \partial_x^n u_{\tau \wedge k\delta_t} = -\int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x^n L u_s \, ds - \frac{1}{N} \sum_{j=1}^N \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x^{n+1} u_s^j \, dW_s^j,$$

and hence

$$\mathbf{E}|\partial_x^n u_{\tau \wedge t} - \partial_x^n u_{\tau \wedge k\delta_t}|^2$$

$$\le 2\mathbf{E}\left(\int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x^n L u_s \, ds\right)^2$$

$$+ 2\mathbf{E}\left(\frac{1}{N} \sum_{j=1}^N \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x^{n+1} u_s^j \, dW_s^j\right)^2.$$

Note that

$$\int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x^{n+1} u_s^j \, dW_s^j = \int_{k\delta_t}^t \chi_{\{k\delta_t \le \tau\}} \chi_{\{s \le \tau\}} \partial_x^{n+1} u_s^j \, dW_s^j.$$

Thus using (2.7) for both u and u^i, for any $t \in (k\delta_t, (k+1)\delta_t]$ we have

$$(4.7) \qquad \sup_{x \in \mathbb{T}} \mathbf{E}|\partial_x^n u_{\tau \wedge t}(x) - \partial_x^n u_{\tau \wedge k\delta_t}(x)|^2 \le C\left(\delta_t^2 + \frac{1}{N^2} \sum_{j=1}^N \delta_t\right) \le C\delta_t,$$

where as usual the constant C may change from line to line, provided it only depends on β, U and T_0.

Similarly, using (4.2) and the above argument we have

$$(4.8) \qquad \sup_{x \in \mathbb{T}} \mathbf{E}\chi_{\{k\delta_t \le \tau\}}|\partial_x^n u_{\tau \wedge t}^i(x) - \partial_x^n u_{k\delta_t}^i(x)|^2 \le C\delta_t$$

for any $t \in (k\delta_t, (k+1)\delta_t]$ and $n \le 2 + \beta$.

Now, from the definition of ε' and equation (4.3) we have

$$
\varepsilon_k' = -\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t} Lu_s \, ds - \frac{1}{N}\sum_{j=1}^{N}\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t} \partial_x u_s^j \, dW_s^j
$$

$$
+ Lu_{\tau\wedge k\delta_t}\Delta_k^\tau t + \partial_x u_{\tau\wedge k\delta_t}\left(\frac{1}{N}\sum_{j=1}^{N}\Delta_k^\tau W^j\right)
$$

$$
(4.9) \qquad = \int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t}(Lu_{k\delta_t} - Lu_s)\, ds
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t}(\partial_x u_{k\delta_t} - \partial_x u_s^j)\, dW_s^j
$$

almost surely. For the Itô integrals in the second term above,

$$
\mathbf{E}_{\mathcal{F}_{k\delta_t}}\left(\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t}(\partial_x u_{k\delta_t} - \partial_x u_s^j)\, dW_s^j\right)
$$

$$
= \mathbf{E}_{\mathcal{F}_{k\delta_t}}\left(\int_{k\delta_t}^{(k+1)\delta_t}\chi_{\{\tau\geq k\delta_t\}}\chi_{\{s\leq\tau\}}(\partial_x u_{k\delta_t} - \partial_x u_{\tau\wedge s}^j)\, dW_s^j\right) = 0,
$$

and hence

$$
\mathbf{E}|\mathbf{E}_{\mathcal{F}_{k\delta_t}}\partial_x^\beta \varepsilon_k'|^2 = \mathbf{E}\left(\partial_x^\beta\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t}(Lu_{k\delta_t} - Lu_s)\, ds\right)^2
$$

$$
\leq \delta_t\int_{k\delta_t}^{(k+1)\delta_t}\mathbf{E}\big[\chi_{\{k\delta_t\leq\tau\}}\chi_{\{s\leq\tau\}}\partial_x^\beta(Lu_{k\delta_t} - Lu_{\tau\wedge s})\big]^2 ds
$$

$$
= \delta_t\int_{k\delta_t}^{(k+1)\delta_t}\mathbf{E}\big[\chi_{\{s\leq\tau\}}\partial_x^\beta(Lu_{\tau\wedge k\delta_t} - Lu_{\tau\wedge s})\big]^2 ds
$$

$$
\leq C\delta_t^3,
$$

where the last inequality follows from (4.7) with $n = 2 + \beta$. This proves (4.6).

For (4.5), note that the expected value of the square of the first term in (4.9) has already been bounded by $C\delta_t^3 < C\delta_t^2$. For the second term, the Itô isometry gives

$$
\mathbf{E}\left(\frac{1}{N}\sum_{j=1}^{N}\int_{\tau\wedge k\delta_t}^{\tau\wedge(k+1)\delta_t}\partial_x^\beta(\partial_x u_{k\delta_t} - \partial_x u_s^j)\, dW_s^j\right)^2
$$

$$
= \frac{1}{N}\sum_{j=1}^{N}\int_{k\delta_t}^{(k+1)\delta_t}\mathbf{E}\big[\chi_{\{k\delta_t\leq\tau\}}\chi_{\{s\leq\tau\}}\partial_x^\beta(\partial_x u_{k\delta_t} - \partial_x u_s^j)\big]^2 ds,
$$

and using (4.8) with $n = 1 + \beta$ the proof is complete. $\quad\square$

We now prove that a time split version of the SPDE (2.6) satisfies the same error estimates as in Lemma 4.3.

LEMMA 4.4.   *Suppose* (2.8) *holds for some* $\beta \in \mathbb{N} \cup \{0\}$, *and let* ε_k'' *be defined by*

$$(4.10) \qquad \varepsilon_k'' = \Delta_k^\tau v + L v_{k\delta_t} \Delta_k^\tau + \partial_x v_{k\delta_t} \left(\frac{1}{N} \sum_{j=1}^N \Delta_k^\tau W^j \right).$$

Then bounds (4.5) *and* (4.6) *hold for* ε_k''.

PROOF.   First note that

$$v_{\tau \wedge t} - v_{\tau \wedge k\delta_t} = -\int_{\tau \wedge k\delta_t}^{\tau \wedge t} L v_s \, ds - \frac{1}{N} \sum_{j=1}^N \int_{\tau \wedge k\delta_t}^{\tau \wedge t} \partial_x v_s \, dW_s^j$$

almost surely. Thus for any $n \le 2 + \beta$ and $t \in [k\delta_t, (k+1)\delta_t]$ using (2.8) gives

$$(4.11) \qquad \sup_{x \in \mathbb{T}} \mathbf{E} |\partial_x^n v_{\tau \wedge t}(x) - \partial_x^n v_{\tau \wedge k\delta_t}(x)|^2 \le C\delta_t.$$

Similar to the derivation of (4.9) we obtain

$$\varepsilon_k'' = \int_{\tau \wedge k\delta_t}^{\tau \wedge (k+1)\delta_t} (L v_{k\delta_t} - L v_s) \, ds + \frac{1}{N} \sum_{j=1}^N \int_{\tau \wedge k\delta_t}^{\tau \wedge (k+1)\delta_t} (\partial_x v_{k\delta_t} - \partial_x v_s) \, dW_s^j$$

from definition (4.10). The remainder of the proof is now identical to the proof of Lemma 4.3.   □

We are now ready to prove Lemma 2.5. We remark that assumptions (2.7) and (2.8) are stronger than necessary. We only need

$$(4.12) \qquad \sup_{0 \le t \le \tau} (\|u_t\|_{C^{1+\beta}} + \|v_t\|_{C^{1+\beta}}) \le U \qquad \text{a.s.,}$$

$$(4.13) \qquad \sup_{x \in \mathbb{T}} \sup_{0 \le t \le \tau} \mathbf{E}(|\partial_x^{2+\beta} u_t(x)|^2 + |\partial_x^{2+\beta} v_t(x)|^2) \le U,$$

and the bounds on ε', ε'' provided by Lemmas 4.3 and 4.4 above. The proof we provide below depends only on these weaker assumptions.

PROOF OF LEMMA 2.5.   Let $\varepsilon_k = \varepsilon_k' - \varepsilon_k''$, where ε_k', ε_k'' are defined by Lemmas 4.3 and 4.4, respectively. Using (4.5), (4.6) and the corresponding estimates for ε_k'', we have

$$(4.14) \qquad \sup_{x \in \mathbb{T}} \mathbf{E} \, \partial_x^\beta \varepsilon_k(x)^2 \le C\delta_t^2,$$

$$(4.15) \qquad \sup_{x \in \mathbb{T}} \mathbf{E} |\mathbf{E}_{\mathcal{F}_{k\delta_t}} \partial_x^\beta \varepsilon_k(x)|^2 \le C\delta_t^3,$$

for all $k \leq \frac{T_0}{\delta_t}$. As before, we assume C is a constant that only depends on β, U and T_0, which may change from line to line. Now, estimates (4.14) and (4.15) imply

$$\text{(4.16)} \qquad \mathbf{E}\|\varepsilon_k\|_{H^\beta}^2 \leq C\delta_t^2,$$

$$\text{(4.17)} \qquad \mathbf{E}\|\mathbf{E}_{\mathcal{F}_{k\delta_t}}\varepsilon_k\|_{H^\beta}^2 \leq C\delta_t^3.$$

For the remainder of the proof we will use the weaker estimates, (4.16) and (4.17).

Now, recall $w_t = w_{\tau \wedge t} = u_{\tau \wedge t} - v_{\tau \wedge t}$, and we know $w_0 = 0$. Thus

$$\begin{aligned}
\partial_x^\beta \Delta_k^\tau w &= \partial_x^\beta \Delta_k^\tau u - \partial_x^\beta \Delta_k^\tau v \\
\text{(4.18)} \qquad &= -\partial_x^\beta (Lu_{\tau \wedge k\delta_t} - Lv_{\tau \wedge k\delta_t})\Delta_k^\tau t \\
&\quad - \partial_x^{\beta+1} w_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N \Delta_k^\tau W^j\right) + \partial_x^\beta \varepsilon_k.
\end{aligned}$$

We first estimate $\mathbf{E}(\partial_x^\beta \Delta_k^\tau w)^2$ where k is any integer such that $k\delta_t \leq T_0$.

For this, independence of W^i, the mean square of the matringale term in (4.18) is bounded by

$$\begin{aligned}
\text{(4.19)} \qquad \mathbf{E}\left[\partial_x^{\beta+1} w_{k\delta_t}\left(\frac{1}{N}\sum_{j=1}^N \Delta_k^\tau W^j\right)\right]^2 &= \mathbf{E}(\partial_x^{\beta+1} w_{k\delta_t})^2 \mathbf{E}\left(\frac{1}{N}\sum_{j=1}^N \Delta_k^\tau W^j\right)^2 \\
&\leq \frac{\delta_t}{N}\mathbf{E}(\partial_x^{\beta+1} w_{k\delta_t})^2.
\end{aligned}$$

Next, for the mean square of the first term in (4.18)

$$\begin{aligned}
\mathbf{E}\big(\partial_x^\beta &(Lu_{\tau \wedge k\delta_t}(x) - Lv_{\tau \wedge k\delta_t}(x))\big)^2 \\
&\leq C\mathbf{E}[(\partial_x^{\beta+2} u_{\tau \wedge k\delta_t})^2 + (\partial_x^{\beta+2} v_{\tau \wedge k\delta_t})^2] \\
&\quad + C\mathbf{E}[(\partial_x^\beta (u_{\tau \wedge k\delta_t}\partial_x u_{\tau \wedge k\delta_t}))^2 + (\partial_x^\beta (v_{\tau \wedge k\delta_t}\partial_x v_{\tau \wedge k\delta_t}))^2] \\
&\leq C\sup_{x\in\mathbb{T}}\max_{0\leq k\leq T_0/\delta_t} \mathbf{E}\big(|\partial_x^{2+\beta} u_{\tau \wedge k\delta_t}(x)|^2 + |\partial_x^{2+\beta} v_{\tau \wedge k\delta_t}(x)|^2\big) \\
&\quad + C\max_{0\leq k\leq T_0/\delta_t} (\|u_{\tau \wedge k\delta_t}\|_{C^{1+\beta}}^4 + \|v_{\tau \wedge k\delta_t}\|_{C^{1+\beta}}^4).
\end{aligned}$$

Hence, using (4.12) and (4.13) we obtain

$$\text{(4.20)} \qquad \mathbf{E}\big(\partial_x^\beta (Lu_{\tau \wedge k\delta_t} - Lv_{\tau \wedge k\delta_t})\Delta_k^\tau t\big)^2 \leq \delta_t^2 C.$$

By (4.16) the mean square of the last term in (4.18) is also bounded by $C\delta_t^2$. Thus, squaring (4.18), taking expected values and using Young's inequality gives

$$\text{(4.21)} \qquad \mathbf{E}(\partial_x^\beta \Delta_k^\tau w)^2 \leq \frac{3\delta_t}{N}\mathbf{E}(\partial_x^{\beta+1} w_{k\delta_t})^2 + C\delta_t^2 + 3\mathbf{E}(\partial_x^\beta \varepsilon_k)^2.$$

Now for any $K \leq \frac{T_0}{\delta_t}$,

$$(\partial_x^\beta w_{K\delta_t})^2 = (\partial_x^\beta w_{\tau \wedge K\delta_t})^2 = 2 \sum_{k=0}^{K-1} \partial_x^\beta w_{k\delta_t} \, \partial_x^\beta \Delta_k^\tau w + \sum_{k=0}^{K-1} (\partial_x^\beta \Delta_k^\tau w)^2$$

$$= 2 \sum_{k=0}^{K-1} \partial_x^\beta w_{k\delta_t} \left(-\partial_x^\beta (Lu_{\tau \wedge k\delta_t} - Lv_{\tau \wedge k\delta_t}) \Delta_k^\tau t \right.$$

$$\left. - \partial_x^{\beta+1} \Delta_k^\tau w \left(\frac{1}{N} \sum_{j=1}^N \Delta_k^\tau W^j \right) + \partial_x^\beta \varepsilon_k \right)$$

$$+ \sum_{k=0}^{K-1} (\partial_x^\beta \Delta_k^\tau w)^2.$$

Taking expected values, integrating in space using (4.21) and (4.16) gives

$$\mathbf{E} \|\partial_x^\beta w_{K\delta_t}\|_{L^2}^2 \leq -2\delta_t \sum_{k=0}^{K-1} \mathbf{E} \int_{\mathbb{T}} \partial_x^\beta w_{k\delta_t} \partial_x^\beta (u_{\tau \wedge k\delta_t} \partial_x u_{\tau \wedge k\delta_t}$$

(4.22)
$$- v_{\tau \wedge k\delta_t} \partial_x v_{\tau \wedge k\delta_t}) \, dx$$

$$+ 2 \sum_{k=0}^{K-1} \mathbf{E} \int_{\mathbb{T}} \partial_x^\beta w_{k\delta_t} \partial_x^\beta \varepsilon_k \, dx$$

$$- \left(1 - \frac{1}{N} \right) \delta_t \sum_{k=0}^{K-1} \mathbf{E} \int_{\mathbb{T}} (\partial_x^{\beta+1} w_{k\delta_t})^2 \, dx + CK\delta_t^2.$$

For the first term on the right-hand side of inequality (4.22) note

(4.23)
$$\partial_x^\beta w_{k\delta_t} \partial_x^\beta (u_{\tau \wedge k\delta_t} \partial_x u_{\tau \wedge k\delta_t} - v_{\tau \wedge k\delta_t} \partial_x v_{\tau \wedge k\delta_t})$$
$$= \partial_x^\beta w_{k\delta_t} \partial_x^\beta (w_{k\delta_t} \partial_x u_{\tau \wedge k\delta_t} - v_{\tau \wedge k\delta_t} \partial_x w_{k\delta_t}).$$

Observe that the mass (spatial mean) of solutions to (2.6) is constant in time. The same is true for solutions to (4.3). Thus, for all $t \leq T_0$, $\int_{\mathbb{T}} u_{\tau \wedge t} \, dx = \int_{\mathbb{T}} u_0 \, dx = \int_{\mathbb{T}} v_{\tau \wedge t} \, dx$, and hence $\int_{\mathbb{T}} w_t \, dx = 0$. Thus integrating (4.23) in space and using the Poincaré inequality, the term involving u above is bounded by

$$\left| \int_{\mathbb{T}} \partial_x^\beta w_{k\delta_t} \, \partial_x^\beta (w_{k\delta_t} \partial_x u_{\tau \wedge k\delta_t}) \, dx \right| \leq C \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 \|u_{\tau \wedge k\delta_t}\|_{C^{\beta+1}}.$$

For the term involving v in (4.23), when all the derivatives fall on w we have

$$\partial_x^\beta w_{k\delta_t} v_{\tau \wedge k\delta_t} \partial_x^{\beta+1} w_{k\delta_t} = \tfrac{1}{2} v_{\tau \wedge k\delta_t} \, \partial_x (\partial_x^\beta w_{k\delta_t})^2,$$

and if we integrate by parts, we can avoid the extra derivative on w. Thus

$$\left| \int_{\mathbb{T}} \partial_x^\beta w_{k\delta_t} \, \partial_x^\beta (v_{\tau \wedge k\delta_t} \, \partial_x w_{k\delta_t}) \, dx \right| \leq C \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 \|v_{\tau \wedge k\delta_t}\|_{C^\beta}.$$

Thus using (4.12) and the above estimates, the first term on the right-hand side of (4.22) is bounded by

$$
(4.24) \quad
\begin{aligned}
-2\delta_t \mathbf{E} \int_{\mathbb{T}} & \partial_x^\beta w_{k\delta_t} \, \partial_x^\beta (u_{\tau \wedge k\delta_t} \, \partial_x u_{\tau \wedge k\delta_t} - v_{\tau \wedge k\delta_t} \, \partial_x v_{\tau \wedge k\delta_t}) \, dx \\
& \leq C\delta_t \mathbf{E} \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 .
\end{aligned}
$$

For the second term in (4.22), we know $w_{k\delta_t}$ is $\mathcal{F}_{k\delta_t}$-measurable. Thus using (4.17) and the Cauchy–Schwarz inequality we obtain

$$
(4.25) \quad \int_{\mathbb{T}} |\mathbf{E}\partial_x^\beta w_{k\delta_t} \, \partial_x^\beta \varepsilon_k| \, dx = \int_{\mathbb{T}} |\mathbf{E}(\partial_x^\beta w_{k\delta_t} \, \mathbf{E}_{\mathcal{F}_{k\delta_t}} \partial_x^\beta \varepsilon_k)| \, dx \leq C\delta_t^{3/2} .
$$

The third term on the right-hand side of (4.22) is always nonpositive, and can be ignored. Thus, recalling $K \leq \frac{T_0}{\delta_t}$, and using (4.24) and (4.25) in (4.22) we have

$$
\mathbf{E} \|\partial_x^\beta w_{K\delta_t}\|_{L^2}^2 \leq C\delta_t^{1/2} + C \sum_{k=0}^{K-1} \mathbf{E} \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 \delta_t .
$$

The remainder of the proof is an elementary discrete Gronwall argument. Let

$$
y_K = C\delta_t^{1/2} + C \sum_{k=0}^{K-1} \mathbf{E} \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 \delta_t .
$$

Then

$$
y_{k+1} - y_k = C\delta_t \mathbf{E} \|\partial_x^\beta w_{k\delta_t}\|_{L^2}^2 \leq C\delta_t y_k
$$

and hence

$$
y_{k+1} \leq (1 + C\delta_t) y_k .
$$

Iterating, and using $y_0 = C\delta_t^{1/2}$ gives

$$
y_k \leq (1 + C\delta_t)^k C\delta_t^{1/2} .
$$

Since $k \leq \frac{T_0}{\delta_t}$ this gives

$$
(4.26) \quad \max_{k \leq T_0/\delta_t} y_k \leq C\delta_t^{1/2} \sup_{\delta_t' > 0} (1 + C\delta_t')^{T_0/\delta_t'} \leq C\delta_t^{1/2} e^{CT_0} .
$$

This proves (2.9) for all times t which are an integer multiple of δ_t. Since for any $x \in \mathbb{T}$, and $k \leq \frac{T_0}{\delta_t}$ we elementarily have

$$
\sup_{k\delta_t \leq t \leq (k+1)\delta_t} \mathbf{E} |\partial_x^\beta v_{\tau \wedge t}(x) - \partial_x^\beta v_{\tau \wedge k\delta_t}(x)|^2 \leq C\delta_t
$$

and

$$
\sup_{k\delta_t \leq t \leq (k+1)\delta_t} \mathbf{E} |\partial_x^\beta u_{\tau \wedge t}(x) - \partial_x^\beta u_{\tau \wedge k\delta_t}(x)|^2 \leq C\delta_t
$$

completing the proof. $\square$

5. Proof of Lemma 2.6. In this section we establish uniform in time bounds for the solution of (2.6) and prove as in Lemma 2.6. We do this via the following two lemmas:

LEMMA 5.1. *Let $u_0 \in C^\infty(\mathbb{T})$, $T > 0$, and suppose $v \in C^\infty([0, T]; \mathbb{T})$ is a solution to (2.6) with initial data u_0 and periodic boundary conditions. Then for any $s \in \mathbb{Z}^+$, there exists a constant $V_s = V_s(s, T, \|u_0\|_{H^s})$ such that*

$$\sup_{0 \leq t \leq T} \|v_t\|_{H^s} \leq V_s$$

almost surely.

LEMMA 5.2. *Let $u_0 \in C^\infty(\mathbb{T})$, and suppose $v \in C^\infty([0, \infty), \mathbb{T})$ is a solution to (2.6) with initial data u_0 and periodic boundary conditions. Then for any $s \in \mathbb{Z}^+$, $T > 0$, there exists a constant $V_s = V_s(s, T, \|u_0\|_{L^2})$ such that*

$$(5.1) \qquad\qquad \sup_{t \geq T} \|v_t\|_{H^s} \leq V_s$$

almost surely.

We draw attention to the fact that a priori bounds are almost sure! Indeed, applying Itô's formula to $\|v_t\|_{L^2}^2$ immediately yields an equation with no martingale part [see (5.2) below].

Given Lemmas 5.1 and 5.2, the proof of Lemma 2.6 is now immediate.

PROOF OF LEMMA 2.6. Given the almost sure a priori bounds in Lemmas 5.1 and 5.2, existence of solutions to (2.6) follows via standard methods. The time global bound (2.10) is also an immediate consequence of Lemmas 5.1 and 5.2.
□

We devote the remainder of this section to proving Lemmas 5.1 and 5.2.

PROOF OF LEMMA 5.1. We prove Lemma 5.1 via energy estimates. First note that Itô's formula and (2.6) give

$$d(v_t)^2 = 2v_t \, dv_t + \frac{1}{N^2} \sum_{j=1}^N (\partial_x v_t)^2 \, dt$$

$$= -2v_t^2 \, \partial_x v_t \, dt + v_t \, \partial_x^2 v_t \, dt - 2\frac{v_t \, \partial_x v_t}{N} \sum_{j=1}^N dW_t^j$$

$$+ \frac{1}{N} (\partial_x v_t)^2 \, dt.$$

Integrating in space, and using $\int_{\mathbb{T}} v_t \, \partial_x v_t \, dx = 0 = \int_{\mathbb{T}} v_t^2 \, \partial_x v_t \, dx$ gives

$$(5.2) \qquad \partial_t \|v_t\|_{L^2}^2 = -\left(1 - \frac{1}{N}\right) \|\partial_x v_t\|_{L^2}^2 \quad \implies \quad \|v_t\|_{L^2} \leq \|u_0\|_{L^2}$$

almost surely.

A similar calculation shows $\|v_t\|_{L^p} \leq \|u_0\|_{L^p}$ for all $p \geq 2$, and hence[11] $\|v_t\|_{L^\infty} \leq \|u_0\|_{L^\infty}$. Recall $s \geq 1$ by assumption, and so the Sobolev embedding theorem shows $\|u_0\|_{L^\infty} \leq c\|u_0\|_{H^s}$ for some absolute constant c.

Now, differentiating (2.6) with respect to x and applying Itô's formula to $(\partial_x v_t)^2$ we obtain

$$d(\partial_x v_t)^2 = 2 \, \partial_x v_t \, d(\partial_x v_t) + \frac{1}{N} |\partial_x^2 v_t|^2 \, dt$$

$$= -2 \, \partial_x v_t \left(\partial_x (v_t \partial_x v_t) \, dt - \frac{1}{2} \partial_x^3 v_t \, dt + \frac{\partial_x^2 v_t}{N} \sum_{j=1}^{N} dW_t^j \right)$$

$$+ \frac{1}{N} |\partial_x^2 v_t|^2 \, dt.$$

Integrating with respect to x on $[0, 1]$, and noting that $\int_{\mathbb{T}} \partial_x v_t \, \partial_x^2 v_t \, dx = 0$, gives

$$d\|\partial_x v_t\|_{L^2}^2 = -\left(1 - \frac{1}{N}\right) \|\partial_x^2 v_t\|_{L^2}^2 \, dt + \left(2 \int_0^1 \partial_x^2 v_t (v_t \, \partial_x v_t) \, dx\right) dt$$

$$\implies \quad \partial_t \|\partial_x v_t\|_{L^2}^2 \leq -\frac{1}{4} \|\partial_x^2 v_t\|_{L^2}^2 + 8\|v_t \partial_x v_t\|_{L^2}^2$$

(5.3)

$$\leq -\frac{1}{4} \|\partial_x^2 v_t\|_{L^2}^2 + 8\|v_t\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2$$

$$\leq -\frac{1}{4} \|\partial_x^2 v_t\|_{L^2}^2 + 8\|u_0\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2,$$

almost surely. Thus, (5.2), (5.3) and Gronwall's inequality gives

$$(5.4) \qquad \|v_t\|_{H^1} \leq C_1 e^{c_0 t} \quad \text{and} \quad \int_0^t \|v_{t'}\|_{H^2}^2 \, dt' \leq C_1 e^{c_0 t},$$

almost surely, for some constants $C_1 = C_1(\|u_0\|_{H^1})$ and $c_0 = c_0(\|u_0\|_{L^\infty})$.

For the remainder of this proof we adopt the convention that c, C denote absolute constants, $C_s = C_s(s, \|u_0\|_{H^s})$ denotes a constant depending only on s, $\|u_0\|_{H^s}$ and c_0 denotes a constant depending only on $\|u_0\|_{L^\infty}$. The exact value of these constants are immaterial, and we will allow them to change from line to line.

---

[11]This can alternately be shown using a version of the maximum principle [15].

Similar to (5.3), differentiating (2.6) twice with respect to x, applying Itô's formula to $(\partial_x^2 v_t)^2$, integrating in space, noting $\int_{\mathbb{T}} \partial_x^2 v_t \, \partial_x^3 v_t \, dx = 0$ and using Hölder's inequality gives

$$\partial_t \|\partial_x^2 v_t\|_{L^2}^2 \leq -\left(1 - \frac{1}{N}\right) \|\partial_x^3 v_t\|_{L^2}^2 + 2\|\partial_x^3 v_t\|_{L^2} \|\partial_x(v_t \partial_x v_t)\|_{L^2}$$

$$\leq -c\|\partial_x^3 v_t\|_{L^2}^2 + C(\|\partial_x v_t\|_{L^\infty}^2 + \|v_t\|_{L^\infty}^2)\|\partial_x^2 v_t\|_{L^2}^2$$

$$\leq -c\|\partial_x^3 v_t\|_{L^2}^2 + C\|v_t\|_{H^2}^2 \|\partial_x^2 v_t\|_{L^2}^2,$$

almost surely, where the last inequality is obtained by the Sobolev embedding theorem. Using (5.4), this gives

$$\|v_t\|_{H^2} \leq C_2 e^{\int_0^t \|v_t\|_{H^2}^2 \, dt} \leq C_2 e^{C_1 e^{c_0 t}} \quad \text{and} \quad \int_0^t \|v_{t'}\|_{H^3}^2 \, dt' \leq C_2 e^{C_1 e^{c_0 t}},$$

almost surely. Proceeding inductively, suppose we know

$$(5.5) \qquad \begin{cases} \|v_t\|_{H^s} \leq C_s \exp(C_{s-1} \exp(C_{s-2} \cdots \exp(c_0 t) \cdots)), \\ \int_0^t \|v_{t'}\|_{H^{s+1}}^2 \, dt' \leq C_s \exp(C_{s-1} \exp(C_{s-2} \cdots \exp(c_0 t) \cdots)), \end{cases}$$

holds almost surely for some $s \in \mathbb{Z}^+$. Differentiating (2.6) $s+1$ times with respect to x, applying Itô's formula for $(\partial_x^{s+1} v_t)^2$ and integrating in space we obtain

$$d\|\partial_x^{s+1} v_t\|_{L^2}^2 = -\left(1 - \frac{1}{N}\right) \|\partial_x^{s+2} v_t\|_{L^2}^2 \, dt + 2\|\partial_x^{s+2} v_t\|_{L^2} \|\partial_x^s(v_t \partial_x v_t)\|_{L^2} \, dt$$

since $\int_{\mathbb{T}} \partial_x^{s+1} v_t \, \partial_x^{s+2} v_t \, dx = 0$. Thus

$$\partial_t \|\partial_x^{s+1} v_t\|_{L^2}^2 \leq -c\|\partial_x^{s+2} v_t\|_{L^2}^2$$
$$+ C(\|\partial_x^s v_t\|_{L^\infty}^2 \|\partial_x v_t\|_{L^2}^2 + \cdots$$
$$+ \|\partial_x v_t\|_{L^\infty}^2 \|\partial_x^s v_t\|_{L^2}^2 + \|v_t\|_{L^\infty}^2 \|\partial_x^{s+1} v_t\|_{L^2}^2)$$
$$\leq -c\|\partial_x^{s+2} v_t\|_{L^2}^2$$
$$+ C(\|\partial_x^s v_t\|_{L^\infty}^2 + \cdots$$
$$+ \|\partial_x v_t\|_{L^\infty}^2 + \|v_t\|_{L^\infty}^2)\|\partial_x^{s+1} v_t\|_{L^2}^2$$
$$\leq -c\|\partial_x^{s+2} v_t\|_{L^2}^2 + C\|v_t\|_{H^{s+1}}^2 \|\partial_x^{s+1} v_t\|_{L^2}^2,$$

almost surely. Thus by Gronwall's lemma

$$\|v_t\|_{H^{s+1}} \leq C_{s+1} \exp\left(\int_0^t \|v_{t'}\|_{H^{s+1}}^2 \, dt'\right)$$

$$\leq C_{s+1} \exp(C_s \exp(C_{s-1} \cdots \exp(c_0 t) \cdots))$$

almost surely. Further

$$\int_0^t \|v_{t'}\|_{H^{s+2}}^2 \, dt' \le C_{s+1} \exp\big(C_s \exp(C_{s-1}\cdots \exp(c_0 t)\cdots)\big),$$

almost surely, completing the inductive step. By induction, (5.5) holds for all $s \in \mathbb{Z}^+$ completing the proof. $\square$

PROOF OF LEMMA 5.2. We prove Lemma 5.2 via a bootstrapping argument in Fourier space. To fix notation, for $n \in \mathbb{Z}$, we use $\hat{f}(n) = \int_{\mathbb{T}} e^{-2\pi i n x} f(x) \, dx$ to denote the nth Fourier coefficient of f.

On Fourier coefficients, using $u \, \partial_x u = \frac{1}{2} \partial_x u^2$, equation (2.6) reduces to

(5.6)
$$d\hat{v}_t(n) + \frac{2\pi i n}{N} \hat{v}_t(n) \sum_{j=1}^N dW_t^j$$
$$+ 2\pi^2 n^2 \hat{v}_t(n) \, dt + \pi i n \sum_{m \in \mathbb{Z}} \hat{v}_t(n-m)\hat{v}_t(m) \, dt = 0$$

for every $n \in \mathbb{Z}$.

By Itô's formula applied to (5.6)

(5.7)
$$d|\hat{v}_t(n)|^2 = \overline{\hat{v}_t(n)} \, d\hat{v}_t(n) \, dt + \hat{v}_t(n) d\,\overline{\hat{v}_t(n)} + \frac{4\pi^2 n^2}{N} |\hat{v}_t(n)|^2 \, dt$$
$$= -4\pi^2 n^2 \Big(1 - \frac{1}{N}\Big)|\hat{v}_t(n)|^2 \, dt$$
$$+ \pi i n\big(\hat{v}_t(n)\overline{B_t(n)} - \overline{\hat{v}_t(n)}B_t(n)\big) \, dt,$$

where $\overline{\hat{v}_t(n)}$ denotes the complex conjugate of $\hat{v}_t(n)$, and

$$B_t(n) = \sum_{m \in \mathbb{Z}} \hat{v}_t(n-m)\hat{v}_t(m),$$

is the nonlinear Fourier coupling in (5.6). Using $N > 1$ and Young's inequality in (5.7) gives

(5.8)
$$\partial_t |\hat{v}_t(n)|^2 \le -2\pi^2 n^2 |\hat{v}_t(n)|^2 + 2\pi n|\hat{v}_t(n)||B_t(n)|$$
$$\le -cn^2|\hat{v}_t(n)|^2 + C|B_t(n)|^2$$

almost surely, where, as before c, C are absolute constants (independent of u_0, T), which may change from line to line. Thus, for any $t_0' \ge 0$ we have

(5.9)
$$|\hat{v}_t(n)|^2 \le |\hat{v}_{t_0'}(n)|^2 e^{-n^2 ct} + C \int_{t_0'}^t e^{-cn^2(t-t')}|B_{t'}(n)|^2 \, dt'$$

almost surely, by Gronwall's inequality.

By Parseval's identity we know $|B_t(n)| \leq \|v_t\|_{L^2}^2$, and by conservation of energy [equation (5.2)] this gives $|B_t(n)| \leq \|u_0\|_{L^2}^2$ almost surely. Thus the second term in the previous inequality is bounded from above by $\frac{C}{cn^2}\|u_0\|_{L^2}^4$. Since $|\hat{u}_{t_0'}|^2 \leq \|u_{t_0'}\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$, given a lower bound on $t - t_0'$, we can certainly arrange the same inequality for the first term. Thus choosing $t_1 = \frac{T}{2}$, for instance, and applying (5.9) with $t_0' = 0$, we obtain

$$(5.10) \qquad \sup_{t \geq t_1}|\hat{v}_t(n)|^2 \leq \frac{C_0}{n^2}$$

almost surely, where $C_0 = C_0(\|u_0\|_{L^2}, T)$ is some constant.

Now we bootstrap, and use (5.10) to obtain a better estimate on B_t. Assume inductively that for some $\alpha \in \mathbb{Z}^+$, and $t_\alpha = \frac{\alpha}{\alpha+1}T$, we have

$$(5.11) \qquad \sup_{t \geq t_\alpha}|\hat{v}_t(n)|^2 \leq \frac{C_\alpha}{|n|^{\alpha+1}}$$

almost surely. Here $C_\alpha = C_\alpha(\|u_0\|_{L^2}, T, \alpha)$ is a constant which we allow to change from line to line if necessary. We will now establish (5.11) for $\alpha + 1$. Note that almost surely, for any $t > t_\alpha$, we have

$$
\begin{aligned}
|B_t(n)| &\leq \sum_{m \in \mathbb{Z}}|\hat{v}_t(n-m)||\hat{v}_t(m)| \leq 2\sum_{|m| \geq |n|/2}|\hat{v}_t(n-m)||\hat{v}_t(m)| \\
(5.12) \qquad &\leq 2\|v_t\|_{L^2}\left(\sum_{|m| \geq |n|/2}|\hat{v}_t(m)|^2\right)^{1/2} \leq 2\|u_0\|_{L^2}\left(\sum_{|m| \geq |n|/2}\frac{C_\alpha}{m^\alpha}\right)^{1/2} \\
&\leq \frac{C_\alpha}{|n|^{(\alpha-1)/2}}.
\end{aligned}
$$

Now returning to (5.9) and choosing $t_0' = t_\alpha$, we see that the second term is bounded by $\frac{C}{cn^2}\frac{C_0}{n^{\alpha-1}} = \frac{CC_0}{cn^{\alpha+1}}$. For any $t \geq t_{\alpha+1}$, we can certainly arrange the same inequality for the first term, and hence this establishes (5.11) for $\alpha + 1$.

Finally note that if (5.11) holds for α, then (5.1) holds for any $s < \frac{\alpha}{2} - 1$, completing the proof. □

6. Proof of Lemma 2.7. In this section, we prove the almost sure $C^n(\mathbb{T})$ bounds on u stated in Lemma 2.7. We need a few preliminary results first.

PROPOSITION 6.1 (Local existence without resetting). *Let u_{t_0} be a $C^1(\mathbb{T})$ valued $\mathcal{F}_{t_0}$-measurable random variable such that*

$$\|u_{t_0}\|_{C^1} \leq U_1^0$$

almost surely. There exists $T_0 = T_0(U_1^0)$, independent of N, such that the solution to (2.1)–(2.3) exists on the interval $[t_0, t_0 + T_0]$. Further if for some $n \geq 1$, u_{t_0} is a $C^n(\mathbb{T})$ valued, $\mathcal{F}_{t_0}$-measurable random variable with

$$\|u_{t_0}\|_{C^n} \leq U_n^0$$

almost surely, then there exists $U_n = U_n(U_n^0, n)$ such that

$$(6.1) \qquad \sup_{t_0 \leq t \leq t_0 + T_0} \|u_t\|_{C^n} \leq U_n$$

almost surely.

Proposition 6.1 can be proved using a standard Picard iteration. A proof of the analogous result for the Navier–Stokes equations appeared in the Appendix of [9] (see also [7, 8]). The proof of 6.1 is very similar, and we do not provide it here.

LEMMA 6.2. *Let $I : \mathbb{R} \to \mathbb{R}$ denote the identity function, $d \in [0, 1)$ and let $\lambda \in C^n(\mathbb{T})$ be a periodic function such that $\|\partial_x \lambda\|_{L^\infty} \leq d$. Then there exists a constant $c_{n-1} = c_{n-1}(\|\partial_x^{n-1} \lambda\|_{L^\infty}, d, n)$ such that for any $f \in C^n(\mathbb{R})$,*

$$(6.2) \quad \|\partial_x^n [f \circ (I + \lambda)]\|_{L^\infty} \leq \|\partial_x^n f\|_{L^\infty} (1 + \|\partial_x \lambda\|_{L^\infty})^n + c_{n-1} \|\partial_x^n \lambda\|_{L^\infty},$$

$$(6.3) \qquad \|\partial_x^n (I + \lambda)^{-1}\|_{L^\infty} \leq c_{n-1} \|\partial_x^n \lambda\|_{L^\infty},$$

for $n > 1$.

REMARK. Note that since $\|\partial_x \lambda\|_{L^\infty} < 1$, the function $I + \lambda$ is a $C^1(\mathbb{R})$ diffeomorphism of $\mathbb{R}$. The notation $(I + \lambda)^{-1}$ in (6.3) refers to the inverse of the $C^1(\mathbb{R})$ diffeomorphism $I + \lambda$.

PROOF OF LEMMA 6.2. First note that we can view λ as a periodic function (with period 1) in $C^n(\mathbb{R})$. Further, by the mean value theorem, for any $k \geq 1$ there exists $x \in \mathbb{T}$ such that $\partial_x^k \lambda(x) = 0$. Thus for any $k \in \{1, \ldots, n\}$, we have $|\partial_x^k \lambda| \leq c(n) \|\partial_x^n \lambda\|_{L^\infty}$, for some constant $c(n)$ depending only on n. (For $k = 0$, we need to subtract the mean of λ for this bound to be valid.)

Now for any two $f, g \in C^n(\mathbb{R})$, we have

$$(6.4) \qquad \partial_x^n (f \circ g) = \sum_{m=1}^n (\partial_x^m f) \circ g \sum_{\substack{k_1 + \cdots + k_m = n \\ k_i \geq 1}} \prod_{i=1}^m \partial_x^{k_i} g.$$

To prove (6.2), we set $g = I + \lambda$. The term in (6.4) corresponding to $m = n$ gives the first term of 6.2. When $m < n$, we notice that $k_i > 1$ for at least one i, and $k_j \leq n - 1$ for all other j. Thus $\|\partial_x^{k_i} (I + \lambda)\|_{L^\infty} = \|\partial_x^{k_i} \lambda\|_{L^\infty} \leq c(n) \|\partial_x^n \lambda\|_{L^\infty}$. The remaining terms $\partial_x^{k_j} (I + \lambda)$, $j \neq i$ in the product can be bounded by c_{n-1}. This proves (6.2).

For (6.3), set $X = I + \lambda$ and $A = X^{-1}$. Since $n > 1$, $\partial_x^n (A \circ X) \equiv 0$, and using (6.4) we obtain

$$\partial_x^n A|_X = \frac{-1}{(\partial_x X)^n} \sum_{m=1}^{n-1} \partial_x^m A|_X \sum_{\substack{k_1 + \cdots + k_m = n \\ k_i \geq 1}} \prod_{i=1}^m \partial_x^{k_i} A.$$

By induction, one can assume that $\|\partial_x^m A\|_{L^\infty} \leq c_{n-1}$ for all $m \leq n-1$. Since $d < 1$, $\frac{1}{\|\partial_x X\|_{L^\infty}} \leq \frac{1}{1-d}$, and remaining terms can be bounded by the same argument as before. This proves (6.3).   □

LEMMA 6.3.   *Let $n \in \mathbb{N}$, u_{t_0} be a bounded, $C^n(\mathbb{T})$ valued, $\mathcal{F}_{t_0}$-measurable random variable. For $k \in \{0, \ldots, k\}$, let U_k^0 be a constant such that $\|u_{t_0}\|_{C^k} \leq U_k^0$ almost surely. Let u be the solution of (2.1)–(2.3) with initial data $u_t = u_{t_0}$ when $t = t_0$. If $n > 1$, there exists $\Omega' \in \mathcal{F}_{t_0}$ with $P(\Omega') = 1$, $T_0 = T_0(U_1^0) > t_0$ and a constant $c_{n-1} = c_{n-1}(U_{n-1}^0, n)$ such that*

$$(6.5) \qquad \|\partial_x^n u_t(\omega')\|_{L^\infty} \leq U_n^0 (1 + c_{n-1}(t - t_0))$$

for all $\omega' \in \Omega'$, $t \in [t_0, t_0 + T_0]$. For $n = 1$, (6.5) holds with c_0 to be an absolute constant.

PROOF.   For simplicity, we assume $t_0 = 0$. One can check that this assumption does not affect our proof below. Our first step is to obtain almost sure $C^1(\mathbb{T})$ estimates on the Eulerian and Lagrangian displacements. Throughout this section, we use the convention that $c_{n-1} = c_{n-1}(U_{n-1}^0, n)$ is a constant depending only on n and U_{n-1}^0 (or an absolute constant for $n = 1$), which can change from line to line.

Let $T_0 = T_0(U_1^0)$ be the local existence time given by Proposition 6.1, and $c_1 = c_1(U_1^0)$ the almost sure bound on $\|u_t\|_{C^1}$ from (6.1). Let $I : \mathbb{R} \to \mathbb{R}$ be the identity map, X^i, A^i, respectively, be as in (2.1), (2.2), with $\tau = T_0$. Define $\lambda_t^i = X_t^i - I$, $\ell_t^i = A_t^i - I$.

Differentiating (2.1) with respect to x we obtain

$$\|\partial_x \lambda_t^i\|_{L^\infty} \leq \int_0^t \|\partial_x u_s\|_{L^\infty}(1 + \|\partial_x \lambda_t^i\|_{L^\infty})$$

almost surely, for $t \in [0, T_0]$. By Gronwall's lemma,

$$\|\partial_x \lambda_t^i\|_{L^\infty} \leq e^{c_1 t} \int_0^t \|\partial_x u_s\|_{L^\infty} \, ds \qquad \text{a.s.}$$

for $t \in [0, T_0]$. Recall $t \leq T_0$, c_1 only depends on U_0^1, and for all $s \leq T_0$, $\|\partial_x u_s\|_{L^\infty} \leq U_1^0$ almost surely. Thus, as T_0 is allowed to depend on U_1^0, by making T_0 smaller if necessary we can arrange

$$(6.6) \qquad \|\partial_x \lambda_t^i\|_{L^\infty} \leq c_0 \int_0^t \|\partial_x u_s\|_{L^\infty} \, ds \quad \text{and} \quad \sup_{0 \leq t \leq T_0} \|\partial_x \lambda_t^i\|_{L^\infty} \leq \frac{1}{2}$$

almost surely, for some absolute constant c_0. Now

$$\partial_x \ell_t^i = \partial_x A_t^i - 1 = \frac{1}{(\partial_x X_t^i) \circ A_t^i} - 1$$

$$= -\frac{(\partial_x \lambda_t^i) \circ A_t^i}{1 + (\partial_x \lambda_t^i) \circ A_t^i}$$

almost surely. Thus we must have

(6.7)
$$\|\partial_x \ell_t^i\|_{L^\infty} \le 2\|\partial_x \lambda_t^i\|_{L^\infty}$$

almost surely for $t \in [0, T_0]$. Using (2.3) and (6.7) we have

$$\|\partial_x u_t\|_{L^\infty} \le \frac{1}{N} \sum_{i=1}^{N} \|\partial_x u_0\|_{L^\infty} (1 + \|\partial_x \ell_t^i\|_{L^\infty})$$

$$\le \frac{1}{N} \sum_{i=1}^{N} \|\partial_x u_0\|_{L^\infty} (1 + 2\|\partial_x \lambda_t^i\|_{L^\infty})$$

$$\le \|\partial_x u_0\|_{L^\infty} + 2c_0 \int_0^t \|\partial_x u_s\|_{L^\infty} \, ds$$

almost surely for $t \in [0, T_0]$. This proves (6.5) for $n = 1$.

For $n > 1$, local existence (Proposition 6.1) guarantees that $\|u_t\|_{C^{n-1}} \le c_{n-1}$ almost surely for $t \in [0, T_0]$, where $c_{n-1} = c_{n-1}(U_{n-1}^0, n)$. Assume by induction that the bound (6.5) holds for some integer $n - 1$. This bound and equation (2.1) immediately imply that $\|\partial_x \lambda_t^i\|_{C^{n-2}} \le c_{n-1}$ almost surely[12] for $t \in [0, T_0]$. Equations (6.6) and (6.3) will imply $\|\partial_x \ell_t^i\|_{C^{n-2}} \le c_{n-1}$ almost surely for $t \in [0, T_0]$.

Thus using equations (2.1) and (6.2) we obtain

$$\|\partial_x^n \lambda_t^i\|_{L^\infty} \le \int_0^t \|\partial_x^n [u_s \circ (I + \lambda_s^i)]\|_{L^\infty} \, ds$$

$$\le c_{n-1} \int_0^t [\|\partial_x^n u_s\|_{L^\infty} + \|\partial_x^n \lambda_s^i\|_{L^\infty}] \, ds$$

almost surely. Using Gronwall's lemma this implies

(6.8)
$$\|\partial_x^n \lambda_t^i\|_{L^\infty} \le c_{n-1} \int_0^t \|\partial_x^n u_s\|_{L^\infty} \, ds$$

---

[12]We remark that our somewhat unusual notation $\|\partial_x \lambda_t^i\|_{C^{n-2}}$ instead of $\|\lambda_t^i\|_{C^{n-1}}$ is necessary. This is because it is impossible to obtain almost sure bounds on $\|\lambda_t^i\|_{L^\infty}$. However, as our argument shows, we can obtain almost sure bounds on $\|\partial_x^k \lambda_t^i\|_{L^\infty}$ for any $k \ge 1$.

almost surely. Here we absorbed the constant $e^{c_{n-1}t}$ into c_{n-1}, which is valid as $t \leq T_0 = T_0(U_1^0)$. Now

$$\|\partial_x^n u_t\|_{L^\infty} \leq \frac{1}{N} \sum_{i=1}^N (\|\partial_x^n u_0\|_{L^\infty}(1 + \|\partial_x \ell_t^i\|_{L^\infty})^n + c_{n-1}\|\partial_x^n \ell_t^i\|_{L^\infty})$$

$$\leq \frac{1}{N} \sum_{i=1}^N (\|\partial_x^n u_0\|_{L^\infty}(1 + 2\|\partial_x \lambda_t^i\|_{L^\infty})^n + c_{n-1}\|\partial_x^n \lambda_t^i\|_{L^\infty})$$

$$\leq \|\partial_x^n u_0\|_{L^\infty}(1 + c_{n-1}t) + c_{n-1} \int_0^t \|\partial_x^n u_s\|_{L^\infty}\, ds$$

almost surely, where we used (6.3) and (6.7) to obtain the second inequality, and equations (6.6) and (6.8) to obtain the third inequality. Now Gronwall's lemma gives (6.5), where we again absorb the exponential factor $e^{c_{n-1}t}$ into $(1 + c_{n-1}t)$, by replacing c_{n-1} with a larger constant, which by our convention we still denote by c_{n-1}. $\quad\square$

PROOF OF LEMMA 2.7. By Proposition 6.1, existence will follow if we establish (2.11) for $n = 1$. We prove (2.11) by induction. Since the constant c_0 in Lemma 6.3 is absolute, the proof for $n = 1$ is identical to the proof of the inductive step. Thus we only prove the inductive step.

Assume that (2.11) holds for $n - 1$, choose $c_{n-1} = c_{n-1}(U_{n-1})$ to be the constant from Lemma 6.3. Thus whenever $\delta_t < T_0$,

(6.9) $$\|\partial_x^n u_{(k+1)\delta_t}^{\delta_t}\|_{L^\infty} \leq (1 + c_{n-1}\delta_t)\|\partial_x^n u_{k\delta_t}^{\delta_t}\|_{L^\infty} \qquad \text{a.s.}$$

holds for all $k \leq \frac{T_0}{\delta_t}$. Iterating this we have

$$\|\partial_x^n u_t^{\delta_t}\|_{L^\infty} \leq (1 + c_{n-1}\delta_t)^{T_0/\delta_t}\|\partial_x^n u_0\|_{L^\infty} \qquad \text{a.s.}$$

for all $t \leq T_0$. Thus we choose U_n to be given by

$$U_n = \|\partial_x^n u_0\|_{L^\infty} \sup_{\delta > 0}(1 + c_{n-1}\delta)^{T_0/\delta}.$$

From (4.3) we see that $\int_x u_t^{\delta_t}$ is conserved almost surely. Since $u_t^{\delta_t}$ is periodic, a bound on $\|\partial_x^n u_t^{\delta_t}\|_{L^\infty}$ will give us a bound on $\|u_t^{\delta_t}\|_{C^n}$, completing the proof. $\quad\square$

7. Proof of Proposition 2.8. In this section we prove Proposition 2.8. We reintroduce an N as a superscript to explicitly keep track of the dependence of our processes on N, and prove convergence as $N \to \infty$.

PROOF OF PROPOSITION 2.8. Let $w_t^N = v_t^N - u_t^b$. Then (1.1) and (2.6) give

(7.1) $$dw_t^N + w_t^N \partial_x v_t^N \, dt + u_t^b \partial_x w_t^N \, dt - \frac{1}{2}\partial_x^2 w_t^N \, dt + \frac{\partial_x v_t^N}{N} \sum_{j=1}^N dW_t^j = 0.$$

Thus, by Itô's formula

$$
\frac{1}{2} d\|w_t^N\|_{L^2}^2 + \left(\int_{\mathbb{T}} (w_t^N)^2 \, \partial_x v_t^N dx \right) dt
$$

$$
+ \left(\int_{\mathbb{T}} u_t^b w_t^N \, \partial_x w_t^N dx \right) dt + \frac{1}{2} \|\partial_x w_t^N\|_{L^2}^2 \, dt
$$

$$
+ \left(\int_{\mathbb{T}} w_t^N \, \partial_x v_t^N dx \right) \left(\frac{1}{N} \sum_{j=1}^{N} dW_t^j \right) dt
$$

$$
= \frac{1}{2N} \|\partial_x v_t^N\|_{L^2}^2 \, dt.
$$

Taking expectations and integrating by parts we obtain

$$
\partial_t \mathbf{E}\|w_t^N\|_{L^2}^2 + \mathbf{E}\left[\int_{\mathbb{T}} (w_t^N)^2 (2 \, \partial_x v_t^N - \partial_x u_t^b) \, dx \right] + \mathbf{E}\|\partial_x w_t^N\|_{L^2}^2
$$

$$
= \frac{1}{N} \mathbf{E}\|\partial_x v_t^N\|_{L^2}^2.
$$

By Lemma 2.6 and the Sobolev embedding theorem, there exists a constant $C = C(s, \|u_0\|_{H^s})$, independent of N, such that

$$
\sup_{t \geq 0} \|\partial_x v_t\|_{L^\infty} \leq C
$$

almost surely. It is well known that the same estimate holds for $\partial_x u_t^b$. Further, since $\mathbf{E}\|\partial_x v_t\|_{L^2}^2 \leq \sup_\Omega \|\partial_x v\|_{L^\infty}^2$, making C larger if necessary we have

$$
\sup_{t \geq 0} \mathbf{E}\|\partial_x v_t\|_{L^2}^2 \leq C.
$$

Thus

$$
\partial_t \mathbf{E}\|w_t^N\|_{L^2}^2 \leq V \left(\mathbf{E}\|w_t^N\|_{L^2}^2 + \frac{1}{N} \right).
$$

and, since $w_0 = 0$, Gronwall's lemma gives

$$
\mathbf{E}\|w_t^N\|_{L^2}^2 \leq \frac{1}{CN} (e^{Ct} - 1)
$$

finishing the proof. $\square$

REFERENCES

[1] ALIBAUD, N., DRONIOU, J. and VOVELLE, J. (2007). Occurrence and non-appearance of shocks in fractal Burgers equations. *J. Hyperbolic Differ. Equ.* **4** 479–499. MR2339805

[2] BEALE, J. T., KATO, T. and MAJDA, A. (1984). Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.* **94** 61–66. MR0763762

[3] CONSTANTIN, P. and IYER, G. (2008). A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations. *Comm. Pure Appl. Math.* **61** 330–345. MR2376844

[4] DE LELLIS, C., OTTO, F. and WESTDICKENBERG, M. (2004). Minimal entropy conditions for Burgers equation. *Quart. Appl. Math.* **62** 687–700.

[5] EVANS, L. C. (1998). *Partial Differential Equations. Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR1625845

[6] FATHI, A. (1997). Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.* **324** 1043–1046. MR1451248

[7] IYER, G. (2006). A stochastic perturbation of inviscid flows. *Comm. Math. Phys.* **266** 631–645. MR2238892

[8] IYER, G. (2006). A stochastic Lagrangian formulation of the Navier–Stokes and related transport equations. Ph.D. thesis, Univ. Chicago.

[9] IYER, G. and MATTINGLY, J. (2008). A stochastic-Lagrangian particle system for the Navier–Stokes equations. *Nonlinearity* **21** 2537–2553. MR2448230

[10] JAUSLIN, H. R., KREISS, H. O. and MOSER, J. (1999). On the forced Burgers equation with periodic boundary conditions. In *Differential Equations*: *La Pietra* 1996 (*Florence*). *Proc. Sympos. Pure Math.* **65** 133–153. Amer. Math. Soc., Providence, RI. MR1662751

[11] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[12] KISELEV, A., NAZAROV, F. and SHTERENBERG, R. (2008). Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.* **5** 211–240. MR2455893

[13] KRYLOV, N. V. (1999). An analytic approach to SPDEs. In *Stochastic Partial Differential Equations*: *Six Perspectives*. *Math. Surveys Monogr.* **64** 185–242. Amer. Math. Soc., Providence, RI. MR1661766

[14] KRYLOV, N. V. and ROZOVSKIĬ, B. L. (1982). Stochastic partial differential equations and diffusion processes. *Uspekhi Mat. Nauk* **37** 75–95. MR0683274

[15] KRYLOV, N. V. (2007). Maximum principle for SPDEs and its applications. In *Stochastic Differential Equations*: *Theory and Applications*. *Interdiscip. Math. Sci.* **2** 311–338. World Sci. Publ., Hackensack, NJ. MR2393582

[16] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations. Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge. MR1070361

[17] MAJDA, A. J. and BERTOZZI, A. L. (2002). *Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics* **27**. Cambridge Univ. Press, Cambridge. MR1867882

[18] METROPOLIS, N. and ULAM, S. (1949). The Monte Carlo method. *J. Amer. Statist. Assoc.* **44** 335–341. MR0031341

[19] ROBERT, C. P. and CASELLA, G. (2004). *Monte Carlo Statistical Methods*, 2nd ed. Springer, New York. MR2080278

[20] ROZOVSKIĬ, B. L. (1990). *Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering*. Translated from the Russian by A. Yarkho. *Mathematics and Its Applications* (*Soviet Series*) **35**. Kluwer, Dordrecht. MR1135324

[21] WEINAN, E., KHANIN, K., MAZEL, A. and SINAI, Y. (2000). Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math.* (2) **151** 877–960. MR1779561

[22] YUDOVICH, V. I. (1963). Non-stationary flows of an ideal incompressible fluid. *Z. Vychisl. Mat. i Mat. Fiz.* **3** 1032–1066. MR0158189

DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213
USA
E-MAIL: gautam@math.cmu.edu

DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16803
USA
E-MAIL: anovikov@math.psu.edu