

# Confidence sets based on penalized maximum likelihood estimators in Gaussian regression\*

Benedikt M. Pötscher

*Department of Statistics, University of Vienna  
Universitätsstraße 5, 1010 Vienna, Austria  
e-mail: [benedikt.poetscher@univie.ac.at](mailto:benedikt.poetscher@univie.ac.at)*

and

Ulrike Schneider

*Institute for Mathematical Stochastics, University of Göttingen  
Goldschmidtstraße 7, 37077 Göttingen, Germany  
e-mail: [ulrike.schneider@math.uni-goettingen.de](mailto:ulrike.schneider@math.uni-goettingen.de)*

**Abstract:** Confidence intervals based on penalized maximum likelihood estimators such as the LASSO, adaptive LASSO, and hard-thresholding are analyzed. In the known-variance case, the finite-sample coverage properties of such intervals are determined and it is shown that symmetric intervals are the shortest. The length of the shortest intervals based on the hard-thresholding estimator is larger than the length of the shortest interval based on the adaptive LASSO, which is larger than the length of the shortest interval based on the LASSO, which in turn is larger than the standard interval based on the maximum likelihood estimator. In the case where the penalized estimators are tuned to possess the ‘sparsity property’, the intervals based on these estimators are larger than the standard interval by an order of magnitude. Furthermore, a simple asymptotic confidence interval construction in the ‘sparse’ case, that also applies to the smoothly clipped absolute deviation estimator, is discussed. The results for the known-variance case are shown to carry over to the unknown-variance case in an appropriate asymptotic sense.

**AMS 2000 subject classifications:** Primary 62F25; secondary 62C25, 62J07.

**Keywords and phrases:** Penalized maximum likelihood, penalized least squares, Lasso, adaptive Lasso, hard-thresholding, soft-thresholding, confidence set, coverage probability, sparsity, model selection.

Received October 2009.

## 1. Introduction

Recent years have seen an increased interest in penalized maximum likelihood (least squares) estimators. Prominent examples of such estimators are the LASSO

---

\*Earlier versions of this paper were circulated under the title “Confidence Sets Based on Penalized Maximum Likelihood Estimators”.

estimator (Tibshirani, 1996) and its variants like the adaptive LASSO (Zou, 2006), the Bridge estimators (Frank and Friedman, 1993), or the smoothly clipped absolute deviation (SCAD) estimator (Fan and Li, 2001). In linear regression models with orthogonal regressors, the hard- and soft-thresholding estimators can also be reformulated as penalized least squares estimators, with the soft-thresholding estimator then coinciding with the LASSO estimator.

The asymptotic distributional properties of penalized maximum likelihood (least squares) estimators have been studied in the literature, mostly in the context of a finite-dimensional linear regression model; see Knight and Fu (2000), Fan and Li (2001), and Zou (2006). Knight and Fu (2000) study the asymptotic distribution of Bridge estimators and, in particular, of the LASSO estimator. Their analysis concentrates on the case where the estimators are tuned in such a way as to perform conservative model selection, and their asymptotic framework allows for dependence of parameters on sample size. In contrast, Fan and Li (2001) for the SCAD estimator and Zou (2006) for the adaptive LASSO estimator concentrate on the case where the estimators are tuned to possess the ‘sparsity’ property. They show that, with such tuning, these estimators possess what has come to be known as the ‘oracle property’. However, their results are based on a fixed-parameter asymptotic framework only. Pötscher and Leeb (2009) and Pötscher and Schneider (2009) study the finite-sample distribution of the hard-thresholding, the soft-thresholding (LASSO), the SCAD, and the adaptive LASSO estimator under normal errors; they also obtain the asymptotic distributions of these estimators in a general ‘moving parameter’ asymptotic framework. The results obtained in these two papers clearly show that the distributions of the estimators studied are often highly non-normal and that the so-called ‘oracle property’ typically paints a misleading picture of the actual performance of the estimator. [In the wake of Fan and Li (2001) a considerable literature has sprung up establishing the so-called ‘oracle property’ for a variety of estimators. All these results are fixed-parameter asymptotic results only and can be very misleading. See Leeb and Pötscher (2008) and Pötscher (2009) for more discussion.]

A natural question now is what all these distributional results mean for confidence intervals that are based on penalized maximum likelihood (least squares) estimators. This is the question we address in the present paper in the context of a normal linear regression model with orthogonal regressors. In the known-variance case we obtain formulae for the finite-sample infimal coverage probabilities of fixed-width confidence intervals based on the following estimators: hard-thresholding, LASSO (soft-thresholding), and adaptive LASSO. We show that among those intervals the symmetric ones are the shortest, and we show that hard-thresholding leads to longer intervals than the adaptive LASSO, which in turn leads to longer intervals than the LASSO. All these intervals are longer than the standard confidence interval based on the maximum likelihood estimator, which is in line with Joshi (1969). In case the estimators are tuned to possess the ‘sparsity’ property, explicit asymptotic formulae for the length of the confidence intervals are furthermore obtained, showing that in this case the intervals based on the penalized maximum likelihood estimators are larger by an

order of magnitude than the standard maximum likelihood based interval. This refines, for the particular estimators considered, a general result for confidence sets based on ‘sparse’ estimators (Pötscher, 2009). Additionally, in the ‘sparsely’ tuned case a simple asymptotic construction of confidence intervals is provided that also applies to other penalized maximum likelihood estimators such as the SCAD estimator. Furthermore, we show how the results for the known-variance case carry over to the unknown-variance case in an asymptotic sense.

The plan of the paper is as follows: After introducing the model and estimators in Section 2, the known-variance case is treated in Section 3 whereas the unknown-variance case is dealt with in Section 4. All proofs as well as some technical lemmata are relegated to the Appendix.

## 2. The model and estimators

For a normal linear regression model with orthogonal regressors, distributional properties of penalized maximum likelihood (least squares) estimators with a separable penalty can be reduced to the case of a Gaussian location problem; for details see, e.g., Pötscher and Schneider (2009). Since we are only interested in confidence sets for individual components of the parameter vector in the regression that are based on such estimators, we shall hence suppose that the data  $y_1, \dots, y_n$  are independent identically distributed as  $N(\theta, \sigma^2)$ ,  $\theta \in \mathbb{R}$ ,  $0 < \sigma < \infty$ . [This entails no loss of generality in the known-variance case. In the unknown-variance case an explicit treatment of the orthogonal linear model would differ from the analysis in the present paper only in that the estimator  $\hat{\sigma}^2$  defined below would be replaced by the usual residual variance estimator from the least-squares regression; this would have no substantial effect on the results.] We shall be concerned with confidence sets for  $\theta$  based on penalized maximum likelihood estimators such as the hard-thresholding estimator, the LASSO (reducing to soft-thresholding in this setting), and the adaptive LASSO estimator. The hard-thresholding estimator  $\tilde{\theta}_H$  is given by

$$\tilde{\theta}_H := \tilde{\theta}_H(\eta_n) = \bar{y} \mathbf{1}(|\bar{y}| > \hat{\sigma} \eta_n)$$

where the threshold  $\eta_n$  is a positive real number,  $\bar{y}$  denotes the maximum likelihood estimator, i.e., the arithmetic mean of the data, and  $\hat{\sigma}^2 = (n - 1)^{-1} \sum_{i=1}^n (y_i - \bar{y})^2$ . Also define the infeasible estimator

$$\hat{\theta}_H := \hat{\theta}_H(\eta_n) = \bar{y} \mathbf{1}(|\bar{y}| > \sigma \eta_n)$$

which uses the value of  $\sigma$ . The LASSO (or soft-thresholding) estimator  $\tilde{\theta}_S$  is given by

$$\tilde{\theta}_S := \tilde{\theta}_S(\eta_n) = \text{sign}(\bar{y})(|\bar{y}| - \hat{\sigma} \eta_n)_+$$

and its infeasible version by

$$\hat{\theta}_S := \hat{\theta}_S(\eta_n) = \text{sign}(\bar{y})(|\bar{y}| - \sigma \eta_n)_+.$$

Here  $\text{sign}(x)$  is defined as  $-1$ ,  $0$ , and  $1$  in case  $x < 0$ ,  $x = 0$ , and  $x > 0$ , respectively, and  $z_+$  is shorthand for  $\max\{z, 0\}$ . The adaptive LASSO estimator  $\tilde{\theta}_A$  in this simple model is given by

$$\tilde{\theta}_A := \tilde{\theta}_A(\eta_n) = \bar{y}(1 - \hat{\sigma}^2 \eta_n^2 / \bar{y}^2)_+ = \begin{cases} 0 & \text{if } |\bar{y}| \leq \hat{\sigma} \eta_n \\ \bar{y} - \hat{\sigma}^2 \eta_n^2 / \bar{y} & \text{if } |\bar{y}| > \hat{\sigma} \eta_n, \end{cases}$$

and its infeasible counterpart by

$$\hat{\theta}_A := \hat{\theta}_A(\eta_n) = \bar{y}(1 - \sigma^2 \eta_n^2 / \bar{y}^2)_+ = \begin{cases} 0 & \text{if } |\bar{y}| \leq \sigma \eta_n \\ \bar{y} - \sigma^2 \eta_n^2 / \bar{y} & \text{if } |\bar{y}| > \sigma \eta_n. \end{cases}$$

It coincides with the nonnegative Garotte in this simple model. For the feasible estimators we always need to assume  $n \geq 2$ , whereas for the infeasible estimators also  $n = 1$  is admissible.

Note that  $\eta_n$  plays the rôle of a tuning parameter and it is most natural to let the estimators depend on the tuning parameter only via  $\sigma \eta_n$  and  $\hat{\sigma} \eta_n$ , respectively, in order to take account of the scale of the data. This makes the estimators mentioned above scale equivariant. We shall often suppress dependence of the estimators on  $\eta_n$  in the notation. In the following let  $P_{n,\theta,\sigma}$  denote the distribution of the sample when  $\theta$  and  $\sigma$  are the true parameters. Furthermore, let  $\Phi$  denote the standard normal cumulative distribution function.

We also note the following obvious fact: Since hard- and soft-thresholding operate in a coordinatewise fashion, the results given below also apply mutatis mutandis to linear regressions with non-orthogonal regressors. Of course, the soft-thresholding estimator then no longer coincides with the LASSO estimator. We refrain from spelling out details.

### 3. Confidence intervals: Known-variance case

In this section we consider the case where the variance  $\sigma^2$  is known,  $n \geq 1$  holds, and we are interested in the finite-sample coverage properties of intervals of the form  $[\hat{\theta} - \sigma a_n, \hat{\theta} + \sigma b_n]$  where  $a_n$  and  $b_n$  are nonnegative real numbers and  $\hat{\theta}$  stands for any one of the estimators  $\hat{\theta}_H = \hat{\theta}_H(\eta_n)$ ,  $\hat{\theta}_S = \hat{\theta}_S(\eta_n)$ , or  $\hat{\theta}_A = \hat{\theta}_A(\eta_n)$ . We shall also consider one-sided intervals  $(-\infty, \hat{\theta} + \sigma c_n]$  and  $[\hat{\theta} - \sigma c_n, \infty)$  with  $0 \leq c_n < \infty$ . Let  $p_n(\theta; \sigma, \eta_n, a_n, b_n) = P_{n,\theta,\sigma}(\theta \in [\hat{\theta} - \sigma a_n, \hat{\theta} + \sigma b_n])$  denote the coverage probability. Due to the above-noted scale equivariance of the estimator  $\hat{\theta}$ , it is obvious that

$$p_n(\theta; \sigma, \eta_n, a_n, b_n) = p_n(\theta/\sigma; 1, \eta_n, a_n, b_n)$$

holds, and the same is true for the one-sided intervals. In particular, it follows that the infimal coverage probabilities  $\inf_{\theta \in \mathbb{R}} p_n(\theta; \sigma, \eta_n, a_n, b_n)$  do not depend on  $\sigma$ . Therefore, we shall assume without loss of generality that  $\sigma = 1$  for the remainder of this section and we shall write  $P_{n,\theta}$  for  $P_{n,\theta,1}$ .

### 3.1. Infimal coverage probabilities in finite samples

We begin with soft-thresholding. Let  $C_{S,n}$  denote the interval  $[\hat{\theta}_S - a_n, \hat{\theta}_S + b_n]$ . We first determine the infimum of the coverage probability  $p_{S,n}(\theta) := P_{n,\theta}(\theta \in C_{S,n})$  of this interval.

**Proposition 1.** *For every  $n \geq 1$ , the infimal coverage probability of the interval  $C_{S,n}$  is given by*

$$\inf_{\theta \in \mathbb{R}} p_{S,n}(\theta) = \begin{cases} \Phi(n^{1/2}(a_n - \eta_n)) - \Phi(n^{1/2}(-b_n - \eta_n)) & \text{if } a_n \leq b_n \\ \Phi(n^{1/2}(b_n - \eta_n)) - \Phi(n^{1/2}(-a_n - \eta_n)) & \text{if } a_n > b_n. \end{cases} \quad (3.1)$$

As a point of interest we note that  $p_{S,n}(\theta)$  is a piecewise constant function with jumps at  $\theta = -a_n$  and  $\theta = b_n$ .

Next we turn to hard-thresholding. Let  $C_{H,n}$  denote the interval  $[\hat{\theta}_H - a_n, \hat{\theta}_H + b_n]$ . The infimum of the coverage probability  $p_{H,n}(\theta) := P_{n,\theta}(\theta \in C_{H,n})$  of this interval has been obtained in Proposition 3.1 in Pötscher (2009), which we repeat for convenience.

**Proposition 2.** *For every  $n \geq 1$ , the infimal coverage probability of the interval  $C_{H,n}$  is given by*

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} p_{H,n}(\theta) & (3.2) \\ = & \begin{cases} \Phi(n^{1/2}(a_n - \eta_n)) - \Phi(-n^{1/2}b_n) & \text{if } \eta_n \leq a_n + b_n \text{ and } a_n \leq b_n \\ \Phi(n^{1/2}(b_n - \eta_n)) - \Phi(-n^{1/2}a_n) & \text{if } \eta_n \leq a_n + b_n \text{ and } a_n > b_n \\ 0 & \text{if } \eta_n > a_n + b_n. \end{cases} \end{aligned}$$

For later use we observe that the interval  $C_{H,n}$  has positive infimal coverage probability if and only if the length of the interval  $a_n + b_n$  is larger than  $\eta_n$ . As a point of interest we also note that the coverage probability  $p_{H,n}(\theta)$  is discontinuous (with discontinuity points at  $\theta = -a_n$  and  $\theta = b_n$ ). Furthermore, as discussed in Pötscher (2009), the infimum in (3.2) is attained if  $\eta_n > a_n + b_n$ , but not in case  $\eta_n \leq a_n + b_n$ .

Finally, we consider the adaptive LASSO. Let  $C_{A,n}$  denote the interval  $[\hat{\theta}_A - a_n, \hat{\theta}_A + b_n]$ . The infimum of the coverage probability  $p_{A,n}(\theta) := P_{n,\theta}(\theta \in C_{A,n})$  of this interval is given next.

**Proposition 3.** *For every  $n \geq 1$ , the infimal coverage probability of  $C_{A,n}$  is given by*

$$\inf_{\theta \in \mathbb{R}} p_{A,n}(\theta) = \Phi(n^{1/2}(a_n - \eta_n)) - \Phi\left(n^{1/2}\left((a_n - b_n)/2 - \sqrt{((a_n + b_n)/2)^2 + \eta_n^2}\right)\right)$$

if  $a_n \leq b_n$ , and by

$$\inf_{\theta \in \mathbb{R}} p_{A,n}(\theta) = \Phi(n^{1/2}(b_n - \eta_n)) - \Phi\left(n^{1/2}\left((b_n - a_n)/2 - \sqrt{((a_n + b_n)/2)^2 + \eta_n^2}\right)\right)$$

if  $a_n > b_n$ .

We note that  $p_{A,n}$  is continuous except at  $\theta = b_n$  and  $\theta = -a_n$  and that the infimum of  $p_{A,n}$  is not attained which can be seen from a simple refinement of the proof of Proposition 3.

**Remark 4.** (i) If we consider the open interval  $C_{S,n}^o = (\hat{\theta}_S - a_n, \hat{\theta}_S + b_n)$  the formula for the coverage probability becomes

$$\begin{aligned} P_{n,\theta}(\theta \in C_{S,n}^o) &= [\Phi(n^{1/2}(a_n - \eta_n)) - \Phi(n^{1/2}(-b_n - \eta_n))]\mathbf{1}(\theta \leq -a_n) \\ &+ [\Phi(n^{1/2}(a_n + \eta_n)) - \Phi(n^{1/2}(-b_n - \eta_n))]\mathbf{1}(-a_n < \theta < b_n) \\ &+ [\Phi(n^{1/2}(a_n + \eta_n)) - \Phi(n^{1/2}(-b_n + \eta_n))]\mathbf{1}(b_n \leq \theta). \end{aligned}$$

As a consequence, the infimal coverage probability of  $C_{S,n}^o$  is again given by (3.1). A fortiori, the half-open intervals  $(\hat{\theta}_n - a_n, \hat{\theta}_n + b_n]$  and  $[\hat{\theta}_n - a_n, \hat{\theta}_n + b_n)$  then also have infimal coverage probability given by (3.1).

(ii) For the open interval  $C_{H,n}^o = (\hat{\theta}_H - a_n, \hat{\theta}_H + b_n)$  the coverage probability satisfies

$$\begin{aligned} P_{n,\theta}(\theta \in C_{H,n}^o) &= P_{n,\theta}(\theta \in C_{H,n}) \\ &- [\mathbf{1}(\theta = b_n) + \mathbf{1}(\theta = -a_n)][\Phi(n^{1/2}(-\theta + \eta_n)) - \Phi(n^{1/2}(-\theta - \eta_n))]. \end{aligned}$$

Inspection of the proof of Proposition 3.1 in Pötscher (2009) then shows that  $C_{H,n}^o$  has the same infimal coverage probability as  $C_{H,n}$ . However, now the infimum is always a minimum. Furthermore, the half-open intervals  $(\hat{\theta}_H - a_n, \hat{\theta}_H + b_n]$  and  $[\hat{\theta}_H - a_n, \hat{\theta}_H + b_n)$  then a fortiori have infimal coverage probability given by (3.2); for these intervals the infimum is attained if  $\eta_n > a_n + b_n$ , but not necessarily if  $\eta_n \leq a_n + b_n$ .

(iii) If  $C_{A,n}^o$  denotes the open interval  $(\hat{\theta}_A - a_n, \hat{\theta}_A + b_n)$ , the formula for the coverage probability becomes

$$\begin{aligned} P_{n,\theta}(\theta \in C_{A,n}^o) &= \\ &\begin{cases} \Phi(n^{1/2}\gamma^{(-)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(-)}(\theta, b_n)) & \text{if } \theta \leq -a_n \\ \Phi(n^{1/2}\gamma^{(+)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(-)}(\theta, b_n)) & \text{if } -a_n < \theta < b_n \\ \Phi(n^{1/2}\gamma^{(+)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(+)}(\theta, b_n)) & \text{if } \theta \geq b_n, \end{cases} \end{aligned}$$

where  $\gamma^{(-)}$  and  $\gamma^{(+)}$  are defined in (A.3) and (A.4) in the Appendix. Again the coverage probability is continuous except at  $\theta = b_n$  and  $\theta = -a_n$  (and is continuous everywhere in the trivial case  $a_n = b_n = 0$ ). It is now easy to see that the infimal coverage probability of  $C_{A,n}^o$  coincides with the infimal coverage probability of the closed interval  $C_{A,n}$ , the infimum of the coverage probability of  $C_{A,n}^o$  now always being a minimum. Furthermore, the half-open intervals  $(\hat{\theta}_A - a_n, \hat{\theta}_A + b_n]$  and  $[\hat{\theta}_A - a_n, \hat{\theta}_A + b_n)$  a fortiori have the same infimal coverage probability as  $C_{A,n}$  and  $C_{A,n}^o$ .

(iv) The one-sided intervals  $(-\infty, \hat{\theta}_S + c_n]$ ,  $(-\infty, \hat{\theta}_S + c_n)$ ,  $[\hat{\theta}_S - c_n, \infty)$ ,  $(\hat{\theta}_S - c_n, \infty)$ ,  $(-\infty, \hat{\theta}_H + c_n]$ ,  $(-\infty, \hat{\theta}_H + c_n)$ ,  $[\hat{\theta}_H - c_n, \infty)$ ,  $(\hat{\theta}_H - c_n, \infty)$ ,  $(-\infty, \hat{\theta}_A +$

$c_n]$ ,  $(-\infty, \hat{\theta}_A + c_n)$ ,  $(\hat{\theta}_A - c_n, \infty)$ , and  $[\hat{\theta}_A - c_n, \infty)$ , with  $c_n$  a nonnegative real number, have infimal coverage probability  $\Phi(n^{1/2}(c_n - \eta_n))$ . This is easy to see for soft-thresholding, follows from the reasoning in Pötscher (2009) for hard-thresholding, and for the adaptive LASSO follows by similar, but simpler, reasoning as in the proof of Proposition 3.

### 3.2. Symmetric intervals are shortest

For the two-sided confidence sets considered above, we next show that given a prescribed infimal coverage probability the symmetric intervals are shortest. We then show that these shortest intervals are longer than the standard interval based on the maximum likelihood estimator and quantify the excess length of these intervals.

**Theorem 5.** For every  $n \geq 1$  and every  $\delta$  satisfying  $0 < \delta < 1$  we have:

(a) Among all intervals  $C_{S,n}$  with infimal coverage probability not less than  $\delta$  there is a unique shortest interval  $C_{S,n}^* = [\hat{\theta}_S - a_{n,S}^*, \hat{\theta}_S + b_{n,S}^*]$  characterized by  $a_{n,S}^* = b_{n,S}^*$  with  $a_{n,S}^*$  being the unique solution of

$$\Phi(n^{1/2}(a_n - \eta_n)) - \Phi(n^{1/2}(-a_n - \eta_n)) = \delta. \tag{3.3}$$

The interval  $C_{S,n}^*$  has infimal coverage probability equal to  $\delta$  and  $a_{n,S}^*$  is positive.

(b) Among all intervals  $C_{H,n}$  with infimal coverage probability not less than  $\delta$  there is a unique shortest interval  $C_{H,n}^* = [\hat{\theta}_H - a_{n,H}^*, \hat{\theta}_H + b_{n,H}^*]$  characterized by  $a_{n,H}^* = b_{n,H}^*$  with  $a_{n,H}^*$  being the unique solution of

$$\Phi(n^{1/2}(a_n - \eta_n)) - \Phi(-n^{1/2}a_n) = \delta. \tag{3.4}$$

The interval  $C_{H,n}^*$  has infimal coverage probability equal to  $\delta$  and  $a_{n,H}^*$  satisfies  $a_{n,H}^* > \eta_n/2$ .

(c) Among all intervals  $C_{A,n}$  with infimal coverage probability not less than  $\delta$  there is a unique shortest interval  $C_{A,n}^* = [\hat{\theta}_A - a_{n,A}^*, \hat{\theta}_A + b_{n,A}^*]$  characterized by  $a_{n,A}^* = b_{n,A}^*$  with  $a_{n,A}^*$  being the unique solution of

$$\Phi(n^{1/2}(a_n - \eta_n)) - \Phi\left(-n^{1/2}\sqrt{a_n^2 + \eta_n^2}\right) = \delta. \tag{3.5}$$

The interval  $C_{A,n}^*$  has infimal coverage probability equal to  $\delta$  and  $a_{n,A}^*$  is positive.

In the statistically uninteresting case  $\delta = 0$  the interval with  $a_n = b_n = 0$  is the unique shortest interval in all three cases. However, for the case of the hard-thresholding estimator also any interval with  $a_n = b_n$  and  $a_n \leq \eta_n/2$  has infimal coverage probability equal to zero.

Given that the distributions of the estimation errors  $\hat{\theta}_S - \theta$ ,  $\hat{\theta}_H - \theta$ , and  $\hat{\theta}_A - \theta$  are not symmetric (see Pötscher and Leeb, 2009 and Pötscher and Schneider, 2009), it may seem surprising at first glance that the shortest confidence intervals are symmetric. Some intuition for this phenomenon can be gained on the grounds

that the distributions of the estimation errors under  $\theta = \tau$  and  $\theta = -\tau$  are mirror-images of one another.

The above theorem shows that given a prespecified  $\delta$  ( $0 < \delta < 1$ ), the shortest confidence set with infimal coverage probability equal to  $\delta$  based on the soft-thresholding (LASSO) estimator is shorter than the corresponding interval based on the adaptive LASSO estimator, which in turn is shorter than the corresponding interval based on the hard-thresholding estimator. All three intervals are longer than the corresponding standard confidence interval based on the maximum likelihood estimator. That is,

$$a_{n,H}^* > a_{n,A}^* > a_{n,S}^* > n^{-1/2}\Phi^{-1}((1 + \delta)/2).$$

Figure 1 below shows  $n^{1/2}$  times the half-length of the shortest  $\delta$ -level confidence intervals based on hard-thresholding, adaptive LASSO, soft-thresholding, and the maximum likelihood estimator, respectively, as a function of  $n^{1/2}\eta_n$  for various values of  $\delta$ . The graphs illustrate that the intervals based on hard-thresholding, adaptive LASSO, and soft-thresholding substantially exceed the length of the maximum likelihood based interval except if  $n^{1/2}\eta_n$  is very small. For large values of  $n^{1/2}\eta_n$  the graphs suggest a linear increase in the length of the intervals based on the penalized estimators. This is formally confirmed in Section 3.2.1 below.

### 3.2.1. Asymptotic behavior of the length

It is well-known that as  $n \rightarrow \infty$  two different regimes for the tuning parameter  $\eta_n$  can be distinguished. In the first regime  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow e$ ,  $0 < e < \infty$ . This choice of tuning parameter leads to estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , and  $\hat{\theta}_A$  that perform conservative model selection. In the second regime  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow \infty$ , leading to estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , and  $\hat{\theta}_A$  that perform consistent model selection (also known as the ‘sparsity property’); that is, with probability approaching 1, the estimators are exactly zero if the true value  $\theta = 0$ , and they are different from zero if  $\theta \neq 0$ . See Pötscher and Leeb (2009) and Pötscher and Schneider (2009) for a detailed discussion. We now discuss the asymptotic behavior, under the two regimes, of the half-length  $a_{n,S}^*$ ,  $a_{n,H}^*$ , and  $a_{n,A}^*$  of the shortest intervals  $C_{S,n}^*$ ,  $C_{H,n}^*$ , and  $C_{A,n}^*$  with a fixed infimal coverage probability  $\delta$ ,  $0 < \delta < 1$ .

If  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow e$ ,  $0 < e < \infty$ , then it follows immediately from Theorem 5 that  $n^{1/2}a_{n,S}^*$ ,  $n^{1/2}a_{n,H}^*$ , and  $n^{1/2}a_{n,A}^*$  converge to the unique solutions of

$$\Phi(a - e) - \Phi(-a - e) = \delta, \tag{3.6}$$

$$\Phi(a - e) - \Phi(-a) = \delta, \tag{3.7}$$

and

$$\Phi\left(\sqrt{a^2 + e^2}\right) - \Phi(-a + e) = \delta, \tag{3.8}$$

respectively. [Actually, this is even true if  $e = 0$ .] Hence, while  $a_{n,H}^*$ ,  $a_{n,A}^*$ , and  $a_{n,S}^*$  are larger than the half-length  $n^{-1/2}\Phi^{-1}((1 + \delta)/2)$  of the standard interval, they are of the same order  $n^{-1/2}$ .

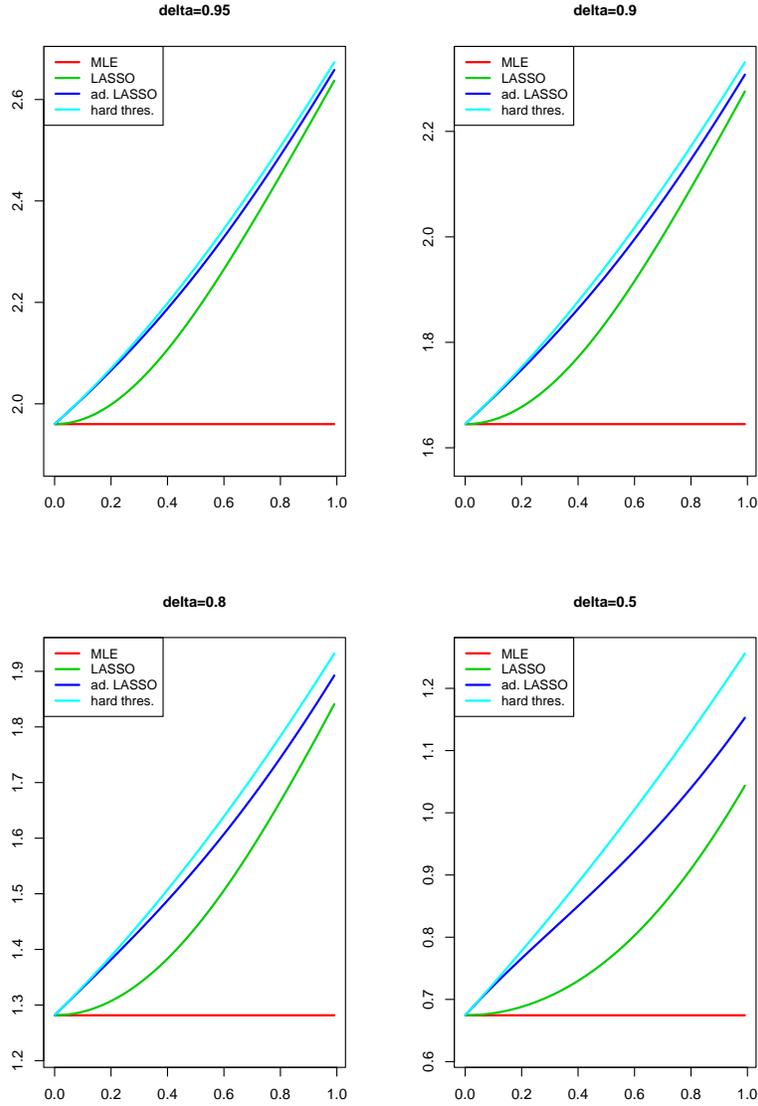


FIG 1.  $n^{1/2}a_{n,H}^*$ ,  $n^{1/2}a_{n,A}^*$ ,  $n^{1/2}a_{n,S}^*$  as a function of  $n^{1/2}\eta_n$  for coverage probabilities  $\delta = 0.5, 0.8, 0.9, 0.95$ . The horizontal line at height  $\Phi^{-1}((1 + \delta)/2)$  indicates  $n^{1/2}$  times the half-length of the standard maximum likelihood based interval.

The situation is different, however, if  $\eta_n \rightarrow 0$  but  $n^{1/2}\eta_n \rightarrow \infty$ . In this case Theorem 5 shows that

$$\Phi(n^{1/2}(a_{n,S}^* - \eta_n)) \rightarrow \delta$$

since  $n^{1/2}(-a_{n,S}^* - \eta_n) \leq -n^{1/2}\eta_n \rightarrow -\infty$ . In other words,

$$a_{n,S}^* = \eta_n + n^{-1/2}\Phi^{-1}(\delta) + o(n^{-1/2}). \tag{3.9}$$

Similarly, noting that  $n^{1/2}a_{n,H}^* > n^{1/2}\eta_n/2 \rightarrow \infty$ , we get

$$a_{n,H}^* = \eta_n + n^{-1/2}\Phi^{-1}(\delta) + o(n^{-1/2}); \tag{3.10}$$

and since  $n^{1/2}\sqrt{a_n^2 + \eta_n^2} \geq n^{1/2}\eta_n \rightarrow \infty$  we obtain

$$a_{n,A}^* = \eta_n + n^{-1/2}\Phi^{-1}(\delta) + o(n^{-1/2}). \tag{3.11}$$

[Actually, the condition  $\eta_n \rightarrow 0$  has not been used in the derivation of (3.9)–(3.11).] Hence, the intervals  $C_{S,n}^*$ ,  $C_{H,n}^*$ , and  $C_{A,n}^*$  are asymptotically of the same length. They are also longer than the standard interval by an order of magnitude: the ratio of each of  $a_{n,S}^*$  ( $a_{n,H}^*$ ,  $a_{n,A}^*$ , respectively) to the half-length of the standard interval is  $n^{1/2}\eta_n$ , which diverges to infinity. Hence, when the estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , and  $\hat{\theta}_A$  are tuned to possess the ‘sparsity property’, the corresponding confidence sets become very large. For the particular intervals considered here this is a refinement of a general result in Pötscher (2009) for confidence sets based on arbitrary estimators possessing the ‘sparsity property’. [We note that the sparsely tuned hard-thresholding estimator or the sparsely tuned adaptive LASSO (under an additional condition on  $\eta_n$ ) are known to possess the so-called ‘oracle property’. In light of the ‘oracle property’ it is sometimes argued in the literature that valid confidence intervals based on these estimators with length proportional to  $n^{-1/2}$  can be obtained. However, in light of the above discussion such intervals necessarily have infimal coverage probability that converges to zero and thus are not valid. This once more shows that *fixed-parameter* asymptotic results like the ‘oracle’ property can be dangerously misleading.]

### 3.3. A simple asymptotic confidence interval

The results for the finite-sample confidence intervals given in Section 3.1 required a detailed case by case analysis based on the finite-sample distribution of the estimator on which the interval is based. If the estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , and  $\hat{\theta}_A$  are tuned to possess the ‘sparsity property’, i.e., if the tuning parameter satisfies  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow \infty$ , a simple asymptotic confidence interval construction relying on asymptotic results obtained in Pötscher and Leeb (2009) and Pötscher and Schneider (2009) is possible as shown below. An advantage of this construction is that it easily extends to other estimators like the smoothly clipped absolute deviation (SCAD) estimator when tuned to possess the ‘sparsity property’.

As shown in Pötscher and Leeb (2009) and Pötscher and Schneider (2009), the uniform rate of consistency of the ‘sparsely’ tuned estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , and  $\hat{\theta}_A$  is not  $n^{1/2}$ , but only  $\eta_n^{-1}$ ; furthermore, the limiting distributions of these estimators under the appropriate  $\eta_n^{-1}$ -scaling and under a moving-parameter asymptotic framework are always concentrated on the interval  $[-1, 1]$ . These facts can be used to obtain the following result.

**Proposition 6.** *Suppose  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow \infty$ . Let  $\hat{\theta}$  stand for any of the estimators  $\hat{\theta}_S(\eta_n)$ ,  $\hat{\theta}_H(\eta_n)$ , or  $\hat{\theta}_A(\eta_n)$ . Let  $d$  be a real number, and define the*

interval  $D_n = [\hat{\theta} - d\eta_n, \hat{\theta} + d\eta_n]$ . If  $d > 1$ , the interval  $D_n$  has infimal coverage probability converging to 1, i.e.,

$$\lim_{n \rightarrow \infty} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in D_n) = 1.$$

If  $d < 1$ ,

$$\lim_{n \rightarrow \infty} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in D_n) = 0.$$

The asymptotic distributional results in the above proposition do not provide information on the case  $d = 1$ . However, from the finite-sample results in Section 3.1 we see that in this case the infimal coverage probability of  $D_n$  converges to 1/2.

Since the interval  $D_n$  for  $d > 1$  has asymptotic infimal coverage probability equal to one, one may wonder how much cruder this interval is compared to the finite-sample intervals  $C_{S,n}^*$ ,  $C_{H,n}^*$ , and  $C_{A,n}^*$  constructed in Section 3.2, which have infimal coverage probability equal to a prespecified level  $\delta$ ,  $0 < \delta < 1$ : The ratio of the half-length of  $D_n$  to the half-length of the corresponding interval  $C_{S,n}^*$ ,  $C_{H,n}^*$ , and  $C_{A,n}^*$  is

$$d(1 + O(n^{-1/2}\eta_n^{-1})) = d(1 + o(1))$$

as can be seen from equations (3.9), (3.10), and (3.11). Since  $d$  can be chosen arbitrarily close to one, this ratio can be made arbitrarily close to one. This may sound somewhat strange, since we are comparing an interval with asymptotic infimal coverage probability 1 with the shortest finite-sample confidence intervals that have a fixed infimal coverage probability  $\delta$  less than 1. The reason for this phenomenon is that, in the relevant moving-parameter asymptotic framework, the distribution of  $\hat{\theta} - \theta$  is made up of a bias-component which in the worst case is of the order  $\eta_n$  and a random component which is of the order  $n^{-1/2}$ . Since  $\eta_n \rightarrow 0$  and  $n^{1/2}\eta_n \rightarrow \infty$ , the deterministic bias-component dominates the random component. This can also be gleaned from equations (3.9), (3.10), and (3.11), where the level  $\delta$  enters the formula for the half-length only in the lower order term.

We note that using Theorem 19 in Pötscher and Leeb (2009) the same proof immediately shows that Proposition 6 also holds for the smoothly clipped absolute deviation (SCAD) estimator when tuned to possess the ‘sparsity property’. In fact, the argument in the proof of the above proposition can be applied to a large class of post-model-selection estimators based on a consistent model selection procedure.

**Remark 7.** (i) Suppose  $D'_n = [\hat{\theta} - d_1\eta_n, \hat{\theta} + d_2\eta_n]$  where  $\hat{\theta}$  stands for any of the estimators  $\hat{\theta}_S$ ,  $\hat{\theta}_H$ , or  $\hat{\theta}_A$ . If  $\min(d_1, d_2) > 1$ , then the limit of the infimal coverage probability of  $D'_n$  is 1; if  $\max(d_1, d_2) < 1$  then this limit is zero. This follows immediately from an inspection of the proof of Proposition 6.

(ii) Proposition 6 also remains correct if  $D_n$  is replaced by the corresponding open interval. A similar comment applies to the open version of  $D'_n$ .

#### 4. Confidence intervals: Unknown-variance case

In this section we consider the case where the variance  $\sigma^2$  is unknown,  $n \geq 2$ , and we are interested in the coverage properties of intervals of the form  $[\tilde{\theta} - \hat{\sigma}a_n, \tilde{\theta} + \hat{\sigma}a_n]$  where  $a_n$  is a nonnegative real number and  $\tilde{\theta}$  stands for any one of the estimators  $\tilde{\theta}_H = \tilde{\theta}_H(\eta_n)$ ,  $\tilde{\theta}_S = \tilde{\theta}_S(\eta_n)$ , or  $\tilde{\theta}_A = \tilde{\theta}_A(\eta_n)$ . For brevity we only consider symmetric intervals. A similar argument as in the known-variance case shows that we can assume without loss of generality that  $\sigma = 1$ , and we shall do so in the sequel; in particular, this argument shows that the infimum with respect to  $\theta$  of the coverage probability does not depend on  $\sigma$ .

##### 4.1. Soft-thresholding

Consider the interval  $E_{S,n} = [\tilde{\theta}_S - \hat{\sigma}a_n, \tilde{\theta}_S + \hat{\sigma}a_n]$  where  $a_n$  is a nonnegative real number and  $\tilde{\theta}_S = \tilde{\theta}_S(\eta_n)$ . We then have

$$P_{n,\theta}(\theta \in E_{S,n}) = \int_0^\infty P_{n,\theta}(\theta \in E_{S,n} | \hat{\sigma} = s) h_n(s) ds$$

where  $h_n$  is the density of  $\hat{\sigma}$ , i.e.,  $h_n$  is the density of the square root of a chi-square distributed random variable with  $n - 1$  degrees of freedom divided by the degrees of freedom. In view of independence of  $\hat{\sigma}$  and  $\bar{y}$  we obtain the following representation of the finite-sample coverage probability

$$\begin{aligned} P_{n,\theta}(\theta \in E_{S,n}) &= \int_0^\infty P_{n,\theta}(\theta \in [\hat{\theta}_S(s\eta_n) - sa_n, \hat{\theta}_S(s\eta_n) + sa_n]) h_n(s) ds \\ &= \int_0^\infty p_{S,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \end{aligned} \tag{4.1}$$

where  $p_{S,n}$  is given in (A.1) in the Appendix.

We next determine the infimal coverage probability of  $E_{S,n}$  in finite samples: It follows from (A.1), the dominated convergence theorem, and symmetry of the standard normal distribution that

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{S,n}) &\leq \lim_{\theta \rightarrow \infty} \int_0^\infty p_{S,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &= \int_0^\infty \lim_{\theta \rightarrow \infty} p_{S,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &= \int_0^\infty [\Phi(n^{1/2}s(a_n - \eta_n)) - \Phi(n^{1/2}s(-a_n - \eta_n))] h_n(s) ds \\ &= T_{n-1}(n^{1/2}(a_n - \eta_n)) - T_{n-1}(n^{1/2}(-a_n - \eta_n)), \end{aligned} \tag{4.2}$$

where  $T_{n-1}$  is the cdf of a Student  $t$ -distribution with  $n - 1$  degrees of freedom. Furthermore, (3.1) shows that

$$p_{S,n}(\theta; 1, s\eta_n, sa_n, sa_n) \geq \Phi(n^{1/2}s(a_n - \eta_n)) - \Phi(n^{1/2}s(-a_n - \eta_n))$$

holds and whence we obtain from (4.1) and (4.2) the following expression for the infimal coverage probability of  $E_{S,n}$ :

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{S,n}) = T_{n-1}(n^{1/2}(a_n - \eta_n)) - T_{n-1}(n^{1/2}(-a_n - \eta_n)) \quad (4.3)$$

for every  $n \geq 2$ . Remark 4 shows that the same relation is true for the corresponding open and half-open intervals.

Relation (4.3) shows the following: suppose  $1/2 \leq \delta < 1$  and  $a_{n,S}^*$  solves (3.3), i.e., the corresponding interval  $C_{S,n}^*$  has infimal coverage probability equal to  $\delta$ . Let  $a_{n,S}^{**}$  be the (unique) solution to

$$T_{n-1}(n^{1/2}(a_n - \eta_n)) - T_{n-1}(n^{1/2}(-a_n - \eta_n)) = \delta,$$

i.e., the corresponding interval  $E_{S,n}^{**} = [\tilde{\theta}_S - \hat{\sigma}a_{n,S}^{**}, \tilde{\theta}_S + \hat{\sigma}a_{n,S}^{**}]$  has infimal coverage probability equal to  $\delta$ . Then  $a_{n,S}^{**} \geq a_{n,S}^*$  holds in view of Lemma 14 in the Appendix. I.e., given the same infimal coverage probability  $\delta \geq 1/2$ , the expected length of the interval  $E_{S,n}^{**}$  based on  $\tilde{\theta}_S$  is not smaller than the length of the interval  $C_{S,n}^*$  based on  $\hat{\theta}_S$ .

Since  $\|\Phi - T_{n-1}\|_\infty = \sup_{x \in \mathbb{R}} |\Phi(x) - T_{n-1}(x)| \rightarrow 0$  for  $n \rightarrow \infty$  holds by Polya's theorem, the following result is an immediate consequence of (4.3), Proposition 1, and Remark 4.

**Theorem 8.** *For every sequence  $a_n$  of nonnegative real numbers we have with  $E_{S,n} = [\tilde{\theta}_S - \hat{\sigma}a_n, \tilde{\theta}_S + \hat{\sigma}a_n]$  and  $C_{S,n} = [\hat{\theta}_S - a_n, \hat{\theta}_S + a_n]$  that*

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{S,n}) - \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{S,n}) \rightarrow 0$$

as  $n \rightarrow \infty$ . The analogous results hold for the corresponding open and half-open intervals.

We discuss this theorem together with the parallel results for hard-thresholding and adaptive LASSO based intervals in Section 4.4.

### 4.2. Hard-thresholding

Consider the interval  $E_{H,n} = [\tilde{\theta}_H - \hat{\sigma}a_n, \tilde{\theta}_H + \hat{\sigma}a_n]$  where  $a_n$  is a nonnegative real number and  $\tilde{\theta}_H = \tilde{\theta}_H(\eta_n)$ . We then have analogously as in the preceding subsection that

$$P_{n,\theta}(\theta \in E_{H,n}) = \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds.$$

Note that  $p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n)$  is symmetric in  $\theta$  and for  $\theta \geq 0$  is given by (see Pötscher (2009))

$$\begin{aligned} & p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) \\ = & \left\{ \Phi(n^{1/2}(-\theta + s\eta_n)) - \Phi(n^{1/2}(-\theta - s\eta_n)) \right\} \mathbf{1}(0 \leq \theta \leq sa_n) \\ & + \max \left[ 0, \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta + s\eta_n)) \right] \mathbf{1}(sa_n < \theta \leq s\eta_n + sa_n) \\ & + \left\{ \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n) \right\} \mathbf{1}(s\eta_n + sa_n < \theta) \end{aligned}$$

in case  $\eta_n > 2a_n$ , by

$$\begin{aligned} & p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) \\ &= \left\{ \Phi(n^{1/2}(-\theta + s\eta_n)) - \Phi(n^{1/2}(-\theta - s\eta_n)) \right\} \mathbf{1}(0 \leq \theta \leq s\eta_n - sa_n) \\ & \quad + \left\{ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta - s\eta_n)) \right\} \mathbf{1}(s\eta_n - sa_n < \theta \leq sa_n) \\ & \quad + \left\{ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta + s\eta_n)) \right\} \mathbf{1}(sa_n < \theta \leq s\eta_n + sa_n) \\ & \quad + \left\{ \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n) \right\} \mathbf{1}(s\eta_n + sa_n < \theta) \end{aligned}$$

if  $a_n \leq \eta_n \leq 2a_n$ , and by

$$\begin{aligned} & p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) \\ &= \left\{ \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n) \right\} \left\{ \mathbf{1}(0 \leq \theta \leq sa_n - s\eta_n) + \mathbf{1}(s\eta_n + sa_n < \theta) \right\} \\ & \quad + \left\{ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta - s\eta_n)) \right\} \mathbf{1}(sa_n - s\eta_n < \theta \leq sa_n) \\ & \quad + \left\{ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta + s\eta_n)) \right\} \mathbf{1}(sa_n < \theta \leq s\eta_n + sa_n) \end{aligned}$$

if  $\eta_n < a_n$ . In the subsequent theorems we consider only the case where  $\eta_n \rightarrow 0$  as this is the only interesting case from an asymptotic perspective: note that any of the penalized maximum likelihood estimators considered in this paper is inconsistent for  $\theta$  if  $\eta_n$  does not converge to zero.

**Theorem 9.** *Suppose  $\eta_n \rightarrow 0$ . For every sequence  $a_n$  of nonnegative real numbers we have with  $E_{H,n} = [\tilde{\theta}_H - \hat{\sigma}a_n, \tilde{\theta}_H + \hat{\sigma}a_n]$  and  $C_{H,n} = [\hat{\theta}_H - a_n, \hat{\theta}_H + a_n]$  that*

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n}) - \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) \rightarrow 0$$

as  $n \rightarrow \infty$ . The analogous results hold for the corresponding open and half-open intervals.

### 4.3. Adaptive LASSO

Consider the interval  $E_{A,n} = [\tilde{\theta}_A - \hat{\sigma}a_n, \tilde{\theta}_A + \hat{\sigma}a_n]$  where  $a_n$  is a nonnegative real number and  $\tilde{\theta}_A = \tilde{\theta}_A(\eta_n)$ . We then have analogously as in the preceding subsections that

$$P_{n,\theta}(\theta \in E_{A,n}) = \int_0^\infty p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds$$

where  $p_{A,n}$  is given in (A.2) in the Appendix.

**Theorem 10.** *Suppose  $\eta_n \rightarrow 0$ . For every sequence  $a_n$  of nonnegative real numbers we have with  $E_{A,n} = [\tilde{\theta}_A - \hat{\sigma}a_n, \tilde{\theta}_A + \hat{\sigma}a_n]$  and  $C_{A,n} = [\hat{\theta}_A - a_n, \hat{\theta}_A + a_n]$  that*

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{A,n}) - \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{A,n}) \rightarrow 0$$

as  $n \rightarrow \infty$ . The analogous results hold for the corresponding open and half-open intervals.

#### 4.4. Discussion

Theorems 8, 9, and 10 show that the results in Section 3 carry over to the unknown-variance case in an asymptotic sense: For example, suppose  $0 < \delta < 1$ , and  $a_{n,S}$  ( $a_{n,H}$ ,  $a_{n,A}$ , respectively) is such that  $E_{S,n}$  ( $E_{H,n}$ ,  $E_{A,n}$ , respectively) has infimal coverage probability converging to  $\delta$ . Then, for a regime where  $n^{1/2}\eta_n \rightarrow e$  with  $0 \leq e < \infty$ , it follows that  $n^{1/2}a_{n,S}$ ,  $n^{1/2}a_{n,H}$ , and  $n^{1/2}a_{n,A}$  have limits that solve (3.6)–(3.8), respectively; that is, they have the same limits as  $n^{1/2}a_{n,S}^*$ ,  $n^{1/2}a_{n,H}^*$ , and  $n^{1/2}a_{n,A}^*$ , which are  $n^{1/2}$  times the half-length of the shortest  $\delta$ -confidence intervals  $C_{S,n}^*$ ,  $C_{H,n}^*$ , and  $C_{A,n}^*$ , respectively, in the known-variance case. Furthermore, for a regime where  $n^{1/2}\eta_n \rightarrow \infty$  it follows that  $a_{n,S}$ ,  $a_{n,H}$ , and  $a_{n,A}$  satisfy (3.9)–(3.11), respectively (where we also assume  $\eta_n \rightarrow 0$  for hard-thresholding and the adaptive LASSO). Hence,  $a_{n,S}$ ,  $a_{n,H}$ , and  $a_{n,A}$  on the one hand, and  $a_{n,S}^*$ ,  $a_{n,H}^*$ , and  $a_{n,A}^*$  on the other hand have again the same asymptotic behavior. Furthermore, Theorems 8, 9, and 10 show that Proposition 6 immediately carries over to the unknown-variance case.

#### Appendix A

**Proof of Proposition 1.** Using the expression for the finite sample distribution of  $n^{1/2}(\hat{\theta}_S - \theta)$  given in Pötscher and Leeb (2009) and noting that this distribution function has a jump at  $-n^{1/2}\theta$  we obtain

$$\begin{aligned} p_{S,n}(\theta) &= [\Phi(n^{1/2}(a_n - \eta_n)) - \Phi(n^{1/2}(-b_n - \eta_n))]\mathbf{1}(\theta < -a_n) \\ &+ [\Phi(n^{1/2}(a_n + \eta_n)) - \Phi(n^{1/2}(-b_n - \eta_n))]\mathbf{1}(-a_n \leq \theta \leq b_n) \\ &+ [\Phi(n^{1/2}(a_n + \eta_n)) - \Phi(n^{1/2}(-b_n + \eta_n))]\mathbf{1}(b_n < \theta). \end{aligned} \quad (\text{A.1})$$

It follows that  $\inf_{\theta \in \mathbb{R}} p_{S,n}(\theta)$  is as given in the proposition.  $\square$

**Proof of Proposition 3.** The distribution function  $F_{A,n,\theta}(x) = P_{n,\theta}(n^{1/2}(\hat{\theta}_A - \theta) \leq x)$  of the adaptive LASSO estimator is given by

$$\begin{aligned} &\mathbf{1}(x + n^{1/2}\theta \geq 0)\Phi\left(-((n^{1/2}\theta - x)/2) + \sqrt{((n^{1/2}\theta + x)/2)^2 + n\eta_n^2}\right) + \\ &\mathbf{1}(x + n^{1/2}\theta < 0)\Phi\left(-((n^{1/2}\theta - x)/2) - \sqrt{((n^{1/2}\theta + x)/2)^2 + n\eta_n^2}\right) \end{aligned}$$

(see Pötscher and Schneider (2009)). Hence, the coverage probability  $p_{A,n}(\theta) = F_{A,n,\theta}(n^{1/2}a_n) - \lim_{x \rightarrow (-n^{1/2}b_n)_-} F_{A,n,\theta}(x)$  is

$$p_{A,n}(\theta) = \begin{cases} \Phi(n^{1/2}\gamma^{(-)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(-)}(\theta, b_n)) & \text{if } \theta < -a_n \\ \Phi(n^{1/2}\gamma^{(+)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(-)}(\theta, b_n)) & \text{if } -a_n \leq \theta \leq b_n \\ \Phi(n^{1/2}\gamma^{(+)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(+)}(\theta, b_n)) & \text{if } \theta > b_n. \end{cases} \quad (\text{A.2})$$

Here

$$\gamma^{(-)}(\theta, x) = -((\theta + x)/2) - \sqrt{((\theta - x)/2)^2 + \eta_n^2} \quad (\text{A.3})$$

$$\gamma^{(+)}(\theta, x) = -((\theta + x)/2) + \sqrt{((\theta - x)/2)^2 + \eta_n^2}, \quad (\text{A.4})$$

which are clearly smooth functions of  $(\theta, x)$ . Observe that  $\gamma^{(-)}$  and  $\gamma^{(+)}$  are nonincreasing in  $\theta \in \mathbb{R}$  (for every  $x \in \mathbb{R}$ ). As a consequence, we obtain for  $-a_n \leq \theta \leq b_n$  the lower bound

$$\begin{aligned} p_{A,n}(\theta) &\geq \Phi\left(n^{1/2}\gamma^{(+)}(b_n, -a_n)\right) - \Phi\left(n^{1/2}\gamma^{(-)}(-a_n, b_n)\right) \\ &= \Phi\left(n^{1/2}\left((a_n - b_n)/2 + \sqrt{((a_n + b_n)/2)^2 + \eta_n^2}\right)\right) \\ &\quad - \Phi\left(n^{1/2}\left((a_n - b_n)/2 - \sqrt{((a_n + b_n)/2)^2 + \eta_n^2}\right)\right). \end{aligned} \quad (\text{A.5})$$

Consider first the case where  $a_n \leq b_n$ . We then show that  $p_{A,n}(\theta)$  is nonincreasing on  $(-\infty, -a_n)$ : The derivative  $dp_{A,n}(\theta)/d\theta$  is given by

$$dp_{A,n}(\theta)/d\theta = n^{1/2}[\phi(n^{1/2}\gamma^{(-)}(\theta, -a_n))\partial\gamma^{(-)}(\theta, -a_n)/\partial\theta - \phi(n^{1/2}\gamma^{(-)}(\theta, b_n))\partial\gamma^{(-)}(\theta, b_n)/\partial\theta]$$

where  $\phi$  denotes the standard normal density function. Using the relation  $a_n \leq b_n$ , elementary calculations show that

$$\partial\gamma^{(-)}(\theta, -a_n)/\partial\theta \leq \partial\gamma^{(-)}(\theta, b_n)/\partial\theta \quad \text{for } \theta \in (-\infty, -a_n).$$

Furthermore, given  $a_n \leq b_n$ , it is not too difficult to see that  $|\gamma^{(-)}(\theta, -a_n)| \leq |\gamma^{(-)}(\theta, b_n)|$  for  $\theta \in (-\infty, -a_n)$  (cf. Lemma 11 below), which implies that

$$\phi(n^{1/2}\gamma^{(-)}(\theta, -a_n)) \geq \phi(n^{1/2}\gamma^{(-)}(\theta, b_n)).$$

The last two displays together with the fact that  $\partial\gamma^{(-)}(\theta, -a_n)/\partial\theta$  as well as  $\partial\gamma^{(-)}(\theta, b_n)/\partial\theta$  are less than or equal to zero, imply that  $dp_{A,n}(\theta)/d\theta \leq 0$  on  $(-\infty, -a_n)$ . This proves that

$$\inf_{\theta < -a_n} p_{A,n}(\theta) = \lim_{\theta \rightarrow (-a_n)_-} p_{A,n}(\theta) = c$$

with

$$c = \Phi\left(n^{1/2}(a_n - \eta_n)\right) - \Phi\left(n^{1/2}\left((a_n - b_n)/2 - \sqrt{((a_n + b_n)/2)^2 + \eta_n^2}\right)\right). \quad (\text{A.6})$$

Since the lower bound given in (A.5) is not less than  $c$ , we have

$$\inf_{\theta \leq b_n} p_{A,n}(\theta) = \inf_{\theta < -a_n} p_{A,n}(\theta) = c.$$

It remains to show that  $p_{A,n}(\theta) \geq c$  for  $\theta > b_n$ . From (A.2) and (A.6) after rearranging terms we obtain for  $\theta > b_n$

$$\begin{aligned} p_{A,n}(\theta) - c &= \left[ \Phi(n^{1/2}\gamma^{(+)}(\theta, -a_n)) - \Phi(n^{1/2}\gamma^{(-)}(-a_n, -a_n)) \right] - \\ &\quad \left[ \Phi(n^{1/2}\gamma^{(+)}(\theta, b_n)) - \Phi(n^{1/2}\gamma^{(-)}(-a_n, b_n)) \right]. \end{aligned}$$

It is elementary to show that  $\gamma^{(+)}(\theta, -a_n) \geq \gamma^{(-)}(-a_n, -a_n) = a_n - \eta_n$  and  $\gamma^{(+)}(\theta, b_n) \geq \gamma^{(-)}(-a_n, b_n)$ . We next show that

$$\gamma^{(+)}(\theta, -a_n) - \gamma^{(-)}(-a_n, -a_n) \geq \gamma^{(+)}(\theta, b_n) - \gamma^{(-)}(-a_n, b_n). \quad (\text{A.7})$$

To establish this note that (A.7) can equivalently be rewritten as

$$f(0) + f((\theta + a_n)/2) \geq f((\theta - b_n)/2) + f((a_n + b_n)/2) \quad (\text{A.8})$$

where  $f(x) = (x^2 + \eta_n^2)^{1/2}$ . Observe that  $0 \leq (\theta - b_n)/2 \leq (\theta + a_n)/2$  holds since  $0 \leq a_n \leq b_n < \theta$ . Writing  $(\theta - b_n)/2$  as  $\lambda(\theta + a_n)/2 + (1 - \lambda)0$  with  $0 \leq \lambda \leq 1$  gives  $(a_n + b_n)/2 = (1 - \lambda)(\theta + a_n)/2 + \lambda 0$ . Because  $f$  is convex, the inequality (A.8) and hence (A.7) follows.

Next observe that in case  $a_n \geq \eta_n$  we have (using monotonicity of  $\gamma^{(+)}(\theta, b_n)$ )

$$0 \leq \gamma^{(-)}(-a_n, -a_n) = a_n - \eta_n \leq b_n - \eta_n = -\gamma^{(+)}(b_n, b_n) \leq -\gamma^{(+)}(\theta, b_n) \quad (\text{A.9})$$

for  $\theta > b_n$ . In case  $a_n < \eta_n$  we have (using  $\gamma^{(-)}(\theta, x) = \gamma^{(-)}(x, \theta)$  and monotonicity of  $\gamma^{(-)}$  in its first argument)

$$\gamma^{(-)}(-a_n, b_n) \leq \gamma^{(-)}(-a_n, -a_n) = a_n - \eta_n < 0, \quad (\text{A.10})$$

and (using monotonicity of  $\gamma^{(+)}$ )

$$\gamma^{(-)}(-a_n, b_n) \leq -\gamma^{(+)}(b_n, -a_n) \leq -\gamma^{(+)}(\theta, -a_n) \quad (\text{A.11})$$

for  $\theta > b_n$ . Applying Lemma 12 below with  $\alpha = n^{1/2}\gamma^{(-)}(-a_n, -a_n)$ ,  $\beta = n^{1/2}\gamma^{(+)}(\theta, -a_n)$ ,  $\gamma = n^{1/2}\gamma^{(-)}(-a_n, b_n)$ , and  $\delta = n^{1/2}\gamma^{(+)}(\theta, b_n)$  and using (A.7)–(A.11), establishes  $p_{A,n}(\theta) - c \geq 0$ . This completes the proof in case  $a_n \leq b_n$ .

The case  $a_n > b_n$  follows from the observation that (A.2) remains unchanged if  $a_n$  and  $b_n$  are interchanged and  $\theta$  is replaced by  $-\theta$ .  $\square$

**Lemma 11.** *Suppose  $a_n \leq b_n$ . Then  $|\gamma^{(-)}(\theta, -a_n)| \leq |\gamma^{(-)}(\theta, b_n)|$  holds for  $\theta \in (-\infty, -a_n)$ .*

*Proof.* Squaring both sides of the claimed inequality shows that the claim is equivalent to

$$a_n^2/2 - (a_n - \theta)\sqrt{((a_n + \theta)/2)^2 + \eta^2} \leq b_n^2/2 + (b_n + \theta)\sqrt{((b_n - \theta)/2)^2 + \eta^2}.$$

But, for  $\theta < -a_n$ , the left-hand side of the preceding display is not larger than

$$a_n^2/2 + (a_n + \theta)\sqrt{((a_n - \theta)/2)^2 + \eta^2}.$$

Since  $a_n^2/2 \leq b_n^2/2$ , it hence suffices to show that

$$-(a_n + \theta)\sqrt{((a_n - \theta)/2)^2 + \eta^2} \geq -(b_n + \theta)\sqrt{((b_n - \theta)/2)^2 + \eta^2}$$

for  $\theta < -a_n$ . This is immediately seen by distinguishing the cases where  $-b_n \leq \theta < -a_n$  and where  $\theta < -b_n$ , and observing that  $a_n \leq b_n$ .  $\square$

The following lemma is elementary to prove.

**Lemma 12.** *Suppose  $\alpha, \beta, \gamma$ , and  $\delta$  are real numbers satisfying  $\alpha \leq \beta, \gamma \leq \delta$ , and  $\beta - \alpha \geq \delta - \gamma$ . If  $0 \leq \alpha \leq -\delta$ , or if  $\gamma \leq \alpha \leq 0$  and  $\gamma \leq -\beta$ , then  $\Phi(\beta) - \Phi(\alpha) \geq \Phi(\delta) - \Phi(\gamma)$ .*

**Proof of Theorem 5.** (a) Since  $\delta$  is positive, any solution to (3.3) has to be positive. Now the equation (3.3) has a unique solution  $a_{n,S}^*$ , since (3.3) as a function of  $a_n \in [0, \infty)$  is easily seen to be strictly increasing with range  $[0, 1)$ . Furthermore, the infimal coverage probability (3.1) is a continuous function of the pair  $(a_n, b_n)$  on  $[0, \infty) \times [0, \infty)$ . Let  $K \subseteq [0, \infty) \times [0, \infty)$  consist of all pairs  $(a_n, b_n)$  such that (i) the corresponding interval  $[\hat{\theta}_S - a_n, \hat{\theta}_S + b_n]$  has infimal coverage probability not less than  $\delta$ , and (ii) the length  $a_n + b_n$  is less than or equal  $2a_{n,S}^*$ . Then  $K$  is compact. It is also nonempty as the pair  $(a_{n,S}^*, a_{n,S}^*)$  belongs to  $K$ . Since the length  $a_n + b_n$  is obviously continuous, it follows that there is a pair  $(a_n^o, b_n^o) \in K$  having minimal length within  $K$ . Since confidence sets corresponding to pairs not belonging to  $K$  always have length larger than  $2a_{n,S}^*$ , the pair  $(a_n^o, b_n^o)$  gives rise to an interval with shortest length within the set of all intervals with infimal coverage probability not less than  $\delta$ . We next show that  $a_n^o = b_n^o$  must hold: Suppose not, then we may assume without loss of generality that  $a_n^o < b_n^o$ , since (3.1) remains invariant under permutation of  $a_n^o$  and  $b_n^o$ . But now increasing  $a_n^o$  by  $\varepsilon > 0$  and decreasing  $b_n^o$  by the same amount such that  $a_n^o + \varepsilon < b_n^o - \varepsilon$  holds, will result in an interval of the same length with infimal coverage probability

$$\Phi(n^{1/2}(a_n^o + \varepsilon - \eta_n)) - \Phi(n^{1/2}(-(b_n^o - \varepsilon) - \eta_n)).$$

This infimal coverage probability will be strictly larger than

$$\Phi(n^{1/2}(a_n^o - \eta_n)) - \Phi(n^{1/2}(-b_n^o - \eta_n)) \geq \delta$$

provided  $\varepsilon$  is chosen sufficiently small. But then, by continuity of the infimal coverage probability as a function of  $a_n$  and  $b_n$ , the interval  $[\hat{\theta}_S - a_n^o - \varepsilon, \hat{\theta}_S + b_n^o - \varepsilon]$  with  $\varepsilon < b_n^o - a_n^o$  will still have infimal coverage probability not less than  $\delta$  as long as  $b_n^o$  is sufficiently close to  $a_n^o$ ; at the same time this interval will be shorter than the interval  $[\hat{\theta}_S - a_n^o, \hat{\theta}_S + b_n^o]$ . This leads to a contradiction and establishes  $a_n^o = b_n^o$ . By what was said at the beginning of the proof, it is now obvious that  $a_n^o = b_n^o = a_{n,S}^*$  must hold, thus also establishing uniqueness. The last claim is obvious in view of the construction of  $a_{n,S}^*$ .

(b) Since  $\delta$  is positive, any solution to (3.4) has to be larger than  $\eta_n/2$ . Now equation (3.4) has a unique solution  $a_{n,H}^*$ , since (3.4) as a function of  $a_n \in [\eta_n/2, \infty)$  is easily seen to be strictly increasing with range  $[0, 1)$ . Furthermore, define  $K$  similarly as in the proof of part (a). Then, by the same reasoning as in (a), the set  $K$  is compact and non-empty, leading to a pair  $(a_n^o, b_n^o)$  that gives rise to an interval with shortest length within the set of all intervals with infimal coverage probability not less than  $\delta$ . We next show that  $a_n^o = b_n^o$  must hold: Suppose not, then we may again assume without loss of generality that  $a_n^o < b_n^o$ . Note that  $a_n^o + b_n^o > \eta_n$  must hold, since the infimal coverage probability

of the corresponding interval is positive by construction. Since all this entails  $|a_n^o - \eta_n| < b_n^o$ , increasing  $a_n^o$  by  $\varepsilon > 0$  and decreasing  $b_n^o$  by the same amount such that  $a_n^o + \varepsilon < b_n^o - \varepsilon$  holds, will result in an interval of the same length with infimal coverage probability

$$\begin{aligned} \Phi(n^{1/2}(a_n^o + \varepsilon - \eta_n)) - \Phi(-n^{1/2}(b_n^o - \varepsilon)) &> \\ \Phi(n^{1/2}(a_n^o - \eta_n)) - \Phi(-n^{1/2}b_n^o) &\geq \delta \end{aligned}$$

provided  $\varepsilon$  is chosen sufficiently small. By continuity of the infimal coverage probability as a function of  $a_n$  and  $b_n$ , the interval  $[\hat{\theta}_S - a_n^o - \varepsilon, \hat{\theta}_S + b_n^o - \varepsilon]$  with  $\varepsilon < b_n^o - a_n^o$  will still have infimal coverage probability not less than  $\delta$  as long as  $b_n^o$  is sufficiently close to  $a_n^o$ ; at the same time this interval will be shorter than the interval  $[\hat{\theta}_S - a_n^o, \hat{\theta}_S + b_n^o]$ , leading to a contradiction thus establishing  $a_n^o = b_n^o$ . As in (a) it now follows that  $a_n^o = b_n^o = a_{n,H}^*$  must hold, thus also establishing uniqueness. The last claim is then obvious in view of the construction of  $a_{n,H}^*$ .

(c) Since  $\delta$  is positive, it is easy to see that any solution to (3.5) has to be positive. Now equation (3.5) has a unique solution  $a_{n,A}^*$ , since (3.5) as a function of  $a_n \in [0, \infty)$  is strictly increasing with range  $[0, 1)$ . Furthermore, the infimal coverage probability as given in Proposition 3 is a continuous function of the pair  $(a_n, b_n)$  on  $[0, \infty) \times [0, \infty)$ . Define  $K$  similarly as in the proof of part (a). Then by the same reasoning as in (a), the set  $K$  is compact and non-empty, leading to a pair  $(a_n^o, b_n^o)$  that gives rise to an interval with shortest length within the set of all intervals with infimal coverage probability not less than  $\delta$ . We next show that  $a_n^o = b_n^o$  must hold: Suppose not, then we may again assume without loss of generality that  $a_n^o < b_n^o$ . But now increasing  $a_n^o$  by  $\varepsilon > 0$  and decreasing  $b_n^o$  by the same amount such that  $a_n^o + \varepsilon < b_n^o - \varepsilon$  holds, will result in an interval of the same length with infimal coverage probability

$$\begin{aligned} \Phi(n^{1/2}(a_n^o + \varepsilon - \eta_n)) - \Phi\left(n^{1/2}\left(\varepsilon + (a_n^o - b_n^o)/2 - \sqrt{((a_n^o + b_n^o)/2)^2 + \eta_n^2}\right)\right) &> \\ \Phi(n^{1/2}(a_n^o - \eta_n)) - \Phi\left(n^{1/2}\left((a_n^o - b_n^o)/2 - \sqrt{((a_n^o + b_n^o)/2)^2 + \eta_n^2}\right)\right) &\geq \delta, \end{aligned}$$

provided  $\varepsilon$  is chosen sufficiently small. This is so since  $a_n^o < b_n^o$  implies

$$|a_n^o - \eta_n| < \left| (a_n^o - b_n^o)/2 - \sqrt{((a_n^o + b_n^o)/2)^2 + \eta_n^2} \right|$$

as is easily seen. But then, by continuity of the infimal coverage probability as a function of  $a_n$  and  $b_n$ , the interval  $[\hat{\theta}_S - a_n^o - \varepsilon, \hat{\theta}_S + b_n^o - \varepsilon]$  with  $\varepsilon < b_n^o - a_n^o$  will still have infimal coverage probability not less than  $\delta$  as long as  $b_n^o$  is sufficiently close to  $a_n^o$ ; at the same time this interval will be shorter than the interval  $[\hat{\theta}_S - a_n^o, \hat{\theta}_S + b_n^o]$ . This leads to a contradiction and establishes  $a_n^o = b_n^o$ . As in (a) it now follows that  $a_n^o = b_n^o = a_{n,A}^*$  must hold, thus also establishing uniqueness. The last claim is obvious in view of the construction of  $a_{n,A}^*$ .  $\square$

**Proof of Proposition 6.** Let

$$c = \liminf_{n \rightarrow \infty} \inf_{\theta \in \mathbb{R}} P_{n,\theta} \left( -d \leq \eta_n^{-1}(\hat{\theta} - \theta) \leq d \right).$$

By definition of  $c$ , we can find a subsequence  $n_k$  and elements  $\theta_{n_k} \in \mathbb{R}$  such that

$$P_{n_k, \theta_{n_k}} \left( -d \leq \eta_{n_k}^{-1}(\hat{\theta} - \theta_{n_k}) \leq d \right) \rightarrow c$$

for  $k \rightarrow \infty$ . Now, by Theorem 17 (for  $\hat{\theta} = \hat{\theta}_H$ ), Theorem 18 (for  $\hat{\theta} = \hat{\theta}_S$ ), and Remark 12 in Pötscher and Leeb (2009), and by Theorem 6 (for  $\hat{\theta} = \hat{\theta}_A$ ) and Remark 7 in Pötscher and Schneider (2009), any accumulation point of the distribution of  $\eta_{n_k}^{-1}(\hat{\theta} - \theta_{n_k})$  with respect to weak convergence is a probability measure concentrated on  $[-1, 1]$ . Since  $d > 1$ , it follows that  $c = 1$  must hold, which proves the first claim. We next prove the second claim. In view of Theorem 17 (for  $\hat{\theta} = \hat{\theta}_H$ ) and Theorem 18 (for  $\hat{\theta} = \hat{\theta}_S$ ) in Pötscher and Leeb (2009), and in view of Theorem 6 (for  $\hat{\theta} = \hat{\theta}_A$ ) in Pötscher and Schneider (2009) it is possible to choose a sequence  $\theta_n \in \mathbb{R}$  such that the distribution of  $\eta_n^{-1}(\hat{\theta} - \theta_n)$  converges to point mass located at one of the endpoints of the interval  $[-1, 1]$ . But then clearly

$$P_{n, \theta_n} \left( -d \leq \eta_n^{-1}(\hat{\theta} - \theta_n) \leq d \right) \rightarrow 0$$

for  $d < 1$  which implies the second claim. □

**Proof of Theorem 9.** We prove the result for the closed interval. Inspection of the proof together with Remark 4 then gives the result for the open and half-open intervals.

Step 1: Observe that for every  $s > 0$  and  $n \geq 2$  we have from the above formulae for  $p_{H,n}$  that

$$\lim_{\theta \rightarrow \infty} p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) = \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n).$$

By the dominated convergence theorem it follows that for  $\theta \rightarrow \infty$

$$\begin{aligned} P_{n,\theta}(\theta \in E_{H,n}) &= \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &\rightarrow \int_0^\infty \left[ \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n) \right] h_n(s) ds \\ &= T_{n-1}(n^{1/2}a_n) - T_{n-1}(-n^{1/2}a_n). \end{aligned}$$

Hence,

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) \leq \lim_{\theta \rightarrow \infty} p_{H,n}(\theta; 1, \eta_n, a_n, a_n) = \Phi(n^{1/2}a_n) - \Phi(-n^{1/2}a_n)$$

and

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n}) \leq T_{n-1}(n^{1/2}a_n) - T_{n-1}(-n^{1/2}a_n) \leq \Phi(n^{1/2}a_n) - \Phi(-n^{1/2}a_n), \tag{A.12}$$

the last inequality following from well-known properties of  $T_{n-1}$ , cf. Lemma 14 below. This proves the theorem in case  $n^{1/2}a_n \rightarrow 0$  for  $n \rightarrow \infty$ .

Step 2: For every  $s > 0$  and  $n \geq 2$  we have from (3.2)

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) &= \inf_{\theta \in \mathbb{R}} p_{H,n}(\theta; 1, \eta_n, a_n, a_n) \\ &= \max \left[ \Phi(n^{1/2}a_n) - \Phi(-n^{1/2}(a_n - \eta_n)), 0 \right] \end{aligned} \quad (\text{A.13})$$

and

$$\inf_{\theta \in \mathbb{R}} p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) = \max \left[ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-sa_n + s\eta_n)), 0 \right].$$

Furthermore,

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n}) &\geq \int_0^\infty \inf_{\theta \in \mathbb{R}} p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &= \int_0^\infty \max \left[ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-sa_n + s\eta_n)), 0 \right] h_n(s) ds \\ &= \max \left[ \int_0^\infty \left[ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-sa_n + s\eta_n)) \right] h_n(s) ds, 0 \right] \\ &= \max \left[ T_{n-1}(n^{1/2}a_n) - T_{n-1}(-n^{1/2}(a_n - \eta_n)), 0 \right]. \end{aligned} \quad (\text{A.14})$$

If  $n^{1/2}(a_n - \eta_n) \rightarrow \infty$ , then the far right-hand sides of (A.13) and (A.14) converge to 1, since  $\|\Phi - T_{n-1}\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  by Polya's Theorem and since  $n^{1/2}a_n \geq n^{1/2}(a_n - \eta_n)$ . This proves the theorem in case  $n^{1/2}(a_n - \eta_n) \rightarrow \infty$ .

Step 3: If  $n^{1/2}\eta_n \rightarrow 0$ , then (A.13) and the fact that  $\Phi$  is globally Lipschitz shows that  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n})$  differs from  $\Phi(n^{1/2}a_n) - \Phi(-n^{1/2}a_n)$  only by a term that is  $o(1)$ . Similarly, (A.12), (A.14), the fact that  $\|\Phi - T_{n-1}\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  by Polya's theorem, and the global Lipschitz property of  $\Phi$  show that the same is true for  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n})$ , proving the theorem in case  $n^{1/2}\eta_n \rightarrow 0$ .

Step 4: By a subsequence argument and Steps 1-3 it remains to prove the theorem under the assumption that  $n^{1/2}a_n$  and  $n^{1/2}\eta_n$  are bounded away from zero by a finite positive constant  $c_1$ , say, and that  $n^{1/2}(a_n - \eta_n)$  is bounded from above by a finite constant  $c_2$ , say. It then follows that  $a_n/\eta_n$  is bounded by a finite positive constant  $c_3$ , say. For given  $\varepsilon > 0$  set  $\theta_n(\varepsilon) = a_n(1 + 2c(\varepsilon)n^{-1/2})$  where  $c(\varepsilon)$  is the constant given in Lemma 13. We then have for  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$

$$sa_n < \theta_n(\varepsilon) \leq s(\eta_n + a_n)$$

whenever  $n > n_0(c(\varepsilon), c_3)$ . Without loss of generality we may choose  $n_0(c(\varepsilon), c_3)$  large enough such that also  $1 - c(\varepsilon)n^{-1/2} > 0$  holds for  $n > n_0(c(\varepsilon), c_3)$ . Consequently, we have (observing that  $\max(0, x)$  has Lipschitz constant 1 and  $\Phi$  has Lipschitz constant  $(2\pi)^{-1/2}$ ) for every  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$  and  $n > n_0(c(\varepsilon), c_3)$

$$\begin{aligned}
 & |p_{H,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) - p_{H,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\
 = & \left| \max(0, \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta_n(\varepsilon) + s\eta_n))) - \max(0, \Phi(n^{1/2}a_n) \right. \\
 & \left. - \Phi(n^{1/2}(-\theta_n(\varepsilon) + \eta_n))) \right| \\
 \leq & \left| \left[ \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}(-\theta_n(\varepsilon) + s\eta_n)) \right] \right. \\
 & \left. - \left[ \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-\theta_n(\varepsilon) + \eta_n)) \right] \right| \\
 \leq & (2\pi)^{-1/2}n^{1/2}(a_n + \eta_n) |s - 1| \leq (2\pi)^{-1/2}c(\varepsilon)(a_n + \eta_n) \\
 \leq & (2\pi)^{-1/2}c(\varepsilon)(c_3 + 1)\eta_n.
 \end{aligned}$$

It follows that for every  $n > n_0(c(\varepsilon), c_3)$

$$\begin{aligned}
 & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\
 \leq & \int_0^\infty p_{H,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\
 = & \int_{1-c(\varepsilon)n^{-1/2}}^{1+c(\varepsilon)n^{-1/2}} p_{H,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\
 & + \int_{\{s: |s-1| \geq c(\varepsilon)n^{-1/2}\}} p_{H,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\
 = & B_1 + B_2.
 \end{aligned}$$

Clearly,  $0 \leq B_2 \leq \varepsilon$  holds, cf. Lemma 13, and for  $B_1$  we have

$$\begin{aligned}
 & |B_1 - p_{H,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\
 \leq & \left| \int_{1-c(\varepsilon)n^{-1/2}}^{1+c(\varepsilon)n^{-1/2}} [p_{H,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) \right. \\
 & \left. - p_{H,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)] h_n(s) ds \right| + \varepsilon \\
 \leq & (2\pi)^{-1/2}c(\varepsilon)(c_3 + 1)\eta_n + \varepsilon
 \end{aligned}$$

for  $n > n_0(c(\varepsilon), c_3)$ . It follows that

$$\begin{aligned}
 & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\
 \leq & p_{H,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n) + (2\pi)^{-1/2}c(\varepsilon)(c_3 + 1)\eta_n + 2\varepsilon
 \end{aligned}$$

holds for  $n > n_0(c(\varepsilon), c_3)$ . Now

$$\begin{aligned}
 & p_{H,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n) \\
 = & \max(0, \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-\theta_n(\varepsilon) + \eta_n))) \\
 = & \max(0, \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-a_n(1 + 2c(\varepsilon)n^{-1/2}) + \eta_n))).
 \end{aligned}$$

But this differs from  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) = \max(0, \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-a_n + \eta_n)))$  by at most

$$\begin{aligned} & \left| \Phi(n^{1/2}(-a_n + \eta_n)) - \Phi(n^{1/2}(-a_n(1 + 2c(\varepsilon)n^{-1/2}) + \eta_n)) \right| \\ & \leq (2\pi)^{-1/2}2c(\varepsilon)a_n \leq (2\pi)^{-1/2}2c(\varepsilon)c_3\eta_n. \end{aligned}$$

Consequently, for  $n > n_0(c(\varepsilon), c_3)$

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n}) &= \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &\leq \max(0, \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-a_n + \eta_n))) \\ &\quad + (2\pi)^{-1/2}c(\varepsilon)(3c_3 + 1)\eta_n + 2\varepsilon \\ &= \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) + (2\pi)^{-1/2}c(\varepsilon)(3c_3 + 1)\eta_n + 2\varepsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{H,n}) &= \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &\geq \int_0^\infty \inf_{\theta \in \mathbb{R}} p_{H,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &= \int_0^\infty \max(0, \Phi(n^{1/2}sa_n) - \Phi(n^{1/2}s(-a_n + \eta_n))) h_n(s) ds \\ &= \max(0, T_{n-1}(n^{1/2}a_n) - T_{n-1}(n^{1/2}(-a_n + \eta_n))) \\ &\geq \max(0, \Phi(n^{1/2}a_n) - \Phi(n^{1/2}(-a_n + \eta_n))) - 2\|\Phi - T_{n-1}\|_\infty \\ &= \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{H,n}) - 2\|\Phi - T_{n-1}\|_\infty. \end{aligned}$$

Since  $\eta_n \rightarrow 0$  and  $\|\Phi - T_{n-1}\|_\infty \rightarrow 0$  for  $n \rightarrow \infty$  and since  $\varepsilon$  was arbitrary the proof is complete.  $\square$

**Proof of Theorem 10.** We prove the result for the closed interval. Inspection of the proof together with Remark 4 then gives the result for the open and half-open intervals.

Step 1: Observe that for every  $s > 0$  and  $n \geq 2$  we have from (A.2) that

$$\lim_{\theta \rightarrow \infty} p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) = \Phi(n^{1/2}sa_n) - \Phi(-n^{1/2}sa_n).$$

Then exactly the same argument as in the proof of Theorem 9 shows that  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{A,n})$  as well as  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{A,n})$  converge to zero for  $n \rightarrow \infty$  if  $n^{1/2}a_n \rightarrow 0$ , thus proving the theorem in this case. For later use we note that this reasoning in particular gives

$$\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{A,n}) \leq T_{n-1}(n^{1/2}a_n) - T_{n-1}(-n^{1/2}a_n) \leq \Phi(n^{1/2}a_n) - \Phi(-n^{1/2}a_n). \tag{A.15}$$

Step 2: By Proposition 3 we have for every  $s > 0$  and  $n \geq 1$

$$\inf_{\theta \in \mathbb{R}} p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) = \Phi(n^{1/2}s\sqrt{a_n^2 + \eta_n^2}) - \Phi(n^{1/2}s(-a_n + \eta_n)).$$

Arguing as in the proof of Theorem 9 we then have

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{A,n}) &= \inf_{\theta \in \mathbb{R}} p_{A,n}(\theta; 1, \eta_n, a_n, a_n) \\ &= \Phi(n^{1/2}\sqrt{a_n^2 + \eta_n^2}) - \Phi(n^{1/2}(-a_n + \eta_n)) \end{aligned} \quad (\text{A.16})$$

and

$$\begin{aligned} \inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in E_{A,n}) &\geq \int_0^\infty \inf_{\theta \in \mathbb{R}} p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ &= T_{n-1}(n^{1/2}\sqrt{a_n^2 + \eta_n^2}) - T_{n-1}(n^{1/2}(-a_n + \eta_n)). \end{aligned} \quad (\text{A.17})$$

If  $n^{1/2}(a_n - \eta_n) \rightarrow \infty$ , then the far right-hand sides of (A.16) and (A.17) converge to 1, since  $\|\Phi - T_{n-1}\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  by Polya's Theorem and since  $n^{1/2}\sqrt{a_n^2 + \eta_n^2} \geq n^{1/2}a_n \rightarrow \infty$  and  $n^{1/2}(-a_n + \eta_n) \rightarrow -\infty$ . This proves the theorem in case  $n^{1/2}(a_n - \eta_n) \rightarrow \infty$ .

Step 3: Analogous to the corresponding step in the proof of Theorem 9, using (A.16), (A.15), (A.17), and additionally noting that  $0 \leq n^{1/2}\sqrt{a_n^2 + \eta_n^2} - n^{1/2}a_n \leq n^{1/2}\eta_n$ , the theorem is proved in the case  $n^{1/2}\eta_n \rightarrow 0$ .

Step 4: Similar as in the proof of Theorem 9 it remains to prove the theorem under the assumption that  $n^{1/2}a_n \geq c_1 > 0$ ,  $n^{1/2}\eta_n \geq c_1$ , and that  $n^{1/2}(a_n - \eta_n) \leq c_2 < \infty$ . Again, it then follows that  $0 \leq a_n/\eta_n \leq c_3 < \infty$ . For given  $\varepsilon > 0$  set  $\theta_n(\varepsilon) = a_n(1 + 2c(\varepsilon)n^{-1/2})$  where  $c(\varepsilon)$  is the constant given in Lemma 13. We then have for  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$

$$sa_n < \theta_n(\varepsilon)$$

for all  $n$ . Choose  $n_0(c(\varepsilon))$  large enough such that  $1 - c(\varepsilon)n^{-1/2} > 1/2$  holds for  $n > n_0(c(\varepsilon))$ . Consequently, for every  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$  and  $n > n_0(c(\varepsilon))$  we have from (A.2) (observing that  $\Phi$  has Lipschitz constant  $(2\pi)^{-1/2}$ )

$$\begin{aligned} &|p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) - p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\ &\leq (2\pi)^{-1/2}n^{1/2} \left( |s - 1|a_n + \left| \sqrt{(\theta_n(\varepsilon) + sa_n)^2/4 + s^2\eta_n^2} \right. \right. \\ &\quad \left. \left. - \sqrt{(\theta_n(\varepsilon) + a_n)^2/4 + \eta_n^2} \right| \right. \\ &\quad \left. + \left| \sqrt{(\theta_n(\varepsilon) - sa_n)^2/4 + s^2\eta_n^2} - \sqrt{(\theta_n(\varepsilon) - a_n)^2/4 + \eta_n^2} \right| \right). \end{aligned}$$

We note the elementary inequality  $|x^{1/2} - y^{1/2}| \leq 2^{-1}z^{-1/2}|x - y|$  for positive  $x, y, z$  satisfying  $\min(x, y) \geq z$ . Using this inequality with  $z = (1 - c(\varepsilon)n^{-1/2})^2\eta_n^2$

twice, we obtain for every  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$  and  $n > n_0(c(\varepsilon))$

$$\begin{aligned} & |p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) - p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\ & \leq (2\pi)^{-1/2}n^{1/2}|s - 1| \\ & \quad \times \left( a_n + \left[ (1 - c(\varepsilon)n^{-1/2})^2\eta_n^2 \right]^{-1/2} \left[ \theta_n(\varepsilon)a_n/2 + (s + 1) \left( (a_n^2/4) + \eta_n^2 \right) \right] \right). \end{aligned}$$

Since  $1 - c(\varepsilon)n^{-1/2} > 1/2$  for  $n > n_0(c(\varepsilon))$  by the choice of  $n_0(c(\varepsilon))$  and since  $a_n/\eta_n \leq c_3$  we obtain

$$\begin{aligned} & |p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) - p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\ & \leq (2\pi)^{-1/2}c(\varepsilon) \left( a_n + 2\eta_n^{-1} \left[ a_n^2 + (5/2) \left( (a_n^2/4) + \eta_n^2 \right) \right] \right) \\ & \leq (2\pi)^{-1/2}c(\varepsilon) \left( c_3 + (13/4)c_3^2 + 5 \right) \eta_n = c_4(\varepsilon)\eta_n \end{aligned} \tag{A.18}$$

for every  $n > n_0(c(\varepsilon))$  and  $s \in [1 - c(\varepsilon)n^{-1/2}, 1 + c(\varepsilon)n^{-1/2}]$ .

Now,

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n)h_n(s)ds \\ & \leq \int_0^\infty p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n)h_n(s)ds \\ & = \int_{1-c(\varepsilon)n^{-1/2}}^{1+c(\varepsilon)n^{-1/2}} p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n)h_n(s)ds \\ & \quad + \int_{|s-1| \geq c(\varepsilon)n^{-1/2}} p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n)h_n(s)ds \\ & =: B_1 + B_2. \end{aligned}$$

Clearly,  $0 \leq B_2 \leq \varepsilon$  holds by the choice of  $c(\varepsilon)$ , see Lemma 13. For  $B_1$  we have using (A.18)

$$\begin{aligned} & |B_1 - p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)| \\ & \leq \int_{1-c(\varepsilon)n^{-1/2}}^{1+c(\varepsilon)n^{-1/2}} |p_{A,n}(\theta_n(\varepsilon); 1, s\eta_n, sa_n, sa_n) \\ & \quad - p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)|h_n(s)ds + \varepsilon \\ & \leq c_4(\varepsilon)\eta_n + \varepsilon \end{aligned}$$

for  $n > n_0(c(\varepsilon))$ . It follows that

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n)h_n(s)ds \\ & \leq p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n) + c_4(\varepsilon)\eta_n + 2\varepsilon \end{aligned}$$

holds for  $n > n_0(c(\varepsilon))$ . Furthermore, the absolute difference between  $p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n)$  and  $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\theta \in C_{A,n})$  can be bounded as follows: Using Proposition 3, (A.2), observing that  $\Phi$  has Lipschitz constant  $(2\pi)^{-1/2}$ , and using the

elementary inequality noted earlier twice with  $z = \eta_n^2$  we obtain

$$\begin{aligned} & \left| p_{A,n}(\theta_n(\varepsilon); 1, \eta_n, a_n, a_n) - \Phi\left(n^{1/2}\sqrt{a_n^2 + \eta_n^2}\right) + \Phi\left(n^{1/2}(-a_n + \eta_n)\right) \right| \\ \leq & (2\pi)^{-1/2} n^{1/2} \left| -a_n c(\varepsilon) n^{-1/2} + \sqrt{a_n^2(1 + c(\varepsilon)n^{-1/2})^2 + \eta_n^2} - \sqrt{a_n^2 + \eta_n^2} \right| \\ & + (2\pi)^{-1/2} n^{1/2} \left| \sqrt{(a_n c(\varepsilon) n^{-1/2})^2 + \eta_n^2} - \sqrt{(a_n c(\varepsilon) n^{-1/2} + \eta_n)^2} \right| \\ \leq & (2\pi)^{-1/2} \left( 2a_n c(\varepsilon) + (2\eta_n)^{-1} a_n^2 (2c(\varepsilon) + c(\varepsilon)^2 n^{-1/2}) \right) \\ \leq & (2\pi)^{-1/2} (2c_3 c(\varepsilon) + 2^{-1} c_3^2 (2c(\varepsilon) + c(\varepsilon)^2)) \eta_n = c_5(\varepsilon) \eta_n. \end{aligned}$$

Consequently, for  $n > n_0(c(\varepsilon))$

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ & \leq \Phi(n^{1/2}\sqrt{a_n^2 + \eta_n^2}) - \Phi(n^{1/2}(-a_n + \eta_n)) \\ & \quad + (c_4(\varepsilon) + c_5(\varepsilon)) \eta_n + 2\varepsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} \int_0^\infty p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ & \geq \int_0^\infty \inf_{\theta \in \mathbb{R}} p_{A,n}(\theta; 1, s\eta_n, sa_n, sa_n) h_n(s) ds \\ & = \int_0^\infty \left[ \Phi(n^{1/2} s \sqrt{a_n^2 + \eta_n^2}) - \Phi(n^{1/2} s (-a_n + \eta_n)) \right] h_n(s) ds \\ & = T_{n-1}(n^{1/2}\sqrt{a_n^2 + \eta_n^2}) - T_{n-1}(n^{1/2}(-a_n + \eta_n)) \\ & \geq \Phi(n^{1/2}\sqrt{a_n^2 + \eta_n^2}) - \Phi(n^{1/2}(-a_n + \eta_n)) - 2\|\Phi - T_{n-1}\|_\infty. \end{aligned}$$

Since  $\eta_n \rightarrow 0$  and  $\|\Phi - T_{n-1}\|_\infty \rightarrow 0$  for  $n \rightarrow \infty$  and since  $\varepsilon$  was arbitrary the proof is complete.  $\square$

**Lemma 13.** *Suppose  $\sigma = 1$ . Then for every  $\varepsilon > 0$  there exists a  $c = c(\varepsilon) > 0$  such that*

$$\int_{\max(0, 1 - cn^{-1/2})}^{1 + cn^{-1/2}} h_n(s) ds \geq 1 - \varepsilon$$

*holds for every  $n \geq 2$ .*

*Proof.* By the central limit theorem and the delta-method we have that  $n^{1/2}(\hat{\sigma} - 1)$  converges to a normal distribution. It follows that  $n^{1/2}(\hat{\sigma} - 1)$  is (uniformly) tight. In other words, for every  $\varepsilon > 0$  we can find a real number  $c > 0$  such that for all  $n \geq 2$  holds

$$\Pr\left(\left|n^{1/2}(\hat{\sigma} - 1)\right| \leq c\right) \geq 1 - \varepsilon.$$

$\square$

**Lemma 14.** *Suppose  $n \geq 2$  and  $x \geq y \geq 0$ . Then*

$$T_{n-1}(x) \leq \Phi(x)$$

and

$$T_{n-1}(x-y) - T_{n-1}(-x-y) \leq \Phi(x-y) - \Phi(-x-y).$$

*Proof.* The first claim is well-known, see, e.g., [Kagan and Nagaev \(2008\)](#). The second claim follows immediately from the first claim, since by symmetry of  $\Phi$  and  $T_{n-1}$  we have

$$\begin{aligned} & \Phi(x-y) - \Phi(-x-y) - (T_{n-1}(x-y) - T_{n-1}(-x-y)) \\ = & [\Phi(x-y) - T_{n-1}(x-y)] + [\Phi(x+y) - T_{n-1}(x+y)] \geq 0. \end{aligned}$$

## References

□

- FAN, J. and LI, R. (2001). Variable Selection Via Nonconcave Penalized Likelihood and Its Oracle Properties. *Journal of the American Statistical Association* **96** 1348–1360. [MR1946581](#)
- FRANK, I. E. and FRIEDMAN, J. H. (1993). A Statistical View of Some Chemometrics Regression Tools (with discussion). *Technometrics* **35** 109–148.
- JOSHI, V. M. (1969). Admissibility of the usual confidence sets for the mean of a univariate or bivariate normal population. *Annals of Mathematical Statistics* **40** 1042–1067. [MR0264811](#)
- KAGAN, A. and NAGAEV, A. V. (2008). A lemma on stochastic majorization and properties of the Student distribution. *Theory of Probability and its Applications* **52** 160–164. [MR2354580](#)
- KNIGHT, K. and FU, W. (2000). Asymptotics of Lasso-Type Estimators. *Annals of Statistics* **28** 1356–1378. [MR1805787](#)
- LEEB, H. and PÖTSCHER, B. M. (2008). Sparse Estimators and the Oracle Property, or the Return of Hodges' Estimator. *Journal of Econometrics* **142** 201–211. [MR2394290](#)
- PÖTSCHER, B. M. (2009). Confidence Sets Based on Sparse Estimators Are Necessarily Large. *Sankhya* **71-A** 1–18.
- PÖTSCHER, B. M. and LEEB, H. (2009). On the Distribution of Penalized Maximum Likelihood Estimators: The LASSO, SCAD, and Thresholding. *Journal of Multivariate Analysis* **100** 2065–2082. [MR2543087](#)
- PÖTSCHER, B. M. and SCHNEIDER, U. (2009). On the Distribution of the Adaptive LASSO Estimator. *Journal of Statistical Planning and Inference* **139** 2775–2790. [MR2523666](#)
- TIBSHIRANI, R. (1996). Regression Shrinkage and Selection Via the Lasso. *Journal of the Royal Statistical Society Series B* **58** 267–288. [MR1379242](#)
- ZOU, H. (2006). The Adaptive Lasso and Its Oracle Properties. *Journal of the American Statistical Association* **101** 1418–1429. [MR2279469](#)