The Annals of Probability
2011, Vol. 39, No. 4, 1332–1364
DOI: 10.1214/10-AOP583

MARKOV PROCESSES ON TIME-LIKE GRAPHS

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We study Markov processes where the "time" parameter is replaced by paths in a directed graph from an initial vertex to a terminal one. Along each directed path the process is Markov and has the same distribution as the one along any other directed path. If two directed paths do not interact, in a suitable sense, then the distributions of the processes on the two paths are conditionally independent, given their values at the common endpoint of the two paths. Conditions on graphs that support such processes (e.g., hexagonal lattice) are established. Next we analyze a particularly suitable family of Markov processes, called harnesses, which includes Brownian motion and other Lévy processes, on such time-like graphs. Finally we investigate continuum limits of harnesses on a sequence of time-like graphs that admits a limit in a suitable sense.

1. Introduction. Classical stochastic processes are families of random variables $\{X_t, t \in \mathcal{T}\}$, where $\mathcal{T}$ is a subset of $\mathbb{R}$, for example, positive integers or $[0, \infty)$. A notable exception is the family of Gaussian processes, for which the structure of the parameter space $\mathcal{T}$ can be virtually arbitrary. For the other two most popular families of stochastic processes, that is, Markov processes and martingales, the situation is much more complicated. The theories of Markov processes and martingales with the parameter set $\mathcal{T}$ equal to an orthant in $\mathbb{R}^d$ are hard, less developed, less popular and less frequently applied than the original theories with one-dimensional $\mathcal{T}$. A book by Khoshnevisan [9] is an excellent monograph devoted to this field of stochastic processes.

This article introduces a class of stochastic processes where the "time" parameter has been replaced by paths in a directed graph. Our goal is to construct a time structure that matches the Markov property better than other nonrectilinear time sets known in the literature. A number of models with tree-like time parameter sets have been studied, for example, branching Brownian motion (see [3], Section 1.1), and its much more complex version known as Le Gall's Brownian snake (see [3], Section 3.6). Other examples include the Brownian web and the Brownian net (see [4, 13, 14]). In all these models, stochastic processes are defined, in a sense, on random graphs. In contrast, we will be concerned with a *deterministic* parameter space.

Received December 2009; revised April 2010.
[1]Supported in part by NSF Grant DMS-09-06743 and by Grant N N201 397137, MNiSW, Poland.
[2]Supported partly by NSF Grant DMS-10-07563.
MSC2010 subject classifications. 60J60, 60G60, 60J99.
Key words and phrases. Harness, graphical Markov model, time-like graphs.

In statistics, graphical models are widely used (see [6, 10]) in areas such as Bayesian data analysis, modeling of causal relationship, information retrieval and language processing. These are probability distributions of a countable number of random variables indexed by the vertices of a graph where the graph structure induces a set of conditional independence constraints (called the graphical Markov property). One can think of our models as a continuous time analogue of such discrete graphical models where the classical Markov property is preserved.

Informally speaking, the following describes our set-up. Consider the law of a classical Markov process $\mathcal{P}$ in the interval [0, 1]. Consider a finite graph with two distinguished vertices marked "0" and "1," and every other vertex is labeled by a real number between zero and one. Consider the collection of all paths (i.e., a sequence of vertices) starting at 0 and ending at 1 such that successive vertices are increasing and share an edge in the graph. Such paths can be seen as homeomorphic images of the unit interval [0, 1]. Thus every such path indexes a copy of the Markov process with law $\mathcal{P}$. We require the additional constraint that the process is defined uniquely at every vertex.

Barring trivial example, it is not easy to even claim that such processes exist. In fact, the existence and uniqueness of the process depends critically on the structure of the underlying graph. In Section 2 we define a collection of graphs which support such stochastic processes. We call these time-like graphs with no co-terminal cells. In Section 3 we construct "natural Markov process on a time-like graph" and prove its uniqueness in law. In Section 4 we provide examples of graphs that do not satisfy our conditions and do not support a "natural Brownian motion."

In the rest of the sections we focus on a class of laws $\mathcal{P}$ which are called harnesses. Harnesses, defined in Section 5, include all integrable Lévy processes and their corresponding bridges. The final Section 6, is devoted to Brownian motion on the honeycomb graph, and its limit as the diameter of hexagonal cells goes to zero.

2. Time-like graphs.

Intuitively speaking, a time-like graph is a directed graph with Jordan arcs as edges. We will first consider graphs with finite numbers of vertices and edges. We will generalize our definitions to infinite graphs at the end of Section 2.1.

DEFINITION 2.1. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ will be called a *time-like graph* (TLG) if its sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$ satisfy the following properties.

The set $\mathcal{V}$ contains at least two elements, $\mathcal{V} = \{t_0, t_1, \dots, t_N\}$, where $t_0 = 0$, $t_N = 1$, $t_k \in (0, 1)$ for $k = 1, \dots, N-1$ and $t_k \le t_{k+1}$ for $k = 0, \dots, N-1$. We do not exclude the case $\mathcal{V} = \{t_0 = 0, t_N = 1\}$.

Formally, we should say that elements of $\mathcal{V}$ have the form (k, t_k), so that (k, t_k) and $(k+1, t_{k+1})$ are distinct even if $t_k = t_{k+1}$. This would make the notation very complicated, so we will write t_k instead of (k, t_k). This should not cause any confusion.

An edge between t_j and t_k will be denoted E_{jk}. We assume that there is no edge between t_j and t_k if $t_j = t_k$. In particular, a TLG has no loops, that is, edges of the form E_{jj}. By convention, the notation E_{jk} indicates that $t_j < t_k$. We assume that the extreme vertices, t_0 and t_N, have degree 1, and all other vertices t_k have degree 3. We assume that for every vertex t_k, except for t_0 and t_N, there exist edges E_{jk} and E_{kn} with $j < k < n$.

If there is a unique edge between t_j and t_k, $j < k$, then it will be denoted E_{jk}. If there are two edges between t_j and t_k, $j < k$, then they will be denoted E'_{jk} and E''_{jk}. We will write E_{jk} to refer to one of the edges E'_{jk} and E''_{jk} when it is irrelevant which of the two edges is used.

See Figures 1, 2 and 3 below for examples of TLGs. We will next define a "representation" of a TLG, that is, a convenient geometric way to think about such a graph. The choice of space for the representation is not significant. We will limit ourselves to representations in $\mathbb{R}^3$ because it is easy to see that every TLG has a representation in $\mathbb{R}^3$.

DEFINITION 2.2. By abuse of notation, let $E_{jk} : [t_j, t_k] \to \mathbb{R}^2$ denote a continuous function. Assume that the images of the open sets (t_j, t_k) under the maps $t \to (t, E_{jk}(t)) \in \mathbb{R}^3$, where $E_{jk} \in \mathcal{E}$, are disjoint. Suppose that $E_{jk}(t_k) = E_{kn}(t_k)$ if $E_{jk}, E_{kn} \in \mathcal{E}$, and $E_{jk}(t_k) = E_{nk}(t_k)$ if $E_{jk}, E_{nk} \in \mathcal{E}$. We will call the set $\mathcal{R}(\mathcal{G}) = \{(t, E_{jk}(t)) \in [0, 1] \times \mathbb{R}^2 : E_{jk} \in \mathcal{E}, t \in [t_j, t_k]\}$ a *representation* of $\mathcal{G}$. We will say that $\mathcal{G}_1$ is a subgraph of $\mathcal{G}_2$ and write $\mathcal{G}_1 \subset \mathcal{G}_2$ if there exist representations of the two TLGs such that $\mathcal{R}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G}_2)$. We will call $\mathcal{G}$ a *planar* TLG if it has a representation $\mathcal{R}(\mathcal{G}) \subset \mathbb{R}^2$.

REMARK 2.3. There are many representations for a given TLG, but there is a unique TLG corresponding to a given representation.

DEFINITION 2.4. We will call a sequence of edges $(E_{k_1k_2}, E_{k_2k_3}, \dots, E_{k_{n-1}k_n})$ a *time path* if $E_{k_j,k_{j+1}} \in \mathcal{E}$ for every j; note that according to our conventions, $k_1 < k_2 < \cdots < k_n$. We will write $\sigma(k_1, k_2, \dots, k_n)$ to denote a time path $(E_{k_1k_2}, E_{k_2k_3}, \dots, E_{k_{n-1}k_n})$. A time path $\sigma(k_1, k_2, \dots, k_n)$ will be called a *full time path* if $k_1 = 0$ and $k_n = N$.

Let $\bar{t}_j = (t_j, E_{jk}(t_j))$ for $j < N$ and $\bar{t}_N = (t_N, E_{N-1,N}(t_N))$. Note that the definition does not depend on the choice of k.

REMARK 2.5. (i) Note that for every $k \in \{0, \dots, N\}$, there exists at least one full time path $\sigma(k_1, k_2, \dots, k_n)$ such that $k_m = k$ for some m. This follows easily from the assumption that for every vertex t_k, except for t_0 and t_N, there exist edges E_{jk} and E_{kn} with $j < k < n$.

(ii) If $\mathcal{G} = (\mathcal{V}, \mathcal{E}) \subset \mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $t_j \in \mathcal{V}_1$, then $\bar{t}_j \in \mathcal{R}(\mathcal{G})$ does not have the same meaning as $t_j \in \mathcal{V}$.

(iii) By abuse of language, for a time path $\sigma_1 = \sigma(k_1, k_2, \dots, k_n)$, we will call the subset $\{(t, E_{jk}(t)) : E_{jk} \in \sigma_1, t \in [t_j, t_k]\}$ of a representation $\mathcal{R}(\mathcal{G})$ a time path as well. If $\mathcal{G} = (\mathcal{V}, \mathcal{E}) \subset \mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $E_{jk} \in \mathcal{E}_1$ and $\{(t, E_{jk}(t)), t \in [t_j, t_k]\} \subset \mathcal{R}(\mathcal{G})$, then we will write $E_{jk} \subset \mathcal{R}(\mathcal{G})$. Note that $E_{jk} \subset \mathcal{R}(\mathcal{G})$ does not imply that $E_{jk} \in \mathcal{E}$.

2.1. *Time-like graphs with no co-terminal cells* (*NCC-graphs*). We will define a subfamily of time-like graphs, with properties that fit well with a probabilistic structure, to be presented later. We will give two definitions of time-like graphs with no co-terminal cells (NCC-graphs) and then we will show that the definitions are equivalent. Each definition is more useful than the other one in some technical arguments.

DEFINITION 2.6. (i) We will say that time paths $\sigma(j_1, j_2, \dots, j_n)$ and $\sigma(k_1, k_2, \dots, k_m)$ are *co-terminal* if $j_1 = k_1$ and $j_n = k_m$.

(ii) A pair of co-terminal time paths $\sigma(j_1, j_2, \dots, j_n)$ and $\sigma(k_1, k_2, \dots, k_m)$ will be called a *cell* if $\{j_2, \dots, j_{n-1}\} \cap \{k_2, \dots, k_{m-1}\} = \varnothing$. We will call t_{j_1} the *start* of the cell and t_{j_n} will be called the *end* of the cell.

(iii) We will call a cell $(\sigma(j_1, j_2, \dots, j_n), \sigma(k_1, k_2, \dots, k_m))$ *simple* if there does not exist time path $\sigma(i_1, i_2, \dots, i_r)$ such that $i_1 \in \{j_2, \dots, j_{n-1}\}$ and $i_r \in \{k_2, \dots, k_{m-1}\}$, or $i_1 \in \{k_2, \dots, k_{m-1}\}$ and $i_r \in \{j_2, \dots, j_{n-1}\}$.

(iv) If t_j is the start of a cell, let t_{j^*} be the smallest time t_k such that there exists a cell with the start t_j and end t_k. A cell with a start t_j and end t_{j^*} will be called *forward-minimal*. Similarly, if t_k is the end of a cell, let $t_{k'}$ be the largest time t_j such that there exists a cell with the start t_j and end t_k. A cell with a start $t_{k'}$ and end t_k will be called *backward-minimal*. We will call two cells *minimal co-terminal cells* if either they are forward-minimal and have different starts and the same end, or they are backward-minimal and they have different ends but the same start.

(v) We will call a TLG an *NCC-graph* if it does not contain any minimal co-terminal cells.

REMARK 2.7. It is easy to see that if a cell is forward-minimal or backward-minimal then it is simple. For example, suppose that a cell $(\sigma(j_1, j_2, \dots, j_n), \sigma(k_1, k_2, \dots, k_m))$ is forward-minimal, and there is a time path $\sigma(i_1, i_2, \dots, i_r)$ such that $i_1 = j_{n_1} \in \{j_2, \dots, j_{n-1}\}$ and $i_r = k_{m_1} \in \{k_2, \dots, k_{m-1}\}$. Then $\sigma(j_1, \dots, j_{n_1}, i_2, \dots, i_r)$ and $\sigma(k_1, \dots, k_{m_1})$ form a cell with the end $k_{m_1} < k_m$, contradicting the assumption that $k_1^* = k_m$.

The next definition, of the family of NCC*-graphs, is inductive and can be explained as follows. The simplest TLG $\mathcal{G}$, with a representation $\mathcal{R}(\mathcal{G}) = [0, 1] \times \{0\}$, is included in this family. If a graph $\mathcal{G}_1$ already belongs to the family of NCC*-graphs, then we add a time path to $\mathcal{R}(\mathcal{G}_1)$, such that the endpoints of this new path lie on a time path already in $\mathcal{R}(\mathcal{G}_1)$ and neither endpoint is a vertex already present in $\mathcal{R}(\mathcal{G}_1)$. Thus amended representation corresponds to a TLG $\mathcal{G}_2$ which we add to the family of NCC*-graphs.

DEFINITION 2.8. We will define *NCC*-graphs* in an inductive way.

(i) The minimal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{t_0 = 0, t_N = 1\}$ and $\mathcal{E} = \{E_{0N}\}$, is an NCC*-graph.

(ii) Suppose that a graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is NCC*, where $\mathcal{V}_1 = \{t_0, t_1, \dots, t_N\}$. Suppose that $t_j, t_k \notin \mathcal{V}_1$, $t_j < t_k$, and for some $E_{j_1 j_2}, E_{k_1 k_2} \in \mathcal{E}_1$, we have $t_{j_1} < t_j < t_{j_2}$ and $t_{k_1} < t_k < t_{k_2}$. Let $\mathcal{V}_2 = \mathcal{V}_1 \cup \{t_j, t_k\}$. Assume that there exists a time path $\sigma(m_1, m_2, \dots, m_n)$ such that $j_1 = m_{n_1}, j_2 = m_{n_1+1}, k_1 = m_{n_2}$, and $k_2 = m_{n_2+1}$, for some $1 \le n_1 \le n_2 \le n_2 + 1 \le n$. If $n_1 < n_2$, then we let $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E_{jk}, E_{j_1 j}, E_{j j_2}, E_{k_1 k}, E_{k k_2}\}) \setminus \{E_{j_1 j_2}, E_{k_1 k_2}\}$. If $n_1 = n_2$, then we let $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E'_{jk}, E''_{jk}, E_{j_1 j}, E_{k j_2}\}) \setminus \{E_{j_1 j_2}\}$. We add $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ to the family of NCC*-graphs.

(iii) We will say that a sequence $\{\mathcal{G}_j\}_{1 \le j \le k}$ is a *tower* of NCC*-graphs if all graphs in the sequence are NCC*, and for every $j > 1$, $\mathcal{G}_j$ is constructed from $\mathcal{G}_{j-1}$ as in part (ii) of the definition.

THEOREM 2.9. (i) *A TLG is an NCC-graph if and only if it is an NCC*-graph.*
(ii) *Every planar TLG is NCC.*
(iii) *There exists a nonplanar NCC-graph.*
(iv) *There exists a non-NCC-graph.*

PROOF. (i) *Step* 1 (*NCC* $\Rightarrow$ *NCC**). Suppose that an NCC-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is given. We will show how to construct it in an inductive way.

We let $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ be the minimal graph with $\mathcal{V}_1 = \{t_0 = 0, t_N = 1\}$ and $\mathcal{E}_1 = \{E_{0N}\}$. Note that $\mathcal{G}_1$ is an NCC*-graph and we can find representations for $\mathcal{G}$ and $\mathcal{G}_1$ such that $\mathcal{R}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G})$.

Suppose that an NCC*-graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ has been constructed and there exist representations such that $\mathcal{R}(\mathcal{G}_k) \subset \mathcal{R}(\mathcal{G})$. Moreover, assume that if two edges $E_{m_1 m}$ and $E_{m_2 m}$ belong to $\mathcal{G}_k$, then t_m is the end of a forward-minimal cell in $\mathcal{G}$. We will prove that there exists an NCC*-graph $\mathcal{G}_{k+1} \ne \mathcal{G}_k$, such that $\mathcal{R}(\mathcal{G}_k) \subset \mathcal{R}(\mathcal{G}_{k+1}) \subset \mathcal{R}(\mathcal{G})$. We will construct $\mathcal{G}_{k+1}$ in such a way that if two edges $E_{m_1 m}$ and $E_{m_2 m}$ belong to $\mathcal{G}_{k+1}$, then t_m is the end of a forward-minimal cell in $\mathcal{G}$. Since $\mathcal{G}$ has a finite number of edges, we must have $\mathcal{R}(\mathcal{G}_{k+1}) = \mathcal{R}(\mathcal{G})$ for some k, so all we have to do to finish the proof is to complete the inductive step.

Suppose that $\mathcal{G}_k \ne \mathcal{G}$. There exists $\bar{t}_n \in \mathcal{R}(\mathcal{G}_k)$ such that $t_n \in \mathcal{V} \setminus \mathcal{V}_k$ and $E_{nm} \not\subset \mathcal{R}(\mathcal{G}_k)$ for some m, because there is at least one edge in $\mathcal{R}(\mathcal{G}) \setminus \mathcal{R}(\mathcal{G}_k)$ that is connected to $\mathcal{R}(\mathcal{G}_k)$, and it is impossible for all such edges to leave $\mathcal{R}(\mathcal{G}_k)$ in the negative direction. Let t_{j_1} be the largest t_n with this property. Let $t_{j_1^*}$ be defined as in Definition 2.6(iv), relative to $\mathcal{G}$.

Case (a). Suppose that $\bar{t}_{j_1^*} \in \mathcal{R}(\mathcal{G}_k)$. Then there exists a time path $\sigma_1 = \sigma(q_0, q_1, \dots, q_{n_1})$ in $\mathcal{G}$, with $q_0 = j_1$, $q_{n_1} = j_1^*$, and $E_{q_0 q_1} \not\subset \mathcal{R}(\mathcal{G}_k)$.

We will now show that σ_1 does not intersect $\mathcal{R}(\mathcal{G}_k)$, except for its endpoints. Suppose otherwise. Then σ_1 intersects $\mathcal{R}(\mathcal{G}_k)$ at some $\bar{t}_m$ such that $t_{j_1} < t_m < t_{j_1^*}$, and for some t_{m_1}, we have $E_{mm_1} \in \sigma_1$ and $E_{mm_1} \not\subset \mathcal{R}(\mathcal{G}_k)$. Let t_{m_2} be the largest t_m with these properties. Then $\bar{t}_{m_2} \in \mathcal{R}(\mathcal{G}_k)$, $t_{m_2} \in \mathcal{V} \setminus \mathcal{V}_k$ and $E_{m_2m_3} \not\subset \mathcal{R}(\mathcal{G}_k)$ for some m_3. Since $t_{j_1} < t_{m_2}$, this contradicts the definition of t_{j_1}.

We add $\{(t, E_{q_rq_{r+1}}(t)), 0 \le r \le n_1 - 1, t \in [t_{q_r}, t_{q_{r+1}}]\}$ to $\mathcal{R}(\mathcal{G}_k)$, and we let this new set to be the representation of $\mathcal{G}_{k+1}$.

We have assumed that if two edges E_{m_1m} and E_{m_2m} belong to $\mathcal{G}_k$, then t_m is the end of a forward-minimal cell in $\mathcal{G}$. This implies that $t_{j_1^*}$ cannot be a vertex of $\mathcal{G}_k$ because it is the end of a forward-minimal cell in $\mathcal{G}$ which is not in $\mathcal{G}_k$, and the assumption that $\mathcal{G}$ is NCC implies that there are no two forward-minimal cells in $\mathcal{G}$ with the same endpoint.

The TLG $\mathcal{G}_{k+1}$ is an NCC*-graph because it was constructed from an NCC*-graph as in Definition 2.8(ii). It is clear that $\mathcal{R}(\mathcal{G}_{k+1}) \subset \mathcal{R}(\mathcal{G})$. Since $t_{j_1^*}$ is the end of a forward-minimal cell in $\mathcal{G}$, all vertices in $\mathcal{G}_{k+1}$ satisfy the property that if two edges E_{m_1m} and E_{m_2m} belong to $\mathcal{G}_{k+1}$ then t_m is the end of a forward-minimal cell in $\mathcal{G}$.

Case (b). Next suppose that $\bar{t}_{j_1^*} \notin \mathcal{R}(\mathcal{G}_k)$. The vertex $t_{j_1^*}$ is the end of a cell (σ_3, σ_4) in $\mathcal{G}$, with the start at t_{j_1}. Since $\bar{t}_{j_1} \in \mathcal{R}(\mathcal{G}_k)$, one and only one of the time paths σ_3 and σ_4 (say, σ_3) has an edge E_{j_1,j_2} that belongs to $\mathcal{R}(\mathcal{G}_k)$. Let E_{j_3,j_4} be the first edge in σ_3 that does not belong to $\mathcal{R}(\mathcal{G}_k)$. Then $\bar{t}_{j_3} \in \mathcal{R}(\mathcal{G}_k)$, $t_{j_3} \in \mathcal{V} \setminus \mathcal{V}_k$ and $E_{j_3j_4} \not\subset \mathcal{R}(\mathcal{G}_k)$. Since $t_{j_3} > t_{j_1}$, this contradicts the definition of t_{j_1}. Hence, it cannot happen that $\bar{t}_{j_1^*} \notin \mathcal{R}(\mathcal{G}_k)$.

This completes the proof of the inductive step and shows that NCC-graphs are NCC*-graphs.

Step 2 (*NCC** $\Rightarrow$ *NCC*). The proof will be inductive. The minimal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{t_0 = 0, t_N = 1\}$ and $\mathcal{E} = \{E_{0N}\}$, is an NCC*-graph, and it is also an NCC-graph.

In Definition 2.8, new graphs in the family of NCC*-graphs are created from other graphs in the same family. Suppose that $\mathcal{G}_1$ is the minimal graph defined above, and $(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M)$ is any tower of NCC*-graphs. Suppose that a vertex t_k is added to $\mathcal{G}_{n-1}$ so that $\mathcal{G}_n$ is the first graph in the sequence that has the vertex t_k. Suppose that t_k is the end of a cell. We will show that t_k is the end of only one forward-minimal cell in $\mathcal{G}_n$, and that it will not be the end of any other forward-minimal cell in any graph $\mathcal{G}_m$ for $n \le m \le M$.

Note that the definitions of NCC*-graphs and NCC-graphs are invariant under time reversal, so the analysis of forward-minimal cells can be applied to backward-minimal cells. Hence, we will limit our argument to forward-minimal cells.

Let us recall the construction given in Definition 2.8(ii). Suppose that $\mathcal{G}_{n-1} = (\mathcal{V}_{n-1}, \mathcal{E}_{n-1})$ is NCC*, where $\mathcal{V}_{n-1} = \{t_0, t_1, \dots, t_N\}$. There exist $t_j, t_k \notin \mathcal{V}_{n-1}$, $t_j < t_k$ such that $E_{j_1j_2}, E_{k_1k_2} \in \mathcal{E}_1$ for some $t_{j_1} < t_j < t_{j_2}$ and $t_{k_1} < t_k < t_{k_2}$. We have $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $\mathcal{V}_n = \mathcal{V}_{n-1} \cup \{t_j, t_k\}$. There exists a time path

$\sigma(m_1, m_2, \ldots, m_n)$ such that $j_1 = m_{n_1}, j_2 = m_{n_1+1}, k_1 = m_{n_2}$, and $k_2 = m_{n_2+1}$, for some $1 \le n_1 \le n_2 \le n_2 + 1 \le n$. There are two possible cases: (a) If $n_1 < n_2$, then $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E_{jk}, E_{j_1 j}, E_{j j_2}, E_{k_1 k}, E_{k k_2}\}) \setminus \{E_{j_1 j_2}, E_{k_1 k_2}\}$; (b) if $n_1 = n_2$, then $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E'_{jk}, E''_{jk}, E_{j_1 j}, E_{k j_2}\}) \setminus \{E_{j_1 j_2}\}$.

First we will show that t_k is the end of only one forward-minimal cell in $\mathcal{G}_n$. In case (b), it is obvious that there is only one cell $(\{E'_{jk}\}, \{E''_{jk}\})$ that is forward-minimal and has the end at t_k. Consider case (a) and let $\sigma_1 = \sigma(j, m_{n_1+1}, \ldots, m_{n_2}, k)$. The cell $(\sigma_1, \{E_{jk}\})$ is forward-minimal and has the end at t_k. Suppose that some other cell (σ_2, σ_3) in $\mathcal{G}_n$ has t_k as its end, and call its start t_r. Then one of the time paths σ_2 or σ_3 (say, σ_2) must pass through t_j, and the other one, σ_3, must pass through t_{k_1}. Recall that σ_1 is a time path that goes through t_j and t_{k_1}. Let the concatenation of the part of σ_2 between t_r and t_j and the part of σ_1 between t_j and t_{k_1} be called σ_4. The time path σ_4 starts at t_r with an edge different from the first edge of σ_3. The paths σ_3 and σ_4 contain t_{k_1} so a forward-minimal cell with start t_r must have the end at t_{k_1} or an earlier time. Therefore, it cannot have the end at $t_k > t_{k_1}$.

Next we will show that t_k cannot be the end of two forward-minimal cells in any graph $\mathcal{G}_m$, $m > n$. Suppose to the contrary that t_k is the end of two different forward-minimal cells in $\mathcal{G}_q$ for some $q > n$, but it is not the end of two different forward-minimal cells in $\mathcal{G}_m$, $m < q$. Recall that, according to our construction, $t_k = t_{j^*}$, that is, when we added t_k to the set of vertices, we also created a forward-minimal cell with start t_j and end t_k. Suppose that $\mathcal{G}_q$ was constructed by adding an edge $E_{\ell_1 \ell_2}$ to $\mathcal{G}_{q-1}$, and this procedure created a new forward-minimal cell $(\sigma_1, \sigma_2) = (\sigma(q_1, \ldots, q_{s_1}, k), \sigma(r_1, \ldots, r_{s_2}, k))$ in $\mathcal{G}_q$ with $q_1 = r_1 \neq j$. The edge $E_{\ell_1 \ell_2}$ must belong to one of the time paths in this cell, say, $\ell_1 = q_{s_3}$, $\ell_2 = q_{s_3+1}$ and $q_{s_3+1} \neq k$. According to Definition 2.8(ii), $\mathcal{G}_{q-1}$ must contain either a time path $\sigma(q_{s_3-1}, u_1, \ldots, u_{s_4}, q_{s_3+2})$ or $\sigma(q_{s_3-1}, q_{s_3+2})$. Then $\mathcal{G}_{q-1}$ contains the cell $(\sigma(q_1, \ldots, q_{s_3-1}, u_1, \ldots, u_{s_4}, q_{s_3+2}, \ldots, q_{s_1}, k), \sigma(r_1, \ldots, r_{s_2}, k))$, possibly with $u_1, \ldots, u_{s_4}$ missing in the first path. If this is a forward-minimal cell, then this contradicts the assumption that there is only one forward-minimal cell in $\mathcal{G}_{q-1}$ with end t_k. If this cell is not forward-minimal, then q_1^*, defined as in Definition 2.6(iv), satisfies $q_1^* < k$, relative to $\mathcal{G}_{q-1}$. This implies that $q_1^* < k$, relative to $\mathcal{G}_q$, which contradicts the assumption that (σ_1, σ_2) is a forward minimal cell. This completes the proof of part (i).

(ii) It is easy to see that if a TLG $\mathcal{G}$ is planar, then the region enclosed by $\mathcal{R}(\mathcal{G})$ is divided by $\mathcal{R}(\mathcal{G})$ into nonintersecting cells that are both forward-minimal and backward-minimal. Therefore every vertex, except t_0 and t_N, is either the start or the end of a single cell that is forward-minimal and backward-minimal.

(iii) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{t_j = j/7, j = 0, 1, \ldots, 7\}$ and

$$\mathcal{E} = \{E_{0,1}, E_{1,2}, E_{2,3}, E_{3,4}, E_{4,5}, E_{5,6}, E_{6,7}, E_{1,4}, E_{2,5}, E_{3,6}\}.$$

It is elementary to check that $\mathcal{G}$ is an NCC-graph and that it is not planar. See Figure 1.

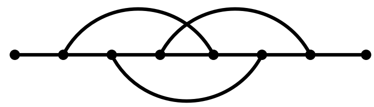

FIG. 1. *A nonplanar NCC-graph.*

(iv) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{t_j = j/7, j = 0, 1, \dots, 7\}$ and

$$\mathcal{E} = \{E_{0,1}, E_{1,2}, E_{1,3}, E_{2,4}, E_{2,5}, E_{3,4}, E_{3,5}, E_{4,6}, E_{5,6}, E_{6,7}\}.$$

The cells $(\sigma(3,4,6), \sigma(3,5,6))$ and $(\sigma(2,5,6), \sigma(2,4,6))$ are minimal and co-terminal. Hence, $\mathcal{G}$ is not an NCC-graph. See Figure 2. □

REMARK 2.10. The analysis of NCC-graphs is somewhat complicated due to the following facts.

(i) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are NCC-graphs and $\mathcal{R}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G}_2)$, then it does not necessarily follow that $\mathcal{G}_1$ and $\mathcal{G}_2$ belong to a tower of NCC-graphs. For example, let $\mathcal{G}_1$ be obtained from the graph in Figure 2 by removing E_{34} and E_{25}, and let $\mathcal{G}_2$ be obtained from the graph in Figure 2 by removing E_{34}. Adding E_{25} to $\mathcal{G}_1$ does not conform to the rules of Definition 2.8.

(ii) It is quite obvious that there exist TLGs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{R}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G}_2)$, $\mathcal{G}_1$ is NCC and $\mathcal{G}_2$ is not NCC. For example, take $\mathcal{G}_1$ to be a single full path and $\mathcal{G}_2$ to be the graph in Figure 2. It is less obvious that there exist TLGs $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{R}(\mathcal{G}_1) \subset \mathcal{R}(\mathcal{G}_2)$, $\mathcal{G}_2$ is NCC and $\mathcal{G}_1$ is not NCC. For example, let $\mathcal{G}_2$ be the graph in Figure 3. To see that $\mathcal{G}_2$ is NCC, note that one can construct it as in Definition 2.8 by starting with the full path $\sigma(t_0, t_1, t_4, t_5, t_6, t_7, t_{10}, t_{11})$ and adding edges in this order: $\sigma(t_1, t_2, t_3, t_4)$, $\sigma(t_3, t_6)$, $\sigma(t_7, t_8, t_9, t_{10})$, $\sigma(t_5, t_8)$, $\sigma(t_2, t_9)$. Let $\mathcal{G}_1$ be the graph obtained by removing E_{34} and E_{78} from $\mathcal{G}_2$. The graph $\mathcal{G}_1$ is topologically the same as that in Figure 2 so it is non-NCC.

We will extend the definition of TLGs to graphs with infinitely many vertices. First, we present two simple generalizations of TLGs with finite $\mathcal{V}$. It will be convenient to allow TLGs (with finitely many vertices) in which t_0 and t_N take values in $\mathbb{R} \cup \{-\infty, \infty\}$ with the restriction that $t_0 < t_N$. Clearly, all theorems proved so

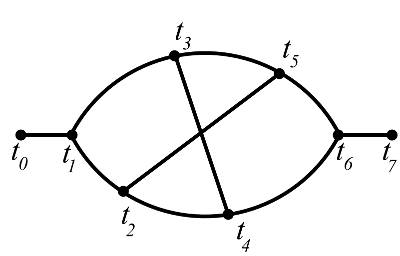

FIG. 2. *A non-NCC-graph.*

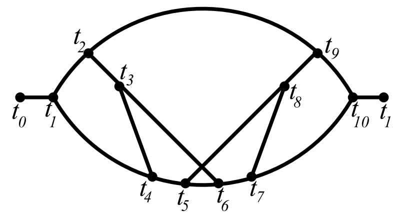

FIG. 3. *An NCC-graph that contains a non-NCC-graph.*

far apply to thus enlarged family of TLGs. Note that allowing t_0 and t_N to take infinite values does not add anything significant to the model because we can rescale the graph by the deterministic function $t \to \arctan t$. We allow for infinite values of t_0 and t_N to be able to study standard examples of Markov processes on the real line.

DEFINITION 2.11. (i) Suppose that the vertex set of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is infinite. We will call $\mathcal{G}$ a *time-like graph* (TLG) if it satisfies the following conditions. (a) There exists a sequence of TLGs $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $n \geq 1$, such that each $\mathcal{V}_n$ is finite, and for some representations of $\mathcal{G}_n$'s and $\mathcal{G}$ we have $\mathcal{R}(\mathcal{G}_n) \subset \mathcal{R}(\mathcal{G}_{n+1})$ for every n, and $\bigcup_n \mathcal{R}(\mathcal{G}_n) = \mathcal{R}(\mathcal{G})$. (b) The graph $\mathcal{G}$ is *locally finite*, that is, it has a representation $\mathcal{R}(\mathcal{G}) \subset \mathbb{R}^3$ such that for any compact set $K \subset \mathbb{R}^3$, only a finite number of edges intersect K.

(ii) We will call a TLG $\mathcal{G}$ with infinite vertex set an NCC-graph if it satisfies the following conditions. (a) The sequence $\{\mathcal{G}_n\}_{n\geq 1}$ in part (i) of the definition can be chosen so that it is a tower of NCC-graphs in the sense of Definition 2.8(iii). (b) Let $\mathcal{V}_n = \{t_{0,n}, t_{1,n}, \dots, t_{N_n,n}\}$. The initial vertices $t_{0,n} \in \mathcal{V}_n$ and terminal vertices $t_{N_n,n} \in \mathcal{V}_n$ are the same for all $\mathcal{G}_n$, that is, $t_{0,n} = t_{0,m}$ and $t_{N_n,n} = t_{N_m,m}$ for all n and m.

REMARK 2.12. (i) Recall the notation from Definition 2.11(ii). It follows from conditions (a) and (b) of that definition that the initial edges form a decreasing sequence, that is, $E_{t_{0,n},t_{1,n}} \subset E_{t_{0,m},t_{1,m}}$ if $n > m$. Similarly, terminal edges form a decreasing sequence, that is, $E(t_{N_n-1,n}, t_{N_n,n}) \subset E(t_{N_m-1,m}, t_{N_m,m})$ if $n > m$.

(ii) It is easy to see that if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a TLG with infinite number of vertices, then all vertices have degree 3, except for at most two vertices with degree 1.

3. Markov processes on time-like graphs. Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a TLG and $\mathcal{V}$ is finite. Let $\mathcal{P}$ denote the distribution of a Markov process $\{Y(t), t \in [t_0, t_N]\}$. We do not assume that the Markov process is necessarily time-homogeneous, that is, that its transition probabilities are invariant under time shifts.

The regular conditional distribution of $\{Y(t), t \in [t_1, t_2]\}$ given $\{Y(t_1) = y_1, Y(t_2) = y_2\}$ exists for $\mathcal{P}$-almost all values of $(Y(t_1), Y(t_2))$, under mild assumptions on the state space of Y (see Section 21.4 in [5] for a discussion of conditional probabilities). The conditional distribution of $\{Y(t), t \in [t_1, t_2]\}$ given

$\{Y(t_1) = y_1, Y(t_2) = y_2\}$ will be called a *Markov bridge*. The Markov bridge is a (time-inhomogeneous) Markov process on the interval $[t_1, t_2]$.

DEFINITION 3.1. Let X be a collection of random variables $X_{jk}(t)$, for all $E_{jk} \in \mathcal{E}$ and $t \in [t_j, t_k]$. If $E_{jk}, E_{kn} \in \mathcal{E}$, then we assume that $X_{jk}(t_k) = X_{kn}(t_k)$, and similarly, if $E_{jk}, E_{nk} \in \mathcal{E}$, then $X_{jk}(t_k) = X_{nk}(t_k)$.

Recall that we may have two edges E'_{jk} and E''_{jk} with the same endpoints t_j and t_k. Then the collection of random variables $X_{jk}(t)$ contains separate families $\{X'_{jk}(t), t \in [t_j, t_k]\}$ and $\{X''_{jk}(t), t \in [t_j, t_k]\}$ corresponding to each edge.

Consider a time path $\sigma_1 = \sigma(k_1, k_2, \dots, k_n)$ and let $X_{\sigma_1}(t) = X_{k_1,k_2,\dots,k_n}(t) = X_{k_j,k_{j+1}}(t)$ for all $j = 1, 2, \dots, n-1$ and $t \in [t_{k_j}, t_{k_{j+1}}]$. We will call X a $\mathcal{P}$-*process on* $\mathcal{G}$ if for every full time path σ, the process $\{X_\sigma(t), t \in [t_0, t_N]\}$ has distribution $\mathcal{P}$. We will write $X(t)$ instead of $X_{jk}(t)$ or $X_\sigma(t)$ when no confusion may arise.

We extend the notion of a $\mathcal{P}$-process on a TLG (with finite $\mathcal{V}$) to processes that are defined for all $t \in E$, $E \in \mathcal{E}$, except t_0 and t_N. For example, we can take $t_0 = -\infty$, $t_N = \infty$ and let $\mathcal{P}$ be the distribution of a two-sided Brownian motion conditioned to have value 0 at time 0. This extension does not pose any technical problems but allows us to consider natural examples.

Note that if X is a $\mathcal{P}$-process and $\sigma_1 = \sigma(k_1, k_2, \dots, k_n)$ then conditionally on $X(t_{k_j}) = x_j$, $1 \le j \le n$, the path $\{X(t), t \in \sigma_1\}$ has the same distribution as the concatenation of independent Markov bridges from $(t_{k_j}, X(t_{k_j}))$ to $(t_{k_{j+1}}, X(t_{k_{j+1}}))$, $1 \le j \le n$.

For every TLG $\mathcal{G}$ and every $\mathcal{P}$, there exists a $\mathcal{P}$-process on $\mathcal{G}$. A trivial example of a $\mathcal{P}$-process on a TLG can be constructed by taking a Markov process $\{Y(t), t \in [t_0, t_N]\}$ with distribution $\mathcal{P}$ and then letting $X_{jk}(t) = Y(t)$ for all $E_{jk} \in \mathcal{E}$ and $t \in [t_j, t_k]$.

DEFINITION 3.2. Suppose that $\mathcal{W} \subset \mathcal{R}(\mathcal{G})$ is a finite nonempty set such that $\mathcal{R}(\mathcal{G}) \setminus \mathcal{W}$ is disconnected. Some edges of $\mathcal{G}$ are cut by $\mathcal{W}$ into two or more subedges; let us call this new collection of edges $\mathcal{E}_0$. Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint sets of edges with the union equal to $\mathcal{E}_0$. Each set $\mathcal{E}_1$ and $\mathcal{E}_2$ may consist of several connected components of $\mathcal{R}(\mathcal{G}) \setminus \mathcal{W}$. We will call a process X on a TLG $\mathcal{G}$ a *graph-Markovian* process if for all $\mathcal{W}$, $\mathcal{E}_1$ and $\mathcal{E}_2$, the conditional distribution of $\{X(t), t \in E, E \in \mathcal{E}_1\}$ given $\{X(t), t \in E, E \in \mathcal{E}_2\}$ depends only on $\{X(t), t \in \mathcal{W}\}$.

DEFINITION 3.3. For a point t in $\mathcal{G}$, let $F(t)$ ("the future of t") be the set of all points $s \ge t$ such that there is a full path passing through t and s. Similarly, let $P(t)$ ("the past of t") be the set of all points $s \le t$ such that there is a full path passing through t and s. We will say that a process X on a TLG $\mathcal{G}$ is *time-Markovian* if for every t, the conditional distributions of $\{X(s), s \in F(t)\}$ and $\{X(s), s \in P(t)\}$ given $X(t)$ are independent.

REMARK 3.4. (i) Suppose that a process X on a TLG $\mathcal{G}$ is time-Markovian, and s and t lie on a full path σ, with $s < t$. It is easy to see that the conditional distributions of $\{X(u), s \le u \le t, u \in \sigma\}$, $\{X(u), u \in F(t)\}$ and $\{X(u), u \in P(s)\}$ given $X(s)$ and $X(t)$ are jointly independent. Moreover, the conditional distribution of $\{X(u), s \le u \le t, u \in \sigma\}$ given $X(s)$ and $X(t)$ is that of a Markov bridge between $(s, X(s))$ and $(t, X(t))$.

(ii) It is easy to see that if a process X on a TLG $\mathcal{G}$ is graph-Markovian, then the families of random variables $\{X(t), t \in E\}$, $E \in \mathcal{E}$, are conditionally independent given $\{X(t), t \in \mathcal{V}\}$, and for every $E_{jk} \in \mathcal{E}$, the conditional distribution of $\{X(t), t \in E_{jk}\}$ given $\{X(t), t \in \mathcal{V}\}$ is a Markov bridge between $(t_j, X(t_j))$ and $(t_k, X(t_k))$.

DEFINITION 3.5. We will say that a $\mathcal{P}$-process X on a TLG $\mathcal{G}$ with finite vertex set $\mathcal{V}$ is *natural* if it is time-Markovian and graph-Markovian.

Recall that we call a cell $(\sigma(j_1, j_2, \dots, j_{n_1}), \sigma(k_1, k_2, \dots, k_{n_2}))$ simple if there is no time path $\sigma(m_1, m_2, \dots, m_{n_3})$ such that we have $m_1 \in \{j_2, \dots, j_{n_1-1}\}$ and $m_{n_3} \in \{k_2, \dots, k_{n_2-1}\}$, or $m_1 \in \{k_2, \dots, k_{n_2-1}\}$ and $m_{n_3} \in \{j_2, \dots, j_{n_1-1}\}$.

DEFINITION 3.6. We will say that a process X on a TLG $\mathcal{G}$ is *cell-Markovian* if for any simple cell consisting of $\sigma(j_1, j_2, \dots, j_{n_1})$ and $\sigma(k_1, k_2, \dots, k_{n_2})$, the processes $\{X_{j_1, j_2, \dots, j_{n_1}}(t), t \in [t_{j_1}, t_{j_{n_1}}]\}$ and $\{X_{k_1, k_2, \dots, k_{n_2}}(t), t \in [t_{k_1}, t_{k_{n_2}}]\}$ are conditionally independent, given the values of $X_{j_1, j_2, \dots, j_{n_1}}(t_{j_1})$ and $X_{j_1, j_2, \dots, j_{n_1}}(t_{j_{n_1}})$ [these are the same as $X_{k_1, k_2, \dots, k_{n_2}}(t_{k_1})$ and $X_{k_1, k_2, \dots, k_{n_2}}(t_{k_{n_2}})$].

Note that there is no direct logical relation between the notions of time-Markovian, graph-Markovian and cell-Markovian processes.

THEOREM 3.7. (i) *For every NCC-graph $\mathcal{G}$ with finite vertex set $\mathcal{V}$ and every Markov process $\mathcal{P}$, there exists a natural $\mathcal{P}$-process X on $\mathcal{G}$, and the distribution of such a process is unique. The natural $\mathcal{P}$-process is cell-Markovian.*

(ii) *Suppose that for some TLG $\mathcal{G}$ with $\mathcal{V} = \{t_0 = 0, t_1, \dots, t_N = 1\}$, there exist simple coterminal cells (σ_1, σ_2) with endpoints $t_1 < t_2$, and (σ_3, σ_4) with endpoints $t_3 < t_4$. Assume that either $t_1 < t_3$ or $t_2 < t_4$. Then there is no natural Brownian motion on $\mathcal{G}$.*

REMARK 3.8. (i) Part (ii) of Theorem 3.7 cannot be generalized to say that "Then there is no natural Markov process on $\mathcal{G}$." The reason is that the process identically equal to 0 is a natural Markov processes on every TLG. There are also less trivial examples.

(ii) If we take $t_2 = 3/7$ instead of $2/7$ in Figure 2, then we will have an example of a TLG with coterminal cells for which neither $t_1 < t_3$ nor $t_2 < t_4$ holds. The starts t_2 and t_3 of the two cells correspond to the same time $3/7$.

(iii) If $\mathcal{G}_1 \subset \mathcal{G}_2$, both graphs are NCC, X is a natural $\mathcal{P}$-process on $\mathcal{G}_1$ and X' is a natural $\mathcal{P}$-process on $\mathcal{G}_2$, then it is not necessarily true that the distribution of X

is that of X' restricted to $\mathcal{G}_1$. To see this, let $\mathcal{P}$ be the distribution of Brownian motion, $\mathcal{G}_2$ be the graph in Figure 3 and let $\mathcal{G}_1$ be the graph obtained by deleting the edges E_{36}, E_{23} and E_{34}. One can check that the joint distribution of $(X(t_2), X(t_4))$ is different from that of $(X'(t_2), X'(t_4))$. This can be shown by applying Proposition 4.1 to $(X(t_2), X(t_4))$. To determine the distribution of $(X'(t_2), X'(t_4))$, note that t_2 and t_4 lie on a full time path in $\mathcal{G}_2$.

(iv) Suppose that an NCC-graph $\mathcal{G}$ is the last element of a tower of NCC-graphs $\{\mathcal{G}_k\}_{1\le k\le n}$. Then the restriction of a natural process X on $\mathcal{G}$ to any $\mathcal{G}_j$, $1 \le j \le n$, is a natural process on $\mathcal{G}_j$. This follows from the proof of Theorem 3.7 below and from the uniqueness of the natural $\mathcal{P}$-process.

(v) Does uniqueness in Theorem 3.7(i) hold true if we replace "natural" with "graph-Markovian?" We leave this as an open problem.

We will prove part (i) of Theorem 3.7 in this section and part (ii) in the next section.

PROOF OF THEOREM 3.7(i). We assume that $\mathcal{V}$ is finite, $t_0 = 0$ and $t_N = 1$. Fix any Markov process distribution $\mathcal{P}$. We will use induction, since according to Theorem 2.9, the family of all NCC-graphs can be constructed inductively, as in Definition 2.8.

It is obvious that there exists a unique in law natural $\mathcal{P}$-process on the minimal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{t_1 = 0, t_N = 1\}$ and $\mathcal{E} = \{E_{1N}\}$. It is also easy to see that this process is cell-Markovian.

Suppose that $\mathcal{G}_1$ is an NCC-graph. We make the inductive assumption that there exists a natural $\mathcal{P}$-process X on $\mathcal{G}_1$, it is unique in law and it is cell-Markovian.

Recall how a new graph $\mathcal{G}_2$ is constructed in part (ii) of Definition 2.8. Suppose that $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ with $\mathcal{V}_1 = \{t_1, t_2, \dots, t_N\}$. Suppose that $t_j, t_k \notin \mathcal{V}_1$, $t_j < t_k$, and for some $E_{j_1 j_2}, E_{k_1 k_2} \in \mathcal{E}_1$, we have $t_{j_1} < t_j < t_{j_2}$ and $t_{k_1} < t_k < t_{k_2}$. Let $\mathcal{V}_2 = \mathcal{V}_1 \cup \{t_j, t_k\}$. Assume that there exists a time path $\sigma(m_1, m_2, \dots, m_n)$ such that $j_1 = m_{n_1}$, $j_2 = m_{n_1+1}$, $k_1 = m_{n_2}$, and $k_2 = m_{n_2+1}$, for some $1 \le n_1 \le n_2 \le n_2 + 1 \le n$. (a) If $n_1 < n_2$, then we let $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E_{jk}, E_{j_1 j}, E_{j j_2}, E_{k_1 k}, E_{k k_2}\}) \setminus \{E_{j_1 j_2}, E_{k_1 k_2}\}$. (b) If $n_1 = n_2$, then we let $\mathcal{E}_2 = (\mathcal{E}_1 \cup \{E'_{jk}, E''_{jk}, E_{j_1 j}, E_{k j_2}\}) \setminus \{E_{j_1 j_2}\}$. Then $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$.

It will suffice to show that there exists a natural $\mathcal{P}$-process X on $\mathcal{G}_2$; it is unique in law and it is cell-Markovian.

In case (a), we effectively add only one edge E_{jk} to graph $\mathcal{G}_1$. Other "new" edges $E_{j_1 j}, E_{j j_2}, E_{k_1 k}$ and $E_{k k_2}$ are created by subdividing $E_{j_1 j_2}$ and $E_{k_1 k_2}$. Let $Z_1 = X_{j_1 j_2}(t_j)$ and $Z_2 = X_{k_1 k_2}(t_k)$. We define $\{X'_{jk}(t), t \in [t_j, t_k]\}$ to be a Markov bridge between (t_j, Z_1) and (t_k, Z_2), otherwise independent of $\{X(t), t \in \mathcal{G}_1\}$. In other words, $X'_{jk}(t_j) = Z_1$, $X'_{jk}(t_k) = Z_2$, and the distribution of $\{X'_{jk}(t), t \in [t_j, t_k]\}$ is the same as that of the process $\{Y(t), t \in [t_j, t_k]\}$ under $\mathcal{P}$, conditioned by $Y(t_j) = Z_1$ and $Y(t_k) = Z_2$. We define a process X' on TLG $\mathcal{G}_2$ by letting it have the same values as X on $\mathcal{R}(\mathcal{G}_1)$, and using the above definition on E_{jk}.

In case (b), we add two edges E'_{jk} and E''_{jk} to $\mathcal{G}_1$. The other new edges $E_{j_1 j}$ and E_{kj_2} are created by subdividing $E_{j_1 j_2}$. Let $Z_1 = X_{j_1 j_2}(t_j)$ and $Z_2 = X_{j_1 j_2}(t_k)$. We define $\{X'_{jk}(t), t \in [t_j, t_k]\}$ and $\{X''_{jk}(t), t \in [t_j, t_k]\}$ to be independent Markov bridges between (t_j, Z_1) and (t_k, Z_2), otherwise independent of $\{X(t), t \in \mathcal{G}_1\}$. We choose the representations $\mathcal{R}(\mathcal{G}_1)$ and $\mathcal{R}(\mathcal{G}_2)$ so that they agree on $\mathcal{R}(\mathcal{G}_1)$ with the part of E_{j_1, j_2} between t_j and t_k removed. We define a process X' on TLG $\mathcal{G}_2$ by first letting it have the same values as X on $\mathcal{R}(\mathcal{G}_1) \setminus E_{jk}$. The process $\{X'_{jk}(t), t \in [t_j, t_k]\}$ represents the values of X' on the edge E'_{jk} and $\{X''_{jk}(t), t \in [t_j, t_k]\}$ represents the values of X' on the edge E''_{jk}.

In the rest of the proof, we will focus on case (a). Case (b) requires minor modifications and is left to the reader.

Recall that $\mathcal{G}_1$ contains a time path $\sigma(m_1, m_2, \dots, m_n)$ with $j_1 = m_{n_1}, j_2 = m_{n_1+1}, k_1 = m_{n_2}$ and $k_2 = m_{n_2+1}$, for some $n_1 < n_2$. This implies that $\mathcal{G}_2$ must contain a time path $\sigma(m_1, \dots, j_1, j, j_2, \dots, k_1, k, k_2, \dots, m_n)$. There is a Markov bridge between (t_j, Z_1) and (t_k, Z_2) in the representation $\mathcal{R}(\mathcal{G}_1)$. The construction of $\{X'_{jk}(t), t \in [t_j, t_k]\}$ consists of generating an independent Markov bridge between the same points. By the Markov property of $\mathcal{P}$, the distribution of X'_{j_1, j, k, k_2} on the graph $\mathcal{G}_2$ is the same as the distribution of $X_{j_1, j_2, \dots, k_1, k_2}$ on the graph $\mathcal{G}_1$. This implies that for every full path $\sigma(r_1, \dots, j_1, j, k, k_2, \dots, r_n)$ in $\mathcal{G}_2$, the distribution of $X'_{r_1, \dots, j_1, j, k, k_2, \dots, r_n}$ is $\mathcal{P}$. Hence, X' is a $\mathcal{P}$-process on $\mathcal{G}_2$.

Next we will show that X' is cell-Markovian. Consider a simple cell (σ_1, σ_2) in $\mathcal{G}_2$. Suppose that the paths σ_1 and σ_2 do not contain the new edge E_{jk}. Then (σ_1, σ_2) is a simple cell in $\mathcal{G}_1$. By the inductive assumption, the processes X on σ_1 and X on σ_2 are conditionally independent given their values at the end and start of the cell. Since X' is equal to X on (σ_1, σ_2), the same claim holds for X'.

Now consider a simple cell (σ_1, σ_2) in $\mathcal{G}_2$ such that σ_1 contains E_{jk}. Then we have $\sigma_1 = \sigma(r_1, \dots, j_1, j, k, k_2, \dots, r_{n_1})$ and $\sigma_2 = \sigma(q_1, \dots, q_{n_2})$. We will show that processes $X'_{r_1, \dots, j_1, j, k, k_2, \dots, r_{n_1}}$ and $X'_{q_1, \dots, q_{n_2}}$ are conditionally independent given their values at the start and end.

First, we will argue that the cell consisting of time paths $\sigma_3 = \sigma(r_1, \dots, j_1, j_2, \dots, k_1, k_2, \dots, r_{n_1})$ and $\sigma_4 = \sigma(q_1, \dots, q_{n_2})$ is simple in $\mathcal{G}_1$. Suppose otherwise, that is, there exists a time path $\sigma_5 = \sigma(s_1, \dots, s_{n_3})$ in $\mathcal{G}_1$ which connects σ_3 and σ_4. We will consider several cases. If $s_1 \in \{r_2, \dots, j_1\}$ and $s_{n_3} \in \{q_2, \dots, q_{n_2 - 1}\}$ then σ_5 connects σ_1 and σ_2 in $\mathcal{G}_2$, a contradiction. We arrive at a contradiction for a similar reason if we assume that $s_1 \in \{k_2, \dots, r_{n_1 - 1}\}$ and $s_{n_3} \in \{q_2, \dots, q_{n_2 - 1}\}$; or if we assume that $s_1 \in \{q_2, \dots, q_{n_2 - 1}\}$ and $s_{n_3} \in \{r_2, \dots, j_1\}$; or $s_1 \in \{q_2, \dots, q_{n_2 - 1}\}$ and $s_{n_3} \in \{k_2, \dots, r_{n_1 - 1}\}$. Next suppose that $s_1 \in \{j_2, \dots, k_1\}$ and $s_{n_3} \in \{q_2, \dots, q_{n_2 - 1}\}$. Let $\sigma_6 = \sigma(j_1, \dots, s_1)$ be the sub-path of σ_3. Then the concatenation of σ_6 and σ_5 connects σ_1 and σ_2 in $\mathcal{G}_2$, a contradiction once again. Finally, suppose that $s_1 \in \{q_2, \dots, q_{n_2 - 1}\}$ and $s_{n_3} \in \{j_2, \dots, k_1\}$. Let $\sigma_7 = \sigma(s_{n_3}, \dots, k_1)$ be the sub-path of σ_3. Then the concatenation of σ_5 and σ_7 connects σ_1 and σ_2 in $\mathcal{G}_2$, which is a contradiction.

By the inductive assumption, $X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}$ and $X_{q_1,\dots,q_{n_2}}$ are conditionally independent given the values at the start and end of the corresponding cell. This and the Markov property imply that the process $\{X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}(t),\ t \in [t_j, t_k]\}$ is conditionally independent from the processes $\{X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}(t), t \in [t_{r_1}, t_{r_{n_1}}] \setminus [t_j, t_k]\}$ and $X_{q_1,\dots,q_{n_2}}$ given the values of $X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}(t_j)$ and $X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}(t_k)$. The claim remains valid if we replace $\{X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}(t), t \in [t_j, t_k]\}$ with $\{X'_{jk}(t), t \in [t_j, t_k]\}$, and this in turn shows that the joint distribution of $X'_{r_1,\dots,j_1,j,k,k_2,\dots,r_{n_1}}$ and $X'_{q_1,\dots,q_{n_2}}$ is the same as that of $X_{r_1,\dots,j_1,j_2,\dots,k_1,k_2,\dots,r_{n_1}}$ and $X_{q_1,\dots,q_{n_2}}$. Hence, $X'_{r_1,\dots,j_1,j,k,k_2,\dots,r_{n_1}}$ and $X'_{q_1,\dots,q_{n_2}}$ are conditionally independent given their values at the start and end. We have shown that X' is cell-Markovian.

Next we will show that X' is time-Markovian. Suppose that $t \notin E_{jk}$. If $F(t)$ and $P(t)$ in $\mathcal{G}_2$ are the same as $F(t)$ and $P(t)$ in $\mathcal{G}_1$, then the time-Markov property obviously holds for t in $\mathcal{G}_2$. Suppose that the future $F_2(t)$ of t in $\mathcal{G}_2$ is the union of the future $F(t)$ of t in $\mathcal{G}_1$ and E_{jk}. Since $\{X'(t), t \in E_{jk}\}$ is the Markov bridge between $(t_j, X(t_j))$ and $(t_k, X(t_k))$ otherwise independent of $\{X'(t), t \in \mathcal{E}_2 \setminus \mathcal{E}_1\}$, and, by the inductive assumption, $\{X(s), s \in F(t)\}$ and $\{X(s), s \in P(t)\}$ are conditionally independent given $X(t)$, it follows easily that $\{X'(s), s \in F_2(t)\}$ and $\{X'(s), s \in P(t)\}$ are conditionally independent given $X'(t)$. A similar argument applies when the past $P_2(t)$ of t in $\mathcal{G}_2$ is the union of the past $P(t)$ of t in $\mathcal{G}_1$ and E_{jk}.

Consider the case when $t \in E_{jk}$. Let σ be a full path disjoint from E_{jk} except for t_j and t_k. By Remark 3.4, the conditional distributions of $\{X(u), t_j \le u \le t_k, u \in \sigma\}$, $\{X(u), u \in F(t_k)\}$ and $\{X(u), u \in P(t_j)\}$ are independent given $X(t_j)$ and $X(t_k)$. Moreover, the conditional distribution of $\{X(u), t_j \le u \le t_k, u \in \sigma\}$ is that of a Markov bridge between $(t_j, X(t_j))$ and $(t_k, X(t_k))$. This and the fact that $\{X'(t), t \in E_{jk}\}$ is the Markov bridge between $(t_j, X(t_j))$ and $(t_k, X(t_k))$ otherwise independent of $\{X'(t), t \in \mathcal{E}_2 \setminus \mathcal{E}_1\}$ imply that the joint distribution of $\{X(u), t_j \le u \le t_k, u \in E_{jk}\}$, $\{X(u), u \in F(t_k)\}$ and $\{X(u), u \in P(t_j)\}$ is the same as the joint distribution of $\{X(u), t_j \le u \le t_k, u \in \sigma\}$, $\{X(u), u \in F(t_k)\}$ and $\{X(u), u \in P(t_j)\}$. By the inductive assumption, the time-Markovian property holds for X, $\mathcal{G}_1$ and the point $t_* \in \sigma$ with the same time coordinate as t, so we conclude that the time-Markovian property holds for X', $\mathcal{G}_2$ and t. This completes the proof of the time-Markovian property for X'.

We will now show that X' is graph-Markovian. Suppose that $\mathcal{W} \subset \mathcal{R}(\mathcal{G}_2)$ is finite, and $\mathcal{E}_0, \mathcal{E}_1$ and $\mathcal{E}_2$ are as in Definition 3.2. Let $\mathcal{W}^* = \mathcal{W} \cap \mathcal{R}(\mathcal{G}_1)$, and, assuming that $\mathcal{W}^* \neq \varnothing$, let $\mathcal{E}_0^*, \mathcal{E}_1^*$ and $\mathcal{E}_2^*$ be defined relative to $\mathcal{G}_1$ as in Definition 3.2. By the induction assumption, the conditional distribution of $\{X'(t), t \in E, E \in \mathcal{E}_1^*\}$ given $\{X'(t), t \in E, E \in \mathcal{E}_2^*\}$ depends only on $\{X'(t), t \in \mathcal{W}^*\}$. Since $\{X'(t), t \in E_{jk}\}$ is a Markov bridge between t_j and t_k, independent of the values of X' except for $X'(t_j)$ and $X'(t_k)$, it is easy to see that the conditional distribution of

$\{X(t), t \in E, E \in \mathcal{E}_1\}$ given $\{X(t), t \in E, E \in \mathcal{E}_2\}$ depends only on $\{X(t), t \in \mathcal{W}\}$. If $\mathcal{W}^* = \varnothing$, the same conclusion is also evident. Hence, $\mathcal{G}_2$ is graph-Markovian.

It remains to prove uniqueness in law of a natural $\mathcal{P}$-process on an NCC-graph. Once again, we use induction. The distribution of a natural $\mathcal{P}$-process on the "minimal" graph described above is obviously unique. Suppose that we have shown uniqueness in law for natural $\mathcal{P}$-processes on all NCC-graphs with the number of edges equal to $1 + 3r$ or less. Any NCC-graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ with $1 + 3(r + 1)$ edges can be constructed from an NCC-graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ with $1 + 3r$ edges by adding an edge, say E_{jk}, as in Definition 2.8(ii). Consider a natural $\mathcal{P}$-process X' on $\mathcal{G}_2$. Its restriction X to $\mathcal{G}_1$ is a $\mathcal{P}$-process. We will argue that X is a natural $\mathcal{P}$-process on $\mathcal{G}_1$.

First, we will prove that X on $\mathcal{G}_1$ is time-Markovian. Consider any point t in $\mathcal{G}_1$, and let $F_1(t)$ and $P_1(t)$ be the future and past of t relative to $\mathcal{G}_1$, defined as in Definition 3.3. Let $F_2(t)$ and $P_2(t)$ be the future and past of t relative to $\mathcal{G}_2$ and note that $F_1(t) \subset F_2(t)$ and $P_1(t) \subset P_2(t)$. Since X' on $\mathcal{G}_2$ is assumed to be natural, the conditional distributions of $\{X'(s), s \in F_2(t)\}$ and $\{X'(s), s \in P_2(t)\}$ are independent given $X'(t)$. This clearly implies that the conditional distributions of $\{X(s), s \in F_1(t)\}$ and $\{X(s), s \in P_1(t)\}$ are independent given $X(t)$. We see that X on $\mathcal{G}_1$ is time-Markovian.

Next we will show that X is graph-Markovian on $\mathcal{G}_1$. Let $\mathcal{W} \subset \mathcal{R}(\mathcal{G}_1)$ be as in Definition 3.2 and let $\widehat{\mathcal{E}}_1$ and $\widehat{\mathcal{E}}_2$ play the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$ in the same definition (in this proof, $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the sets of edges of $\mathcal{G}_1$ and $\mathcal{G}_2$). Since X' is natural, the conditional distribution of $\{X'(t), t \in E_{jk}\}$ given $\{X'(t), t \in E, E \in \mathcal{E}_2 \setminus \{E_{jk}\}\}$ is that of a Markov bridge between $(t_j, X'(t_j))$ and $(t_k, X'(t_k))$. For future reference, let us call this property (A).

Suppose that $t_j \in E_1, t_k \in E_2$ for some $E_1, E_2 \in \widehat{\mathcal{E}}_1$, $t_j, t_k \notin \mathcal{W}$ and let $\widetilde{\mathcal{E}}_1 = \widehat{\mathcal{E}}_1 \cup \{E_{jk}\}$. We have assumed that X' is graph-Markovian, so the conditional distribution of $\{X'(t), t \in E, E \in \widetilde{\mathcal{E}}_1\}$ given $\{X'(t), t \in E, E \in \widehat{\mathcal{E}}_2\}$ depends only on $\{X'(t), t \in \mathcal{W}\}$. This and (A) easily imply that the conditional distribution of $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ given $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_2\}$ depends only on $\{X(t), t \in \mathcal{W}\}$.

The same argument applies when $t_j \in E_1, t_k \in E_2$ for some $E_1, E_2 \in \widehat{\mathcal{E}}_1$, $t_j \notin \mathcal{W}$ and $t_k \in \mathcal{W}$, and also in the case when $t_j \in E_1, t_k \in E_2$ for some $E_1, E_2 \in \widehat{\mathcal{E}}_1$, $t_j \in \mathcal{W}$ and $t_k \notin \mathcal{W}$.

Consider the case when $t_j \in E_1, t_k \in E_2$ for some $E_1, E_2 \in \widehat{\mathcal{E}}_2$ and let $\widetilde{\mathcal{E}}_2 = \widehat{\mathcal{E}}_2 \cup \{E_{jk}\}$. We have assumed that X' is graph-Markovian so the conditional distribution of $\{X'(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ given $\{X'(t), t \in E, E \in \widetilde{\mathcal{E}}_2\}$ depends only on $\{X'(t), t \in \mathcal{W}\}$. This and (A) easily imply that the conditional distribution of $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ given $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_2\}$ depends only on $\{X(t), t \in \mathcal{W}\}$.

Note that if $t \in \mathcal{W}$, then $t \in E_1$ for some $E_1 \in \widehat{\mathcal{E}}_1$ and $t \in E_2$ for some $E_2 \in \widehat{\mathcal{E}}_2$. Hence, the only case that remains to be analyzed is when $t_j \in E_1, t_k \in E_2$ for some $E_1 \in \widehat{\mathcal{E}}_1, E_2 \in \widehat{\mathcal{E}}_2$, $t_j, t_k \notin \mathcal{W}$. Since $t_j \in E_1$ for some $E_1 \in \widehat{\mathcal{E}}_1$, $t_j \notin \mathcal{W}$, and taking into account how E_{jk} was added to $\mathcal{G}_1$, it follows that there exist $E_{t_1,t_j}, E_{t_j,t_2} \in \widehat{\mathcal{E}}_1$

such that t_1, t_j, t_2, t_k lie on a full time path σ_1. Since the process X' is natural, the conditional distributions of $\{X'(u), t_1 \le u \le t_2, u \in \sigma_1\}$, $\{X'(u), u \in F(t_2)\}$ and $\{X'(u), u \in P(t_1)\}$ are independent given $X'(t_1)$ and $X'(t_2)$, and, moreover, the conditional distribution of $\{X'(u), t_1 \le u \le t_2, u \in \sigma_1\}$ given $X'(t_1)$ and $X'(t_2)$ is that of a Markov bridge between $(t_1, X'(t_1))$ and $(t_2, X'(t_2))$. We will need the following two facts in the next step of the argument. The first is property (A) defined above. The second is an application of the graph-Markovian property for X'. Let $\mathcal{E}_3$ be the union of all edges that comprises $\{u : t_1 \le u \le t_2, u \in \sigma_1\}$, $F(t_2)$, $P(t_1)$ and E_{jk}. Let $\mathcal{E}_4$ be the union of all edges such that the union of $\mathcal{E}_3$ and $\mathcal{E}_4$ represents the whole graph $\mathcal{G}_2$, and $\mathcal{W}_1$ is a finite set of points that $\mathcal{E}_3$ and $\mathcal{E}_4$ have in common. Note that $t_j, t_k \notin \mathcal{W}_1$ because all edges that end at these points belong to $\mathcal{E}_3$. By the graph-Markovian property of X', the conditional distribution of $\{X'(t), t \in E, E \in \mathcal{E}_4\}$ given $\{X'(t), t \in E, E \in \mathcal{E}_3\}$ depends only on $\{X'(t), t \in \mathcal{W}_1\}$.

Let $\mathcal{D}_1$ be the distribution of $\{X'(t), t \in P(t_1) \cup F(t_2)\}$. Let $\mathcal{D}_2(x_1, x_2)$ be the conditional distribution of $\{X'(u), t_1 \le u \le t_2, u \in \sigma_1\}$ given $\{X'(t_1) = x_1, X'(t_2) = x_2\}$. Let $\mathcal{D}_3(x_j, x_k)$ be the conditional distribution of $\{X'(u), u \in E_{jk}\}$ given $\{X'(t_j) = x_j, X'(t_k) = x_k\}$. Let $\mathcal{D}_4(\overline{x})$ be the conditional distribution of $\{X'(u), u \in E, E \in \mathcal{E}_4\}$ given the sequence $\overline{x}$ of values of X' at all points in $\mathcal{W}_1$.

We can construct a process Y on $\mathcal{G}_2$ with the same distribution as X' as follows. First, define a process $\{Y(t), t \in P(t_1) \cup F(t_2)\}$ with distribution $\mathcal{D}_1$ on some probability space. Then define a process $\{Y(u), u \in E, E \in \mathcal{E}_4\}$ with distribution $\mathcal{D}_4(\overline{y})$, independent of $\{Y(t), t \in P(t_1) \cup F(t_2)\}$, except that $\overline{y}$ is the already generated sequence of values of Y on $\mathcal{W}_1$. Next define an independent (except for the endpoints) Markov bridge $\{Y(u), t_1 \le u \le t_2, u \in \sigma_1\}$ between $(t_1, Y(t_1))$ and $(t_2, Y(t_2))$. This process has distribution $\mathcal{D}_2(Y(t_1), Y(t_2))$. Finally define an independent (except for the endpoints) Markov bridge $\{Y(u), u \in E_{jk}\}$ between $(t_j, Y(t_j))$ and $(t_k, Y(t_k))$. This process has distribution $\mathcal{D}_3(Y(t_j), Y(t_k))$. It follows from our earlier remarks that Y has the same distribution as X' on $\mathcal{G}_2$. The point of this construction is that it shows that given $\{Y(t_1), Y(t_2)\}$, the distribution of Y on $P(t_1) \cup F(t_2) \cup \mathcal{E}_4$ is independent of $\{Y(u), t_1 \le u \le t_2, u \in \sigma_1\}$. Hence, the distribution of X on $P(t_1) \cup F(t_2) \cup \mathcal{E}_4$ is independent of $\{X(u), t_1 \le u \le t_2, u \in \sigma_1\}$ given $\{X(t_1), X(t_2)\}$.

Let $\widetilde{\mathcal{E}}_2 = \widehat{\mathcal{E}}_2 \cup \{E_{jk}\}$. Since X' is graph-Markovian, the conditional distribution of $\{X'(t), t \in E, E \in \widetilde{\mathcal{E}}_2\}$ given $\{X'(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ depends only on $\{X'(t), t \in \mathcal{W} \cup \{t_j\}\}$. It follows that the conditional distribution of $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_2\}$ given $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ depends only on $\{X(t), t \in \mathcal{W} \cup \{t_j\}\}$. Note that the values of $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ include the values of $X(t_1)$ and $X(t_2)$. Since the distribution of X on $P(t_1) \cup F(t_2) \cup \mathcal{E}_4$ is independent of $\{X(u), t_1 \le u \le t_2, u \in \sigma_1\}$ given $\{X(t_1), X(t_2)\}$, we conclude that the conditional distribution of $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_1\}$ given $\{X(t), t \in E, E \in \widehat{\mathcal{E}}_2\}$ depends only on $\{X(t), t \in \mathcal{W}\}$. This completes the discussion of the last remaining case of graph-Markovian property for X on $\mathcal{G}_1$.

By assumption, the families of random variables $\{X'(t), t \in E\}$, $E \in \mathcal{E}_2$, are conditionally independent given $\{X'(t), t \in \mathcal{V}_2\}$, and for every $E_{jk} \in \mathcal{E}_2$, the conditional distribution of $\{X'(t), t \in E_{jk}\}$ given $\{X'(t), t \in \mathcal{V}\}$ is a Markov bridge between $(t_j, X'(t_j))$ and $(t_k, X'(t_k))$. It is obvious that this implies that the analogous property holds for X on $\mathcal{G}_1$. We have already shown that X is time-Markovian and graph-Markovian on $\mathcal{G}_1$, so X is natural on $\mathcal{G}_1$. By the induction assumption, X has a unique distribution. Rephrasing what we said earlier in this paragraph, for $E_{jk} \in \mathcal{E}_2 \setminus \mathcal{E}_1$, the conditional distribution of $\{X'(t), t \in E_{jk}\}$ given $\{X'(t), t \in E, E \in \mathcal{E}_1\}$ is a Markov bridge between $(t_j, X'(t_j))$ and $(t_k, X'(t_k))$. This determines the distribution of X' uniquely. $\square$

Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an NCC TLG, and $\mathcal{V}$ is infinite. According to the definition of an NCC TLG with an infinite vertex set, there exists a tower of NCC-graphs $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $n \geq 1$, such that each $\mathcal{V}_n$ is finite, $\mathcal{V} = \bigcup_{n\geq 1} \mathcal{V}_n$ and $\mathcal{E} = \bigcup_{n\geq 1} \mathcal{E}_n$. Let X_n be the natural $\mathcal{P}$-process on $\mathcal{G}_n$. By Remark 3.8(iv), the restriction of X_n to $\mathcal{G}_k$, for $k < n$, has the same distribution as that of X_k. A routine application of Kolmogorov's consistency theorem shows that there exists a $\mathcal{P}$-process X on $\mathcal{G}$ such that its restriction to any $\mathcal{G}_k$ has the same distribution as that of X_k. Note that the distribution of X may depend, in principle, on the sequence $\{\mathcal{G}_n\}$. We will show that it does not if $\mathcal{G}$ is planar. We conjecture that the result holds for all NCC TLGs with infinite $\mathcal{V}$.

THEOREM 3.9. *Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a planar NCC TLG, and $\mathcal{V}$ is infinite. If X and X' are two $\mathcal{P}$-processes on $\mathcal{G}$ constructed using two towers of NCC-graphs $\{\mathcal{G}_n\}_{n\geq 1}$ and $\{\mathcal{G}'_n\}_{n\geq 1}$, then X and X' have the same distributions.*

PROOF. Suppose that we can prove that for any $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$ and $\mathcal{G}'_k = (\mathcal{V}'_k, \mathcal{E}'_k)$ there exist $n \geq \max(j, k)$ and graphs $\mathcal{H}_m$, $m = j+1, \dots, n$, and $\mathcal{H}'_m$, $m = k+1, \dots, n'$ such that $\mathcal{H}_n = \mathcal{H}'_{n'} \subset \mathcal{G}$ and $\mathcal{G}_1, \dots, \mathcal{G}_j, \mathcal{H}_{j+1}, \dots, \mathcal{H}_n$ and $\mathcal{G}'_1, \dots, \mathcal{G}'_k, \mathcal{H}'_{k+1}, \dots, \mathcal{H}'_{n'}$ are towers of NCC-graphs. By Theorem 3.7(i) and its proof, we can construct a natural process Y on $\mathcal{H}_n$ such that its restriction to $\mathcal{G}_j$ is X, and we can construct a natural process Y' on $\mathcal{H}'_{n'}$ such that its restriction to $\mathcal{G}'_k$ is X'. By the uniqueness in distribution of the natural process on an NCC-graph, the distributions of Y and Y' are identical. Hence, the distributions of X and X' agree on $\mathcal{E}_j \cap \mathcal{E}'_k$. Letting $j, k \to \infty$, we conclude that the distributions of X and X' agree on $\mathcal{E}$.

It remains to prove that we can construct sequences $\{\mathcal{H}_m\}$ and $\{\mathcal{H}'_m\}$ with the properties listed above. First suppose that the initial vertices $t_{0,n} \in \mathcal{V}_n$ and $t'_{0,n} \in \mathcal{V}'_n$ and terminal vertices $t_{N_n,n} \in \mathcal{V}_n$ and $t'_{N_n,n} \in \mathcal{V}'_n$ are the same for all $\mathcal{G}_n$ and $\mathcal{G}'_n$, that is, $t_0 := t_{0,n} = t'_{0,m}$ and $t_\infty := t_{N_n,n} = t'_{N_m,m}$ for all n and m. Assume also that the initial edges for both sequences overlap, that is, $E_{t_{0,n},t_{1,n}} \subset E_{t'_{0,m},t'_{1,m}}$ or $E_{t'_{0,m},t'_{1,m}} \subset E_{t_{0,n},t_{1,n}}$ for all n and m. Similarly, assume that terminal edges overlap, that is,

$E(t_{N_n-1,n}, t_{N_n,n}) \subset E(t'_{N_m-1,m}, t'_{N_m,m})$ or $E(t'_{N_m-1,m}, t'_{N_m,m}) \subset E(t_{N_n-1,n}, t_{N_n,n})$ for all n and m. Moreover, we assume that $\mathcal{G}_1 = \mathcal{G}'_1$ is the same full time path.

Consider a (planar) representation $\mathcal{R}(\mathcal{G})$ of $\mathcal{G}$ and suppose that $\mathcal{R}(\mathcal{G}_j) \subset \mathcal{R}(\mathcal{G})$ and $\mathcal{R}(\mathcal{G}'_k) \subset \mathcal{R}(\mathcal{G})$. Recall that a representation of a planar graph is a set of points $(t, x) \in \mathbb{R}^2$. It is easy to see that the upper boundary of $\mathcal{R}(\mathcal{G}_j)$ is the graph of a continuous function $f_j : [t_0, t_\infty] \to \mathbb{R}$, that is, $f_j(t) = \max\{x : (t, x) \in \mathcal{R}(\mathcal{G}_j)\}$. We similarly define f'_k relative to $\mathcal{G}'_k$ and let $f = \max(f_j, f'_k)$. We then define the lower boundary of $\mathcal{R}(\mathcal{G}_j)$ as the graph of a continuous function $g_j : [t_0, t_\infty] \to \mathbb{R}$, that is, $g_j(t) = \min\{x : (t, x) \in \mathcal{R}(\mathcal{G}_j)\}$, g'_k as the lower boundary of $\mathcal{R}(\mathcal{G}'_k)$ and $g = \min(g_j, g'_k)$. For any real functions $a(t)$ and $b(t)$, let the graph $\mathcal{K}_{a,b}$ be defined by $\mathcal{R}(\mathcal{K}_{a,b}) = \{(t, x) \in \mathcal{R}(\mathcal{G}) : a(t) \le x \le b(t)\}$. Let $\mathcal{H}_n = \mathcal{H}'_{n'} = \mathcal{K}_{f,g}$.

Since $\mathcal{G}_j$ is an element of an infinite tower of graphs $\{\mathcal{G}_m\}$ such that $\bigcup_{m\ge 1} \mathcal{R}(\mathcal{G}_m) = \mathcal{R}(\mathcal{G})$ and $\mathcal{G}$ is locally finite, we must have $\mathcal{R}(\mathcal{K}_{f_j,g_j}) \subset \mathcal{R}(\mathcal{G}_{m_1})$ for some $m_1 \ge j$. Let $E_1, E_2, \dots, E_{m_2}$ be edges added during the inductive construction of the tower $\{\mathcal{G}_m\}_{j\le m\le m_1}$ and such that their representations are in $\mathcal{R}(\mathcal{G}_{m_1}) \setminus \mathcal{R}(\mathcal{K}_{f_j,g_j})$, listed in the order in which they are added during the inductive construction. We construct a tower $\mathcal{G}_j, \mathcal{H}_{j+1}, \dots, \mathcal{H}_{j+m_2} = \mathcal{K}_{f_j,g_j}$ by adding edges $E_1, E_2, \dots, E_{m_2}$ in the same order (and no other edges). This construction can proceed according to the rules of the inductive construction of NCC graphs because edges $E_1, E_2, \dots, E_{m_2}$ are shielded by the graphs of the functions f_j and g_j from all other edges added during the construction of $\{\mathcal{G}_m\}_{j\le m\le m_1}$. We construct a tower $\mathcal{G}'_k, \mathcal{H}'_{k+1}, \dots, \mathcal{H}_{k+m_3} = \mathcal{K}_{f'_k,g'_k}$ in an analogous way.

It remains to define $\mathcal{H}_{j+m_2+1}, \dots, \mathcal{H}_n$. For future reference, we label the next part of the proof "Step (I)." If $f'_k(t) \le f_j(t)$ and $g'_k(t) \ge g_j$ for all t then we let $\mathcal{H}_{j+m_2+1} = \mathcal{H}_{j+m_2}$. Otherwise, suppose without loss of generality that $f'_k(t) > f_j(t)$ for some t. Let $t_1, t_2, \dots, t_{m_4}$ be all vertices in the graph of f_j such that there is an edge in $\mathcal{H}_n \setminus \mathcal{H}_{j+m_2}$ ending in t_r, for $r = 1, \dots, m_4$. The first such edge must go from t_1 forward in time, and the last such edge must end in t_{m_4}. Hence, there must be a pair of vertices t_r and t_{r+1} such that there is an edge E_r in $\mathcal{H}_n \setminus \mathcal{H}_{j+m_2}$ starting from t_r and an edge E_{r+1} (possibly the same edge) ending in t_{r+1}. By the planarity of $\mathcal{G}$, there must be a time path σ from t_r to t_{r+1} in $\mathcal{H}_n$ containing E_r and E_{r+1}. We add σ (treated as a single edge) to $\mathcal{H}_{j+m_2}$ and thus obtain $\mathcal{H}_{j+m_2+1}$.

If $f'_k(t) \ge f_j(t)$ and $g'_k(t) \le g_j(t)$ for all t, then we let $\mathcal{H}'_{j+m_3+1} = \mathcal{H}'_{j+m_3}$. Otherwise, we generate $\mathcal{H}'_{j+m_3+1}$ in a way analogous to that used to construct $\mathcal{H}_{j+m_2+1}$.

If $f'_k(t) = f_j(t)$ and $g'_k(t) = g_j(t)$ for all t, then we let $\mathcal{H}_n = \mathcal{H}'_{n'} = \mathcal{H}_{j+m_2}$. In this case, we are done. Otherwise, we have constructed towers of NCC-graphs

$$\mathcal{G}_1, \dots, \mathcal{G}_j, \qquad \mathcal{H}_{j+1}, \dots, \mathcal{H}_{j+m_2+1}$$

and

$$\mathcal{G}'_1, \dots, \mathcal{G}'_k, \qquad \mathcal{H}'_{k+1}, \dots, \mathcal{H}'_{k+m_3+1}$$

such that either $\mathcal{H}_{j+m_2+1}$ is strictly greater than $\mathcal{G}_j$ or $\mathcal{H}'_{k+m_3+1}$ is strictly greater than $\mathcal{G}'_k$, or both. Moreover, the TLG analogous to $\mathcal{K}_{f,g}$ but defined relative to $\mathcal{H}_{j+m_2+1}$ and $\mathcal{H}'_{k+m_3+1}$ in place of $\mathcal{G}_j$ and $\mathcal{G}'_k$ is the same as $\mathcal{K}_{f,g}$.

We now proceed in an inductive way. Suppose that we constructed $\mathcal{H}_{r_1}$ and $\mathcal{H}'_{r_2}$. Let f_{r_1} represent the upper boundary of $\mathcal{H}_{r_1}$, let f'_{r_2} represent the upper boundary of $\mathcal{H}'_{r_2}$, let g_{r_1} represent the lower boundary of $\mathcal{H}_{r_1}$ and let g'_{r_2} represent the lower boundary of $\mathcal{H}'_{r_2}$. We now repeat Step (I) with f_j replaced by f_{r_1}, f'_k replaced by f'_{r_2}, g_k replaced by g_{r_2} and g'_k replaced by g'_{r_2}. This will generate towers

$$\mathcal{G}_1, \dots, \mathcal{G}_j, \qquad \mathcal{H}_{j+1}, \dots, \mathcal{H}_{r_1+1} \quad \text{and} \quad \mathcal{G}'_1, \dots, \mathcal{G}'_k, \qquad \mathcal{H}'_{k+1}, \dots, \mathcal{H}'_{r_2+1}.$$

If $\mathcal{H}_{r_1+1} = \mathcal{H}'_{r_2+1} = \mathcal{H}_n = \mathcal{H}'_{n'}$, then we are done. Otherwise $\mathcal{H}_{r_1+1}$ is strictly greater than $\mathcal{H}_{r_1}$, or $\mathcal{H}'_{r_2+1}$ is strictly greater than $\mathcal{H}'_{r_2}$, or both. The growth cannot continue forever because $\mathcal{K}_{f,g}$ has a finite number of edges, so eventually we will have $\mathcal{H}_{r_1+1} = \mathcal{H}'_{r_2+1} = \mathcal{H}_n = \mathcal{H}'_{n'}$.

Next we will argue that one can drop the assumption that $\mathcal{G}_1 = \mathcal{G}'_1$ is the same full time path (but we keep the assumption about overlapping of the initial edges and terminal edges of $\mathcal{G}_j$'s and $\mathcal{G}'_k$'s). Suppose that $\mathcal{R}(\mathcal{G}_1) \cup \mathcal{R}(\mathcal{G}'_1)$ contains only one cell. Then the cell has no edges inside. Then the argument given above will work under this weakened assumption because $\mathcal{R}(\mathcal{H}_{j+m_2}) \cup \mathcal{R}(\mathcal{H}'_{k+m_3})$ will contain all edges between the graphs of f and g.

A rather easy but tedious argument based on ideas used earlier in this proof shows that for any two graphs (full time paths) $\mathcal{G}_1$ and $\mathcal{G}'_1$ with initial edges and terminal edges overlapping there exists a sequence of graphs $\mathcal{J}_1 = \mathcal{G}_1, \mathcal{J}_2, \dots, \mathcal{J}_q = \mathcal{G}'_1$ such that $\mathcal{R}(\mathcal{J}_r) \cup \mathcal{R}(\mathcal{J}_{r+1})$ contains only one cell, for every r. This shows that X and X' have the same distributions if $\mathcal{G}_1$ and $\mathcal{G}'_1$ have overlapping initial and terminal edges.

Finally, we will show how to eliminate the assumption that the initial and terminal edges of $\mathcal{G}_1$ and $\mathcal{G}'_1$ are overlapping. Suppose that $\mathcal{E}_* \subset \mathcal{E}$ is a finite set, and $\{\mathcal{G}_n\}$ and $\{\mathcal{G}'_n\}$ are two towers of NCC-graphs increasing to $\mathcal{G}$. Let n_1 be such that $\mathcal{R}(\mathcal{E}_*) \subset \mathcal{R}(\mathcal{G}_{n_1}) \cap \mathcal{R}(\mathcal{G}'_{n_1})$. Let t_* be a vertex such that $t_* \in \mathcal{V}$, $t_* \in \mathcal{R}(\mathcal{G}_1)$, and t_* lies to the left of all the vertices in $\mathcal{V}_{n_1}$ and $\mathcal{V}'_{n_1}$, except the initial vertices. Note that there exists n_2 so large that $t_* \in \mathcal{V}'_{n_2}$. This shows that there is a time path σ' with the initial edge overlapping with the initial edge of $\mathcal{G}'_1$ and ending at t_*. Let σ be the initial part of $\mathcal{G}_1$, between t_0 and t_*. Let $\mathcal{G}''_{n_1}$ be the graph $\mathcal{G}'_{n_1}$ with σ' replaced by σ. Note that $\mathcal{G}''_{n_1}$ is an NCC-graph because we only changed the second coordinate of the representation of $\mathcal{G}'_{n_1}$ for a part of the graph. Let X'' be the natural process on $\mathcal{G}''_{n_1}$. The distribution of X'' on $\mathcal{E}_*$ is the same as that of X' because, once again, we only changed the second coordinate of the representation of $\mathcal{G}'_{n_1}$ for a part of the graph. We now modify the terminal part of $\mathcal{G}''_{n_1}$ to obtain an NCC-graph $\mathcal{G}'''_{n_1}$ such that the initial and terminal edges of $\mathcal{G}'''_{n_1}$ and $\mathcal{G}_{n_1}$ are overlapping. Let X''' be the natural process on $\mathcal{G}'''_{n_1}$. The distribution of X''' on $\mathcal{E}_*$ is the same as that of X'. And this is the same distribution as the distribution of X on $\mathcal{E}_*$, by the first part of the proof. Since $\mathcal{E}_*$ is an arbitrary finite subset of $\mathcal{E}$, we see that X and X' have the same distributions. $\square$

DEFINITION 3.10. If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an NCC TLG with infinite $\mathcal{V}$ and X is a process on $\mathcal{G}$ with the distribution as in Theorem 3.9, then we will call X *natural*.

4. Brownian motion on time-like graphs.

In this section $\mathcal{P}$ refers to the distribution of standard Brownian motion. We will consider a TLG with a finite vertex set $\mathcal{V}$, $t_0 = 0$ and $t_N = 1$. The $\mathcal{P}$-process X on a TLG $\mathcal{G}$ is a mean zero Gaussian process so it is completely specified by its covariance structure.

PROPOSITION 4.1. *If $(\sigma(j, j_{n_1}, \dots, j_{n_2}, k, j_{n_3}, \dots, j_{n_4}, n), \sigma(j, j_{n_5}, \dots, j_{n_6}, m, j_{n_7}, \dots, j_{n_8}, n))$ is a simple cell of a TLG $\mathcal{G}$, and X is a natural Brownian motion on $\mathcal{G}$, then*

$$\mathbb{E}(X_{j,j_{n_1},\dots,j_{n_2},k,j_{n_3},\dots,j_{n_4},n}(t_k) X_{j,j_{n_5},\dots,j_{n_6},m,j_{n_7},\dots,j_{n_8},n}(t_m)) \\ = t_j + \frac{(t_k - t_j)(t_m - t_j)}{(t_n - t_j)}.$$

PROOF. We will abbreviate $X' = X_{j,j_{n_1},\dots,j_{n_2},k,j_{n_3},\dots,j_{n_4},n}$ and

$$X'' = X_{j,j_{n_5},\dots,j_{n_6},m,j_{n_7},\dots,j_{n_8},n}.$$

By the cell-Markovian property of the process, we can represent the joint distribution of X' and X'' as follows. Let W and W' be independent Brownian bridges on the interval $[t_j, t_n]$; in other words, W and W' are independent Brownian motions conditioned by $W_{t_j} = W'_{t_j} = 0$ and $W_{t_n} = W'_{t_n} = 0$. Let $\alpha = (t_k - t_j)/(t_n - t_j)$ and $\beta = (t_m - t_j)/(t_n - t_j)$. Then the conditional distribution of $X'(t_k)$ given $\{X'(t_j) = x_j, X'(t_n) = x_n\}$ is the same as the distribution of $(1-\alpha)x_j + \alpha x_n + W_{t_k}$. Moreover, the joint distribution of $(X'(t_k), X''(t_m))$ given $\{X'(t_j) = x_j, X'(t_n) = x_n\}$ is the same as the distribution of $((1-\alpha)x_j + \alpha x_n + W_{t_k}, (1-\beta)x_j + \beta x_n + W'_{t_m})$. Therefore,

$$\begin{aligned} &\mathbb{E}\big(X'(t_k)X''(t_m) \mid X'(t_j) = x_j, X''(t_n) = x_n\big) \\ &\quad = \mathbb{E}\big(\big((1-\alpha)x_j + \alpha x_n + W_{t_k}\big)\big((1-\beta)x_j + \beta x_n + W'_{t_m}\big)\big) \\ &\quad = \big((1-\alpha)x_j + \alpha x_n\big)\big((1-\beta)x_j + \beta x_n\big) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}(X'(t_k)X''(t_m)) \\ &\quad = \mathbb{E}\big(\big((1-\alpha)X'(t_j) + \alpha X'(t_n)\big)\big((1-\beta)X'(t_j) + \beta X'(t_n)\big)\big) \\ &\quad = \big((1-\alpha)(1-\beta) + (1-\alpha)\beta + \alpha(1-\beta)\big)t_j + \alpha\beta t_n \\ &\quad = (1-\alpha\beta)t_j + \alpha\beta t_n = t_j + \alpha\beta(t_n - t_j) \end{aligned}$$

$$= t_j + \frac{(t_k - t_j)(t_m - t_j)}{(t_n - t_j)^2}(t_n - t_j)$$

$$= t_j + \frac{(t_k - t_j)(t_m - t_j)}{(t_n - t_j)}. \qquad \square$$

PROOF OF THEOREM 3.7(ii). Suppose that for a TLG $\mathcal{G}$, there exist simple coterminal cells (σ_1, σ_2) with endpoints $t_1 < t_2$, and (σ_3, σ_4) with endpoints $t_3 < t_4$. Moreover, either $t_1 < t_3$ or $t_2 < t_4$. Assume that there exists a natural Brownian motion X on $\mathcal{G}$. We will show that this assumption leads to a contradiction.

We will assume without loss of generality that $t_2 < t_4$ and $t_1 = t_3$. The first edge of σ_1, say, E_{1k}, must be the same as the first edge of σ_3 or the first edge of σ_4. Suppose without loss of generality that the first edge of σ_1 is the same as the first edge of σ_3. Then the first edge of σ_2, say E_{1m}, is the same as the first edge of σ_4. Then we can use Proposition 4.1 to express the covariance of X at vertices t_k and t_m. If we use the formula relative to the cell (σ_1, σ_2), then the answer is

$$t_1 + \frac{(t_k - t_1)(t_m - t_1)}{(t_2 - t_1)}.$$

If we apply the same proposition relative to the cell (σ_3, σ_4), then we obtain a different answer,

$$t_1 + \frac{(t_k - t_1)(t_m - t_1)}{(t_4 - t_1)}.$$

This contradiction shows that there is no natural Brownian motion on $\mathcal{G}$. $\square$

5. Graph martingales and Harnesses.

The previous computation of Brownian covariance in Section 4 can be extended to a class of processes called harnesses. This class of processes, which includes all integrable Lévy processes and their bridges, was introduced originally by Hammersley [7]. We follow the definition given in the article by Mansuy and Yor [11].

DEFINITION 5.1. Suppose that $\mathcal{T} \subset \mathbb{R}$ is a bounded or unbounded interval, and let $\{H(t), t \in \mathcal{T}\}$ be an integrable process for all t whose sample paths are RCLL (right continuous with left limits) almost surely. Consider a past-future filtration $(\mathcal{H}_{t,T}, t < T; t, T \in \mathcal{T})$, with the property that

$$\sigma\{H(s); s \le t \text{ and } s \ge T\} \subset \mathcal{H}_{t,T} \quad \text{and} \quad \mathcal{H}_{t_1,T_1} \subseteq \mathcal{H}_{t,T}, \qquad t_1 \le t < T \le T_1.$$

The process H is said to be a *harness* with respect to the filtration $(\mathcal{H}_{t,T}, t < T; t, T \in \mathcal{T})$ if, for all $a < b < c < d$, we have

$$\mathbb{E}\left(\frac{H(c) - H(b)}{c - b} \,\middle|\, \mathcal{H}_{a,d}\right) = \frac{H(d) - H(a)}{d - a}. \tag{5.1}$$

The equality in (5.1) may also be reformulated as: H is a harness if and only if for all $s < t < u$, we get

$$\mathbb{E}(H(t) \mid \mathcal{H}_{s,u}) = \frac{t-s}{u-s} H(u) + \frac{u-t}{u-s} H(s). \tag{5.2}$$

The following lemma establishes more path properties.

LEMMA 5.2. *Let $\{Y(t), t \in [0,1]\}$ be a harness with respect to some past-future filtration $(\mathcal{H}_{t,T}, t < T; t, T \in [0,1])$. Then the following properties hold*:

(i) *The set of random variables $\{Y(s), 0 \le s \le 1\}$ is uniformly integrable*;
(ii) *Y is continuous in probability, that is, for any $0 \le t \le 1$ we have*

$$P\Big(\lim_{s \to t} Y(s) = Y(t)\Big) = 1. \tag{5.3}$$

PROOF. To prove (i), consider the collection of random variables $\{Y(s), 0 \le s \le 1/2\}$. By (5.2), for $t = 1/2$, $u = 1$, we get

$$\frac{1-2s}{2(1-s)} Y(1) + \frac{1}{2(1-s)} Y(s) = \mathbb{E}\big(Y(1/2) \mid \mathcal{H}_{s,1}\big).$$

As s varies between 0 and $1/2$, the collection of conditional expectations on the right is clearly uniformly integrable. Thus, by rearranging terms and noting that Y_1 is integrable, we get $\{Y(s), 0 \le s \le 1/2\}$ is also uniformly integrable. By a similar argument one gets uniform integrability of $\{Y(s), 1/2 \le s \le 1\}$, and this shows uniform integrability of the entire process.

For (ii), recall that we consider only right continuous harnesses. So it remains to prove that Y is continuous in probability from the left at time t. By applying (5.2), for any $s < u < t < T$ we get

$$\mathbb{E}(Y(u) \mid \mathcal{H}_{s,T}) = \frac{u-s}{T-s} Y(T) + \frac{T-u}{T-s} Y(s). \tag{5.4}$$

Now we take u approaching t from the left. By uniform integrability we get

$$\begin{aligned}\lim_{u \uparrow t} \mathbb{E}(Y(u) \mid \mathcal{H}_{s,T}) = \mathbb{E}(Y(t-) \mid \mathcal{H}_{s,T}) &= \frac{t-s}{T-s} Y(T) + \frac{T-t}{T-s} Y(s) \\ &= \mathbb{E}(Y(t) \mid \mathcal{H}_{s,T}).\end{aligned}$$

In other words, for all $s < t < T$ we get $\mathbb{E}(Y(t-) \mid \mathcal{H}_{s,T}) = \mathbb{E}(Y(t) \mid \mathcal{H}_{s,T})$. Now we take $T \downarrow t$ and use martingale convergence theorem (see [8], page 18) to claim $\mathbb{E}(Y(t-) \mid \mathcal{H}_{s,t+}) = \mathbb{E}(Y(t) \mid \mathcal{H}_{s,t+})$, where $\mathcal{H}_{s,t+} = \bigcap_{T>t} \mathcal{H}_{s,T}$. Finally we take $s \uparrow t$ and the martingale convergence theorem to claim

$$\mathbb{E}(Y(t-) \mid \mathcal{H}_{t-,t+}) = \mathbb{E}(Y(t) \mid \mathcal{H}_{t-,t+}).$$

Here $\mathcal{H}_{t-,t+} = \bigvee_{s<t} \mathcal{H}_{s,t+}$. But $Y(t-)$ is obviously measurable with respect to $\mathcal{H}_{t-,t+}$ and $Y(t+)$ is also measurable due to assumed right continuity. Hence

$Y(t-) = Y(t)$ almost surely. This shows (5.3) and completes the proof of the lemma. □

To discuss the properties of a harness on a TLG $\mathcal{G}$ we need to introduce a few definitions. Recall that a path in a graph is any sequence of vertices $(k_1, \dots, k_m)$ such that adjacent vertices (k_j, k_{j+1}) have edges in the graph $\mathcal{G}$. The only difference between a path and a time-path is that we do not require the vertices to be increasing.

DEFINITION 5.3. Let $\mathcal{G}$ be an NCC TLG with finite $\mathcal{V}$. Consider a full time path $\sigma^* = \sigma(k_1, \dots, k_n)$ and a point $t^* \in E_{j^*k^*}$. Consider the subgraph $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ where $\mathcal{V}^*$ consists of all vertices $v \in \mathcal{G}$ such that there exists a path starting at v and ending at j^* or k^*, and the path does not include any vertex in σ^*. The edges of this subgraph are the edges in $\mathcal{G}$ such that both its vertices are included in $\mathcal{V}^*$. The full time path σ^* will be called a *support* for t^* if the subgraph $\mathcal{G}^*$ is a tree. In other words, if we remove the time path σ^* from the graph $\mathcal{G}$, then the connected component of the remaining subgraph that contains t^* is a tree.

Let $\mathcal{P}$ denote the law of a Markovian harness in $[0, 1]$. Consider a natural $\mathcal{P}$-process on an NCC TLG $\mathcal{G}$. Suppose a full time path $\sigma^* = \sigma(k_1, \dots, k_n)$ is a support for a nonvertex point t^* on an edge $E_{j^*k^*}$. We want to know what $E(X_{j*k*}(t^*) \mid X(t), t \in \sigma^*)$ is. The answer will be expressed using a filtration constructed as follows.

Let $\mathcal{W}_1$ denote the two vertices $\{t_{j^*}, t_{k^*}\}$. Let $\mathcal{G}_1$ denote the subgraph of $\mathcal{G}$ with the edge $E_{j^*k^*}$ removed. Or, equivalently, in any representation of $\mathcal{G}$, we remove the interior of the set $E_{j^*k^*}$. Let $\mathcal{H}_1$ denote the σ-algebra generated by the set of all random variables $\{X_E(t), t \in E, E \in \mathcal{G}_1\}$. Note that the vertices j^* and k^* have degree two in the graph $\mathcal{G}_1$ since the common edge gets deleted.

Now we proceed by induction. Suppose we have constructed $\mathcal{W}_m, \mathcal{G}_m$, and $\mathcal{H}_m$ such that every $t_i \in \mathcal{W}_m$ has degree two in the graph $\mathcal{G}_m$. To construct $\mathcal{W}_{m+1}$, consider sequentially every vertex $t_i \in \mathcal{W}_m$. If t_i is a vertex in σ^* (i.e., t_i is one of $\{t_{k_1}, \dots, t_{k_n}\}$), then t_i continues to be in $\mathcal{W}_{m+1}$. This, in particular, holds true if t_i is 0 or 1 which are in σ^*. In this case we define the set of *descendants of* t_i, $\mathcal{N}(t_i)$, as the singleton set $\{t_i\}$. Otherwise, $\mathcal{N}(t_i)$ consists of the two distinct neighbors of t_i in the graph $\mathcal{G}_m$. We define the set $\mathcal{W}_{m+1}$ as

$$\mathcal{W}_{m+1} = \bigcup_{t_i \in \mathcal{W}_m} \mathcal{N}(t_i).$$

The subgraph $\mathcal{G}_{m+1}$ is obtained from $\mathcal{G}_m$ by deleting all the vertices of $\mathcal{W}_m$ not included in $\mathcal{W}_{m+1}$ and all their incident edges. The σ-algebra $\mathcal{H}_{m+1}$ is defined to be the one generated by all the random variables $\{X_E(t), t \in E, E \in \mathcal{G}_{m+1}\}$.

We stop the inductive process at the first K when all vertices in $\mathcal{W}_K$ are in σ^*, which gives us a backward filtration

$$\mathcal{H}_K \subset \mathcal{H}_{K-1} \subset \cdots \subset \mathcal{H}_1.$$

LEMMA 5.4. *Suppose that σ^* is a support of t^* so, by definition, $\mathcal{G}^*$ is a tree.*

(i) *Unless t_i is in σ^*, it cannot have a descendant already present in $\mathcal{W}_m$.*

(ii) *Any $v \in \mathcal{W}_{m+1}$ which is not included in σ^* has exactly two neighbors in the graph $\mathcal{G}_{m+1}$.*

(iii) *There exists a tower of NCC graphs $(\mathcal{G}'_1, \mathcal{G}'_2, \dots, \mathcal{G}'_M, \mathcal{G}_K, \dots, \mathcal{G}_{K-1}, \dots, \mathcal{G}_{K-2}, \dots, \mathcal{G}_1, \mathcal{G})$, where $\mathcal{G}'_1$ is a graph with a single time path. In other words, every graph $\mathcal{G}_m$, $m = 1, \dots, K$, is an element of this tower of NCC graphs. It is not necessarily true that $\mathcal{G}_m$'s are consecutive elements in this tower.*

PROOF. To see (i), consider two vertices $t_1 < t_2$ in $\mathcal{W}_m$. Note that there is always a path of the form $(t_1, u_1, \dots, u_k, t_2)$ such that $\{u_1, \dots, u_k\}$ is in $\bigcup_{i=1}^{m-1} \mathcal{W}_i$. If t_1 and t_2 are neighbors, then that creates a loop in the graph $\mathcal{G}^*$ in Definition 5.3. Since we have assumed the graph $\mathcal{G}^*$ to be a tree, this is impossible.

For (ii), note that, any $v \in \mathcal{W}_{m+1}$ which is not in σ^* is a neighbor to some distinct vertex in $\mathcal{W}_m$ and that edge has been deleted in $\mathcal{G}_{m+1}$. Since the degree of every nonterminal vertex is three, it remains to show that v cannot be a neighbor to two (or three) vertices in $\mathcal{W}_m$.

Assume on the contrary that there is a vertex $v \in \mathcal{W}_{m+1}$ which is a neighbor of both $u_1 < u_2$, where $u_1, u_2 \in \mathcal{W}_m$. Since $v \notin \sigma^*$, this produces another loop in the graph $\mathcal{G}^*$ the possibility of which has been ruled out by our assumption.

(iii) Let A be the connected component of $\mathcal{R}(\mathcal{G}) \setminus \sigma^*$ that contains t^*. Then $\mathcal{R}(\mathcal{G}) \setminus A = \mathcal{R}(\mathcal{G}_K)$, by construction. We can reverse the construction presented before the lemma based on deleting edges. In the reversed construction we add edges one at a time, not in batches, to obtain a tower of graphs $(\mathcal{G}_K, \dots, \mathcal{G}_{K-1}, \dots, \mathcal{G}_{K-2}, \dots, \mathcal{G}_1, \mathcal{G})$. Every graph $\mathcal{G}_m$, $m = 1, \dots, K$, is an element of this tower of graphs, but $\mathcal{G}_m$'s are not necessarily consecutive elements.

We will argue that $\mathcal{G}_K$ is an NCC-graph. Suppose that $\mathcal{G}_K$ is not an NCC graph. Then, according to Definition 2.6(v), there are minimal co-terminal cells (σ_1, σ_2) and (σ_3, σ_4) in $\mathcal{G}_K$. Since A is a connected component of $\mathcal{R}(\mathcal{G}) \setminus \sigma^*$, it is easy to see that both cells (σ_1, σ_2) and (σ_3, σ_4) will stay minimal if we add A to $\mathcal{G}_K$. Hence, these cells will be minimal co-terminal cells in $\mathcal{G}$. This contradicts the assumption that $\mathcal{G}$ is NCC and finishes the proof that $\mathcal{G}_K$ is NCC. Hence, there exists a tower of NCC graphs $(\mathcal{G}'_1, \mathcal{G}'_2, \dots, \mathcal{G}'_M, \mathcal{G}_K)$, where $\mathcal{G}'_1$ contains only one full time path. We can concatenate this tower and $(\mathcal{G}_K, \dots, \mathcal{G}_{K-1}, \dots, \mathcal{G}_{K-2}, \dots, \mathcal{G}_1, \mathcal{G})$ to obtain a single tower of NCC graphs $(\mathcal{G}'_1, \mathcal{G}'_2, \dots, \mathcal{G}'_M, \mathcal{G}_K, \dots, \mathcal{G}_{K-1}, \dots, \mathcal{G}_{K-2}, \dots, \mathcal{G}_1, \mathcal{G})$. □

PROPOSITION 5.5. *Let $\mathcal{G}$ be a TLG with a full time path σ^* that is a support for a time point $t \in E_{j^*k^*}$. Let X be a natural $\mathcal{P}$-Markovian harness on $\mathcal{G}$.*

Let $\{\beta(u), u \geq 0\}$ be a one-dimensional Brownian motion independent of the $\mathcal{P}$-harness X, with $\beta(0) = t$. We define the sequence of stopping times

$$\sigma_1 = \inf\{u \geq 0 : \beta(u) \in \{t_{j*}, t_{k*}\}\},$$

and then inductively,

$$\sigma_{m+1} = \inf\{u \geq \sigma_m : \beta(u) \in \mathcal{N}(\beta(\sigma_m))\}.$$

Then, for any $m = 1, 2, \ldots, K$, we get

$$\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_m) = \mathbb{E}_\beta[X(\beta(\sigma_m))].$$

Here $\mathbb{E}_\beta$ is the expectation with respect to the law of β, when the values of the process X are given.

PROOF. Consider the case of $m = 1$. By the graph-Markovian property of the process X, it is clear that

$$\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_1) = \mathbb{E}(X_{j^*k^*} \mid X(t_{j^*}), X(t_{k^*})).$$

Now, applying the harness property (5.2), we get

$$\begin{aligned}\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_1) &= \frac{t - t_{j^*}}{t_{k^*} - t_{j^*}} X(t_{k^*}) + \frac{t_{k^*} - t}{t_{k^*} - t_{j^*}} X(t_{j^*}) \\ &= \mathbb{P}\big(\beta(\sigma_1) = t_{k^*}\big) X(t_{k^*}) + \mathbb{P}\big(\beta(\sigma_1) = t_{j^*}\big) X(t_{j^*}) \\ &= \mathbb{E}_\beta[X(\beta(\sigma_1))].\end{aligned}$$

We now proceed by induction. Suppose that

$$\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_m) = \sum_{t_i \in \mathcal{W}_m} \mathbb{P}\big(\beta(\sigma_m) = t_i\big) X(t_i). \tag{5.5}$$

Then, by the the tower property of conditional expectations, we get

$$\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_{m+1}) = \sum_{t_i \in \mathcal{W}_m} \mathbb{P}\big(\beta(\sigma_m) = t_i\big) \mathbb{E}(X(t_i) \mid \mathcal{H}_{m+1}). \tag{5.6}$$

Now there are two cases to consider. First suppose that t_i is in the fixed full time path σ^*, in which case it is measurable with respect to $\mathcal{H}_{m+1}$, and thus $\mathbb{E}(X(t_i) \mid \mathcal{H}_{m+1}) = X(t_i)$.

The other case is when $t_i \notin \sigma^*$. Note that, since the degree of the vertex t_i is exactly two in the graph $\mathcal{G}_m$, there are two vertices v_1 and v_2 such that if we remove these two vertices, t_i is disconnected from the rest of the graph. By Lemma 5.4(iii) and Remark 3.8(iv), the restriction of X to $\mathcal{G}_m$ is a natural $\mathcal{P}$-process. Thus, from the graph-Markovian property of X on $\mathcal{G}_m$ and harness property

$$\begin{aligned}\mathbb{E}(X(t_i) \mid \mathcal{H}_{m+1}) &= \mathbb{E}(X(t_i) \mid X_{v_1}, X_{v_2}) \\ &= \mathbb{P}\big(\beta(\sigma_{m+1}) = t_{v_1} \mid \beta(\sigma_m) = t_i\big) X(t_{v_1}) \\ &\quad + \mathbb{P}\big(\beta(\sigma_{m+1}) = t_{v_2} \mid \beta(\sigma_m) = t_i\big) X(t_{v_2}).\end{aligned}$$

Substituting this expression back in (5.6) and (5.5) we get

$$\mathbb{E}(X_{j^*k^*}(t) \mid \mathcal{H}_{m+1}) = \sum_{t_i \in \mathcal{W}_{m+1}} \mathbb{P}\big(\beta(\sigma_{m+1}) = t_i\big) X(t_i) = \mathbb{E}_\beta[X(\beta(\sigma_{m+1}))].$$

This completes the proof of the proposition. $\square$

THEOREM 5.6. *Let μ be any probability distribution on* $[0,1]$. *Let Y be a Markovian harness with law $\mathcal{P}$.*

(i) *Given any $\varepsilon > 0$ and any metric ρ which induces the topology of weak convergence, it is possible construct a NCC TLG $\mathcal{G}$, a time point $t^* \in E_{j^*k^*}$, and a full time path $\sigma^* = \sigma(t_{k_1}, t_{k_2}, \dots, t_{k_n})$ such that for a natural harness X on $\mathcal{G}$ with law $\mathcal{P}$, the difference between the laws of the random variables*

$$E(X_{j^*k^*}(t^*) \mid X_E(s), 0 \le s \le 1, E \in \sigma^*) \quad \textit{and} \quad \int_0^1 Y(s)\mu(ds)$$

is less than ε in the metric ρ.

(ii) *If $\mathcal{P}$ is the Wiener measure on* $[0,1]$, *it follows that for any time point $u \in \sigma^*$, one can make the difference between*

$$E(X_{j*k*}(t^*)X_{\sigma^*}(u)) \quad \textit{and} \quad \int_0^u s\mu(ds) + u\mu(u,1] \tag{5.7}$$

smaller than ε.

PROOF. We use Dubins's solution to the Skorokhod embedding problem. Please see the original article by Dubins [2] for more details, or page 332 in the survey article by Obłój [12] (which treats the case when μ is continuous).

Given a measure μ with support in $[0,1]$, the Skorokhod problem asks for a stopping time τ with respect to the Brownian filtration such that a standard one-dimensional Brownian motion β stopped at τ has law μ. The following is a solution proposed by Lester Dubins.

Consider any probability measure ν supported on $[0,1]$. For any finite sequence s of 0's and 1's starting with 0, we will define a probability measure ν_s. Let $\nu_{(0)} = \nu$. Suppose that $s_1 = (s,0)$ and $s_2 = (s,1)$ (this notation is not quite rigorous but it is quite clear). It will suffice to define ν_{s_1} and ν_{s_2} as functions of ν_s. If ν_s is supported on exactly one point then we let $\nu_{s_1} = \nu_{s_2} = \nu_s$. Otherwise we consider ν_s restricted to intervals $[0, \mathbb{E}(\nu_s))$ and $[\mathbb{E}(\nu_s), 1]$. We renormalize both measures and thus we obtain ν_{s_1} and ν_{s_2}.

Let $H_n(\mu)$ be the set of all numbers $\mathbb{E}(\nu_s)$ for all sequences s of length $n+1$, $n \ge 0$, where $\nu_{(0)} = \mu$. The sequence $\{H_n(\mu), n = 0, 1, 2, \dots\}$ can be naturally represented as a tree where every vertex has two descendants unless it is a vertex that is repeated forever.

Let β denote a one-dimensional Brownian motion such that $\beta(0) = \mathbb{E}(\mu)$. Define $\tau_0 \equiv 0$ and define the successive stopping times

$$\tau_{n+1} = \inf\{t \ge \tau_n : \beta(t) \in H_{n+1}(\mu)\}, \qquad n = 0, 1, 2, \dots.$$

Then Dubins shows that the distribution μ_n of $\beta(\tau_n)$ is supported on at most 2^n many atoms, and moreover μ_n converges to μ weakly as n tends to infinity.

We will later show that

$$\mathbb{E}_\beta(Y(\beta(\tau_n))) \to \int Y(s)\,d\mu(s) \tag{5.8}$$

weakly as n tends to infinity.

Assuming that (5.8) is true, for any $\varepsilon > 0$, there exist a large enough N such that the ρ-distance between $\int Y(s)\,d\mu_n(s)$ and $\int Y(s)\,d\mu(s)$ is smaller than ε. This is enough to prove part (i) of the proposition since we can construct a tree $\mathbb{T}_N$ with vertices $\bigcup_{i=0}^{N} H_i(\mu)$ with an obvious tree structure. We add to $\mathbb{T}_N$ a full time path σ^* by connecting $\{0, 1\}$ with all the elements in H_N. Finally we delete the vertex at $\mathbb{E}(\mu)$ and name this time point t^*. Then σ^* is a support for the point t^*. This and Proposition 5.5 imply part (i) of the proposition.

For part (ii) we note that when Y is Brownian motion, the weak convergence (5.8) entails convergence in $\mathbb{L}^2$. This is a standard result for linear combination of Gaussian processes that follows by considering pointwise convergence of the characteristic function. In other words, one can construct a tree as above with an N large enough such that the $\mathbb{L}^2$ distance between

$$\mathbb{E}(X_{j*k*}(t) \mid \mathcal{H}_N) \quad \text{and} \quad \int X_{\sigma*}(s)\,d\mu(s)$$

is appropriately small. Now part (ii) follows by applying the Cauchy–Schwarz inequality since the right-hand side of (5.7) is the covariance between

$$\int X_{\sigma*}(s)\,d\mu(s) \quad \text{and} \quad X_{\sigma*}(u).$$

We return to the proof of (5.8). It follows from Dubins's construction that there is a limiting stopping time τ such that $\lim_{n\to\infty} \tau_n = \tau$ almost surely, and $\lim_{n\to\infty} \beta(\tau_n) = \beta(\tau)$, where $\beta(\tau)$ has law μ. Since β is independent of Y which is continuous in probability (Lemma 5.2), it follows that

$$\lim_{n\to\infty} Y(\beta(\tau_n)) = Y(\beta(\tau)) \qquad \text{with probability one.}$$

By Lemma 5.2 we know that $\{Y(s), 0 \le s \le 1\}$ is uniformly integrable, so the above shows that

$$\lim_{n\to\infty} \mathbb{E}_\beta(Y(\beta(\tau_n))) = \mathbb{E}_\beta Y(\beta(\tau)) \qquad \text{with probability one.}$$

This completes the proof of the theorem. □

6. Brownian motion on honeycomb graph. We will prove a limit theorem for natural Brownian motion on the honeycomb graph, when the diameter of hexagonal cells goes to zero. We will use the term "Brownian motion" to denote the two-sided Brownian motion on the real line conditioned to be equal to 0 at time 0.

Let $\mathcal{R}(\mathcal{G}_\rho^*)$ consist of the boundary of a single hexagon with diameter $\rho > 0$, with two of its sides parallel to the first axis, and the leftmost vertex at $(0, 0)$. Let $\mathcal{R}(\mathcal{G}_\rho)$ be the usual hexagonal lattice in the whole plane, containing $\mathcal{R}(\mathcal{G}_\rho^*)$ as a subset. It is easy to see that there exists a tower $\{\mathcal{G}_\rho^n\}$ of NCC TLGs with the limit $\mathcal{G}_\rho$, satisfying Definition 2.11(ii) so $\mathcal{G}_\rho$ is NCC. Hence, there exists a natural Brownian motion X on $\mathcal{G}_\rho$.

Recall from Theorem 3.9 that the distribution of X does not depend on the tower of NCC-graphs used in the inductive construction. Images of elements of a tower of NCC-graphs under the symmetry with respect to the horizontal axis form another tower converging to $\mathcal{G}_\rho$. The same can be said about images under vertical shifts by the hexagonal cell height. This implies that the natural Brownian motion X on $\mathcal{G}_\rho$ is invariant under the symmetry with respect to the horizontal axis and under vertical shifts.

THEOREM 6.1. *Let X be the natural Brownian motion on $\mathcal{G}_\rho$. Consider $(u,0),(v,x)\in\mathbb{R}^2$ with $u,v,x>0$. Let $(u_\rho,0_\rho)$ be one of the vertices in $\mathcal{R}(\mathcal{G}_\rho)$ with the smallest distance to $(u,0)$, and let E_u^ρ be an edge of $\mathcal{G}_\rho$ that contains this vertex. Let (v_ρ,x_ρ) be one of the vertices in $\mathcal{R}(\mathcal{G}_\rho)$ with the smallest distance to $(v,4x/(\sqrt{3}\rho))$, and let E_v^ρ be an edge of $\mathcal{G}_\rho$ that contains this vertex. Let Φ be the standard normal cumulative distribution function, that is, $\Phi(a)=(1/\sqrt{2\pi})\int_{-\infty}^a e^{-s^2/2}\,ds$. Then*

$$
\begin{aligned}
\lim_{\rho\to 0}&\,\mathbb{E}(X_{E_u^\rho}(u_\rho)X_{E_v^\rho}(v_\rho))\\
&=\frac{\sqrt{5x}}{8\sqrt{\pi}}\big(e^{-16(u+v)^2/(5x)}-e^{-16(u-v)^2/(5x)}\big)\\
&\quad-(1/2)(u-v)\big(2\Phi\big(4(u-v)/\sqrt{5x}\big)-1\big)\\
&\quad+(1/2)(u+v)\big(2\Phi\big(4(u+v)/\sqrt{5x}\big)-1\big).
\end{aligned}
$$

REMARK 6.2. The above formula can be slightly simplified, but we leave it in the present form to show that the expression is symmetric in u and v. This is evident once we recall that $2\Phi(a)-1$ is an odd function. Symmetry in u and v is something that we expect because of the invariance of X under the symmetry with respect to the horizontal axis and invariance under vertical shifts. Note that the formula does not depend only on $|u-v|$. This is because X is not invariant under horizontal shifts. The reason is that $X(t)=0$, a.s., for every t of the form $(0,y)\in\mathcal{R}(\mathcal{G}_\rho)$; this is not true for any other vertex.

PROOF OF THEOREM 6.1. The core of our argument is based on the harness idea, just like the arguments in Section 5.

In this proof we will distinguish between points in the representation of a graph and their projections on the real axis. So far, this distinction was not very helpful, so it was ignored in most of the paper. We will identify edges E_{jk} with sets $E_{jk}([t_j,t_k])\subset\mathbb{R}^2$. Recall the following convention introduced after Definition 2.4: $\overline{t}_j=(t_j,E_{jk}(t_j))$. As a first application of this notation, we write $\overline{v}_\rho=(v_\rho,x_\rho)$ and $\overline{u}_\rho=(u_\rho,0_\rho)$. The meaning of $X(\overline{t})$ for $\overline{t}\in\mathcal{G}_\rho$ is clear.

Recall that hexagonal cells in $\mathcal{R}(\mathcal{G}_\rho)$ have diameter ρ, and one of the vertices is located at $(0, 0)$. Let $\rho_1 = \rho\sqrt{3}/4$.

We will construct a tower of finite NCC graphs. Let $\Gamma_1 \subset \mathcal{R}(\mathcal{G}_\rho)$ be the graph of a nondecreasing function with a starting point $(y_1, 0)$ on the horizontal axis and endpoint at $\overline{v}_\rho$. It is easy to check that this defines Γ_1 uniquely. Similarly, let $\Gamma_2 \subset \mathcal{R}(\mathcal{G}_\rho)$ be the graph of a nonincreasing function with the starting point at $\overline{v}_\rho$ and an endpoint $(y_2, 0)$ on the horizontal axis. Let $\Gamma_3 \subset \mathcal{R}(\mathcal{G}_\rho)$ be the graph of a function on the interval $(-\infty, \infty)$ with values in $[-\rho_1, 0]$; such a function is unique. Let $\Gamma_4 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and note that $\mathcal{R}(\mathcal{G}_\rho) \setminus \Gamma_4$ has two unbounded connected components, say, Γ_5 and Γ_6. Let $\Gamma_7 = \mathcal{R}(\mathcal{G}_\rho) \setminus (\Gamma_5 \cup \Gamma_6)$. It is easy to see that Γ_7 is a representation of a TLG $\mathcal{G}_*$.

All vertices of $\mathcal{R}(\mathcal{G}_\rho)$ lie on lines $L_j := \{(t, x) : x = j\rho_1\}$. Let $\mathcal{W}_j$ be the set of all vertices of $\mathcal{R}(\mathcal{G}_\rho)$ that lie on $\bigcup_{n \le j} L_n$. Let $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$ be the graph obtained from $\mathcal{G}_*$ by deleting all vertices in $\mathcal{R}(\mathcal{G}_*)$ which are not in $\mathcal{W}_j$ and all corresponding edges. Note that $\mathcal{R}(\mathcal{G}_j) \cap L_j \neq \varnothing$ but there are no vertices in $\mathcal{V}_j$ with representation in L_j.

It is easy to see that $\mathcal{G}_j$ are elements of a tower of NCC graphs that starts from a graph with a single full time path represented by Γ_3 and ends with $\mathcal{G}_*$. To construct such a tower, we add edges, one at a time, at the top layer of $\mathcal{G}_j$, until we obtain $\mathcal{G}_{j+1}$. A similar idea can be used to continue the construction of the tower beyond $\mathcal{G}_*$, so that the union of all the graphs in the tower is $\mathcal{G}_\rho$. This construction, Theorem 3.9 and Remark 3.8(iv) show that the restriction of X to $\mathcal{G}_j$ is a natural Brownian motion on $\mathcal{G}_j$.

Note that x_ρ is within distance ρ_1 of $[\sqrt{3}x/(4\rho)]$. We define j_* by $(v_\rho, x_\rho) \in L_{j_*}$. Let $\mathcal{W}^-$ be the set of all vertices of the form $(s, x) \in \mathcal{R}(\mathcal{G}_\rho)$ with $s \le 0$. Let $\mathcal{H}_k$ be the σ-field generated by $\{X_\sigma, \sigma \in \mathcal{E}_k\}$ and by $\{X(t), t \in \mathcal{W}^-\}$. The family $\{\mathcal{H}_k\}$ is a filtration, that is, $\mathcal{H}_k \subset \mathcal{H}_{k+1}$ for all k. Consider a vertex $\overline{t}$ of $\mathcal{G}_*$ whose representation belongs to L_k. Recall that $\overline{t}$ does not belong to $\mathcal{V}_k$. Hence, $\overline{t}$ is in the interior of an edge in $\mathcal{E}_k$ connecting two vertices in $\mathcal{V}_k \cap L_{k-1}$. Let $\overline{\mathcal{N}}(\overline{t})$ denote the set of endpoints of this edge and let $\mathcal{N}(\overline{t})$ be the projection of $\overline{\mathcal{N}}(\overline{t})$ on the time axis.

Let $\{\beta_u, u \ge 0\}$ be a one-dimensional Brownian motion independent of X, starting at $\beta_0 = v_\rho$. We define a sequence of stopping times, starting with $\tau_0 = 0$ and

$$\tau_1 = \inf\{u \ge 0 : \beta_u \in \mathcal{N}(\overline{v}_\rho)\}.$$

Let $\overline{\beta}(\tau_1)$ be the point in $\overline{\mathcal{N}}(\overline{v}_\rho)$ with the time coordinate $\beta(\tau_1)$ and note that $\overline{\beta}(\tau_1) \in L_{j_*-1}$.

If $\beta(\tau_m) \le 0$, then we let $\tau_{m+1} = \tau_m$. Otherwise we let

$$\tau_{m+1} = \inf\{u \ge \tau_m : \beta_u \in \mathcal{N}(\overline{\beta}_{\tau_m})\}.$$

Let $\overline{\beta}(\tau_{m+1})$ be the point in $\overline{\mathcal{N}}(\overline{\beta}_{\tau_m})$ with the time coordinate $\beta(\tau_{m+1})$; then $\overline{\beta}(\tau_{m+1}) \in L_{j_*-m-1}$. It follows from our definition of $\mathcal{G}_*$ that $\overline{\mathcal{N}}(\overline{\beta}_{\tau_m}) \subset \mathcal{R}(\mathcal{G}_*)$. See Figure 4.

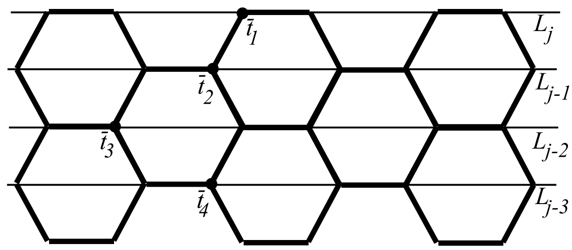

FIG. 4. *The figure represents a fragment of the upper part of $\mathcal{G}_j$. The points $\bar{t}_k$, $k = 1, 2, 3, 4$, represent a possible path of $\overline{\beta}_{\tau_m}$, $m = j_* - j, j_* - j + 1, \ldots$.*

Recall that for a random variable Z, $\mathbb{E}_\beta Z$ denotes the expectation with respect to the law of β, that is, a function of X. We have shown that the restriction of X to $\mathcal{G}_j$ is a unique in law natural Brownian motion on $\mathcal{G}_j$.

Let $\overline{\mathcal{N}}(\overline{v}_\rho) = \{\bar{t}_1, \bar{t}_2\} \subset \mathcal{W}_{j_*-1}$, with $t_1 < t_2$. By the graph-Markovian property of X on the graph $\mathcal{G}_{j_*-1}$,

$$\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_{j_*-1}) = \mathbb{E}(X(\overline{v}_\rho) \mid X(\bar{t}_1), X(\bar{t}_2)).$$

Applying the harness property (5.2), we get

$$\begin{aligned}
\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_{j_*-1}) &= \frac{v_\rho - t_1}{t_2 - t_1} X(\bar{t}_2) + \frac{t_2 - v_\rho}{t_2 - t_1} X(\bar{t}_1) \\
&= \mathbb{P}\big(\beta(\tau_1) = t_2\big) X(\bar{t}_2) + \mathbb{P}\big(\beta(\tau_1) = t_1\big) X(\bar{t}_1) \\
&= \mathbb{P}\big(\overline{\beta}(\tau_1) = \bar{t}_2\big) X(\bar{t}_2) + \mathbb{P}\big(\overline{\beta}(\tau_1) = \bar{t}_1\big) X(\bar{t}_1) \\
&= \mathbb{E}_\beta(X(\overline{\beta}_{\tau_1})).
\end{aligned}$$

We now proceed by induction. Suppose that

$$\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_{j_*-m}) = \sum_{\bar{t}_i \in \mathcal{W}_{j_*-m} \cup \mathcal{W}^-} \mathbb{P}\big(\overline{\beta}(\tau_m) = \bar{t}_i\big) X(\bar{t}_i). \tag{6.1}$$

Then, by the the tower property, we get

$$\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_{j_*-m-1}) = \sum_{\bar{t}_i \in \mathcal{W}_{j_*-m} \cup \mathcal{W}^-} \mathbb{P}\big(\overline{\beta}(\tau_m) = \bar{t}_i\big) \mathbb{E}(X(\bar{t}_i) \mid \mathcal{H}_{j_*-m-1}). \tag{6.2}$$

If $\bar{t}_i \in \mathcal{W}^-$ then $\mathbb{E}(X(\bar{t}_i) \mid \mathcal{H}_{j_*-m-1}) = X(\bar{t}_i)$. If $\bar{t}_i \in \mathcal{W}_{j_*-m} \setminus \mathcal{W}^-$ then let $\overline{\mathcal{N}}(\bar{t}_i) = \{\bar{t}_{i_1}, \bar{t}_{i_2}\} \subset \mathcal{W}_{j_*-m-1}$. From the graph-Markovian property and harness property applied to X restricted to $\mathcal{G}_{j_*-m}$,

$$\begin{aligned}
\mathbb{E}(X(\bar{t}_i) \mid \mathcal{H}_{j_*-m-1}) &= \mathbb{E}(X(\bar{t}_i) \mid X(\bar{t}_{i_1}), X(\bar{t}_{i_2})) \\
&= \mathbb{P}\big(\overline{\beta}(\tau_{m+1}) = \bar{t}_{i_1} \mid \overline{\beta}(\tau_m) = \bar{t}_i\big) X(\bar{t}_{i_1}) \\
&\quad + \mathbb{P}\big(\overline{\beta}(\tau_{m+1}) = \bar{t}_{i_2} \mid \overline{\beta}(\tau_m) = \bar{t}_i\big) X(\bar{t}_{i_2}).
\end{aligned}$$

Substituting this expression back in (6.1) and (6.2) we get

$$\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_{j_*-m-1}) = \sum_{\overline{t}_i \in \mathcal{W}_{j_*-m-1} \cup \mathcal{W}^-} \mathbb{P}(\overline{\beta}(\tau_{m+1}) = \overline{t}_i) X(t_i) = \mathbb{E}_\beta(X(\overline{\beta}_{\tau_{m+1}})).$$

We are interested in the case when $j_* - m - 1 = 0$, that is,

$$\mathbb{E}(X(\overline{v}_\rho) \mid \mathcal{H}_0) = \mathbb{E}_\beta(X(\overline{\beta}_{\tau_{j_*}})). \tag{6.3}$$

For integer $m \geq 1$, let $\overline{B}_{m\rho^2} := \overline{\beta}_{\tau_m}$ and define $\overline{B}_s$ for other values of $s \geq 0$ by $\overline{B}_s = \overline{B}_{[s/\rho^2]\rho^2}$. Let $\overline{C}_k = \overline{B}_{(k+1)\rho^2} - \overline{B}_{k\rho^2}$. Let B_s be the projection of $\overline{B}_s$ on the time axis, and similarly, let C_k be the projection of $\overline{C}_k$ on the time axis. Let us ignore for the moment the possibility that $\overline{\beta}$ hits $\mathcal{W}^-$. The random variables C_k are not independent but they form a Markov chain. By [1], Example 2, page 167, and [1], Theorem 20.1, when $\rho \to 0$, the process $B_s = \sum_{k \leq [s/\rho^2]-1} C_k$ converges weakly in the Skorokhod space to Brownian motion W_s with a diffusion coefficient **s**. We will next calculate **s**.

The possible values of C_k's are $-3\rho/4$, $-\rho/4$, $\rho/4$ and $3\rho/4$. If we list all states of C_k in this order then the transition matrix for this Markov chain is

$$\begin{pmatrix} 1/4 & 0 & 3/4 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 3/4 & 0 & 1/4 \end{pmatrix}.$$

The stationary distribution for C_k is $(1/8, 3/8, 3/8, 1/8)$. Hence, in the stationary regime,

$$\begin{aligned}\mathbb{E}C_k^2 &= (-3\rho/4)^2(1/8) + (-\rho/4)^2(3/8) + (\rho/4)^2(3/8) + (3\rho/4)^2(1/8) \\ &= 5\rho^2/32.\end{aligned}$$

Since the process $B_{k\rho^2}$ is a martingale, it follows that

$$\operatorname{Var}(B_{k\rho^2} - B_0) = \sum_{0 \leq n \leq k-1} \operatorname{Var} C_n = \sum_{0 \leq n \leq k-1} 5\rho^2/32 = k5\rho^2/32.$$

Hence, the diffusion coefficient **s** of W_s is $\sqrt{5/32}$.

Note that although we suppressed ρ in the notation for the process B_s, the distribution of this process depends on ρ. Recall that $\rho_1 = \rho\sqrt{3}/4$, and let $A_s^\rho = x_\rho - s\rho_1$ for $s \geq 0$. Heuristically, $\overline{\beta}_{\tau_m} = (B_{m\rho^2}, A_m^\rho)$, $m \in \mathbb{Z}$, $m \geq 0$, is a space–time discrete time Markov chain, with the "time" A_m^ρ running in the negative direction along the second axis, starting from x_ρ, and the speed of ρ_1 per one step. The "space" component $B_{m\rho^2}$ of this process runs along the first axis, starting from v_ρ. The right-hand side of (6.3) is evaluated by integrating the values of X with respect to the hitting distribution of $\mathcal{W}_0 \cup \mathcal{W}^-$ by $(B_{m\rho^2}, A_m^\rho)$. Since $X(\overline{t}) = 0$ for $\overline{t}$

of the form $(0, y)$, $u_\rho > 0$ and $X(\overline{\beta}_{\tau_{j*}}) \le 0$, we obtain from the graph-Markovian property $\mathbb{E}(X(\overline{u}_\rho)X(\overline{\beta}_{\tau_{j*}})) = 0$. For $\overline{t}_1, \overline{t}_2 \in L_0$, $\mathbb{E}(X(\overline{t}_1)X(\overline{t}_2)) = t_1 \wedge t_2$.

Let $A_s = x - s$ for $s \ge 0$. When $\rho \to 0$, processes $(B_{s\rho^2}, \rho_1 A_s^\rho)$, $s \ge 0$, converge to space–time Brownian motion (W_s, A_s), $s \ge 0$, with $(W_0, A_0) = (v, x)$, stopped at the exit time from the first quadrant, with the "time" component A_s running at the standard speed and the spatial component having diffusion coefficient $\mathbf{s} = \sqrt{5/32}$. Let τ^* be the exit time from the first quadrant by (W_s, A_s), and let τ^{**} be the exit time from the upper half-plane. Let K be the vertical part of the boundary of the first quadrant. Let f be the real valued function defined on the boundary of the first quadrant, with zero values on K and such that $f(t, 0) = u \wedge t$ for $t \ge 0$. Then, by weak convergence, and using (6.3), $\lim_{\rho\to 0} \mathbb{E}(X(\overline{u}_\rho)X(\overline{v}_\rho)) = \mathbb{E} f(A_{\tau^*}, W_{\tau^*})$.

Let $\phi(s)$ be the density of normal random variable with mean v and variance $5x/32$, that is,

$$\phi(s) = \frac{1}{\sqrt{5\pi x/16}} \exp\biggl(-\frac{(s-v)^2}{5x/16}\biggr).$$

Then using the reflection principle at the hitting time of K we obtain

$$\begin{aligned}
&\lim_{\rho\to 0} \mathbb{E}(X(\overline{u}_\rho)X(\overline{v}_\rho)) \\
&\quad = \mathbb{E} f(A_{\tau^*}, W_{\tau^*}) \\
&\quad = \mathbb{E} f(A_{\tau^{**}}, W_{\tau^{**}}) - \mathbb{E}\bigl(f(A_{\tau^{**}}, W_{\tau^{**}})\mathbf{1}_{\{(A_{\tau^*}, W_{\tau^*}) \in K\}}\bigr) \\
&\quad = \biggl(\int_0^u s\phi(s)\,ds + \int_u^\infty u\phi(s)\,ds\biggr) - \biggl(\int_{-u}^0 (-s)\phi(s)\,ds + \int_{-\infty}^{-u} u\phi(s)\,ds\biggr) \\
&\quad = \int_{-u}^u s\phi(s)\,ds + \int_u^\infty u\phi(s)\,ds - \int_{-\infty}^{-u} u\phi(s)\,ds \\
&\quad = \frac{\sqrt{5x}}{8\sqrt{\pi}}\bigl(e^{-16(u+v)^2/(5x)} - e^{-16(u-v)^2/(5x)}\bigr) \\
&\qquad - (1/2)(u-v)\bigl(2\Phi\bigl(4(u-v)/\sqrt{5x}\bigr) - 1\bigr) \\
&\qquad + (1/2)(u+v)\bigl(2\Phi\bigl(4(u+v)/\sqrt{5x}\bigr) - 1\bigr). \qquad \square
\end{aligned}$$

Acknowledgment. We are grateful to the referee for a very careful reading of the original manuscript and very helpful suggestions for improvement.

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