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PERIODIC HOMOGENIZATION WITH AN INTERFACE: THE MULTI-DIMENSIONAL CASE

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We consider a diffusion process with coefficients that are periodic outside of an "interface region" of finite thickness. The question investigated in this article is the limiting long time/large scale behavior of such a process under diffusive rescaling. It is clear that outside of the interface, the limiting process must behave like Brownian motion, with diffusion matrices given by the standard theory of homogenization. The interesting behavior therefore occurs on the interface. Our main result is that the limiting process is a semimartingale whose bounded variation part is proportional to the local time spent on the interface. The proportionality vector can have nonzero components parallel to the interface, so that the limiting diffusion is not necessarily reversible. We also exhibit an explicit way of identifying its parameters in terms of the coefficients of the original diffusion.

Similarly to the one-dimensional case, our method of proof relies on the framework provided by Freidlin and Wentzell [*Ann. Probab.* **21** (1993) 2215–2245] for diffusion processes on a graph in order to identify the generator of the limiting process.

1. Introduction. The theory of periodic homogenization is by now extremely well understood; see, for example, the monographs [4, 23]. Recall that the most basic result states that if X is a diffusion with smooth periodic coefficients, then the diffusively rescaled process $X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$ converges in law to a Brownian motion with an explicitly computable diffusion matrix. If one considers diffusions that are "locally periodic," but with slow modulations over spatial scales of order ε^{-1}, then it was shown in [5] that the rescaled process converges in general to some diffusion process with a computable expression for both its drift and diffusion coefficients.

In this article, we will also consider the "locally periodic" situation, but instead of considering slow modulations of the coefficients, we consider the case of a sharp [i.e., of size $\mathcal{O}(1)$] transition between two periodic structures. In the (much simpler) one-dimensional case, this model was previously studied in [17], where we showed that the rescaled process converges in law to skew Brownian motion with an explicit expression for the skewness parameter. In higher dimensions, this model has not yet been studied to the best of our knowledge. The aim of this article

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is to clarify what is the behavior of X^ε near the interface for very small values of ε. It is important to remark at this stage that we do *not* make the assumption that our diffusion is reversible. As we will see in Section 2, there are then situations in which the limiting process is not reversible either, contrary to the one-dimensional situation.

One feature of the problem at hand is that there is no *finite* invariant measure built into the framework of the problem. This is unlike most other homogenization problems, even those exhibiting rather "bad" ergodic properties, such as the random environment case [21, 24] or the quenched convergence results for the Bouchaud trap model [2]. Since in our case the invariant measure μ of X is only σ-finite, this leads to two problems when trying to compute the effect of the behavior of X near the interface in the limit $\varepsilon \to 0$. Indeed, one would "naïvely" expect that an effective drift along the interface can be described by the quantity

$$\int b(x)\mu(dx). \tag{1.1}$$

One problem with this expression is that there is no obvious natural normalization for μ. Furthermore, since b is periodic away from the interface and the same is (approximately) true for μ, this integral certainly does not converge, even if we consider it as an integral over $\mathbb{R} \times \mathbb{T}^{d-1}$ by making use of the periodic structure in the directions parallel to the interface. See however (2.4) and Proposition 6.3 below for the correct way of interpreting (1.1) and our main result, Theorem 2.4 below, on how this quantity appears in the construction of the limiting process.

Another common feature of many homogenization results is the usage of a globally defined corrector function to compensate for the singular terms appearing in the problem. This is of course the case for standard periodic homogenization [4], but also for a number of stochastic homogenization problems, as, for example, in [21, 22, 24, 25]. For the present problem however, it will be convenient to make use of corrector function that only cancels the singular terms away from the interface and to treat the behavior of the limiting process at the interface by completely different means.

One very recent homogenization result where discontinuous coefficients appear in the limiting equation can be found in [3] (which in turn generalizes [18]). However, their framework is quite different to the one considered here and does not seem to encompass our problem. Much more closely related problems are homogenization problems with the presence of a boundary [1, 14]. Those have been mostly studied by analytical tools so far. In our probabilistic language, what comes closest to the boundary layers studied in these articles is the σ-finite invariant measure of X, which is shown in Proposition 5.5 below to converge exponentially fast to a measure with periodic densities away from the interface.

For simplicity, we will consider the case of a constant diffusion matrix, but it is straightforward to adapt the proofs to cover the case of nonconstant diffusivity as

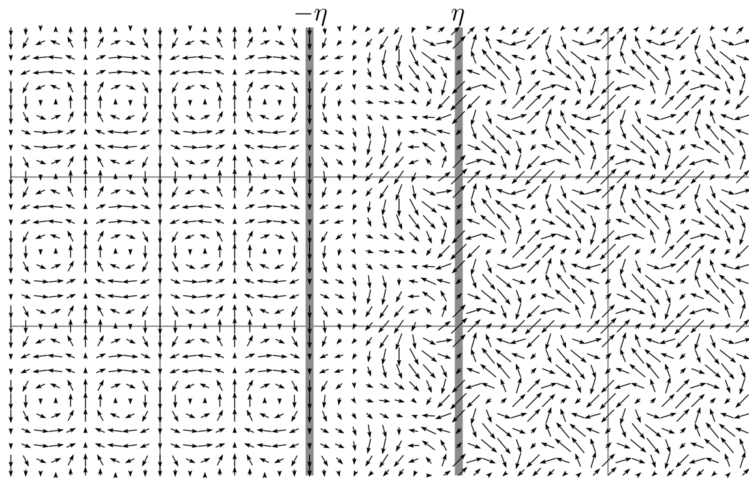

FIG. 1. *Example of a vector field b satisfying our conditions.*

well. More precisely, we consider the family of processes X^ε taking values in $\mathbb{R}^d$, solutions to the stochastic differential equations

$$dX^\varepsilon = \frac{1}{\varepsilon} b\left(\frac{X^\varepsilon}{\varepsilon}\right) ds + dB(s), \qquad X^\varepsilon(0) = x, \tag{1.2}$$

where B is a d-dimensional standard Wiener process. The drift b is assumed to be smooth and such that $b(x + e_i) = b(x)$ for the unit vectors e_i with $i = 2, \dots, d$ (but not for $i = 1$). Furthermore, we assume that there exist smooth vector fields $b_\pm$ with unit period in *every* direction and $\eta > 0$ such that

$$b(x) = b_+(x), \qquad x_1 > \eta, \qquad b(x) = b_-(x), \qquad x_1 < -\eta.$$

Figure 1 is a typical illustration of the type of vector fields that we have in mind.

If we denote by X the same process, but with $\varepsilon = 1$, then the process X^ε given by (1.2) is equal in law to the diffusive rescaling of X by a factor $\frac{1}{\varepsilon}$. In the sequel, we denote the generator of X by $\mathcal{L}$ and the generator of X^ε by $\mathcal{L}_\varepsilon$. We furthermore denote by $\mathcal{L}_\pm$ the generators for the diffusion processes on the torus given by

$$dX^\pm = b_\pm(X^\pm)\, ds + dB(s), \tag{1.3}$$

and by $\mu_\pm$ the corresponding invariant probability measures. With this notation at hand, we impose the centering condition $\int_{\mathbb{T}^d} b_\pm(x)\mu_\pm(x) = 0$.

Under these conditions, our main result formulated in Theorem 2.4 below states that the family X^ε converges in law to a limiting process $\bar{X}$. Furthermore, we give an explicit characterization of $\bar{X}$, both as the unique solution of a martingale problem with some explicitly given generator and as the solution of a stochastic differential equation involving a local time term on the interface $\{x_1 = 0\}$. In addition to the homogenized diffusion coefficients on either side of the interface, this limiting process is characterized by a "transmissivity coefficient," as well as by a "drift vector" pointing along the interface.

The remainder of this article is structured as follows. After formulating our main results in Section 2, we show tightness of the family in Section 3. In Section 4, we then formulate the main tool used in the identification of the limiting process, namely a multidimensional analogue of the tool used by Freidlin and Wentzell in [12] to study homogenization problems where the limiting process takes values in a graph. Section 5 is then devoted to the computation of the transmissivity coefficient, whereas Section 6 contains the computation of the drift vector. Finally, we show in Section 7 that the martingale problem is well-posed and we identify its solution with the solution to a stochastic differential equation.

1.1. *Notation*. We define the "interface" of width K by

$$\mathscr{I}_K = \{x \in \mathbb{R}^d : x_1 \in [-K, K]\}.$$

We also denote by $\partial \mathscr{I}_K$ its boundary.

Frequently throughout the paper we will construct successive escape and subsequent reentry times particularly when constructing invariant measures in terms of the invariant measure of an embedded Markov chain as in [16]. We will denote such pairs of stopping times as σ, ϕ, which denote escape and reentry times, respectively. Other stopping times not part of such a sequence will be denoted by τ.

2. The main result. Before stating the main result, we will first define the various quantities involved and their relevance. It is clear that, in view of standard results from periodic homogenization [4, 23], any limiting process for X^ε should behave like Brownian motion on either side of the interface $\mathscr{I}_0 = \{x_1 = 0\}$, with effective diffusion tensors given by

$$D_{ij}^{\pm} = \int_{\mathbb{T}^d} (\delta_{ik} + \partial_k g_i^{\pm})(\delta_{kj} + \partial_k g_j^{\pm})\, d\mu_{\pm}.$$

(Summation of k is implied.) Here, the corrector functions $g_\pm : \mathbb{T}^d \to \mathbb{R}^d$ are the unique solutions to $\mathcal{L}_\pm g_\pm = -b_\pm$ such that

$$\int_{\mathbb{T}^d} g_\pm(x)\mu_\pm(dx) = 0.$$

Since $b_\pm$ are centered with respect to $\mu_\pm$, such functions do indeed exist.

This justifies the introduction of a differential operator $\bar{\mathcal{L}}$ on $\mathbb{R}^d$ defined in two parts by $\bar{\mathcal{L}}_+$ on $I_+ = \{x_1 > 0\}$ and $\bar{\mathcal{L}}_-$ on $I_- = \{x_1 < 0\}$ with

$$\bar{\mathcal{L}}_\pm = \frac{D_{ij}^\pm}{2} \partial_i \partial_j, \tag{2.1}$$

then one would expect any limiting process to solve a martingale problem associated to $\bar{\mathcal{L}}$. However, the above definition of $\bar{\mathcal{L}}$ is not complete, since we did not specify any boundary condition at the interface $\mathscr{I}_0$.

One of the main ingredients in the analysis of the behavior of the limiting process at the interface is the invariant measure μ for the (original, not rescaled) process X. It is not clear a priori that such an invariant measure exists, since X is not expected to be recurrent in general. However, if we identify points that differ by integer multiples of e_j for $j = 2, \dots, d$, we can interpret X as a process with state space $\mathbb{R} \times \mathbb{T}^{d-1}$. It then follows from the results in [16] that this process admits a σ-finite invariant measure μ on $\mathbb{R} \times \mathbb{T}^{d-1}$.

Note that the invariant measure μ is *not* finite and can therefore not be normalized in a canonical way. However, if we define the "unit cells" $C_j^{\pm}$ by

$$C_j^+ = [j, j+1] \times \mathbb{T}^{d-1}, \qquad C_j^- = [-j-1, -j] \times \mathbb{T}^{d-1},$$

then it is possible to make sense of the quantity $q_{\pm} = \lim_{j\to\infty} \mu(C_j^{\pm})$ (we will show in Proposition 5.5 below that this limit actually exists).

Let now $p_{\pm}$ be given by

$$p_{\pm} = \frac{q_{\pm} D_{11}^{\pm}}{q_+ D_{11}^+ + q_- D_{11}^-},$$

which can also we rewritten in a more suggestive way as

$$\frac{p_+}{p_-} = \frac{q_+ D_{11}^+}{q_- D_{11}^-}. \tag{2.2}$$

This is the homogenized diffusion coefficient in the direction perpendicular to the interface, weighted by the invariant measure of a unit cell. Comparing with the one-dimensional case [17], one would expect this to yield the likelihood for X^ε to exit a small (but still much larger than ε) neighborhood of the interface on a specific side.

REMARK 2.1. The ratio

$$\frac{p_+ \sqrt{D_{11}^-}}{p_- \sqrt{D_{11}^+} + p_+ \sqrt{D_{11}^-}} \tag{2.3}$$

gives the asymptotic probability of the process being located in the rhs (+) of the interface after a long time. This follows from the weak convergence of the first component to a skew Brownian motion with (possibly) different diffusion coefficients on either side of the interface. If we rescale this skew BM on either side of the interface by $\sqrt{D_{11}^{\pm}}$ to obtain a standard skew BM, we can use the scale function of BM to finish the verification of (2.3).

However, unlike in the one-dimensional case, these quantities are not yet sufficient to characterize the limiting process. The reason is that since X^ε is expected to spend time proportional to ε in the interface, but the drift is of order ε^{-1} there, it is not impossible that the limiting process picks up a nontrivial drift along the interface. It turns out that this drift can be described by the coefficients α_j given by

$$\alpha_j = 2\left(\frac{p_+}{D_{11}^+} + \frac{p_-}{D_{11}^-}\right)\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \big(b_j(x) + \mathcal{L}g_j(x)\big)\mu(dx), \tag{2.4}$$

where μ is again normalized in such a way that $q_+ + q_- = 1$ and where g is any smooth function agreeing with $g_\pm$ on either side of the interface (see Section 3).

REMARK 2.2. Since $\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \mathcal{L}\phi(x)\mu(dx) = 0$ for every smooth *compactly supported* function ϕ, one should interpret the integral on the right-hand side of (2.4) as a "renormalized" form of the intuitive more meaningful quantity (1.1).

REMARK 2.3. The expression (2.4) is useful in order to generate examples with nonvanishing values for the coefficients α_i.

Given all of these ingredients, we can construct an operator $\bar{\mathcal{L}}$ as follows. The domain $\mathscr{D}(\bar{\mathcal{L}})$ of $\bar{\mathcal{L}}$ consists of functions $f:\mathbb{R}^d \to \mathbb{R}$ such that:

- The restrictions of f to I_+, I_- and $\mathscr{I}_0$ are smooth.
- The partial derivatives $\partial_i f$ are continuous for $i \geq 2$.
- The partial derivative $\partial_1 f(x)$ has right and left limits $\partial_1 f|_{I_\pm}$ as $x \to \mathscr{I}_0$ and these limits satisfy the gluing condition

$$p_+\,\partial_1 f|_{I_+} - p_-\partial_1 f|_{I_-} + \sum_{j=2}^{d} \alpha_j\,\partial_j f = 0. \tag{2.5}$$

For any $f \in \mathscr{D}(\bar{\mathcal{L}})$, we then set $\bar{\mathcal{L}}f(x) = \mathcal{L}_\pm f(x)$ for $x \in I_\pm$. With these definitions at hand, we can state the main result of the article.

THEOREM 2.4. *The family of processes X^ε converges in law to the unique solution $\bar{X}$ to the martingale problem given by the operator $\bar{\mathcal{L}}$. Furthermore, there exist matrices $M_\pm$ and a vector $K \in \mathbb{R}^d$ such that this solution solves the SDE*

$$d\bar{X}(t) = \mathbf{1}_{\bar{X}_1\leq 0}M_-\,dW(t) + \mathbf{1}_{\bar{X}_1>0}M_+\,dW(t) + K\,dL(t). \tag{2.6}$$

where L denotes the symmetric local time of $\bar{X}_1$ at the origin and W is a standard d-dimensional Wiener process. The matrices $M_\pm$ and the vector K satisfy

$$M_\pm M_\pm^T = D^\pm, \qquad K_1 = p_+ - p_-, \qquad K_j = \alpha_j,$$

for $j = \{2,\dots,d\}$.

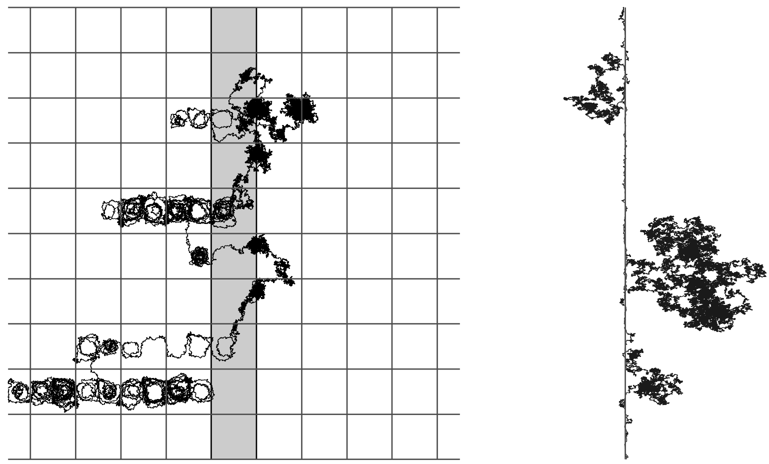

FIG. 2. *Sample paths at small (left) and large (right) scales.*

In Figure 2, we show an example of a numerical simulation of the process studied in this article. The figure on the left shows the small-scale structure (the periodic structure of the drift is drawn as a grid). One can clearly see the periodic structure of the sample path, especially to the left of the interface. One can also see that the effective diffusivity is not necessarily proportional to the identity. In this case, to the left of the interface, the process diffuses much more easily horizontally than vertically.

The picture to the right shows a simulation of the process at a much larger scale. We used a slightly different vector field for the drift in order to obtain a simulation that shows clearly the strong drift experienced by the process when it hits the interface.

REMARK 2.5. Since the quadratic variation of $\bar{X}$ has a discontinuity at $\bar{X}_1 = 0$, we do have to specify which kind of local time L is. Using the symmetric local time yields nicer expressions. See, for example, [19, 26] for a definition of the symmetric local time.

Analyzing what this means for a simple example, we consider the case of a two-dimensional problem where we have $b_1 = 0$ and $b_2 = f(x_1)$ for f a smooth function that is zero outside of $\mathscr{I}_\eta$. Clearly, $p_\pm = \frac{1}{2}$. In this case, the invariant measure μ of the process X is given by $\frac{1}{2}$ times Lebesgue measure on $\mathbb{R} \times S^1$ and we can choose $g = 0$. This implies that we then simply have

$$\alpha_2 = \int_{\mathbb{R}} f(x)\,dx,$$

as one would expect.

3. Tightness of the family. The aim of this section is to prove the following tightness result.

THEOREM 3.1. *Denote by $\mathbb{P}^\varepsilon$ the law of X_x^ε on $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$. Then the family $\{\mathbb{P}^\varepsilon\}_{\varepsilon\in(0,1]}$ is tight.*

Similar to what happens in the classical theory of periodic homogenization, it will be very convenient to construct a "corrected process" Y, obtained by adding to X a corrector function that cancels out to first order the effect of the small oscillations. To this aim, we introduce a smooth function $g:\mathbb{R}^d\to\mathbb{R}^d$ which is periodic in the directions $2,\dots,d$ and such that $g(x)=g_+(x)$ for $x_1\geq\eta$ and similarly for $x_1\leq-\eta$. (Recall that $g_\pm$ was defined in Section 2.) We do not specify the behavior of g inside the interface $\mathscr{I}_\eta$, except that it has to be smooth in the whole space and periodic in the directions parallel to the interface. We fix such a function g once and for all from now on. We furthermore denote by Y^ε the process defined by $Y^\varepsilon=X^\varepsilon+\varepsilon g(\varepsilon^{-1}X^\varepsilon)$, as well as $y=x+\varepsilon g(x/\varepsilon)$ for its initial condition.

Defining the corrected drift $\tilde{b}(x)=(\mathcal{L}g+b)(x)$ and the corrected diffusion coefficient $\tilde{\sigma}_{ij}(x)=\delta_{ij}+\partial_j g_i(x)$, it follows from Itô's formula that the ith component of Y_y^ε satisfies

$$(Y_y^\varepsilon)_i(t)=y_i+\int_0^t\frac{1}{\varepsilon}\tilde{b}_i\biggl(\frac{1}{\varepsilon}X_x^\varepsilon(s)\biggr)\,ds+\int_0^t\tilde{\sigma}_{ij}\biggl(\frac{1}{\varepsilon}X_x^\varepsilon(s)\biggr)\,dW_j(s).\tag{3.1}$$

It is very important to note that the corrected drift $\tilde{b}$ *vanishes* outside of $\mathscr{I}_\eta$, so that the process Y is subject to a large drift only when X is inside the interface.

Our main tool in the proof of Theorem 3.1 is the following result, which is very similar to [28], Theorem 1.4.6.

PROPOSITION 3.2. *Let $\mathscr{P}$ be a family of probability measures on $\Omega=\mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$ and denote by x the canonical process on Ω. Assume that*

$$\lim_{R\nearrow\infty}\sup_{\mathbb{P}\in\mathscr{P}}\mathbb{P}\bigl(|x(0)|\geq R\bigr)=0.$$

Furthermore, for any given $\rho>0$, let $\tau_0=0$, and define recursively $\tau_{i+1}=\inf_{t>\tau_i}|x(t)-x(\tau_i)|>\rho$. Assume that the limit

$$\lim_{\delta\to0}\operatorname{ess\,sup}\mathbb{P}[\tau_{n+1}-\tau_n\leq\delta|\mathscr{F}_{\tau_n}]\to0,\qquad\mathbb{P}\text{ a.s., on }\{\tau_n<\infty\}\tag{3.2}$$

holds uniformly for every $\mathbb{P}\in\mathscr{P}$ and every $n\geq0$. Then the family of probability measures $\mathscr{P}$ is tight on Ω.

PROOF. The proof is similar to that of Theorem 1.4.6 in [28], except that their Lemma 1.4.4 is replaced by (3.2).

Fix an arbitrary final time $T > 0$. Furthermore, denote for $\omega \in \Omega$

$$N_\rho = N_\rho(\omega) = \min\{n : \tau_{n+1} > T\},$$

and the modulus of continuity by δ_ρ,

$$\delta_\rho = \delta_\rho(\omega) = \min\{\tau_n - \tau_{n-1} : 1 \le n \le N_\rho(\omega)\}.$$

Note that this expression depends on ρ via the definition of the stopping times τ_i.

With this notation at hand, tightness follows as in [28] if one can show that $\lim_{\delta\to 0} \sup_{\mathbb{P}\in\mathscr{P}} \mathbb{P}(\delta_\rho \le \delta) = 0$ for every fixed $\rho > 0$. As in [28], one has for every $k > 0$ the bound

$$\mathbb{P}(\delta_\rho \le \delta) \le \sum_{i=1}^{k} \mathbb{E}\big[\mathbb{P}[\tau_{i+1} - \tau_i \le \delta | \mathscr{F}_{\tau_i}]\big] + \mathbb{P}(N_\rho > k).$$

For every fixed $k > 0$, the first term then converges uniformly to 0 by assumption. Since the second term is independent of δ, it remains to verify that converges to 0 as $k \to \infty$, uniformly over $\mathscr{P}$ (convergence for every fixed $\mathbb{P} \in \mathscr{P}$ is trivial but not sufficient for our needs).

This is a consequence of [28], Lemma 1.4.5, provided that one can find $\lambda < 1$ such that $\mathbb{E}[e^{-(\tau_{i+1}-\tau_i)} | \mathscr{F}_{\tau_i}] \le \lambda$. This in turn follows from

$$\begin{aligned}\mathbb{E}\big[e^{-(\tau_{i+1}-\tau_i)} | \mathscr{F}_{\tau_i}\big] &\le \mathbb{P}[\tau_{i+1} - \tau_i \le t_0 | \mathscr{F}_{\tau_i}] + e^{-t_0}\mathbb{P}[\tau_{i+1} - \tau_i > t_0 | \mathscr{F}_{\tau_i}] \\ &\le e^{-t_0} + (1 - e^{-t_0})\mathbb{P}[\tau_{i+1} - \tau_i \le t_0 | \mathscr{F}_{\tau_i}].\end{aligned}$$

Indeed, by choosing t_0 sufficiently small, this term can be made strictly less than 1, provided that $\mathbb{P}[\tau_{i+1} - \tau_i \le t_0 | \mathscr{F}_{\tau_i}]$ tends to zero uniformly (over the members of $\mathscr{P}$ and over i) as t_0 tends to zero, which is precisely our assumption. $\square$

We now turn to the following.

PROOF OF THEOREM 3.1. Recall that we defined the process $Y^\varepsilon = X^\varepsilon + \varepsilon g(\varepsilon^{-1} X^\varepsilon)$ in Section 2. Note then that, just as in [17], Proposition 2.5, the tightness of the laws of X_x^ε is equivalent to that of the laws of Y_x^ε. Therefore, all that remains to be shown is that we have the bound (3.2) for the law of Y^ε, uniformly over $\varepsilon \in (0, 1]$. The approach that we use is to consider separately the martingale part and the bounded variation part for Y_y^ε given by (3.1), and to show that the probability of either of these moving by at least $\frac{\rho}{2}$ during a time interval δ tends to zero uniformly over the initial condition.

Given any fixed $\rho, \gamma > 0$, we want to show that there exists a sufficiently small $\delta > 0$ such that $\mathbb{P}(\tau_{n+1} - \tau_n \le \delta | \mathscr{F}_{\tau_n}) < \gamma$ uniformly over $\mathbb{P} \in \mathscr{P}$ (i.e., uniformly over the laws of Y_x^ε with $\varepsilon \in (0, 1]$) and n. We split the contributions from the martingale and the bounded variation parts in the following way:

$$
\begin{aligned}
&\mathbb{P}(\tau_{n+1} - \tau_n \le \delta | \mathscr{F}_{\tau_n}) \\
&\quad = \mathbb{P}_{X(\tau_n)}\Big(\sup_{t<\delta} |Y(t) - Y(0)| > \rho\Big) \\
&\quad \le \sup_x \mathbb{P}_x \left(\sup_{t<\delta} \left|\frac{1}{\varepsilon}\int_0^t \tilde{b}_i(\varepsilon^{-1} X_x^\varepsilon(s))\, ds\right| > \frac{\rho}{2}\right) \\
&\qquad + \sup_x \mathbb{P}_x \left(\sup_{t<\delta} \left|\int_0^t \tilde{\sigma}_{ij}(\varepsilon^{-1} X_x^\varepsilon(s))\, dW_j(s)\right| > \frac{\rho}{2}\right) \\
&\quad \le \frac{2}{\varepsilon\rho} \sup_x \mathbb{E}_x \int_0^t |\tilde{b}_i(\varepsilon^{-1} X_x^\varepsilon(s))|\, ds \\
&\qquad + \frac{2}{\rho} \sup_x \mathbb{E}_x \sup_{t\le\delta} \left|\int_0^t \tilde{\sigma}_{ij}(\varepsilon^{-1} X_x^\varepsilon(s))\, dW_j(s)\right|.
\end{aligned}
\tag{3.3}
$$

Here, we used the Chebychev's inequality to obtain the last bound. Since the functions $\tilde{\sigma}_{ij}$ are uniformly bounded, the stochastic integral appearing in the second term is easily bounded by $\mathcal{O}(\sqrt{\delta})$ by the Burkholder–Davis–Gundy inequalities. Furthermore, by the definition of the corrector function g, there exists $\tilde{\eta} > 0$ such that $\tilde{b}(x) = 0$ for $x \notin \mathscr{I}_{\tilde{\eta}\varepsilon}$, so that there exists a constant C such that

$$
\mathbb{P}(\tau_{n+1} - \tau_n \le \delta | \mathscr{F}_{\tau_n}) \le \frac{C}{\rho\varepsilon} \sup_x \mathbb{E}_x \left(\int_0^\delta \mathbf{1}_{\mathscr{I}_{\tilde{\eta}\varepsilon}}(X_x^\varepsilon(s))\, ds\right) + \frac{C\sqrt{\delta}}{\rho}.
\tag{3.4}
$$

For fixed $\rho > 0$, the second term obviously goes to 0 as $\delta \to 0$, uniformly in ε, so it remains to consider the first term. As one would expect from the expression for the local time of a Brownian motion, it turns out that the expected time spent by the process in $\mathscr{I}_{\tilde{\eta}\varepsilon}$ scales like $\varepsilon\sqrt{\delta}$, thus showing that this term is also of order $\sqrt{\delta}/\rho$. Once we are able to show this, the proof is complete.

The occupation time of the interface appearing in the first term of (3.4) is bounded by the trivial estimate $C\delta/(\rho\varepsilon)$, which goes to 0 as $\delta \to 0$ provided that we consider $\varepsilon \ge \sqrt{\delta}$, say. We can therefore assume without any loss of generality in the sequel that we consider $\varepsilon < \sqrt{\delta}$.

The idea to bound the occupation time is the following. We decompose the trajectory for the process X^ε into excursions away from the interface, separated by pieces of trajectory inside the interface. We first show that if the process starts inside the interface, then the expected time spent in the interface before making a new excursion is of order ε^2. Then, we show that each excursion has a probability at least $\varepsilon/\sqrt{\delta}$ of being of length δ or more. This shows that in the time interval δ of interest, the process will perform at most of the order of $\sqrt{\delta}/\varepsilon$ excursions, so that the total time spent in the interface is indeed of the order $\varepsilon\sqrt{\delta}$, thus showing that the first term in (3.4) behaves like $\sqrt{\delta}/\rho$, as expected.

More precisely, we first choose two constants $K > 0$ and $\hat{K} > 0$ such that the chain of implications

$$\{X^\varepsilon \in \mathscr{I}_{\tilde{\eta}\varepsilon}\} \Rightarrow \{Y^\varepsilon \in \mathscr{I}_{\hat{K}\varepsilon}\} \Rightarrow \{X^\varepsilon \in \mathscr{I}_{(K-1)\varepsilon}\} \Rightarrow \{X^\varepsilon \in \mathscr{I}_{K\varepsilon}\} \tag{3.5}$$

holds. We then set up a sequence of stopping times in the following way. We set $\phi_0 = 0$ and we set recursively

$$\sigma_n = \inf\{t \geq \phi_n : X^\varepsilon(t) \notin \mathscr{I}_{K\varepsilon}\},$$
$$\phi_n = \inf\{t \geq \sigma_{n-1} : Y^\varepsilon(t) \in \mathscr{I}_{\hat{K}\varepsilon}\}.$$

[Note that we can have $\sigma_0 = 0$ if the initial condition does not belong to $\mathscr{I}_{K\varepsilon}$. Apart from that, the second implication in (3.5) shows that increments from one stopping time to the next are always strictly positive.] This construction was chosen in such a way that the times when $X^\varepsilon \in \mathscr{I}_{\tilde{\eta}\varepsilon}$ always fall between ϕ_n and σ_n for some $n \geq 0$. In particular, if we set

$$N = \inf\{n \geq 0 : \phi_{n+1} - \sigma_n \geq \delta\},$$

then we have the bound

$$\begin{aligned}\sup_x \mathbb{E}_x\left(\int_0^\delta \mathbf{1}_{\mathscr{I}_{\tilde{\eta}\varepsilon}}(X_x^\varepsilon(s))\,ds\right) &\leq \sup_x \mathbb{E}_x\left(\sum_{n=0}^N(\sigma_n - \phi_n)\right) \\ &= \sup_x \sum_{n=0}^\infty \mathbb{E}_x\big((\sigma_n - \phi_n)\mathbf{1}_{N\geq n}\big) \\ &= \sum_{n=0}^\infty \sup_x \mathbb{P}_x(N \geq n) \sup_x \mathbb{E}_x\big(\mathbb{E}_{X^\varepsilon(\phi_n)}\sigma_1\big),\end{aligned}$$

where we used the strong Markov property and the fact that $\{N \geq n\}$ is $\mathscr{F}_{\phi_n}$-measurable in order to obtain the last identity. It follows from the definition of N that this expression is in turn bounded by

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x\sigma_0 \sum_{n\geq 0}\Big(\sup_{x\notin\mathscr{I}_{K\varepsilon}} \mathbb{P}_x(\phi_0 < \delta)\Big)^n = \frac{\sup_{x\in\mathbb{R}^d}\mathbb{E}_x\sigma_0 \sup_{x\notin\mathscr{I}_{K\varepsilon}}\mathbb{P}_x(\phi_0 < \delta)}{\inf_{x\notin\mathscr{I}_{K\varepsilon}}\mathbb{P}_x(\phi_0 \geq \delta)}.$$

We now bound both terms appearing in this expression separately.

First, we turn to the expected escape time from the interface, $\mathbb{E}_x\sigma_0$. The idea is to use a comparison argument just like in [17], Proposition 3.8. We define a "worst-case scenario" process V_x^ε, which is the solution to the SDE with initial condition x, diffusion coefficient 1 and drift coefficient given by b_V^ε, where

$$b_V^\varepsilon(x) = \begin{cases} \dfrac{-b_V}{\varepsilon}, & \text{for } x \geq 0, \\ \dfrac{b_V}{\varepsilon}, & \text{for } x < 0, \end{cases}$$

for some constant $b_V > 0$. We then have the following lemma.

LEMMA 3.3. *There exist* $b_V > 0$ *and* $\tilde{K} > 0$ *such that, if we define* $\tau^{\tilde{K}} = \inf\{t \geq 0 : V_x^\varepsilon(t) \notin \mathscr{I}_{\tilde{K}\varepsilon}\}$, *we have*

$$\mathbb{E}_x \sigma_0 \leq \mathbb{E}_x \tau^{\tilde{K}},$$

for every $x \in \mathbb{R}^d$.

The proof of Lemma 3.3 is almost identical to that of [17], Proposition 3.8, so we are going to omit it. A straightforward calculation using the particular form of the drift coefficient for V allows to check that there exists indeed a constant $C > 0$ such that the bound

$$\sup_x \mathbb{E}_x \tau^{\tilde{K}} \leq C\varepsilon^2,$$

holds so that, combining this with Lemma 3.3, we have $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x \sigma_0 \leq C\varepsilon^2$.

Let us now turn to the bound on $\mathbb{P}_x(\phi_0 \geq \delta)$. The idea here is to look at the process Y^ε instead of X^ε and to time-change it in such a way that we can compare it to a standard Brownian motion. Note first that the last two implications in (3.5) show that if we start with X^ε anywhere outside of $\mathscr{I}_{K\varepsilon}$, then the first component of Y^ε has to travel by at least ε before the process Y^ε can hit $\mathscr{I}_{\hat{K}\varepsilon}$. Furthermore, it follows from (3.1) that the time change C_t such that $Y^\varepsilon(C_t)$ is a standard Brownian motion satisfies $C_t \geq ct$ for some $c > 0$. It therefore follows that, setting $H(z) = \inf_{t>0}\{B_t > z\}$, one has the lower bound

$$\inf_{x\notin\mathscr{I}_{K\varepsilon}} \mathbb{P}_x(\phi_0 \geq \delta) \geq \mathbb{P}\big(H(\varepsilon) \geq \delta/c\big).$$

The explicit expression for the law of $H(z)$ given in [8], page 163, equation 2.02, yields in turn

$$\mathbb{P}\big(H(\varepsilon) \geq \delta/c\big) = \int_{\delta/(c\varepsilon^2)}^\infty \frac{e^{-1/(2t)}}{\sqrt{2\pi}\, t^{3/2}}\, dt.$$

It follows immediately that this in turn is bounded from below by $C\varepsilon/\sqrt{\delta}$ for some $C > 0$, provided that $\varepsilon \leq \sqrt{\delta}$. Collecting these bounds completes the proof of Theorem 3.1. □

4. Main tool for identifying the limit process. Instead of considering a graph as before, we will consider a generalized multidimensional version different from that considered by Freidlin and Wentzell in [12], Section 6. Note that the generalization considered here is different (and actually simpler) than the one considered in [13]. We consider processes in $\mathbb{R}^d$ and we set $I_- = \{x \in \mathbb{R}^d : x_1 < 0\}$, and similarly for I_+. We consider a family of $\mathbb{R}^d$-valued processes X^ε and we denote by τ^ε the first hitting time of $\mathscr{I}_{\varepsilon\eta}$. Correspondingly, τ^δ is the first escape time of the set $\mathscr{I}_\delta$ by X^ε.

With this the main tool, will be the following multidimensional analogue of [12], Theorem 4.1.

THEOREM 4.1. *Let $\bar{\mathcal{L}}_i$ be second order differential operators on I_i with bounded coefficients and let D_i be some sets of test functions over I_i whose members are bounded and have bounded derivatives of all orders. Suppose that for $i \in \{+,-\}$, any function $f \in D_i$ and for any $\lambda > 0$, the bound*

$$
\begin{aligned}
&\mathbb{E}_x\bigg[e^{-\lambda\tau^\varepsilon} f(X^\varepsilon(\tau^\varepsilon)) - f(X^\varepsilon(0)) \\
&\qquad + \int_0^{\tau^\varepsilon} e^{-\lambda t}\big(\lambda f(X^\varepsilon(t)) - \bar{\mathcal{L}}_i f(X^\varepsilon(t))\big)\,dt\bigg] = O(k(\varepsilon)),
\end{aligned}
\tag{4.1}
$$

holds as $\varepsilon \to 0$, uniformly with respect to $x \in I_i$. Assume furthermore that the rate k is such that $\lim_{\varepsilon\to 0} k(\varepsilon) = 0$.

Assume that, for every $\lambda > 0$ and every $i \in \{+,-\}$, there exist functions $u_{i,\lambda} \in D_i$ such that $\bar{\mathcal{L}}_i u_{i,\lambda}(x) = \lambda u_{i,\lambda}(x)$ holds for $x \in I_i$ with $|x_1| \le 1$ and such that $u_{\pm,\lambda}(x) = 1$ for $x_1 = 0$ and $x_1 = \pm 1$.

Assume that there exists a rate $\delta = \delta(\varepsilon) \to 0$ such that $\delta(\varepsilon)/k(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and such that for $\lambda > 0$,

$$
\mathbb{E}_x^\varepsilon\bigg[\int_0^\infty e^{-\lambda t}\mathbf{1}_{(-\delta,\delta)}(X_1^\varepsilon(t))\,dt\bigg] \to 0 \tag{4.2}
$$

as $\varepsilon \to 0$, uniformly in the initial point. Assume the convergence

$$
\mathbb{P}_x^\varepsilon[X^\varepsilon(\tau^\delta) \in I_i] \to p_i, \tag{4.3}
$$

holds uniformly in x in the set $\mathscr{I}_{\varepsilon\eta}$ for some constants $p_\pm$ with $p_+ + p_- = 1$. Assume furthermore that there exist constants α_j and C such that

$$
\frac{1}{\delta}\mathbb{E}_x^\varepsilon[X_j^\varepsilon(\tau^\delta) - x_j] \to \alpha_j, \qquad \frac{1}{\delta^2}\mathbb{E}_x^\varepsilon\big[\big(X_j^\varepsilon(\tau^\delta) - x_j\big)^2\big] \le C, \tag{4.4}
$$

for $j \ge 2$. Again, the limit is assumed to be uniform over $x \in \mathscr{I}_{\varepsilon\eta}$ as $\varepsilon \to 0$, and the inequality is assumed to be uniform over all $\varepsilon \in (0,1]$ and all $x \in \mathscr{I}_{\varepsilon\eta}$.

Let then D be the set of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ such that the restriction of f to I_i belongs to D_i and such that the gluing condition (2.5) holds. Then, for any fixed $f \in D$, $t_0 \ge 0$ and $\lambda > 0$,

$$
\begin{aligned}
\Delta(\varepsilon) = \operatorname{ess\,sup}\bigg|\mathbb{E}_x^\varepsilon\bigg[&\int_{t_0}^\infty e^{-\lambda t}[\lambda f(X^\varepsilon(t)) - \bar{\mathcal{L}} f(X^\varepsilon(t))]\,dt \\
&- e^{-\lambda t_0} f(X^\varepsilon(t_0))\Big|\mathscr{F}_{[0,t_0]}\bigg]\bigg| \to 0
\end{aligned}
\tag{4.5}
$$

as $\varepsilon \to 0$, uniformly with respect to x. In particular, every weak limit of X^ε as $\varepsilon \to 0$ satisfies the martingale problem for $\bar{\mathcal{L}}$.

REMARK 4.2. Note that we did not specify how "large" the sets D_i of admissible test functions need to be. If these sets are too small, then the theorem still holds, but the corresponding martingale problem might become ill-posed.

PROOF OF THEOREM 4.1. Since the proof is virtually identical to that of [12], Theorem 4.1, we only sketch it here. The basic idea behind the proof given by Freidlin and Wentzell is to rewrite (4.5) using the strong Markov property of X^ε as a sum of terms between successive stopping times. To this effect, set, for example, $\sigma_0 = 0$ and then recursively $\phi_n = \inf\{t > \sigma_n : X_1^\varepsilon(t) \in \mathscr{I}_{\varepsilon\eta}\}$, $\sigma_{n+1} = \inf\{t > \phi_n : X_1^\varepsilon(t) \notin \mathscr{I}_\delta\}$. They then break up the term produced from (4.5) into two sums of analogous terms between times σ_n and ϕ_n and those between ϕ_n and σ_{n+1}.

The terms covering the time intervals $[\sigma_n, \phi_n]$ are bounded exactly as in [12], making use of (4.1), together with the bound $\sum_n \mathbb{E}_x e^{-\lambda\sigma_n} = \mathcal{O}(1/\delta)$ which follows from the existence of the functions $u_{i,\lambda}$ just as in [12].

Using assumption (4.2), the terms covering the time intervals $[\phi_n, \sigma_{n+1}]$ are then simplified to

$$\sum_n e^{-\lambda\phi_n}\bigl(f(X^\varepsilon(\sigma_{n+1})) - f(X^\varepsilon(\phi_n))\bigr),$$

modulo contributions that converge to 0 as $\varepsilon \to 0$. Since the expectation of this term is bounded by

$$\sup_{x\in\mathscr{I}_{\eta\varepsilon}} \mathbb{E}_x\bigl(f(X^\varepsilon(\tau^\delta)) - f(x)\bigr)\sum_n \mathbb{E}e^{-\lambda\phi_n},$$

and since we already know that $\sum_n \mathbb{E}e^{-\lambda\phi_n} = \mathcal{O}(1/\delta)$, in remains to show that the supremum is of order $o(\delta)$. It follows from Taylor's expansion and the fact that $f \in \mathcal{C}^2$ outside of the interface, that on the event $\Omega_+ \stackrel{\text{def}}{=} \{X_1^\varepsilon(\tau^\delta) > 0\}$, one has

$$f(X^\varepsilon(\tau^\delta)) - f(x) = \delta\partial_1 f(x)|_{I_+} + \sum_{i=2}^d \partial_i f(x)\bigl(X_i^\varepsilon(\tau^\delta) - x_i\bigr) + \mathcal{O}\bigl(|X_i^\varepsilon(\tau^\delta) - x_i|^2\bigr),$$

and similarly on $\Omega_- = \{X_1^\varepsilon(\tau^\delta) < 0\}$. Combining this with (4.4), we thus have

$$\begin{aligned}\mathbb{E}_x\bigl(f(X^\varepsilon(\tau^\delta)) - f(x)\bigr) &= \delta\,\partial_1 f(x)|_{I_+}\mathbb{P}_x(\Omega_+) + \delta\,\partial_1 f(x)|_{I_-}\mathbb{P}_x(\Omega_-)\\ &\quad + \delta\sum_{i=2}^d \alpha_i\,\partial_i f(x) + o(\delta).\end{aligned}$$

Since we assume that $\mathbb{P}_x(\Omega_\pm) \to p_\pm$ uniformly over $x \in \mathscr{I}_{\eta\varepsilon}$, the required bound now follows from the gluing condition. □

Most of the remainder of this article is devoted to the verification of the assumptions of Theorem 4.1. The bounds (4.1) and (4.2) will be relatively straightforward to verify and this will form the content of the remainder of this section. The convergence (4.3) is the one that is most difficult to obtain and will be the content of Section 5. Finally, we will show that (4.4) holds in Section 6. We start by the following result.

LEMMA 4.3. *Let $\bar{\mathcal{L}}_\pm$ be as in (2.1) and let X^ε be the family of processes from Section 2. Then, the bound (4.1) holds with $k(\varepsilon) = \varepsilon$ for every $\lambda > 0$ and for every smooth bounded function $f : I_i \to \mathbb{R}$ that has bounded derivatives of all orders.*

PROOF. It follows from [17], Lemma 3.4, that, for any initial point x with $x_1 \neq 0$ and for ε sufficiently small so that $x \notin \mathscr{I}_{\varepsilon\eta}$,

$$\mathbb{E}_x\biggl[\int_0^{\tau^\varepsilon} e^{-\lambda s} f(X^\varepsilon(s)) h\biggl(\frac{X^\varepsilon(s)}{\varepsilon}\biggr)\, ds\biggr] = \mathcal{O}(\varepsilon), \tag{4.6}$$

for h centered with respect to μ_+ (resp., μ_- if $x_1 < 0$). We assume that $x_1 > 0$ from now on, but the calculations are identical for the case $x_1 < 0$.

Note now that it suffices to obtain the bound (4.1) for the family of processes Y^ε, since $\|Y^\varepsilon(t) - X^\varepsilon(t)\| = \mathcal{O}(\varepsilon)$, uniformly. Applying Itô's formula to $e^{-\lambda\tau^\varepsilon} \times f(Y^\varepsilon(\tau^\varepsilon))$, we obtain the identity

$$\begin{aligned} e^{-\lambda\tau^\varepsilon} f(Y^\varepsilon(\tau^\varepsilon)) = f(y) &+ \int_0^{\tau^\varepsilon} -\lambda e^{-\lambda s} f(Y^\varepsilon(s))\, ds \\ &+ \frac{1}{2}\int_0^{\tau^\varepsilon} e^{-\lambda s} (\tilde{\sigma}_{ik}\tilde{\sigma}_{kj})\biggl(\frac{X^\varepsilon}{\varepsilon}\biggr) \partial_{ij}^2 f(Y^\varepsilon(s))\, ds \\ &+ \int_0^{\tau^\varepsilon} e^{-\lambda s} \tilde{\sigma}_{ik}\biggl(\frac{X^\varepsilon(s)}{\varepsilon}\biggr) \partial_i f(Y^\varepsilon(s))\, dW_k(s). \end{aligned}$$

Since $|Y^\varepsilon - X^\varepsilon| \leq \mathcal{O}(\varepsilon)$ and since all derivatives of f are assumed to be bounded, it then follows from (4.6) that

$$\begin{aligned} \mathbb{E}(e^{-\lambda\tau^\varepsilon} f(Y^\varepsilon(\tau^\varepsilon))) = f(y) &- \lambda \mathbb{E}\int_0^{\tau^\varepsilon} e^{-\lambda s} f(Y^\varepsilon(s))\, ds \\ &+ \frac{1}{2}\mathbb{E}\int_0^{\tau^\varepsilon} e^{-\lambda s} D_{ij}^+\, \partial_{ij}^2 f(Y^\varepsilon(s))\, ds + \mathcal{O}(\varepsilon), \end{aligned}$$

which is precisely the required result. $\square$

Additionally we have that the solution to $\bar{\mathcal{L}}_i u = \lambda u$ on I_i, $u = 1$ on $\{x_1 = 0\}$ and $\{x_1 = \pm 1\}$, is bounded and has bounded derivatives of all orders. This follows from the fact that u is given explicitly by $u(x) = C_1 e^{\sqrt{\lambda (D_{11}^\pm)^{-1}} x_1} + C_2 e^{-\sqrt{\lambda (D_{11}^\pm)^{-1}} x_1}$ for some constants C_i. We now show that the process Y^ε satisfies the bound (4.2), that is, it does not spend too much time in the vicinity of the interface.

LEMMA 4.4. *If we choose $\delta = \varepsilon^\alpha$ for any $\alpha \in (\frac{1}{2}, 1)$, then (4.2) holds for the family of processes X^ε from Section 2.*

PROOF. Again, it suffices to show the bound for the process Y^ε since it differs from X^ε by $\mathcal{O}(\varepsilon)$. We would like to use an argument similar to what can be used in the one-dimensional case [17], that is, we time-change the corrected process Y^ε in such a way that it becomes a diffusion with diffusion coefficient 1. Its drift then vanishes outside of the interface and is bounded by K/ε for some $K > 0$. At this stage, one compares this process to the "worst-case scenario" process Z^ε given by

$$dZ^\varepsilon = \hat{b}(Z^\varepsilon)\,dt + dB(t),$$

where the drift $\hat{b}$ is given by

$$\hat{b}(z) = \begin{cases} -K\varepsilon^{-1}, & \text{if } z \in [0, l\varepsilon), \\ K\varepsilon^{-1}, & \text{if } z \in (-l\varepsilon, 0), \\ 0, & \text{otherwise}, \end{cases}$$

for some $l \in \mathbb{R}$. It can then be shown that Z^ε spends more time in the interface than Y^ε does, so that the requested bound can be obtained from a simple calculation.

The problem with this argument is that in the multi-dimensional case the time-change required to turn the first component of Y^ε into a diffusion with unit diffusion coefficient is given by

$$T_t = \inf\left\{ s \in \mathbb{R}_+ : \int_0^s \sum_{i=1}^n \big(\delta_{1i} + \partial_i g_1(\varepsilon^{-1} X^\varepsilon(u))\big)^2\,du > t \right\}. \tag{4.7}$$

We do not know of an argument giving a uniform bound *from below* on the quantity appearing under the integral in this expression. Therefore, an upper bound on the time spent by the process Z^ε in the interval $(-\delta, \delta)$ does not give us any control on the time spent by Y^ε (and therefore X^ε) in that interval.

Because of this, we modify our argument in the following way. We break up the integral in (4.2) as

$$\mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} \mathbf{1}_{(-\delta,\delta)}(Y_1^\varepsilon(t))\,dt\right] = \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} \mathbf{1}_{(-c\varepsilon,c\varepsilon)}(Y_1^\varepsilon(t))\,dt\right] \tag{4.8}$$

$$+ \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} \mathbf{1}_{(-\delta,-c\varepsilon)}(Y_1^\varepsilon(t))\,dt\right] \tag{4.9}$$

$$+ \mathbb{E}_x\left[\int_0^\infty e^{-\lambda t} \mathbf{1}_{(c\varepsilon,\delta)}(Y_1^\varepsilon(t))\,dt\right],$$

where Y_1^ε is the first component of Y^ε and c is a value to be determined. By symmetry, the last two terms are of the same order, so that it is sufficient to bound the first two terms. In order to bound the first term, we use the argument outlined above, but we replace Y^ε by the process $\tilde{Y}^\varepsilon$ given by $\tilde{Y}^\varepsilon(t) = X^\varepsilon(t) + \varepsilon\tilde{g}(\varepsilon^{-1}X^\varepsilon(t))$, where the corrector $\tilde{g}$ has the following properties:

1. The function $\tilde{g}(x)$ is smooth, periodic in the variables parallel to the interface, and equal to $g(x)$ for $x \notin \mathscr{I}_{c_1}$ for some c_1.
2. One has the implication $Y^\varepsilon \in \mathscr{I}_{c\varepsilon} \Rightarrow \tilde{Y}^\varepsilon \in \mathscr{I}_{c_2\varepsilon}$ for some $c_2 < c_1$.
3. If $\tilde{Y}^\varepsilon \in \mathscr{I}_{c_2\varepsilon}$, then $\tilde{g}(\varepsilon^{-1}X^\varepsilon) = 0$.

It is always possible to satisfy these properties by choosing c_1 sufficiently large and setting $g = 0$ in a sufficiently wide band around the interface. We now set $\tilde{Z}(t) = \tilde{Y}(\tilde{T}_t)$, where $\tilde{T}_t$ is defined as in (4.7), but with g replaced by $\tilde{g}$, so that it follows from the second property that one has the bound

$$\begin{aligned}\mathbb{E}_x \int_0^\infty e^{-\lambda t} \mathbf{1}_{(-c\varepsilon, c\varepsilon)}(Y_1^\varepsilon(t))\,dt &\le \mathbb{E}_x \int_0^\infty e^{-\lambda t} \mathbf{1}_{(-c_2\varepsilon, c_2\varepsilon)}(\tilde{Y}_1^\varepsilon(t))\,dt \\ &\le \mathbb{E}_x \int_0^\infty e^{-\lambda T_t} \mathbf{1}_{(-c_2\varepsilon, c_2\varepsilon)}(\tilde{Z}_1^\varepsilon(t))\,dT_t.\end{aligned}$$

At this stage, we remark that since the function $\tilde{g}$ has bounded derivatives, there exists a constant K_1 such that $T_t \ge K_1 t$ almost surely. On the other hand, it follows from the last property that one actually has $dT_t = dt$ whenever $\tilde{Y}^\varepsilon \in \mathscr{I}_{c_2\varepsilon}$, so that this expression is bounded by

$$\mathbb{E}_x \int_0^\infty e^{-K_1 t} \mathbf{1}_{(-c_2\varepsilon, c_2\varepsilon)}(\tilde{Z}_1^\varepsilon(t))\,dt.$$

This expression in turn can be bounded by $\mathcal{O}(\varepsilon)$ just as in [17].

We now proceed to bounding the term (4.9). For this, let us first introduce a constant $c_3 < c$ and make c from (4.8) sufficiently large such that:

4. The implication $X^\varepsilon(t) \in \mathscr{I}_{c_3\varepsilon} \Rightarrow Y^\varepsilon(t) \in \mathscr{I}_{c\varepsilon}$ holds.
5. One has $c_3 > \eta + 1$.

Then, we define a series of stopping times $\{\phi'_n\}_n$ and $\{\sigma'_n\}_n$ recursively by $\phi'_{-1} = 0, \ldots, \sigma'_n = \inf\{t \ge \phi'_{n-1} : X_1^\varepsilon(t) \notin (-2\delta, -c_3\varepsilon + \varepsilon)\}$ and $\phi'_n = \inf\{t \ge \sigma'_n : X_1^\varepsilon(t) \in (-\delta, -c_3\varepsilon)\}$.

Now we can use the strong Markov property as in [12], Lemma 4.1, with the stopping times ϕ'_n to obtain the bound $\mathbb{E}_x[\sum_{n=0}^\infty e^{-\lambda\sigma'_n(\varepsilon)}] = O(\frac{1}{\varepsilon})$, uniformly in the initial point x for $x \in \{x : x_1 = -c_3\varepsilon + \varepsilon\} \cup \{x : x_1 = -2\delta\}$. This is a consequence of the fact that $\mathbb{E}_x[e^{-\lambda\sigma'_0}] = 1 - O(\varepsilon)$ uniformly. Furthermore, it follows from the definition of these stopping times, property 4 and the strong Markov property that (4.9) is bounded by

$$\begin{aligned}\mathbb{E}_x \int_0^\infty e^{-\lambda t} \mathbf{1}_{(-\delta, -c_3\varepsilon)}(X_1^\varepsilon(t))\,dt &\le \mathbb{E}_x \sum_{n\ge 0} \int_{\phi'_{n-1}}^{\sigma'_n} e^{-\lambda t}\,dt \\ &\le \lambda^{-1} \mathbb{E}_x \sum_{n\ge 0} e^{-\lambda\phi'_{n-1}} (\sigma'_n - \phi'_{n-1}) \\ &\le \lambda^{-1} \left(\mathbb{E}_x \sum_{n=0}^\infty e^{-\lambda\phi'_n(\varepsilon)}\right) \sup_x \mathbb{E}_x \sigma'_0 \\ &\le \frac{C}{\varepsilon\lambda} \sup_x \mathbb{E}_x \sigma'_0.\end{aligned} \tag{4.10}$$

It follows that it suffices to be able to choose δ in such a way that $\mathbb{E}_x\sigma_0'$ is $o(\varepsilon)$ uniformly in the initial point. Specifically, we will show that (4.10) is $\mathcal{O}(\delta^2)$, so that the claim follows.

This will be a consequence of the following result.

LEMMA 4.5. *Let X^- be as in* (1.3) *and define $X^{-,\varepsilon}(t) = \varepsilon X^-(\varepsilon^{-2}t)$. Let $\tau = \inf\{t > 0 : X_1^{-,\varepsilon}(t) \notin [-1, 0]\}$. Then, there exists a constant C such that*

$$\mathbb{E}_x\tau \le C,$$

independently of $\varepsilon \in (0, 1]$ and independently of $x \in \mathbb{R}^d$.

Before we prove Lemma 4.5, we use it to complete the proof of Lemma 4.4. It follows from property 5. that up to time σ_0', the process X^ε is identical in law to the process $X^{-,\varepsilon}$. Furthermore, the stopping time σ_0' is certainly bounded from above by the first exit time of the first component of $X^{-,\varepsilon}$ from $(-2\delta, 0)$. Rescaling space by a factor 2δ and rescaling time correspondingly by $4\delta^2$, we deduce from Lemma 4.5 that $\mathbb{E}\sigma_0' \le 4C\delta^2$, uniformly in the initial condition as required. □

We now turn to the proof of the lemma.

PROOF OF LEMMA 4.5. Denote by U the region $\{x \in \mathbb{R}^d : x_1 \in [-1, 0]\}$ and define f^ε by $f^\varepsilon(x) = \mathbb{E}_x\tau$. Then f^ε satisfies

$$\mathcal{L}^\varepsilon f^\varepsilon = -1, \qquad f^\varepsilon(x) = 0 \qquad \text{for } x \in \partial U,$$

where $\mathcal{L}^\varepsilon = \frac{1}{2}\Delta + \varepsilon^{-1}b_-(\varepsilon^{-1}\cdot)\nabla_x$. In order to obtain a bound on f, we will give a uniformly bounded (uniformly over ε) function g^ε such that it satisfies

$$\mathcal{L}^\varepsilon g^\varepsilon = -1, \qquad g^\varepsilon(x) \ge 0 \qquad \text{for } x \in \partial U. \tag{4.11}$$

It then follows from the strong maximum principle (which we can apply since our diffusion is periodic in the directions in which U is unbounded) that $g^\varepsilon \ge f^\varepsilon$, so that the requested bound holds.

We use a standard multiscale expansion for g^ε of the form

$$g^\varepsilon = g_0 + \varepsilon g_1 + \varepsilon^2 g_2.$$

Now to find such a g^ε. We proceed by starting off with a constant order term, that is, the typical term one would expect for the escape time if we were dealing with a Brownian motion, then removing the order $\frac{1}{\varepsilon}$ terms that arise when the operator L^ε acts on the constant order term by adding an order ε term. Then finally we add an order ε^2 term to remove the constant order terms that are produced by the action of L^ε on the order ε term. Incidentally, this approach of correction works exactly with the maximum order term in ε being 2 and produces a series of terms that are known and have the right properties to provide a uniform bound.

Taking guidance from the fact that the homogenized process is given by Brownian motion, we make the ansatz $g_0(x) = C_2 - C_1 x_1(1 + x_1)$, for C_1 and C_2 two constants to be determined. Applying $\mathcal{L}^\varepsilon$ to g_0 yields

$$\mathcal{L}^\varepsilon g_0(x) = -C_1 - \frac{C_1}{\varepsilon} b_{-,1}\left(\frac{x}{\varepsilon}\right)(1 + 2x_1)$$

for $b_{-,1}$ the first component of b_-. Our aim now is to choose g_1 in such a way that $\mathcal{L}g_1$ contains a term of order ε^{-1} that precisely cancels out the second term in this expression. Denote as in the introduction by g_- the unique centered solution to the Poisson equation

$$\mathcal{L}g_- = b_-, \tag{4.12}$$

where $\mathcal{L} = \frac{1}{2}\Delta + b_- \nabla_x$ is the generator for the nonrescaled process. We then set $g_1(x) = C_1(1 + 2x_1) g_{-,1}(\varepsilon^{-1}x)$, where $g_{-,1}$ is the first component of g_-, and we note that

$$\begin{aligned}\varepsilon\mathcal{L}^\varepsilon g_1(x) &= \frac{C_1}{\varepsilon} b_{-,1}\left(\frac{x}{\varepsilon}\right)(1 + 2x_1) \\ &\quad + 2C_1 b_{-,1}\left(\frac{x}{\varepsilon}\right) g_{-,1}\left(\frac{x}{\varepsilon}\right) + 2C_1 \frac{\partial g_{-,1}}{\partial x_1}\left(\frac{x}{\varepsilon}\right) \\ &= \frac{C_1}{\varepsilon} b_{-,1}\left(\frac{x}{\varepsilon}\right)(1 + 2x_1) + C_1 F\left(\frac{x}{\varepsilon}\right),\end{aligned} \tag{4.13}$$

for some periodic function F independent of ε and of C_1. The term involving F appearing in this expression is still of order one, so we aim to compensate it by a judicious choice of g_2. It is not necessarily centred with respect to the invariant measure μ of our process, but there exists a periodic centred function h such that

$$\begin{aligned}&\mathcal{L}h = F - K, \\ &K = \int F(x)\mu(dx) = -\int |\nabla g_{-,1}(x)|^2 \mu(dx) + 2\int \frac{\partial g_{-,1}}{\partial x_1}\mu(dx).\end{aligned}$$

Finally, setting $g_2(x) = -h(\varepsilon^{-1}x)$, we obtain

$$\mathcal{L}^\varepsilon g^\varepsilon = C_1(K - 1) = -C_1 \int |e_1 - \nabla g_{-,1}(x)|^2 \mu(dx). \tag{4.14}$$

Since the integral is strictly positive, the right-hand side can be made to be equal to -1. Furthermore, since the corrector terms $\varepsilon g_1 + \varepsilon^2 g_2$ are uniformly bounded for $\varepsilon < 1$, it is straightforward to find a constant C_2 that ensures that $g(x) \geq 0$ for $x \in \partial U$, thus concluding the proof. $\square$

5. Computation of the transmissivity coefficient. The aim of this section is to prove that the following proposition holds.

PROPOSITION 5.1. *The identity* (4.3) *holds for the family of processes* X^ε *in Section* 2 *with* $p_\pm$ *given by* (2.2).

Let us first introduce some notation. Given a starting point $x \in \mathscr{I}_\eta$, we set $p_+^{x,k} = \mathbb{P}_x(X(\tau^{(k)}) > 0)$, and similarly for $p_-^{x,k}$, where $\tau^{(k)}$ is the first hitting time of $\partial\mathscr{I}_k$. We furthermore set

$$\bar{p}_+^k = \sup_{x\in\mathscr{I}_\eta} p_+^{x,k}, \qquad \underline{p}_+^k = \inf_{x\in\mathscr{I}_\eta} p_+^{x,k}, \qquad p_+^{(k)} = \frac{1}{2}(\bar{p}_+^k + \underline{p}_+^k),$$

and similarly for p_-. It is clear that Proposition 5.1 follows if we can show that p_+^k converges to a limit satisfying (2.2) and $\bar{p}_+^k - \underline{p}_+^k \to 0$ as $k \to \infty$.

We will first show the latter, as it is relatively straightforward to show. In order to show the convergence of p_+^k, our main ingredient will be to show that the invariant measure $\mu(dx)$ for the process X looks more and more similar to $\mu_\pm(dx)$ as $x_1 \to \pm\infty$. Note that in this whole section, we will always consider X and $X_\pm$ as processes on $\mathbb{R} \times \mathbb{T}^{d-1}$, obtained by identifying points (x, y) such that $x_1 = y_1$ and $x_j - y_j \in \mathbb{Z}$ for $j \geq 2$. With this interpretation, the interface is compact and we will show that the processes are recurrent. If we were to consider them as processes in $\mathbb{R}^d$, they would *not* be recurrent for $d \geq 3$.

Before we show that indeed $\bar{p}_+^k - \underline{p}_+^k \to 0$, we obtain some recurrence properties of X and ensure that it visits any open set in $\mathscr{I}_\eta$ sufficiently often before the hitting time $\tau^{(k)}$.

LEMMA 5.2. *Fix a neighborhood* $\gamma \subset \mathscr{I}_\eta$. *Then the probability for* X *to enter* γ *before hitting* $\partial\mathscr{I}_k$, *starting from an arbitrary initial point in* $\mathscr{I}_\eta$ *tends to* 1 *uniformly as* $k \to \infty$. *In particular, the process* X *is recurrent.*

Our first step in showing this result is to argue that if the process starts at distance $\mathcal{O}(1)$ of the interface, then it will return to the interface with overwhelming probability before exiting $\mathscr{I}_k$.

LEMMA 5.3. *There exists* $K > 0$ *such that the probability, starting at* x, *for* X *to return to* $\mathscr{I}_\eta$ *before hitting* $\partial\mathscr{I}_k$, *is bounded from above by* $1 - \frac{x-K}{k}$ *and from below by* $1 - \frac{x+K}{k}$.

PROOF. Denote by $f^k(x)$ the probability of hitting $\mathscr{I}_\eta$ before $\partial\mathscr{I}_k$, starting from x. We assume without loss of generality that $x_1 > 0$, since the case $x_1 < 0$ follows using the same argument. The function f^k then satisfies the equation $\mathcal{L}f^k = 0$, endowed with the boundary conditions $f^k(x) = 1$ if $x_1 = \eta$ and

$f^k(x) = 0$ if $x_1 = k$. As in the proof of Lemma 4.5, we aim to construct a function g^k satisfying $\mathcal{L}g^k = 0$ and such that either $g^k(x) \le f^k(x)$ on the two boundaries or $g^k(x) \ge f^k(x)$ on the two boundaries. The claim then follows from the maximum principle.

Let g_+ be as in (4.12) and set

$$g^k(x) = 1 - k^{-1}\big(K + x_1 - g_{+,1}(x)\big),$$

for some constant K to be determined. It is straightforward to check that g^k does indeed satisfy $\mathcal{L}g^k = 0$, as well as the required inequalities on the boundary, provided that K is either sufficiently large or sufficiently small. This concludes the proof. $\square$

We now use the result of Lemma 5.3 to prove Lemma 5.2. This is done using the strong Markov property in conjunction with success/failure trials.

PROOF OF LEMMA 5.2. Consider the two hyperplanes that delimit $\mathscr{I}_\eta$ and two further hyperplanes at distance m from $\mathscr{I}_\eta$, with m a sufficiently large constant to be determined later. We then break the process into excursions from $\partial\mathscr{I}_\eta$ to $\partial\mathscr{I}_{\eta+m}$ and back.

More precisely, we define two sets of stopping times $\{\sigma_n^m\}_n$ and $\{\phi_n^m\}_n$ recursively by $\sigma_1^m = \inf\{t \ge 0 : X(t) \in \partial\mathscr{I}_{\eta+m}\}, \ldots, \phi_n^m = \inf\{t > \sigma_n^m : X(t) \in \mathscr{I}_\eta\}$, $\sigma_{n+1}^m = \inf\{t > \phi_n^m : X(t) \in \partial\mathscr{I}_{\eta+m}\}$. We furthermore denote by $\mathscr{F}_n$ the σ-algebra generated by trajectories of X up to the time ϕ_n^m and by $\bar{\mathscr{F}}_n$ the σ-algebra generated by trajectories of X up to the time σ_{n+1}^m. We also denote by τ_γ the first hitting time of the set γ and by $\tau^{(k)}$ the first hitting time of the set $\partial\mathscr{I}_k$.

It follows from the ellipticity of X and the resulting smoothness of its transition probabilities that there exists some $p > 0$ such that $\inf_{x\in\partial\mathscr{I}_\eta} P_1(x,\gamma) = 2p > 0$. Furthermore, it is straightforward, for instance using a comparison argument with a process with constant drift away from the interface and using the continuity of paths, to show that

$$\lim_{m\to\infty} \sup_{x\in\mathscr{I}_\eta} \mathbb{P}_x(\sigma_1^m \le 1) = 0. \tag{5.1}$$

It follows that we can choose m large enough so that the probability appearing in (5.1) is bounded above by p. As a consequence, for such a choice of m, one has the almost sure bound

$$\mathbb{P}(\tau_\gamma < \sigma_{n+1}^m | \mathscr{F}_n) \ge p. \tag{5.2}$$

On the other hand, it follows from Lemma 5.3 that the probability that the process hits $\partial\mathscr{I}_k$ between σ_n^m and ϕ_n^m is bounded from *above* uniformly by $\beta_k = \mathcal{O}(k^{-1})$ so that, almost surely,

$$\mathbb{P}\big(\tau^{(k)} < \phi_{n+1}^m | \bar{\mathscr{F}}_n\big) \le \beta_k. \tag{5.3}$$

Note furthermore that by construction the event appearing in (5.2) is $\bar{\mathscr{F}}_n$-measurable.

Denote now by Y_n a Markov chain with states $\{-1, 0, 1\}$ such that $\{\pm 1\}$ are absorbing and such that $P(Y_{n+1} = -1 | Y_n = 0) = p$, $P(Y_{n+1} = 1 | Y_n = 0) = \beta_k$. As a consequence of (5.2) and (5.3), it is then possible to couple Y and X in such a way that the following two implications hold almost surely:

$$\{Y_n = 0 \text{ and } Y_{n+1} = -1\} \Rightarrow \{\phi_n^m < \tau_\gamma < \sigma_{n+1}^m < \tau^{(k)}\},$$

$$\{\sigma_{n+1}^m < \tau^{(k)} < \phi_{n+1}^m < \tau_\gamma\} \Rightarrow \{(Y_n = 0 \text{ and } Y_{n+1} = 1)\}.$$

It follows that the probability of entering γ before the hitting time $\tau^{(k)}$ is bounded from below by

$$\mathbb{P}(\tau_\gamma < \tau^{(k)}) \geq \mathbb{P}\Big(\lim_{n\to\infty} Y_n = -1\Big) = \frac{p}{p + \beta_k}.$$

Since p is fixed and $\beta_k = \mathcal{O}(k^{-1})$, this quantity can be made arbitrarily close to 1.

This shows that the set γ is recurrent for X. Since furthermore X has transition probabilities that have strictly positive densities with respect to Lebesgue measure (as a consequence of the ellipticity of the equations describing it), recurrence follows from [20], Theorem 8.0.1. $\square$

We now use this result to prove the following proposition.

PROPOSITION 5.4. $\bar{p}_+^k - \underline{p}_+^k \to 0$ *as* $k \to \infty$.

PROOF. The idea is to use the fact that, before the process exits $\mathscr{I}_k$, it has had sufficient amount of time to forget about its initial condition by visiting a small set on which a strong minorizing condition holds for its transition probabilities.

Fix a value $\beta > 0$. Our aim is to show that there then exists $k_0 > 0$ such that

$$\underline{p}_\pm^k \geq p_\pm^{0,k} - \beta,$$

say, for every $k \geq k_0$. Since $p_+^{x,k} = 1 - p_-^{x,k}$, the claim then follows. We restrict ourselves to the bound for p_+ since the other bound can be obtained in exactly the same way.

The argument is now the following. It follows from the smoothness of transition probabilities that there exists a neighborhood γ of the origin such that the transition probabilities at time 1 for X, starting from γ satisfy the lower bound

$$\rho(y) = \inf_{x\in\gamma} P_1(x, y),$$

with $\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \rho(y)\, dy \geq 1 - \beta/2$. It then follows immediately that for $x \in \gamma$, one has $p_+^{x,k} \geq p_+^{0,k} - \beta/2 - \mathbb{P}_x(\exists t \leq 1 : X(t) \in \partial\mathscr{I}_k)$. For arbitrary x, it therefore follows from the strong Markov property that

$$p_+^{x,k} \geq p_+^{0,k} - \beta/2 - \sup_{y\in\gamma} \mathbb{P}_y\big(\exists t \leq 1 : X(t) \in \mathscr{I}_k\big) - \mathbb{P}_x(X \text{ hits } \partial\mathscr{I}_k \text{ before } \gamma).$$

The last term can be made smaller than $\beta/4$ by Lemma 5.2. The remaining term $\mathbb{P}_y(\exists t \le 1 : X(t) \in \mathscr{I}_k)$ on the other hand was already shown to be arbitrary small in (5.1). $\square$

We next show that the invariant measure of the process converges to that of the relevant periodic process with increasing distance from the interface.

PROPOSITION 5.5. *Let A denote a bounded measurable set and denote by μ the (unique up to scaling) invariant σ-finite measure of the process X. Denote furthermore by $\mu_\pm$ the invariant measure of the relevant periodic process, normalized in such a way that $\mu_\pm([k, k+1] \times \mathbb{T}^{d-1}) = 1$ for every $k \in \mathbb{Z}$. Then there exist normalization constants $q_\pm$ such that*

$$\lim_{k\to\infty} \big(|\mu(A+k) - q_+\mu_+(A)| + |\mu(A-k) - q_-\mu_-(A)|\big) = 0. \tag{5.4}$$

(Here k is an integer.) Furthermore, this convergence is exponential, and uniform over the set A if we restrict its diameter.

REMARK 5.6. We used the shorthand notation $A + k$ for $\{x + k : x \in A\}$.

PROOF OF PROPOSITION 5.5. We restrict ourselves to the estimate of $\mu(A + k)$, since the one on $\mu(A - k)$ is similar. For fixed $k \ge 0$, we introduce the sequence of stopping times given by $\phi_0^{(k)} = \inf\{t \ge 0 : X_1(t) = k\}$ and then recursively $\sigma_n^{(k)} = \inf\{t \ge \phi_n^{(k)} : |X_1(t) - k| = 1\}$, $\phi_{n+1}^{(k)} = \inf\{t \ge \sigma_n^{(k)} : X_1(t) = k\}$. This allows us to define an embedded Markov chain $Z^{(k)}$ on $\mathbb{T}^{d-1}$ by setting $Z_n^{(k)} = \Pi X(\phi_n^{(k)})$, where $\Pi(x, y) = y$ for $(x, y) \in \mathbb{R} \times \mathbb{T}^{d-1}$.

We similarly define an embedded Markov chain Z for the process X^+. (By periodicity of X^+, the choice of k is unimportant for the law of Z, so that we drop its dependence of k.) Denote by $\pi^{(k)}$ the invariant measure for $Z^{(k)}$ and by π the invariant measure for Z. We then define σ-finite measures μ_+ and $\mu^{(k)}$ on $\mathbb{R} \times \mathbb{T}^{d-1}$ through the identities

$$\mu^{(k)}(B) = \int_{\mathbb{T}^{d-1}} \mathbb{E}_{x+ke_1} \int_0^{\phi_1^{(k)}} \mathbf{1}_B(X(s))\, ds\, \pi^{(k)}(dx), \tag{5.5}$$

$$\mu_+(B) = \int_{\mathbb{T}^{d-1}} \mathbb{E}_{x+ke_1} \int_0^{\phi_1^{(k)}} \mathbf{1}_B\big(X^+(s) - k\big)\, ds\, \pi(dx). \tag{5.6}$$

[Here and below we make a slight abuse of notation and identify elements $x \in \mathbb{T}^{d-1}$ with the element $(0, x) \in \mathbb{R} \times \mathbb{T}^{d-1}$.] It follows from [16], Theorem 2.1, that $\mu^{(k)}$ is invariant for the process X and μ_+ is invariant for X^+. Therefore, there exist constants $c_k > 0$ such that $\mu^{(k)} = c_k\mu$ since the invariant measure for X is unique up to normalization. Note that by translation invariance of X^+, μ_+ does not depend on k.

Note that we can assume without any loss of generality that $A \subset \{x : x_1 > 0\}$ [it suffices to shift it by a finite number of steps to the right in (5.4)]. In this case, we can rewrite (5.5) as

$$\mu^{(k)}(A+k) = \int_{\mathbb{T}^{d-1}} \mathbb{E}_{x+ke_1} \int_0^{\phi_1^{(k)}} \mathbf{1}_A\big(X^+(s) - k\big)\, ds\, \pi^{(k)}(dx). \tag{5.7}$$

This is because $X(t) = X^+(t)$ for $t \le \sigma_1^{(k)}$ and, if $X(\sigma_1^{(k)}) < k$, then

$$\int_{\sigma_1^{(k)}}^{\phi_1^{(k)}} \mathbf{1}_A\big(X(s) - k\big)\, ds = 0,$$

whereas if $X(\sigma_1^{(k)}) > k$, then $X(t) = X^+(t)$ for $t \le \phi_1^{(k)}$. This shows that the claim follows if we can show that $\|\pi - \pi^{(k)}\|_{\mathrm{TV}} \to 0$ as $k \to \infty$ and there exists a constant c_∞ such that $c_k \to c_\infty$.

Let us first show that the latter is a consequence of the former. Setting $B_k = [k, k+1] \times \mathbb{T}^{d-1}$, we have $c_{k+1}/c_k = \mu^{(k)}(B_{k+1})/\mu^{(k+1)}(B_{k+1})$. On the other hand a straightforward trial/error argument allows one to show that $\mathbb{E}_x \int_0^{\phi_1^{(0)}} \mathbf{1}_A(X^+(s))\, ds$ is bounded uniformly over $x \in \mathbb{T}^{d-1}$. It then follows immediately from (5.7) that there exists a constant C such that

$$\big|\mu^{(k)}(B_{k+1}) - \mu(B_0)\big| \le C \|\pi - \pi^{(k)}\|_{\mathrm{TV}},$$

and similarly for $|\mu^{(k+1)}(B_{k+1}) - \mu(B_0)|$. It follows that provided that $\sum_{k\ge 0} \|\pi - \pi^{(k)}\|_{\mathrm{TV}} < \infty$, one does indeed have $c_k \to c_\infty$.

Denote now by P the transition probabilities for Z and by $P^{(k)}$ the transition probabilities for $Z^{(k)}$. Then, we can write $P = QR$, where R is the Markov kernel from $\mathbb{T}^{d-1}$ to $\{-1, 1\} \times \mathbb{T}^{d-1}$ given by $R(x, A) = \mathbb{P}_x(X^+(\sigma_1) \in A)$ and Q is the Markov kernel from $\{-1, 1\} \times \mathbb{T}^{d-1}$ to $\mathbb{T}^{d-1}$ given by $Q(x, A) = \mathbb{P}_x(X^+(\phi_0) \in A)$ for $X_1(0) = 0$, $\sigma_1 = \inf\{t > 0 : |X_1(t)| = 1\}$ and $\phi_1 = \inf\{t > \sigma_1 : X_1(t) = 0\}$. Since the diffusion X^+ is elliptic, both Q and R are strong Feller and irreducible. It follows from the Doeblin–Doob–Khas'minskii theorem [10], Proposition 4.1.1, that $P(x, \cdot)$ and $P(y, \cdot)$ are mutually equivalent for any $x, y \in \mathbb{T}^{d-1}$. Furthermore, it follows from the Meyer–Mokobodzki theorem [9, 15, 27] that the map $x \mapsto P(x, \cdot)$ is continuous in the total variation topology. We conclude that the map $(x, y) \mapsto \|P(x, \cdot) - P(y, \cdot)\|_{\mathrm{TV}}$ reaches its maximum and that this is strictly less than 2, so that P satisfies Doeblin's condition. It follows that there exists a constant $\eta < 1$ such that P has the contraction property

$$\|P\nu_1 - P\nu_2\|_{\mathrm{TV}} \le \eta \|\nu_1 - \nu_2\|_{\mathrm{TV}},$$

for any two probability measures ν_1, ν_2 on $\mathbb{T}^{d-1}$. Therefore, if we can find constants ε_k such that

$$\sup_{x \in \mathbb{T}^{d-1}} \big\|P(x, \cdot) - P^{(k)}(x, \cdot)\big\|_{\mathrm{TV}} \le \varepsilon_k, \tag{5.8}$$

then we have

$$\|\pi - \pi^{(k)}\|_{\mathrm{TV}} \le \|P\pi - P\pi^{(k)}\|_{\mathrm{TV}} + \|P\pi^{(k)} - P^{(k)}\pi^{(k)}\|_{\mathrm{TV}} \le \eta\|\pi - \pi^{(k)}\|_{\mathrm{TV}} + \varepsilon_k, \tag{5.9}$$

so that $\|\pi - \pi^{(k)}\|_{\mathrm{TV}} \le \varepsilon_k/(1-\eta)$. The problem thus boils down to obtaining (5.8) for an exponentially decaying sequence ε_k.

It follows from the same calculation as in Lemma 5.3 that the probability that X reaches the interface $\mathscr{I}_\eta$ before time $\phi_1^{(k)}$ when started on the hyperplane $\{x_1 = k\}$ is bounded from above by $\mathcal{O}(1/k)$. This yields the "trivial" bound $\varepsilon_k \le \mathcal{O}(1/k)$, which unfortunately is not even summable. However, a more refined analysis allows to obtain Proposition 5.7 below, thus concluding the proof. □

PROPOSITION 5.7. *There exists a constant $\rho \in (0,1)$ such that $\varepsilon_k \le \mathcal{O}(\rho^k)$.*

PROOF. The intuitive idea behind the proof of Proposition 5.7 is that if the process goes all the way back to the interface then, by the time it reaches again the plane $\{x_1 = k\}$, its hitting distribution depends only very little on its behavior near the interface. In order to formalize this, let us introduce the Markov transition kernel Q_+ from $\mathbb{T}^{d-1}$ to $\mathbb{T}^{d-1}$ which is such that $Q_+(x,\cdot)$ is the hitting distribution of the plane $\{1\} \times \mathbb{T}^{d-1}$ for the process X_+ started at $(0,x)$. Similarly, we denote by $Q^{\ell,k}(x,\cdot)$ the hitting distribution of the plane $\{k\} \times \mathbb{T}^{d-1}$ for the process X started at (ℓ, x).

For a fixed integer $\ell > \eta$, our aim is to show that $Q^{\ell,k}(x,\cdot)$ gets very close to $Q_+^{k-\ell}(x,\cdot)$. Here, we denote by Q_+^k the kth iteration of the Markov transition kernel Q_+. With this notation at hand, define the quantities

$$\alpha_k \equiv \sup_{x\in\mathbb{T}^{d-1}} \|Q^{\ell,k}(x,\cdot) - Q_+^{k-\ell}(x,\cdot)\|_{\mathrm{TV}}.$$

Note now that since, for fixed ℓ, the probability that X reaches the interface $\mathscr{I}_\ell$ before time $\phi_1^{(k)}$ when started on the hyperplane $\{x_1 = k\}$ is bounded from above by $\mathcal{O}(1/k)$, we have

$$\begin{aligned}\varepsilon_k &\le \sup_{x\in\mathbb{T}^{d-1}} \|Q^{k-1,k}(x,\cdot) - Q_+(x,\cdot)\|_{\mathrm{TV}} \\ &\le \frac{C}{k} \sup_{x\in\mathbb{T}^{d-1}} \|Q^{\ell,k}(x,\cdot) - Q_+^{k-\ell}(x,\cdot)\|_{\mathrm{TV}} \le \frac{C}{k}\alpha_k,\end{aligned} \tag{5.10}$$

so that it suffices to obtain an exponentially decaying bound on the α_k's.

We now look for a recursion relation on the α_k's which then yields the required bound. We have the identities $Q^{\ell,k} = Q^{k-1,k}Q^{\ell,k-1}$ and $Q_+^{k-\ell} = Q_+Q_+^{k-\ell-1}$. It follows from the triangle inequality that one has the bound

$$\begin{aligned}\|Q^{\ell,k}\delta_x - Q_+^{k-\ell}\delta_x\|_{\mathrm{TV}} &\le \|(Q^{k-1,k} - Q_+)Q^{\ell,k-1}\delta_x\|_{\mathrm{TV}} \\ &\quad + \|Q_+(Q^{\ell,k-1}\delta_x - Q_+^{k-\ell-1}\delta_x)\|_{\mathrm{TV}}.\end{aligned} \tag{5.11}$$

At this stage, we note that by exactly the same reasoning as for P, the kernel Q_+ satisfies Doeblin's condition. Therefore, there exists a constant $\bar{\eta} < 1$ such that

$$\|Q_+\nu_1 - Q_+\nu_2\|_{\mathrm{TV}} \le \bar{\eta}\|\nu_1 - \nu_2\|_{\mathrm{TV}},$$

for any two probability measures ν_1, ν_2. This and the definition of α_k immediately implies that the second term in (5.11) is uniformly bounded by $\bar{\eta}\alpha_{k-1}$. On the other hand, it follows from (5.10) that the first term is bounded by $\frac{C}{k}\alpha_k$, so that

$$\alpha_k \le \frac{C}{k}\alpha_k + \bar{\eta}\alpha_{k-1},$$

for some fixed constant C. The claim now follows at once. □

Finally, the last estimate that we need is the following. Denote by τ the first hitting time of the interface $\partial\mathscr{I}_\eta$ and fix an arbitrary smooth positive function φ that is supported in the interval $[1, 2]$. Set furthermore $\varphi_n^+(x) = n^{-2}\varphi(n^{-1}x_1)$ and $\varphi_n^-(x) = n^{-2}\varphi(-n^{-1}x_1)$. Then we have the following lemma.

LEMMA 5.8. *With the above notation, setting $\bar{\varphi} = \int_1^2 \varphi(x)\,dx$, we have*

$$\left|\mathbb{E}_x \int_0^\tau \varphi_n^\pm(X_\pm(t))\,dt - \frac{2\bar{\varphi}}{D_{11}^\pm}\right| \to 0,$$

uniformly for all $x \in \{\pm n\} \times \mathbb{T}^{d-1}$ as $n \to \infty$.

PROOF. Again, we only consider the expression for X_+, the one for X_- follows in the same way. It follows from standard homogenization results [4, 23] that the law of $n^{-1}X_+(n^2t)$ converges weakly as $n \to \infty$ to the law of Brownian motion with diffusion coefficient D_{11}^+. It thus follows from [6], Corollary 8.4.2, that the law of $n^{-1}X_+(n^2t)$, where X_+ is stopped at the first hitting time of $\mathscr{I}_\eta$ converges weakly as $n \to \infty$ to the law of Brownian motion stopped when it reaches the hyperplane $\mathscr{I}_0$.

Denoting this limiting process by X_+^∞, an explicit calculation allows to check that $\mathbb{E}_x \int_0^\tau \varphi(X_+^\infty(t))\,dt = \frac{2\bar{\varphi}}{D_{11}^\pm}$ when $x_1 = 1$. Now, for any fixed $T > 0$, the map $\Phi_T : X \mapsto \int_0^{\tau\wedge T} \varphi_n^+(X(t))\,dt$ is continuous, so that $\mathbb{E}_x \int_0^{\tau\wedge T} \varphi_n^+(X_+(t))\,dt$ converges as $n \to \infty$ to $\mathbb{E}_x \int_0^{\tau\wedge T} \varphi(X_+^\infty(t))\,dt$. Letting $T \to \infty$ concludes the proof. □

We now have all the tools that we need to show that the exit probabilities from the interface converge to the desired limiting values.

PROOF OF PROPOSITION 5.1. Similarly to the proof of Proposition 5.5 we use a representation of the invariant measure μ in terms of an embedded Markov chain.

This time, we consider the stopping times $\tilde{\phi}_0^{(k)} = \inf\{t \geq 0 : |X_1(t)| = \eta\}$ and then $\tilde{\sigma}_n^{(k)} = \inf\{t \geq \tilde{\phi}_n^{(k)} : |X_1(t)| = k\}$, $\tilde{\phi}_{n+1}^{(k)} = \inf\{t \geq \tilde{\sigma}_n^{(k)} : |X_1(t)| = \eta\}$. Denoting as similar to before by $\tilde{\pi}^{(k)}$ the invariant measure of the embedded Markov chain $\tilde{Z}_n^{(k)} = X(\tilde{\phi}_n^{(k)})$ (which is now a Markov chain on $\partial \mathscr{I}_\eta$), we set

$$\tilde{\mu}^{(k)}(B) = \int_{\partial \mathscr{I}_\eta} \mathbb{E}_x \int_0^{\tilde{\phi}_1^{(k)}} \mathbf{1}_B(X(s))\, ds\, \tilde{\pi}^{(k)}(dx). \tag{5.12}$$

Again, the measures $\tilde{\mu}^{(k)}$ differ from μ purely through a scaling factor, so that there are constants C_k such that $\tilde{\mu}^{(k)}(B) = \tilde{C}_k \mu(B)$ for every measurable set B.

The idea now is to evaluate $\tilde{\mu}^{(k)}(\varphi_k^{\pm})$ in two different ways and to compare the resulting answers. First, we note from Proposition 5.5 that

$$\mu^{(k)}(\varphi_k^{\pm}) = \frac{C_k}{k}(q_{\pm}\bar{\varphi} + \mathcal{O}(k^{-1})).$$

On the other hand, combining Proposition 5.4 and Lemma 5.8 with the definition (5.12), we see that

$$\mu^{(k)}(\varphi_k^{\pm}) = \frac{2p_{\pm}^{(k)}\bar{\varphi}}{D_{11}^{\pm}} + o(1) \tag{5.13}$$

as $k \to \infty$. Combining these two identities, we see that

$$\frac{p_+^{(k)}}{p_-^{(k)}} = \frac{D_{11}^+ q_+}{D_{11}^- q_-} + o(1),$$

thus concluding the proof. □

6. Computation of the drift along the interface.

This section is devoted to the computation of the drift coefficients α_j along the interface. Denote by τ^n the first hitting time of $\partial \mathscr{I}_n$ by the process X. With this notation, recall that, by (4.4), we have the identity

$$\alpha_j = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_x \int_0^{\tau^n} b_j(X_s)\, ds, \tag{6.1}$$

provided that this limit exists and is independent (and uniform) over starting points $x \in \mathscr{I}_\eta$.

PROPOSITION 6.1. *The expression on the right-hand side in* (6.1) *converges to the expression given by* (2.4), *uniformly in* $x \in \mathscr{I}_\eta$.

In order to show this, we will use the same construction as in the proof of Proposition 5.1. In particular, recall the definition (5.12) of the measures $\tilde{\mu}^{(k)}$, which are nothing but multiples of the invariant measure μ, as well as the sequence of stopping times $\tilde{\phi}_n^{(k)}$ and $\tilde{\sigma}_n^{(k)}$. Denote furthermore by $\tilde{\pi}_n^{(k)}$ the invariant measure for the process on $\partial\mathscr{I}_\eta$ with transition probabilities $P(x, A)$ given by

$$P(x, A) \stackrel{\text{def}}{=} \mathbb{P}_x\big(X\big(\tilde{\phi}_1^{(k)}\big) \in A | \tau^n > \tilde{\phi}_1^{(k)}\big). \tag{6.2}$$

Our proof will proceed in two steps. First, we show that the limit (6.1) exists and is equal to the value (2.4) given in the interface, provided that we start the process X in the stationary measure $\tilde{\pi}_n^{(k)}$ and let $k \to \infty$. In the second step, we then show by a coupling argument similar to the proof of Proposition 5.4 that the expression in (6.1) depends only weakly on the initial condition as n gets large, thus concluding the proof.

Before we proceed with this program, we perform the following preliminary calculation.

LEMMA 6.2. *One has the normalization*

$$\lim_{k\to\infty} k^{-2}\tilde{\mu}^{(k)}([-k, k] \times \mathbb{T}^{d-1}) = 2\left(\frac{p_+}{D_{11}^+} + \frac{p_-}{D_{11}^-}\right) \stackrel{\mathit{def}}{=} \beta,$$

where the coefficients $p_\pm$ are as in (2.2). *In particular, if μ is normalized as in the Introduction, then one has $k^{-1}\tilde{\mu}^{(k)} \approx \beta\mu$ for large values of k.*

PROOF. We know from Proposition 5.5 that $\mu(dx) \to \mu_\pm(dx)$ at exponential rate as $x_1 \to \pm\infty$, so that on large scales μ behaves like a multiple of Lebesgue measure on either side of the interface. Furthermore, we know from Proposition 5.1 that the corresponding normalization constants satisfy the relation (2.2). Combining this with the fact that $\tilde{\mu}^{(k)}$ is just a multiple of μ, the result then follows from (5.13). $\square$

Using this result, we obtain the following proposition.

PROPOSITION 6.3. *The limit*

$$\alpha_j = \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\tilde{\pi}_n^{(k)}}\int_0^{\tau^n} b_j(X_s)\,ds,$$

exists and is equal to

$$\beta\int_{\mathbb{R}\times\mathbb{T}^{d-1}}\big(b_j(x) + \mathcal{L}g_j(x)\big)\mu(dx), \tag{6.3}$$

where g is the function fixed in Section 3 and the constant β is as in Lemma 6.2.

REMARK 6.4. Note that if ϕ is any smooth compactly supported function, then the identity $\int \mathcal{L}\phi(x)\mu(dx) = 0$ holds. As a consequence, the expression (6.3) is independent of the choice of the compensator g.

PROOF OF PROPOSITION 6.3. It follows from the definition of $\tilde{\pi}_n^{(k)}$ and the strong Markov property of X that one has the identity

$$
\begin{aligned}
&\mathbb{E}_{\tilde{\pi}_n^{(k)}} \int_0^{\tau^n} \tilde{b}_j(X_s)\,ds \\
&\quad = \sum_{m\geq 0} \big(\mathbb{P}_{\tilde{\pi}_n^{(k)}}\big(\tilde{\phi}_1^{(k)} < \tau^n\big)\big)^m \mathbb{E}_{\tilde{\pi}_n^{(k)}} \int_0^{\tilde{\phi}_1^{(k)}\wedge\tau^n} \tilde{b}_j(X_s)\,ds \qquad (6.4)\\
&\quad = \frac{\mathbb{E}_{\tilde{\pi}_n^{(k)}} \int_0^{\tilde{\phi}_1^{(k)}\wedge\tau^n} \tilde{b}_j(X_s)\,ds}{\mathbb{P}(\tilde{\phi}_1^{(k)} > \tau^n)}.
\end{aligned}
$$

Note now that it follows from Lemma 5.3 that

$$
\mathbb{P}\big(\tilde{\phi}_1^{(k)} > \tau^n\big) = k/n + \mathcal{O}(1/n). \tag{6.5}
$$

Since $\lim_{n\to\infty} g_j(X(\tau^n))/n = 0$ and furthermore, using the same argument as in (5.9), we have $\lim_{n\to\infty} \|\tilde{\pi}_n^{(k)} - \tilde{\pi}^{(k)}\|_{\mathrm{TV}} = 0$ for every $k > 0$, so that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\tilde{\pi}_n^{(k)}} \int_0^{\tau^n} b_j(X_s)\,ds &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\tilde{\pi}_n^{(k)}} \Big[\int_0^{\tau^n} b_j(X_s)\,ds + g_j(X(\tau_n))\Big] \\
&= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\tilde{\pi}_n^{(k)}} \int_0^{\tau^n} \tilde{b}_j(X_s)\,ds \\
&= \lim_{n\to\infty} \frac{1}{k}\mathbb{E}_{\tilde{\pi}^{(k)}} \int_0^{\tilde{\phi}_1^{(k)}\wedge\tau^n} \tilde{b}_j(X_s)\,ds \qquad (6.6)\\
&= \frac{1}{k}\mathbb{E}_{\tilde{\pi}^{(k)}} \int_0^{\tilde{\phi}_1^{(k)}} \tilde{b}_j(X_s)\,ds \\
&= \frac{1}{k}\int_{\mathbb{R}\times\mathbb{T}^{d-1}} \tilde{b}_j(x)\tilde{\mu}^{(k)}(dx).
\end{aligned}
$$

Here, we used (6.4) and (6.5) to go from the second to the third line and we used the definition of the $\tilde{\mu}^{(k)}$ to obtain the last identity. The claim now follows from Lemma 6.2. □

We can now complete the proof.

PROOF OF PROPOSITION 6.1. In view of Proposition 6.3, it remains to show that

$$
\lim_{n\to\infty} \frac{1}{n}\left|\mathbb{E}_x \int_0^{\tau^n} b(X_s)\,ds - \mathbb{E}_y \int_0^{\tau^n} b(X_s)\,ds\right| = 0,
$$

uniformly over $x, y \in \mathscr{I}_\eta$. Fix an arbitrary value of $k > \eta$ and consider again the transition probabilities P given by (6.2). Since they arise as exit probabilities for an elliptic diffusion, we can show again by the same argument as in the proof of Proposition 5.5 that P satisfies the Doeblin condition for some constant η, namely $\|P\nu_1 - P\nu_2\|_{\mathrm{TV}} \le (1-\eta)\|\nu_1 - \nu_2\|_{\mathrm{TV}}$, uniformly over probability measures ν_1 and ν_2 on $\partial\mathscr{I}_\eta$. Note now that one has the identity

$$
\begin{aligned}
\mathbb{E}_x \int_0^{\tau^n} b(X_s)\,ds &= \sum_{m\ge 0}\Big(\prod_{0\le \ell<m} \mathbb{P}_\ell^x\big(\tilde\phi_1^{(k)} < \tau^n\big)\Big)\mathbb{E}_m^x \int_0^{\tilde\phi_1^{(k)}\wedge\tau^n} b(X_s)\,ds \\
&= \sum_{m\ge 0} \mathbb{P}_x\big(\tilde\phi_m^{(k)} < \tau^n\big)\mathbb{E}_m^x \int_0^{\tilde\phi_1^{(k)}\wedge\tau^n} b(X_s)\,ds,
\end{aligned}
\tag{6.7}
$$

where we denote by $\mathbb{P}_m$ (resp., $\mathbb{E}_m$) the probability (resp., expectation) for the process X started at $P^m(x,\cdot)$.

Note now that we have the identity

$$\mathbb{P}_x\big(\tilde\phi_m^{(k)} < \tau^n\big) = \mathbb{P}_x\big(\tilde\phi_\ell^{(k)} < \tau^n\big) + \mathbb{P}_{P^\ell(x,\cdot)}\big(\tilde\phi_{m-\ell}^{(k)} < \tau^n\big).$$

Also, by choosing k sufficiently large (but independent of n), we can ensure that there exist constants $c, C > 0$ such that

$$1 - \frac{C}{n} \le \mathbb{P}_x\big(\tilde\phi_1^{(k)} < \tau^n\big) \le 1 - \frac{c}{n},$$

uniformly for $x \in \mathscr{I}_\eta$ and for n sufficiently large. It also follows from the contraction properties of P that

$$\big|\mathbb{P}_m^x\big(\tilde\phi_1^{(k)} < \tau^n\big) - \mathbb{P}_m^y\big(\tilde\phi_1^{(k)} < \tau^n\big)\big| \le 2(1-\eta)^m,$$

uniformly over $x, y \in \mathscr{I}_\eta$.

Combining these bounds, we obtain for every $\ell \le m\wedge n$ the estimate

$$\big|\mathbb{P}_x\big(\tilde\phi_m^{(k)} < \tau^n\big) - \mathbb{P}_y\big(\tilde\phi_m^{(k)} < \tau^n\big)\big| \le \frac{K\ell}{n} + 2(1-\eta)^\ell.$$

In particular, there exists a constant K, such that we have the uniform bound

$$\big|\mathbb{P}_x\big(\tilde\phi_m^{(k)} < \tau^n\big) - \mathbb{P}_y\big(\tilde\phi_m^{(k)} < \tau^n\big)\big| \le \frac{K}{\sqrt{n}} \wedge \frac{Km}{n} \wedge \Big(1 - \frac{c}{n}\Big)^m,$$

valid for every $m > 0$ and every n sufficiently large. Summing over m, it follows that

$$\sum_{m\ge 0}\big|\mathbb{P}_x\big(\tilde\phi_m^{(k)} < \tau^n\big) - \mathbb{P}_y\big(\tilde\phi_m^{(k)} < \tau^n\big)\big| \le K\sqrt{n},$$

for a possibly different constant K.

On the other hand, it is possible to check that there exists a constant C (depending on k) such that

$$\Big|\mathbb{E}_x \int_0^{\tilde\phi_1^{(k)}\wedge\tau^n} b(X_s)\,ds\Big| \le C,$$

uniformly over $x \in \mathscr{I}_\eta$, so that

$$\left|\mathbb{E}_m^x \int_0^{\tilde\phi_1^{(k)}\wedge\tau^n} b(X_s)\,ds - \mathbb{E}_m^y \int_0^{\tilde\phi_1^{(k)}\wedge\tau^n} b(X_s)\,ds\right| \le 2C(1-\eta)^m.$$

Inserting these bounds into (6.7), we obtain

$$\left|\mathbb{E}_x \int_0^{\tau^n} b(X_s)\,ds - \mathbb{E}_y \int_0^{\tau^n} b(X_s)\,ds\right| \le 2C\sum_{m\ge 0}(1-\eta)^m + C\sqrt{n},$$

so that the requested bound follows at once. □

6.1. *Bound on the second moment.* In order to conclude the verification of the assumptions of Theorem 2.4, it remains to show that the second bound holds in (4.4). For the nonrescaled process, we can reformulate this as a proposition.

PROPOSITION 6.5. *For every* $\bar\eta > 0$, *there exists a constant* $C > 0$ *such that the bound*

$$\mathbb{E}_y\|Y(\tau^n) - y\|^2 \le Cn^2,$$

holds for every $n \ge 1$ *and every initial condition* $y \in \mathscr{I}_{\bar\eta}$.

PROOF. It follows from (3.1) that

$$\mathbb{E}_y\|Y(\tau^n) - y\|^2 \le 2\mathbb{E}_y\left\|\int_0^{\tau^n} \tilde b(X_s)\,ds\right\|^2 + 2\mathbb{E}_y\left\|\int_0^{\tau^n} \tilde\sigma(X_s)\,dW(s)\right\|^2. \tag{6.8}$$

It follows from Itô's isometry that the second term is bounded by $C\mathbb{E}\tau^n$, which in turn is bounded by $\mathcal{O}(n^2)$ by a calculation virtually identical to that of Lemma 4.5.

It remains to bound the first term, which we will do with the help of a decomposition similar to that used in the proof of Proposition 5.1. For two constants $c > 0$ and $a > 0$ to be determined, we set $\phi_0 = 0$, $\sigma_n = \inf\{t \ge \phi_n : |X_1(t)| = c + a\}$ and $\phi_n = \inf\{t \ge \sigma_{n-1} : |X_1(t)| = c\}$. Define furthermore

$$N = \inf\{k \ge 0 : \sigma_k \ge \tau^n\}.$$

Since $\tilde b$ is supported in a bounded strip around $\mathscr{I}_0$, we can make c sufficiently large so that the first term in (6.8) is bounded by some multiple of

$$\begin{aligned}\mathbb{E}_y\left(\sum_{k=0}^N(\sigma_k - \phi_k)\right)^2 &\le \sqrt{\mathbb{E}_y N^3 \mathbb{E}_y \sum_{k=0}^N (\sigma_k - \phi_k)^4} \\ &\le \sqrt{\mathbb{E}_y N^3 \sum_{k=0}^\infty \mathbb{E}_y\big((\sigma_k - \phi_k)^4 | N \ge k\big)\mathbb{P}_y(N \ge k)}.\end{aligned}$$

Note now that since σ_k is the exit time from a compact region for an elliptic diffusion, there exists a constant C such that $\mathbb{E}_y((\sigma_k - \phi_k)^4 | N \geq k) \leq C$, uniformly in y. Furthermore, it follows from Lemma 5.3 that if a is sufficiently large, then

$$\mathbb{P}_y(N > 1) \leq 1 - \frac{c}{n},$$

for some constant $c > 0$, uniformly in y. The strong Markov property then immediately implies that $\mathbb{P}_y(N > k) \leq (1 - \frac{c}{n})^k$, so that N is stochastically bounded by a Poisson random variable with parameter $\mathcal{O}(n)$ and the claim follows. □

7. Well-posedness of the martingale problem and characterization of the limiting process. The aim of this section is to show that the martingale problem associated to the operator $\bar{\mathcal{L}}$ as defined in Theorem 2.4 is unique and to characterize the corresponding (strong) Markov process. Our main tool is the following general result by Ethier and Kurtz [11], Theorem 4.1.

THEOREM 7.1. *Let E be a separable metric space, and let $A : \mathscr{D}(A) \to \mathcal{B}_b(E)$ be linear and dissipative. Suppose there exists $\lambda > 0$ such that*

$$\mathcal{C} \stackrel{\text{def}}{=} \overline{\mathscr{R}(\lambda - A)} = \overline{\mathscr{D}(A)}, \tag{7.1}$$

and such that $\mathcal{C}$ is separating. Let $\mu \in \mathscr{P}(E)$ and suppose X is a solution of the martingale problem for (A, μ). Then X is a Markov process corresponding to the semigroup on $\mathcal{C}$ generated by the closure of A, and uniqueness holds for the martingale problem for (A, μ).

See also [7] for a more general result on the well-posedness of a martingale problem with discontinuous coefficients. This allows us to finally give the proof of Theorem 2.4.

PROOF OF THEOREM 2.4. Since we already know from the results in the previous two sections that limit points of X^ε solve the martingale problem associated to $\bar{\mathcal{L}}$, it suffices to show that this martingale problem is well-posed and that its solutions are of the form (2.6).

For this, we somehow take the reverse approach: first, we construct a solution to (2.6) and we show that this is a Markov process solving the martingale problem associated to $\bar{\mathcal{L}}$. We then show that this Markov process generates a strongly continuous semigroup on $\mathcal{C}_0(\mathbb{R}^d)$, whose generator is the closure of $\bar{\mathcal{L}}$ in $\mathcal{C}_0$. Since $\mathcal{C}_0$ is separating and since generators of strongly continuous semigroups are dissipative and satisfy (7.1) by the Hille–Yosida theorem, the claim then follows.

In order to construct a solution to (2.6), let $M_\pm$ be matrices satisfying $M_\pm M_\pm^T = D^\pm$ and such that

$$M_\pm = \begin{pmatrix} \sqrt{D_{11}^\pm} & 0 \\ v_\pm & \tilde{M}_\pm \end{pmatrix},$$

for some vectors $v_\pm \in \mathbb{R}^{d-1}$ and some $(d-1)\times(d-1)$ matrices $\tilde{M}_\pm$. (This is always possible by the QR decomposition.) We then first construct a Wiener process W_1 and a process $\bar{X}_1$ such that

$$d\bar{X}_1 = \big(\mathbf{1}_{\bar{X}_1\le 0}\sqrt{D_{11}^-} + \mathbf{1}_{\bar{X}_1>0}\sqrt{D_{11}^+}\big)\,dW(t) + (p_+ - p_-)\,dL(t),$$

where L is the symmetric local time of $\bar{X}_1$ at the origin. This can be achieved, for example, by setting $\bar{X}_1 = g(Z)$, where

$$g(x) = \begin{cases} \sqrt{D_{11}^+}, & \text{if } x > 0,\\ \sqrt{D_{11}^-}, & \text{otherwise,}\end{cases}$$

Z is a skew-Brownian motion with parameter

$$p = \frac{p_+\sqrt{D_{11}^-}}{p_+\sqrt{D_{11}^-} + p_-\sqrt{D_{11}^+}},$$

and W is the martingale part of Z. Given such a pair $(\bar{X}_1, W)$, we then let $\tilde{W}$ be an independent $d-1$-dimensional Wiener process and we define pathwise the $\mathbb{R}^{d-1}$-valued process $\tilde{X}$ by

$$\begin{aligned}\tilde{X}(t) = {}&\int_0^t (\mathbf{1}_{\bar{X}_1\le 0}\tilde{M}_- + \mathbf{1}_{\bar{X}_1>0}\tilde{M}_+)\,d\tilde{W}(t) + \int_0^t (\mathbf{1}_{\bar{X}_1\le 0}v_- + \mathbf{1}_{\bar{X}_1>0}v_+)\,dW(t)\\ &+ \tilde{\alpha}\int_0^t dL(t),\end{aligned}$$

where $\tilde{\alpha}_j = \alpha_{j+1}$. Since we know that skew-Brownian motion enjoys the Markov property, it follows immediately that $\bar{X}_1$ is Markov, so that $\bar{X} = (\bar{X}_1, \tilde{X})$ is also a Markov process. Applying the symmetric Itô–Tanaka formula to $f(\bar{X})$ it is furthermore a straightforward exercise to check that $\bar{X}$ does indeed solve the martingale problem for $\bar{\mathcal{L}}$.

The corresponding Markov semigroup $\{\mathcal{P}_t\}_{t\ge 0}$ maps $\mathcal{C}_0(\mathbb{R}^d)$ into itself as a consequence of the Feller property of skew-Brownian motion [19]. Furthermore, as a consequence of the uniform stochastic continuity of $\bar{X}$, it is strongly continuous, so that its generator must be an extension of $\bar{\mathcal{L}}$. Since the range of $\bar{\mathcal{L}}$ contains $\mathcal{C}_0^\infty(\mathbb{R}^d)$, which is a dense subspace of $\mathcal{C}_0(\mathbb{R}^d)$, the claim follows. □

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