

Asymptotic results for spatial causal ARMA models

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Abstract: The paper establishes a functional central limit theorem for the empirical distribution function of a stationary, causal, ARMA process given by $X_{s,t} = \sum_{i \geq 0} \sum_{j \geq 0} a_{i,j} \xi_{s-i,t-j}$, $(s,t) \in \mathbf{Z}^2$, where the $\xi_{i,j}$ are independent and identically distributed, zero mean innovations. By judicious choice of σ -fields and element enumeration, one dimensional martingale arguments are employed to establish the result.

AMS 2000 subject classifications: Primary 62G30; secondary 60F17, 62M10, 60G60.

Keywords and phrases: Causal stationary processes, empirical distribution function, spatial ARMA processes, central limit theorem, quantile process, martingale.

Received November 2009.

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1. Introduction

The analysis of stationary random processes and random fields is a classic problem in mathematical statistics. The asymptotic behaviour of partial sums and

*Research supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

empirical distributions is of particular interest, with the nature of the limit depending on whether the process has short or long memory. A stationary random field $(X_{i,j} : i, j \in \mathbf{Z})$ on the lattice is said to have short memory or to be short-range dependent if and only if its covariance function is absolutely summable: i.e. $\sum_i \sum_j |Cov(X_{0,0}, X_{i,j})| < \infty$; otherwise it is said to have long memory.

Many results are available for long memory fields; recent articles include [8, 20] and [19], to which the reader is referred for thorough bibliographies. Although there is an extensive literature on asymptotics for random fields satisfying various types of conditions involving mixing or association (cf. [7, 6] and [3] and the references therein), there are only a few papers available on short memory fields without explicit reference to mixing or association. For short memory processes $(X_i : i \in \mathbf{Z})$, such assumptions can often be avoided through the use of the elegant martingale methods developed by Gordin [13], but these techniques are not generally applicable in higher dimensions. Central limit theorems for partial sums of short memory stationary fields over sets have been investigated by Dedecker ([4] and [5]), but to date there seem to be no results on the behaviour of the empirical distribution of a short memory stationary random field without additional assumptions on mixing or association. We note that in [4] and [5], a projective criterion related to that of [13] is assumed, but martingale techniques are not used.

In this paper we will focus on the empirical distribution generated by a causal autoregressive moving average (ARMA) field in two dimensions:

$$X_{s,t} = \sum_{i \geq 0} \sum_{j \geq 0} a_{i,j} \xi_{s-i, t-j}, \quad s, t \in \mathbf{Z}, \quad (1)$$

where $\{\xi_{u,v} : u, v \in \mathbf{Z}\}$ is an array of independent and identically distributed random variables. This model was first introduced by Tjøstheim in [24]; parameter estimation has been studied by a number of authors including [1, 15, 16, 18, 21] and [25]. In the case that only finitely many of the $a_{i,j}$ are non-zero, the field is strongly mixing and the behaviour of the empirical distribution is well understood (cf. [10], for example). Consequently, in what follows it will always be assumed that infinitely many of the $a_{i,j}$ are non-zero.

Although the causal model may not seem as natural in two dimensions as it does in one, it is pointed out in [1] that the spatial causal model provides an appropriate representation of many general patterns for the covariance structure of a stationary random field. See [1] for a detailed discussion and bibliography, including references to applications of causal models to field trial data.

We will prove an invariance principle for the empirical distribution of the ARMA field when infinitely many of the $a_{i,j}$ in equation (1) are non-zero and illustrate some immediate consequences, including a functional central limit theorem for the quantile process. This model is of particular interest since its structure allows us to exploit a novel one-dimensional martingale argument which utilizes a certain total order on the plane. Significantly, we require no projective criteria nor do we make any assumptions about association or mixing properties. Indeed, although our model includes the short memory ARMA field, in the case

of the invariance principle for the empirical process we do not even require that $X_{i,j}$ have a finite mean. We believe that our method is of independent interest and will be applicable to more general causal models in dimensions higher than one.

Invariance principles for the empirical distribution of a causal ARMA process on \mathbf{Z} have been developed by Doukhan and Surgailis [9] and Ho and Hsing [14] under different assumptions. Our technique allows us to combine one-dimensional martingale and two dimensional ergodicity arguments to produce an invariance principle for the empirical process generated by the spatial ARMA field. This is illustrated by following the development of Doukhan and Surgailis [9] to produce a two-dimensional result under conditions analogous to theirs.

Our main result and two applications are presented in Section 2; proofs appear in Section 3.

2. Results and applications

This paper will investigate the asymptotic behaviour of the ARMA model

$$X_{s,t} = \sum_{i \geq 0} \sum_{j \geq 0} a_{i,j} \xi_{s-i,t-j}, \quad (s, t) \in \mathbf{Z}^2,$$

where $\{\xi_{u,v} \mid u, v \in \mathbf{Z}\}$ are independent and identically distributed random variables, If $E[\xi_{0,0}^2] < \infty$ and $\sum_i \sum_j |a_{i,j}| < \infty$, we have a short memory field. We will proceed under the following more general assumptions.

Assumptions 2.1.

1. Let $\{a_{i,j}\}$ be an array of constants, infinitely many of which are non-zero, satisfying

$$\sum_{i \geq 0} \sum_{j \geq 0} |a_{i,j}|^\gamma < \infty,$$

for some $\gamma \in (0, 1]$.

2. There exists a constant $C < \infty$ and $\Delta \in (\frac{1}{2}, 1]$ such that for all $u \in \mathbf{R}$,

$$|E \exp(iu\xi_0)| \leq \frac{C}{(1 + |u|)^\Delta}.$$

3. $E(|\xi_{0,0}|^{2\gamma}) < \infty$.
4. $E(|\xi_{0,0}|^{4\gamma}) < \infty$.

Comments

- Note that the more general the moment condition, Assumptions 2.1.3 or 2.1.4, the more restrictive the summability condition Assumption 2.1.1.
- Assumption 2.1.2, like condition (4) of [9], implies that the distribution function of a partial sum of the $a_{i,j}\xi_{s-i,t-j}$ terms is differentiable with density

bounded by a constant provided a sufficiently large number of terms with non-zero $a_{i,j}$ are included in the sum. It also implies that the associated density satisfies a uniform Lipschitz condition provided sufficient terms are included in the moving average. See Giraitis and Surgailis, [12], for details.

We need to introduce some basic notation. The random variables $X_{i,j}$ and $\xi_{i,j}$ have distribution functions F and G respectively. Let

$$\begin{aligned} R_{i,j}(x) &:= I(X_{i,j} \leq x) - F(x), \\ W_{m,n}(x) &:= \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n R_{i,j}(x), \text{ and} \\ H_{m,n}(x) &:= \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n I(X_{i,j} \leq x). \end{aligned}$$

Theorem 2.2. *Assume Assumptions 2.1.1-2.1.3 hold. For fixed x , as m and n both tend to infinity*

$$W_{m,n}(x) = \sqrt{mn} [H_{m,n}(x) - F(x)] \xrightarrow{\mathcal{D}} N(0, \sigma^2(x))$$

where

$$\begin{aligned} \sigma^2(x) &:= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}(I(X_{0,0} \leq x), I(X_{i,j} \leq x)) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}(R_{0,0}(x), R_{i,j}(x)). \end{aligned}$$

Theorem 2.3. *Assume Assumptions 2.1.1, 2.1.2 and 2.1.4 hold. The convergence in Theorem 2.2 can be extended to*

$$W_{m,n}(\cdot) \xrightarrow{\mathcal{D}} W(\cdot)$$

in the sup norm topology on $D((-\infty, \infty))$, where $W(\cdot)$ is a centred Gaussian process with

$$\sigma(x, y) := \text{Cov}(W(x), W(y)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}(I(X_{0,0} \leq x), I(X_{i,j} \leq y)).$$

By applying the functional delta method, (for example, see van der Vaart and Wellner [26]), we obtain the following two corollaries as straightforward consequences of Theorem 2.3.

Corollary 2.4. *(c.f. [26], Lemma 3.9.17) Assume the $\xi_{u,v}$ have mean zero, variance 1 and Assumptions 2.1.1, 2.1.2 and 2.1.4 hold for some $\gamma \in [\frac{1}{2}, 1]$. Let $\bar{X}_{mn} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n X_{i,j}$. Then*

$$\sqrt{mn} \bar{X}_{mn} \xrightarrow{\mathcal{D}} \int x dW \text{ as } m, n \rightarrow \infty,$$

where W is the limiting Gaussian process in Theorem 2.3. In particular, the limit is a mean 0 normal random variable with variance $\sigma^2 = (\sum_i \sum_j a_{i,j})^2$.

Comment: Convergence of $\sqrt{mn}\bar{X}_{mn}$ to the $N(0, \sigma^2)$ distribution can be proven directly under Assumptions 2.1.1, and 2.1.3 with $\gamma = 1$. The method of proof for the ARMA field is virtually identical to that presented in [11] for the ARMA process.

Next recall Assumption 2.1.2 ensures that F is continuously differentiable. We can now state a functional central limit theorem for the empirical quantile process, $H_{m,n}^{-1}$ associated with $X_{i,j}$, where $H_{m,n}^{-1}(p) = \inf\{x : H_{m,n}(x) \geq p\}$.

Corollary 2.5. (c.f. [26] Section 3.9.4.2) Assume Assumptions 2.1.1, 2.1.2 and 2.1.4 hold. For $0 < p < q < 1$, if the derivative of F , f , is strictly positive on the interval $[F^{-1}(p) - \varepsilon, F^{-1}(q) + \varepsilon]$ for some $\varepsilon > 0$, then

$$\sqrt{mn}(H_{m,n}^{-1}(\cdot) - F^{-1}(\cdot)) \xrightarrow{\mathcal{D}} -W(F^{-1}(\cdot))/f(F^{-1}(\cdot)) \quad (2)$$

in $\ell^\infty[p, q]$, where W is the limiting Gaussian process in Theorem 2.3. In particular, the limit in (2) is a zero mean, Gaussian process with covariance function

$$\frac{\text{Cov}(W(F^{-1}(s)), W(F^{-1}(t)))}{f(F^{-1}(s))f(F^{-1}(t))}, \quad s, t \in [p, q].$$

3. Proofs

Let \leq denote the usual partial order on \mathbf{R}^2 : $(i, j) \leq (i', j') \Leftrightarrow i \leq i'$ and $j \leq j'$. Let

$$\mathcal{F}_{i,j} = \sigma\{\xi_{u,v} : u \leq i, v \leq j\}.$$

The martingale argument will be based on the total order \prec on \mathbf{R}^2 defined as follows:

$$(i, j) \prec (i', j') \Leftrightarrow i + j < i' + j' \quad \underline{\text{or}} \quad i + j = i' + j' \text{ and } i < i'.$$

Define

$$\begin{aligned} \mathcal{G}_{i,j} &= \sigma\{\xi_{u,v} : (u, v) \preceq (i, j)\} \\ &= \sigma\{\xi_{u,v} : u + v < i + j \text{ or } u + v = i + j \text{ and } u \leq i\} \\ \mathcal{G}_\ell^D &= \sigma\{\xi_{u,v} : u + v \leq \ell\}. \end{aligned}$$

To simplify the notation, C will denote a generic constant throughout the paper which may be different at each appearance.

A few observations.

1. Since the model is causal, $X_{i,j}$ is both $\mathcal{F}_{i,j}$ and $\mathcal{G}_{i,j}$ -measurable.

2. The ordering \prec cannot be defined via an enumeration of \mathbf{Z}^2 . It can if we are working on \mathbf{Z}_+^2 because we can start at $(0, 0)$, move to $(0, 1)$, then $(1, 0)$ and progressively down each successive diagonal. Note that each diagonal is finite. We can also count backwards via \prec order on $(-\infty, i] \times (-\infty, j]$, for any (i, j) . Note that this is not true of the lexicographic order employed in [4] and [5].
3. Let X be $\mathcal{F}_{i,j}$ measurable. Then by independence of $\xi_{i,j}$'s

$$\begin{aligned} E[X|\mathcal{G}_{u,v}] &= E[X|\xi_{h,\ell}, (h, \ell) \preceq (u, v)] \\ &= E[X|\xi_{h,\ell} \text{ with } (h, \ell) \preceq (u, v) \text{ and } (h, \ell) \leq (i, j)] \\ &= E[X|\mathcal{G}_{u,v} \cap \mathcal{F}_{i,j}]. \end{aligned}$$

4. \mathcal{G}_ℓ^D is decreasing as $\ell \rightarrow -\infty$. Let $\gamma_\ell = (\xi_{i,\ell-i} : i \in \mathbf{Z})$, $\ell \in \mathbf{Z}$ and note γ_ℓ are iid. Then $\mathcal{G}_\ell^D = \sigma(\gamma_u : u \leq \ell)$ and $\cap \mathcal{G}_\ell^D = \mathcal{T}$ satisfies the 0-1 law (see [2], Theorem 2.2).
5. If X is $\mathcal{F}_{i,j}$ measurable and $\ell < i + j$, then

$$\begin{aligned} E[X|\mathcal{G}_\ell^D] &= E[X|\mathcal{G}_\ell^D \cap \mathcal{F}_{i,j}] \\ &= E[X|\mathcal{G}_{i,\ell-i}]. \end{aligned} \tag{3}$$

6. Note that from (3)

$$\begin{aligned} P[X_{i,j} \leq x|\mathcal{G}_{i,j-u}] &= P[X_{i,j} \leq x|\mathcal{G}_{i+j-u}^D] \\ &= P[X_{i,j} \leq x|\mathcal{G}_{i-u,j+1}] \end{aligned} \tag{4}$$

since $\mathcal{G}_{i-u,j+1} = \mathcal{G}_{i+j-u}^D \vee \sigma\{\xi_{v,i+j+1-u-v} : v \leq i-u\}$ and $X_{i,j}$ is independent of $\xi_{v,w}$ for $w > j$.

For all (i, j) and $h, k \geq 0$ define (suppressing the dependence on x)

$$U_{i,j}(h, k) := \begin{cases} P[X_{i,j} \leq x|\mathcal{G}_{i-h,j-k}] - P[X_{i,j} \leq x|\mathcal{G}_{i-h-1,j-k+1}], & k \geq 1 \\ P[X_{i,j} \leq x|\mathcal{G}_{i-h,j}] - P[X_{i,j} \leq x|\mathcal{G}_{i,j-h-1}], & k = 0 \end{cases} \tag{5}$$

For h, k fixed, the $U_{i,j}(h, k)$ are stationary in i, j . By referring to Figure 1 note that we can write

$$U_{i,j}(h, k) = P[X_{i,j} \leq x|\mathcal{G}_{i-h,j-k}] - P[X_{i,j} \leq x|\mathcal{G}_{i-h-1,j-k+1}], \tag{6}$$

for all $i, j \geq 1$, $h, k \geq 0$. The case $k = 0$ follows from equation (4) by setting $u = h + 1$. We use this unified formula in the sequel.

Thus when we condition $X_{i,j}$ on $\mathcal{G}_{i-h-1,j+1}$, it is the same as conditioning on $\mathcal{G}_{i,j-h-1}$ as the extra $\xi_{u,v}$ terms involved in defining $\mathcal{G}_{i-h-1,j+1}$ in addition to those generating $\mathcal{G}_{i,j-h-1}$ are independent of $X_{i,j}$. As a result, via conditioning under the total order \prec we are able to successively move over each diagonal in the quadrant to the left and below (i, j) . Although many total orders can be defined on the plane, this procedure also enables us to maintain stationarity.

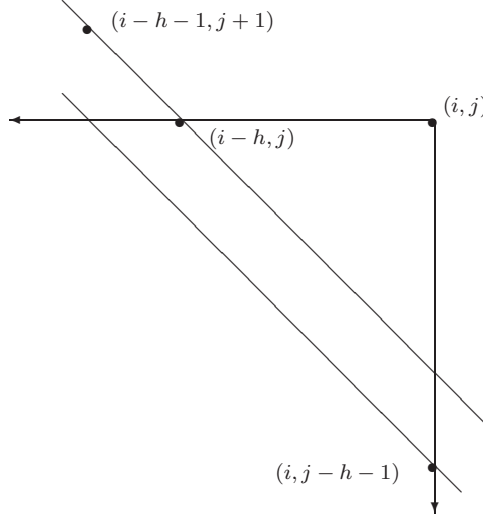


FIG 1. Drawing back to a lower diagonal.

For $N < \infty$, define

$$\begin{aligned}
 R_{i,j}^N(x) &:= \sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{i,j}(h, \ell - h) \\
 &= I(X_{i,j} \leq x) - P(X_{i,j} \leq x | \mathcal{G}_{i,j-(N-1)}) \\
 &= I(X_{i,j} \leq x) - P(X_{i,j} \leq x | \mathcal{G}_{i+j-(N+1)}^D),
 \end{aligned} \tag{7}$$

where the second line follows since the series collapses and the third line follows by (3).

Recall $R_{i,j}(x) = I(X_{i,j} \leq x) - P(X_{i,j} \leq x)$. Observe that

$$\begin{aligned}
 R_{i,j}^N(x) &= I(X_{i,j} \leq x) - P(X_{i,j} \leq x | \mathcal{G}_{i+j-(N+1)}^D), \\
 &\xrightarrow{\text{a.s.}} I(X_{i,j} \leq x) - P(X_{i,j} \leq x | \cap_{\ell} \mathcal{G}_{\ell}^D) \\
 &\quad \text{(by the reversed martingale convergence theorem),} \\
 &= I(X_{i,j} \leq x) - P(X_{i,j} \leq x), \text{ since } \cap_{\ell} \mathcal{G}_{\ell}^D \text{ is trivial,} \\
 &= R_{i,j}(x).
 \end{aligned}$$

Thus, almost surely,

$$R_{i,j}(x) = \lim_{N \rightarrow \infty} R_{i,j}^N(x) = \sum_{\ell=0}^{\infty} \sum_{h=0}^{\ell} U_{i,j}(h, \ell - h).$$

Since $0 \leq P(X_{i,j} \leq x | \mathcal{G}_{i+j-(N+1)}^D) \leq 1$, we also have convergence in L^p for all $p > 0$.

At this point, we observe that by (6) for i, j fixed, $(U_{i,j}(i-h, j-k), \mathcal{G}_{h,k})$ are martingale differences in the total order \prec : $E[U_{i,j}(i-h, j-k)|\mathcal{G}_{h',k'}] = 0$ if $(h', k') \prec (h, k)$. With this one-dimensional martingale structure in place we can now proceed with the proofs of Theorems 2.2 and 2.3 following an approach similar to [9].

Proof of Theorem 2.2. Write $R_{i,j}(x) = R_{i,j}^N(x) + \tilde{R}_{i,j}^N(x)$, where $R_{i,j}^N(x)$ has been defined above and

$$\tilde{R}_{i,j}^N(x) = \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} U_{i,j}(h, \ell-h).$$

Now

$$W_{m,n}(x) = W_{m,n}^N(x) + \tilde{W}_{m,n}^N(x),$$

where

$$W_{m,n}^N(x) = \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n R_{i,j}^N(x) \quad \text{and} \quad \tilde{W}_{m,n}^N(x) = \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n \tilde{R}_{i,j}^N(x).$$

We will show

(a) $W_{m,n}^N(x) \xrightarrow{\mathcal{D}} N(0, \sigma_N^2(x))$, as $m, n \rightarrow \infty$, where

$$\sigma_N^2(x) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}(R_{0,0}^N(x), R_{i,j}^N(x));$$

(b) $\sigma_N^2(x) \rightarrow \sigma^2(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{Cov}(R_{0,0}(x), R_{i,j}(x)) < \infty$, as $N \rightarrow \infty$;

(c) $\text{Var}(\tilde{W}_{m,n}^N(x)) \leq \delta(N)$, where $\delta(N) \rightarrow 0$.

Then,

$$W_{m,n}^N(x) \xrightarrow{\mathcal{D}} N(0, \sigma_N^2(x)) \xrightarrow{\mathcal{D}} N(0, \sigma^2(x)),$$

where the first limit is taken as $m, n \rightarrow \infty$ and the second limit corresponds to $N \rightarrow \infty$. The result follows from [2] Theorem 4.2, as (c) implies that

$$\lim_{N \rightarrow \infty} \limsup_{m, n \rightarrow \infty} P(|\tilde{W}_{m,n}^N(x)| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0$$

and so

$$W_{m,n}(x) \xrightarrow{\mathcal{D}} N(0, \sigma^2(x)) \quad \text{as } m, n \rightarrow \infty.$$

Proof of (a) Define

$$M_{i,j}^N := \sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{i+h, j+\ell-h}(h, \ell-h). \quad (8)$$

Note $M_{i,j}^N$ is $\mathcal{G}_{i,j}$ measurable. If $(i', j') \prec (i, j)$, once again we have from (6)

$$\begin{aligned} & E[U_{i+h, j+\ell-h}(h, \ell-h) | \mathcal{G}_{i', j'}] \\ &= E[P(X_{i+h, j+\ell-h} \leq x | \mathcal{G}_{i, j}) | \mathcal{G}_{i', j'}] - E[P(X_{i+h, j+\ell-h} \leq x | \mathcal{G}_{i-1, j+1}) | \mathcal{G}_{i', j'}] \\ &= 0 \end{aligned}$$

since $(i-1, j+1) = \sup\{(u, v) : (u, v) \prec (i, j)\}$, and so $\mathcal{G}_{u,v} \subseteq \mathcal{G}_{i-1, j+1}$, for all $(u, v) \prec (i, j)$.

Consequently, $M_{i,j}^N$ are 1-d martingale differences in the total order \prec on \mathbf{Z}_+^2 . Also, since $U_{i,j}(h, k)$ is stationary in (i, j) for each (h, k) , we have that $(M_{i,j}^N)$ is stationary under horizontal and vertical shifts.

Henceforth, dependence on x will be suppressed in the notation when no ambiguity arises. Write

$$\begin{aligned} W_{m,n}^N &= \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n R_{i,j}^N \\ &= \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{i,j}(h, \ell-h) \\ &= \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1}^m \sum_{j=1}^n U_{i,j}(h, \ell-h) \\ &= \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i'=1-h}^{m-h} \sum_{j'=1-\ell+h}^{n-\ell+h} U_{i'+h, j'+\ell-h}(h, \ell-h) \\ &= \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1}^m \sum_{j=1}^n U_{i+h, j+\ell-h}(h, \ell-h) \\ &\quad + \left\{ \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1-h}^{m-h} \sum_{j=1-\ell+h}^0 U_{i+h, j+\ell-h}(h, \ell-h) \right. \end{aligned} \quad (9)$$

$$+ \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1-h}^0 \sum_{j=1-\ell+h}^{n-1+h} U_{i+h, j+\ell-h}(h, \ell-h) \quad (10)$$

$$\left. - \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1-h}^0 \sum_{j=1-\ell+h}^0 U_{i+h, j+\ell-h}(h, \ell-h) \right\} \quad (11)$$

$$- \left\{ \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=1}^m \sum_{j=n-\ell+h+1}^n U_{i+h, j+\ell-h}(h, \ell-h) \right. \quad (12)$$

$$+ \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=m-h+1}^m \sum_{j=1}^n U_{i+h, j+\ell-h}(h, \ell-h) \quad (13)$$

$$\left. - \frac{1}{\sqrt{mn}} \sum_{\ell=0}^N \sum_{h=0}^{\ell} \sum_{i=m-h+1}^m \sum_{j=n-\ell+h+1}^n U_{i+h, j+\ell-h}(h, \ell-h) \right\} \quad (14)$$

$$= \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n M_{i,j}^N + \frac{1}{\sqrt{mn}} Q_{m,n}^N.$$

We will apply a martingale central limit theorem to the first term and show that the second term converges to 0 in probability.

To show $\frac{1}{\sqrt{mn}}Q_{m,n}^N \xrightarrow{P} 0$, for fixed N , consider (9). For ℓ, h, j fixed we will show

$$\frac{1}{\sqrt{mn}} \left(\sum_{i=1-h}^{m-h} U_{i+h,j+\ell-h}(h, \ell-h) \right) \xrightarrow{L^2} 0. \quad (15)$$

This implies that (9) consists of a finite sum of terms, each of which converges to 0 in probability. The terms (10), (12), and (13) are similar, and the sums in (11) and (14) are bounded.

Return to (15). First $U_{i+h,j+\ell-h}(h, \ell-h)$ is $\mathcal{G}_{i,j}$ measurable. For j fixed, if i is increased then we move to a higher diagonal, i.e. $\mathcal{G}_{i,j} \subseteq \mathcal{G}_{i',j}$ if $i < i'$. Therefore, the terms are all orthogonal and so

$$\begin{aligned} & E \left[\left(\frac{1}{\sqrt{mn}} \sum_{i=1-h}^{m-h} U_{i+h,j+\ell-h}(h, \ell-h) \right)^2 \right] \\ &= \frac{1}{mn} \sum_{i=1-h}^{m-h} E(U_{i+h,j+\ell-h}(h, \ell-h))^2 \\ &\leq \frac{1}{n}, \quad \text{since } |U_{i,j}(h, k)| \leq 1, \\ &\rightarrow 0. \end{aligned}$$

Thus $\frac{1}{\sqrt{mn}}Q_{m,n}^N \xrightarrow{P} 0$.

A 1-dimensional martingale central limit theorem will now be applied to $\frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n M_{i,j}^N$. We begin by considering $\text{Var}(M_{0,0}^N)$. We have

$$\begin{aligned} \text{Var}(M_{0,0}^N) &= E \left(\sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{h,\ell-h}(h, \ell-h) \times \sum_{\ell'=0}^N \sum_{h'=0}^{\ell'} U_{h',\ell'-h'}(h', \ell'-h') \right) \\ &= E \left(\sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{0,0}(h, \ell-h) \times \sum_{\ell'=0}^N \sum_{h'=0}^{\ell'} U_{h'-h,(\ell'-\ell)-(h'-h)}(h', \ell'-h') \right) \\ &= E \left(\sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \sum_{\ell=0}^N \sum_{\ell'=0}^N \sum_{h=0}^{\ell} \sum_{h'=0}^{\ell'} U_{0,0}(h, \ell-h) U_{i,j}(h', \ell'-h') \right) \\ &= \sum_i \sum_j E \left(\sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{0,0}(h, \ell-h) \sum_{\ell'=0}^N \sum_{h'=0}^{\ell'} U_{i,j}(h', \ell'-h') \right) \\ &= \sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} \text{Cov}(R_{0,0}^N, R_{i,j}^N) \\ &= \sigma_N^2, \end{aligned}$$

where the second equality follows by stationarity and the third equality follows since $E[U_{0,0}(h, \ell - h) U_{i,j}(h', \ell' - h')] = 0$ unless the terms correspond to the same diagonal position, that is, $i = h' - h$ and $j = (\ell' - \ell) - (h' - h)$.

We will now proceed with the proof of the central limit theorem for

$$\frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n M_{i,j}^N$$

where N is fixed. The result follows from the four steps below.

1. $\{M_{i,j}^N, \mathcal{G}_{i,j}\}$ is a martingale difference array in the total order, \prec .
2. Since the $\xi_{i,j}$ are i.i.d. the original array $\{\xi_{i,j}\}$ is ergodic in \mathbf{Z}^2 . Therefore $(M_{i,j}^N)$ is stationary and ergodic on \mathbf{Z}^2 since the invariant σ -field is the intersection of the invariant σ -fields under horizontal and vertical shifts, each of which is trivial (since the rows and columns of $\xi_{i,j}$'s are independent and identically distributed.)

Applying the 2-dimensional ergodic theorem (see Theorem 10.12 in Kallenberg [17]) we have

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (M_{i,j}^N)^2 \rightarrow E[(M_{1,1}^N)^2] = \sigma_N^2 \text{ in } L_2$$

provided $m \rightarrow \infty$ and $n \rightarrow \infty$.

3. For each (m, n) we have one dimensional martingale differences by enumerating $(M_{1,1}^N, \dots, M_{m,n}^N)$ according to the total order, \prec .
4. Apply Theorem 2.3 of McLeish [22]. Note

$$\max_{1 \leq i \leq m, 1 \leq j \leq n} \left| \frac{1}{\sqrt{mn}} M_{i,j}^N \right| < \frac{C}{\sqrt{mn}},$$

as N is fixed and each summand is bounded. This observation implies McLeish's conditions (a) and (b) hold. Condition (c) follows from point 2 above as we have

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (M_{i,j}^N)^2 \xrightarrow{L_2} \sigma_N^2.$$

Thus

$$\frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n M_{i,j}^N \xrightarrow{D} N(0, \sigma_N^2).$$

This completes the proof of (a). \square

To complete the proof of Theorem 2.2 we need the following two lemmas.

Lemma 3.1. *Under the conditions of Theorem 2.2, there exists an ℓ_0 such that, provided $h + k > \ell_0$,*

$$|U_{i,j}(h, k)| \leq C |a_{h,k}|^\gamma (1 + |\xi_{i-h,j-k}|^\gamma), \quad (16)$$

where C does not depend on i, j, h or k .

Proof. The proof follows as in [9] by writing

$$X_{i,j} = X_{i,j}(h, k) + \tilde{X}_{i,j}(h, k),$$

where

$$X_{i,j}(h, k) = \sum_{\ell=0}^{h+k-1} \sum_{u=0}^{\ell} a_{u,\ell-u} \xi_{i-u,j-\ell+u} + \sum_{u=0}^h a_{u,h+k-u} \xi_{i-u,j-h-k+u},$$

and

$$\tilde{X}_{i,j}(h, k) = \sum_{\ell=h+k+1}^{\infty} \sum_{u=0}^{\ell} a_{u,\ell-u} \xi_{i-u,j-\ell+u} + \sum_{u=h+1}^{h+k} a_{u,h+k-u} \xi_{i-u,j-h-k+u}.$$

Let $F_{h,k}(x) := P(X_{i,j}(h, k) \leq x)$ which does not depend on (i, j) . Recalling the definition (5), we have

$$U_{i,j}(h, k) = \begin{cases} F_{h-1,k+1}(x - \tilde{X}_{i,j}(h-1, k+1)) - F_{h,k}(x - \tilde{X}_{i,j}(h, k)), & \text{if } h \geq 1, \\ F_{k-1,0}(x - \tilde{X}_{i,j}(k-1, 0)) - F_{0,k}(x - \tilde{X}_{i,j}(0, k)), & \text{if } h = 0. \end{cases} \quad (17)$$

Note

$$\tilde{X}_{i,j}(h, k) = \begin{cases} \tilde{X}_{i,j}(h-1, k+1) - a_{h,k} \xi_{i-h,j-k}, & \text{if } h \geq 1, \\ \tilde{X}_{i,j}(k-1, 0) - a_{0,k} \xi_{i,j-k}, & \text{if } h = 0. \end{cases} \quad (18)$$

Now observe that if $\xi_{0,0} \sim G$ we have

$$F_{h,k}(x) = \begin{cases} \int F_{h-1,k+1}(x - a_{h,k}u) G(du), & h \geq 1, \\ \int F_{k-1,0}(x - a_{0,k}u) G(du), & h = 0. \end{cases} \quad (19)$$

Substituting (18) and (19) into (17) we get

$$U_{i,j}(h, k) = \begin{cases} \int_{\mathbf{R}} [F_{h-1,k+1}(x - \tilde{X}_{i,j}(h, k) - a_{h,k}u) - F_{h-1,k+1}(x - \tilde{X}_{i,j}(h-1, k+1) - a_{h,k}u)] dG(u), & \text{if } h \geq 1, \\ \int_{\mathbf{R}} [F_{k-1,0}(x - \tilde{X}_{i,j}(0, k) - a_{0,k}u) - F_{k-1,0}(x - \tilde{X}_{i,j}(k-1, 0) - a_{0,k}u)] dG(u), & \text{if } h = 0. \end{cases} \quad (20)$$

Recall Assumption 2.1.2 implies that $F_{u,v}$ is differentiable with density bounded by a constant provided $u + v > \ell_0$ for some ℓ_0 . By the mean value theorem

$$|U_{i,j}(h, k)| \leq C|a_{h,k}| \int_{\mathbf{R}} |\xi_{i-h,j-k} - u| dG(u) \leq C|a_{h,k}|(1 + |\xi_{i-h,j-k}|).$$

Further, $|U_{i,j}(h,k)| \leq 1$ so (16) follows from the above and the fact that $\min(1, |x|) \leq |x|^\gamma$, for $0 < \gamma \leq 1$. \square

Lemma 3.2. *Under Assumptions 2.1.2 and 2.1.3 there exists an array $\{b_{i,j}\}$, $b_{i,j} > 0$, $i, j \in \mathbf{Z}$, independent of N , such that $\sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} b_{i,j} < \infty$ and*

$$|\text{Cov}(R_{0,0}^N(x), R_{i,j}^N(x))| \leq b_{i,j}. \quad (21)$$

Proof. Recall $U_{0,0}(h, \ell - h)$ and $U_{i,j}(h', \ell' - h')$ are orthogonal unless $i = h' - h$ and $j = (\ell' - \ell) - (h' - h)$.

$$\begin{aligned} \text{Cov}(R_{0,0}^N, R_{i,j}^N) &= E \left(\sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{0,0}(h, \ell - h) \sum_{\ell'=0}^N \sum_{h'=0}^{\ell'} U_{i,j}(h', \ell' - h') \right) \\ &= E \left(\sum_{\ell=0}^N \sum_{h=0}^{\ell} U_{0,0}(h, \ell - h) U_{i,j}(i + h, j + \ell - h) \right) \\ &\leq \sum_{\ell=0}^N \sum_{h=0}^{\ell} E^{\frac{1}{2}}(U_{0,0}^2(h, \ell - h)) E^{\frac{1}{2}}(U_{i,j}^2(i + h, j + \ell - h)) \\ &\quad \text{for } i, j \geq 0. \end{aligned}$$

From (16) and since $|U_{0,0}(h, \ell - h)| \leq 1$, for $i + j > \ell_0$

$$\begin{aligned} &\sum_{\ell=0}^{\ell_0} \sum_{h=0}^{\ell} E^{\frac{1}{2}}(U_{0,0}^2(h, \ell - h)) E^{\frac{1}{2}}(U_{i,j}^2(i + h, j + \ell - h)) \\ &\leq C \sum_{\ell=0}^{\ell_0} \sum_{h=0}^{\ell} |a_{i+h, j+\ell-h}|^\gamma (1 + E^{\frac{1}{2}}|\xi_{0,0}|^{2\gamma}) \\ &\leq C \sum_{\ell=0}^{\ell_0} \sum_{h=0}^{\ell} |a_{i+h, j+\ell-h}|^\gamma \\ &= b'_{i,j}, \end{aligned}$$

and $\sum_i \sum_j b'_{i,j} < \infty$, by Assumption 2.1.1, since the number of terms in the sums above are finite. Now

$$\begin{aligned} &\sum_{\ell=\ell_0+1}^N \sum_{h=0}^{\ell} E^{\frac{1}{2}}(U_{0,0}^2(h, \ell - h)) E^{\frac{1}{2}}(U_{i,j}^2(i + h, j + \ell - h)) \\ &\leq C \sum_{\ell=\ell_0+1}^N \sum_{h=0}^{\ell} |a_{h, \ell-h}|^\gamma |a_{i+h, j+\ell-h}|^\gamma (1 + E^{\frac{1}{2}}|\xi_{0,0}|^{2\gamma})^2 \\ &\leq C \sum_{\ell=\ell_0+1}^N \sum_{h=0}^{\ell} |a_{h, \ell-h}|^\gamma |a_{i+h, j+\ell-h}|^\gamma \\ &= b''_{i,j}. \end{aligned}$$

Also

$$\begin{aligned} \sum_i \sum_j b''_{i,j} &\leq C \sum_{\ell=\ell_0+1}^N \sum_{h=0}^{\ell} |a_{h,\ell-h}|^\gamma \sum_i \sum_j |a_{i+h,j+\ell-h}|^\gamma \\ &\leq C \sum_{\ell=1}^{\infty} \sum_{h=0}^{\ell} |a_{h,\ell-h}|^\gamma \\ &< \infty. \end{aligned}$$

Thus $|Cov(R_{0,0}^N(x), R_{i,j}^N(x))| \leq b_{i,j}$, where for $i+j > \ell_0$, $b_{i,j} = b'_{i,j} + b''_{i,j}$ and, for $i+j \leq \ell_0$, $b_{i,j} = 1$ since from (7) $|R_{i,j}^N(x)| \leq 1$ for all N . This completes the proof of Lemma 3.2. \square

Return to the Proof of Theorem 2.2.

Proof of (b) Recall

$$\begin{aligned} &Cov(R_{0,0}^N(x), R_{i,j}^N(x)) \\ &= E \left[\left(I(X_{0,0} \leq x) - P(X_{0,0} \leq x | \mathcal{G}_{-(N+1)}^D) \right) \right. \\ &\quad \left. \times \left(I(X_{i,j} \leq x) - P(X_{i,j} \leq x | \mathcal{G}_{i+j-(N+1)}^D) \right) \right] \end{aligned}$$

and

$$\begin{aligned} R_{0,0}^N(x) &\rightarrow I(X_{0,0} \leq x) - P(X_{0,0} \leq x) \text{ a.s.} \\ R_{i,j}^N(x) &\rightarrow I(X_{i,j} \leq x) - P(X_{i,j} \leq x) \text{ a.s.} \end{aligned}$$

By the bounded convergence theorem

$$Cov(R_{0,0}^N(x), R_{i,j}^N(x)) \rightarrow Cov(R_{0,0}(x), R_{i,j}(x))$$

and since the covariances are absolutely summable, by Lemma 3.2, we can exchange limits and summations to obtain (b).

Proof of (c) Assume $N > \ell_0$.

$$\begin{aligned} Var(\tilde{W}_{m,n}^N) &= \frac{1}{mn} E \left(\sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} U_{i,j}(h, \ell-h) \right)^2 \\ &= \frac{1}{mn} E \left(\sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} U_{i,j}(h, \ell-h) \right. \\ &\quad \left. \times \sum_{i'=1}^m \sum_{j'=1}^n \sum_{\ell'=N+1}^{\infty} \sum_{h'=0}^{\ell'} U_{i',j'}(h', \ell'-h') \right). \end{aligned}$$

However, $E[U_{i,j}(h, \ell-h) U_{i',j'}(h', \ell'-h')] = 0$ unless $i-h = i'-h'$ and $j-(\ell-h) = j'-(\ell'-h')$, and so

$$\begin{aligned}
& \text{Var}(\tilde{W}_{m,n}^N) \\
&= \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \sum_{i'=1}^m \sum_{j'=1}^n \sum_{\substack{\ell=N+1 \\ (i'-i)+(j'-j)+\ell \geq N+1}}^{\infty} \sum_{h=0}^{\ell} E[U_{i,j}(h, \ell-h) \times \\
&\quad U_{i',j'}(i'-i+h, j'-j+\ell-h)] \\
&\leq \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \sum_{i'=1}^m \sum_{j'=1}^n \sum_{\substack{\ell=N+1 \\ (i'-i)+(j'-j)+\ell \geq N+1}}^{\infty} \sum_{h=0}^{\ell} E^{\frac{1}{2}}[U_{i,j}^2(h, \ell-h)] \times \\
&\quad E^{\frac{1}{2}}[U_{i',j'}^2(i'-i+h, j'-j+\ell-h)] \\
&\leq \frac{C}{mn} \sum_{i=1}^m \sum_{j=1}^n \sum_{i'=1}^m \sum_{j'=1}^n \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} |a_{h, \ell-h}|^{\gamma} |a_{i'-i+h, j'-j+\ell-h}|^{\gamma} \\
&\leq \frac{C}{mn} \sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} |a_{h, \ell-h}|^{\gamma} \sum_{i'=1}^{\infty} \sum_{j'=1}^{\infty} |a_{i'-i+h, j'-j+\ell-h}|^{\gamma} \\
&\leq C \sum_{\ell=N+1}^{\infty} \sum_{h=0}^{\ell} |a_{h, \ell-h}|^{\gamma} \\
&= \delta(N) \text{ and } \delta(N) \rightarrow 0 \text{ as } N \rightarrow \infty.
\end{aligned}$$

□

Proof of Theorem 2.3. The finite dimensional convergence follows by using the Cramér Wold technique (see, for example, [17], Corollary 5.5) and arguing as in the proof of Theorem 2.2. Note we only need $\xi_{0,0}$ to have finite moments of order 2γ , that is, Assumption 2.1.3 holds, to obtain finite dimensional convergence.

To obtain the functional limit result we need to show that $\{W_{m,n}\}$ is tight in the sup norm topology on $D[-\infty, \infty]$. As in Shao and Yu [23] it suffices to establish the following moment bound: there exist constants $C < \infty$ and $\delta > 0$ such that for any $x, y \in \mathbf{R}$, with $|x - y| \leq 1$,

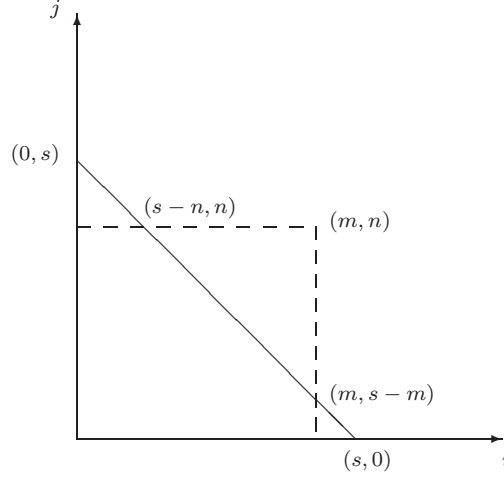
$$E(W_{m,n}(x) - W_{m,n}(y))^4 \leq C(|x - y|^{1+\delta} + (mn)^{-1}|x - y|^{\delta}). \quad (22)$$

For fixed x and y , with $|x - y| \leq 1$, define, suppressing x and y in the notation,

$$\begin{aligned}
T_{i,j}(h, \ell-h) &= P(y < X_{i,j} \leq x \mid \mathcal{G}_{i-h, j-(\ell-h)}) \\
&\quad - P(y < X_{i,j} \leq x \mid \mathcal{G}_{i-h-1, j-(\ell-h)+1}).
\end{aligned}$$

Write

$$W_{m,n}(x) - W_{m,n}(y) = \frac{1}{\sqrt{mn}} \sum_{i=1}^m \sum_{j=1}^n \left(\sum_{\ell=0}^{\infty} \sum_{h=0}^{\ell} T_{i,j}(h, \ell-h) \right).$$


 FIG 2. Tracking terms along the s th diagonal.

Next define $V_{m,n}(s, t) := \sum_{i=1 \vee t}^m \sum_{j=1 \vee (s-t)}^n T_{i,j}(i-t, j-(s-t))$ so that

$$\sqrt{mn}(W_{m,n}(x) - W_{m,n}(y)) = \sum_{s \leq m+n} \sum_{t=s-n}^m V_{m,n}(s, t).$$

Note that $T_{i,j}(i-t, j-(s-t))$ is $\mathcal{G}_{t,s-t} \cap \mathcal{F}_{i,j}$ measurable. Thus $V_{m,n}(s, t)$ is $\mathcal{G}_{t,s-t}$ measurable and, $V_{m,n}(s, t)$ and $V_{m,n}(s', t')$ are orthogonal unless $s = s'$ and $t = t'$. By referring to Figure 2 we see the term preceding $V_{m,n}(s, t)$ in the total order, \prec , is $V_{m,n}(s, t-1)$, where $V_{m,n}(s, s-n-1)$ is defined to be $V_{m,n}(s-1, m)$. Note $V_{m,n}(s, t)$ is measurable with respect to $\mathcal{G}_{t,s-t}$, which is a σ -field corresponding to a position on the diagonal passing through $(0, s)$, $(s, 0)$.

We will use the procedure in [9] to establish (22). First we obtain an expression for the fourth moment like (13) in [9] by expressing the moment via sums in (s, t) and (s', t') , where (s', t') is the term preceding (s, t) in the total order. We have

$$(mn)^2 E[W_{m,n}(x) - W_{m,n}(y)]^4 = 4I_1 + 6I_2 + 4I_3 + I_4, \quad (23)$$

where

$$I_a = E \left[\sum_{s \leq m+n} \sum_{t=s-n}^m V_{m,n}^a(s, t) \left(\sum_{s' < s} \sum_{t'=s'-n}^m V_{m,n}(s', t') + \sum_{t'=s-n}^{t-1} V_{m,n}(s, t') \right)^{4-a} \right],$$

$a = 1, 2, 3, 4$, where the last sum is 0 if $t = s - n$.

$$I_a = E \left[\sum_{s \leq m+n} \sum_{t=s-n}^m E[V_{m,n}^a(s, t) | \mathcal{G}_{t-1, s-t+1}] \times \right.$$

$$\begin{aligned}
& \left(\sum_{s' < s} \sum_{t' = s' - n}^m V_{m,n}(s', t') + \sum_{t' = s - n}^{t-1} V_{m,n}(s, t') \right)^{4-a} \\
& \leq \sum_{s \leq m+n} \sum_{t = s - n}^m E \left[|E(V_{m,n}^a(s, t) | \mathcal{G}_{t-1, s-t+1})| \times \right. \\
& \quad \left. \left| \left(\sum_{s' < s} \sum_{t' = s' - n}^m V_{m,n}(s', t') + \sum_{t' = s - n}^{t-1} V_{m,n}(s, t') \right)^{4-a} \right| \right].
\end{aligned}$$

The term I_1 is 0 as $V_{m,n}(s, t)$ is a sum of martingale differences. For the other terms we will develop a deterministic bound, $b_{m,n}^{(a)}(s, t)$ for the conditional expectation such that $\sum_s \sum_t b_{m,n}^{(a)}(s, t) = O(mn)$. Then

$$I_a \leq \sum_{s \leq m+n} \sum_{t = s - n}^m b_{m,n}^{(a)}(s, t) E \left[\left| \sum_{s' < s} \sum_{t' = s' - n}^m V_{m,n}(s', t') + \sum_{t' = s - n}^{t-1} V_{m,n}(s, t') \right|^{4-a} \right] \quad (24)$$

Assumption 2.1.2 implies that $X_{i,j}$ has a bounded density which satisfies a uniform Lipschitz condition provided sufficient terms are included in the moving average. In our context this means that we include sufficient terms in the weighted sums by pulling back under the total order along a sufficient number of diagonals, ℓ_0 .

Let $f_{h,k}$ be the density associated with $F_{h,k}$. For $h+k \geq \ell_0$ we have $|f_{h,k}(x) - f_{h,k}(y)| \leq C|x - y|$ and $f_{h,k}$ is bounded by K for some constant K , so arguing as in the development of (20)

$$T_{i,j}(h, k) = \begin{cases} \int_{\mathbf{R}} \int_y^x [f_{h-1, k+1}(v - \tilde{X}_{i,j}(h, k) - a_{h,k}\xi_{i-h, j-k}) \\ - f_{h-1, k+1}(v - \tilde{X}_{i,j}(h, k) - a_{h,k}u)] dv dG(u), & \text{if } h \geq 1, \\ \int_{\mathbf{R}} \int_y^x [f_{k-1, 0}(v - \tilde{X}_{i,j}(0, k) - a_{0,k}\xi_{i, j-k}) \\ - f_{k-1, 0}(v - \tilde{X}_{i,j}(0, k) - a_{0,k}u)] dv dG(u), & \text{if } h = 0. \end{cases}$$

Focussing on the $h \neq 0$ case, since the density $f_{h,k}$ is bounded and $\min(1, |x|) \leq |x|^\gamma$ for $0 < \gamma \leq 1$,

$$\begin{aligned}
|T_{i,j}(h, k)| & \leq K \int_{\mathbf{R}} \int_y^x C[(|a_{h,k}||\xi_{i-h, j-k} - u|)/K] \wedge 1 dv dG(u) \\
& \leq C \int_y^x [(|a_{h,k}|(1 + |\xi_{i-h, j-k}|)/K) \wedge 1] dv \\
& \leq C \int_y^x (|a_{h,k}|(1 + |\xi_{i-h, j-k}|)/K)^\gamma dv \\
& \leq C|x - y||a_{h,k}|^\gamma(1 + |\xi_{i-h, j-k}|^\gamma). \quad (25)
\end{aligned}$$

Using the notation in the proof of Lemma 3.1, for clarity we will continue the proof of tightness assuming that g , the density of $\xi_{0,0}$, is bounded and satisfies a Lipschitz condition. This implies the same for the density $f_{h,k}$ of $X_{i,j}(h,k)$ for all h and k . This simplifies the proof and allows us to use the fourth moment criterion given in [2] Theorem 12.3, refer to equation (12.51). However, tightness also holds under the weaker Assumption 2.1.2. The proof requires consideration of the cases $s < -\ell_0$ and $s \geq \ell_0$ separately for each moment bound. The reader is referred to Doukhan and Surgailis [9] for details.

Consider the term I_2 . Notice that

$$\begin{aligned}
E(V_{m,n}^2(s,t)|\mathcal{G}_{t-1,s-t+1}) &= \sum_{i,i'=1 \vee t}^m \sum_{j,j'=1 \vee (s-t)}^n E(T_{i,j}(i-t, j-(s-t)) \times \\
&\quad T_{i',j'}(i'-t, j'-(s-t)) | \mathcal{G}_{t-1,s-t+1}) \\
&\leq \sum_{i,i'=1 \vee t}^m \sum_{j,j'=1 \vee (s-t)}^n C|x-y|^2 |a_{i-t,j-(s-t)}|^\gamma \times \\
&\quad |a_{i'-t,j'-(s-t)}|^\gamma E((1+|\xi_{t,s-t}|^\gamma)^2 | \mathcal{G}_{t-1,s-t+1}) \\
&\leq C|x-y|^2 \left(\sum_{i=1}^m \sum_{j=1}^n |a_{i-t,j-(s-t)}|^\gamma \right)^2 \\
&\leq C|x-y|^2 \sum_{i=1}^m \sum_{j=1}^n |a_{i-t,j-(s-t)}|^\gamma \\
&= b_{m,n}^{(2)}(s,t),
\end{aligned}$$

where the third last line follows because $\xi_{t,s-t}$ is independent of $\mathcal{G}_{t-1,s-t+1}$ and $E|\xi_{0,0}|^{2\gamma} < \infty$. Note $\sum_s \sum_t b_{m,n}^{(2)}(s,t) \leq Cmn|x-y|^2$. Next, by orthogonality,

$$\begin{aligned}
&E \left(\sum_{s' < s} \sum_{t'=s'-n}^m V_{m,n}(s',t') + \sum_{t'=s-n}^{t-1} V_{m,n}(s,t') \right)^2 \\
&= \sum_{s' < s} \sum_{t'=s'-n}^m EV_{m,n}^2(s',t') + \sum_{t'=s-n}^{t-1} EV_{m,n}^2(s,t') \\
&\leq \sum_{s < m+n} \sum_{t=s-n}^m EV_{m,n}^2(s,t) \\
&\leq \sum_{s < m+n} \sum_{t=s-n}^m b_{m,n}^{(2)}(s,t) \\
&\leq C|x-y|^2(mn), \tag{26}
\end{aligned}$$

as $\sum_{s < m+n} \sum_{t=s-n}^m |a_{i-t,j-(s-t)}|^\gamma < \infty$. Thus $EI_2 \leq [C|x-y|^2(mn)]^2$.

For I_3 ,

$$\begin{aligned}
& E \left| \sum_{s' < s} \sum_{t' = s' - n}^m V_{m,n}(s', t') + \sum_{t' = s - n}^{t-1} V_{m,n}(s, t') \right| \\
& \leq E^{\frac{1}{2}} \left(\sum_{s' < s} \sum_{t' = s' - n}^m V_{m,n}(s', t') + \sum_{t' = s - n}^{t-1} V_{m,n}(s, t') \right)^2 \\
& \leq \left(\sum_{s < m+n} \sum_{t = s - n}^m E(V_{m,n}^2(s, t)) \right)^{\frac{1}{2}} \\
& \leq \left(\sum_{s < m+n} \sum_{t = s - n}^m b_{m,n}^{(2)}(s, t) \right)^{\frac{1}{2}} \\
& \leq (C|x - y|^2(mn))^{\frac{1}{2}}
\end{aligned}$$

by (26). Repeatedly using (25), since $E|\xi_{0,0}|^{3\gamma} < \infty$,

$$\begin{aligned}
& |E(V_{m,n}^3(s, t) | \mathcal{G}_{t-1, s-t+1})| \\
& = \left| \sum_{i_1, i_2, i_3 = 1 \vee t}^m \sum_{j_1, j_2, j_3 = 1 \vee (s-t)}^n E(T_{i_1, j_1}(i_1 - t, j_1 - (s - t)) \times \right. \\
& \quad \left. T_{i_2, j_2}(i_2 - t, j_2 - (s - t)) T_{i_3, j_3}(i_3 - t, j_3 - (s - t)) | \mathcal{G}_{t-1, s-t+1}) \right| \\
& \leq C|x - y|^3 \sum_{i_1, i_2, i_3 = 1 \vee t}^m \sum_{j_1, j_2, j_3 = 1 \vee (s-t)}^n |a_{i_1 - t, j_1 - (s-t)}|^\gamma \times \\
& \quad |a_{i_2 - t, j_2 - (s-t)}|^\gamma |a_{i_3 - t, j_3 - (s-t)}|^\gamma \\
& \leq C|x - y|^3 \sum_{i=1}^m \sum_{j=1}^n |a_{i-t, j-(s-t)}|^\gamma.
\end{aligned}$$

Thus $I_3 \leq C|x - y|^4(mn)^2$.

Finally for I_4 ,

$$\begin{aligned}
& |E(V_{m,n}^4(s, t) | \mathcal{G}_{t-1, s-t+1})| \\
& \leq \sum_{i_1, i_2, i_3, i_4 = 1 \vee t}^m \sum_{j_1, j_2, j_3, j_4 = 1 \vee (s-t)}^n |E \left(\prod_{a=1}^4 T_{i_a, j_a}(i_a - t, j_a - (s - t)) | \mathcal{G}_{t-1, s-t+1} \right)| \\
& \leq C|x - y|^4 \sum_{i=1}^m \sum_{j=1}^n |a_{i-t, j-(s-t)}|^\gamma.
\end{aligned}$$

Thus $I_4 \leq C|x - y|^4(mn)$. Substituting the bounds for I_2, I_3 and I_4 into (23) gives the bound required to establish tightness:

$$E[W_{m,n}(x) - W_{m,n}(y)]^4 \leq C|x - y|^4.$$

□

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