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Asymptotic direction in random walks in random environment revisited

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Abstract. Consider a random walk $\{X_n : n \geq 0\}$ in an elliptic i.i.d. environment in dimensions $d \geq 2$ and call P_0 its averaged law starting from 0. Given a direction $l \in \mathbb{S}^{d-1}$, $A_l = \{\lim_{n\to\infty} X_n \cdot l = \infty\}$ is called the event that the random walk is transient in the direction l. Recently Simenhaus proved that the following are equivalent: the random walk is transient in the neighborhood of a given direction; P_0-a.s. there exists a deterministic asymptotic direction; the random walk is transient in any direction contained in the open half space defined by this asymptotic direction. Here we prove that the following are equivalent: $P_0(A_l \cup A_{-l}) = 1$ in the neighborhood of a given direction; there exists an asymptotic direction ν such that $P_0(A_\nu \cup A_{-\nu}) = 1$ and P_0-a.s we have $\lim_{n\to\infty} X_n/|X_n| = \mathbb{1}_{A_\nu}\nu - \mathbb{1}_{A_{-\nu}}\nu$; $P_0(A_l \cup A_{-l}) = 1$ if and only if $l \cdot \nu \neq 0$. Furthermore, we give a review of some open problems.

1 Introduction

For each site $x \in \mathbb{Z}^d$, consider the vector $\omega(x) := \{\omega(x, e) : e \in \mathbb{Z}^d, |e| = 1\}$ such that $\omega(x, e) \in [0, 1]$ and $\sum_{|e|=1} \omega(x, e) = 1$. We call the set of possible values of these vectors $\mathcal{P}$ and define the *environment* $\omega = \{\omega(x) : x \in \mathbb{Z}^d\} \in \Omega := \mathcal{P}^{\mathbb{Z}^d}$. We define a random walk on the random environment ω, as a random walk $\{X_n : n \in \mathbb{N}\}$ with a transition probability from a site $x \in \mathbb{Z}^d$ to a nearest neighbor site $x + e$ with $|e| = 1$ given by $\omega(x, e)$. Let us call $P_{x,\omega}$ the law of this random walk starting from site x in the environment ω. Let $\mathbb{P}$ be a probability measure on Ω such that the coordinates $\{\omega(x)\}$ of ω are i.i.d. and such that the environment ω is *elliptic*, which means that $\mathbb{P}(\min_e \omega(0, e) > 0) = 1$. On the other hand, whenever there is a constant $\kappa > 0$ such that $\mathbb{P}(\min_e \omega(0, e) \geq \kappa) = 1$, we say the environment is *uniformly elliptic*. We call $P_{x,\omega}$ the *quenched* law of the random walk in random environment (RWRE), starting from site x. Furthermore, we define the *averaged* (or *annealed*) law of the RWRE starting from x by $P_x := \int_\Omega P_{x,\omega}\, d\mathbb{P}$. In this note we discuss some aspects of RWRE related to the a.s. existence of an asymptotic direction in dimension $d \geq 2$, briefly reviewing some of the open questions which have been unsolved and proving an improved version of a recent theorem of Simenhaus on the a.s. existence of an asymptotic direction.

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Some very fundamental and natural questions about this model remain open. Given a vector $l \in \mathbb{R}^d \backslash \{0\}$, define the event

$$A_l := \left\{ \lim_{n\to\infty} X_n \cdot l = \infty \right\}.$$

Whenever A_l occurs, we say that the random walk is transient in direction defined by l. Let also

$$B_l := \left\{ \liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0 \right\}.$$

Whenever B_l occurs, we say that the random walk is ballistic in direction defined by l. Recently, Sabot and Tournier showed in [4] and [12], that there exist examples of RWRE in elliptic i.i.d. environments in dimensions $d \geq 2$ wich are transient in a given direction but are not ballistic in that direction. Nevertheless, the following question remains open.

Open Problem 1.1. *For any RWRE in a uniformly elliptic i.i.d. environment in dimensions $d \geq 2$, does transience in direction l already imply ballisticity in direction l?*

Some partial progress related to this question has been achieved by Sznitman and Zerner [10], and later by Sznitman in [7–9], which we will discuss below. Under the assumption of uniform ellipticity the following lemma, which we call Kalikow's zero–one law, was proved by Kalikow in [2] (see also Lemma 1 in Sznitman and Zerner [10]). Thereafter, Zerner and Merkl derived the corresponding result under the assumption of ellipticity only (cf. Proposition 3 in [15]).

Lemma 1.2 (Kalikow, Sznitman–Zerner). *For any RWRE in an elliptic i.i.d. environment and $l \in \mathbb{S}^{d-1}$,*

$$P_0(A_l \cup A_{-l}) = 0 \quad or \quad 1.$$

On the other hand, in dimension $d = 1$ a zero–one law holds, that is, $P_0(A_l) \in \{0, 1\}$. Zerner and Merkl, proved the following (see Theorem 1 in [15] and a simplified proof in [14]).

Theorem 1.3 (Zerner–Merkl). *Consider an RWRE in an elliptic i.i.d. environment in dimension $d = 2$. Then, for $l \in \mathbb{S}^1$,*

$$P_0(A_l) = 0 \quad or \quad 1.$$

Nevertheless, we still have the following open problem.

Open Problem 1.4. *Consider an RWRE in an elliptic i.i.d. environment in dimensions $d \geq 3$. Does*

$$P_0(A_l) \in \{0, 1\}$$

hold for all $l \in \mathbb{S}^{d-1}$.

Combining Kalikow's zero–one law with the directional law of large numbers results of Sznitman and Zerner [10] as well as Zerner [13] one obtains the following theorem.

Theorem 1.5 (Sznitman–Zerner). *Given an RWRE in an elliptic i.i.d. environment in dimensions $d \geq 2$, there exist a direction $\nu \in \mathbb{S}^{d-1}$ and $v_1, v_2 \in [0, 1]$ such that P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_1 \nu \mathbb{1}_{A_\nu} - v_2 \nu \mathbb{1}_{A_{-\nu}}.$$

Indeed, we start with the following version of the directional law of large numbers. Theorem 3.2.2 of [11], the proof of which can be performed in the same manner with the assumption of ellipticity only instead of uniform ellipticity, states that for $l \in \mathbb{S}^{d-1}$ with

$$P_0(A_l \cup A_{-l}) = 1 \tag{1.1}$$

there exist $v_l, v_{-l} \in [0, 1]$ such that P_0-a.s.

$$\lim_{n\to\infty} \frac{X_n \cdot l}{n} = v_l \mathbb{1}_{A_l} - v_{-l} \mathbb{1}_{A_{-l}}. \tag{1.2}$$

Combining this with Theorem 1 of [13] we may omit assumption (1.1) and still obtain (1.2). Having (1.2) for the elements $l = e_1, \dots, e_d$ of the standard basis of $\mathbb{R}^d$, we obtain that $\lim_{n\to\infty} X_n/n$ exists P_0-a.s. and may take values in a set of cardinality 2^d. Employing the same argument as Goergen in page 1112 of [1] we now obtain that P_0-a.s. $\lim_{n\to\infty} X_n/n$ takes two values at most. Indeed, if there existed v_1 and v_2 noncolinear with $P_0(\lim_{n\to\infty} X_n/n = v_i) > 0$ for $i \in \{1, 2\}$, then by (1.2) one obtains that $l \cdot v_1 = v_l = l \cdot v_2$ for each l such that $l \cdot v_1, l \cdot v_2 > 0$. Letting l vary among a sufficiently rich set of such vectors, we conclude $v_1 = v_2$, a contradiction. This yields Theorem 1.5.

Whenever $\lim_{n\to\infty} X_n/|X_n|$ exists P_0-a.s. we call this limit the *asymptotic direction* and we say that a.s. an asymptotic direction exists. The existence of an asymptotic direction can already be established assuming some of the conditions introduced by Sznitman which imply ballisticity. Let $\gamma \in (0, 1)$ and $l \in \mathbb{S}^{d-1}$. By definition, condition $(T)_\gamma$ holds relative to l if for all $l' \in \mathbb{S}^{d-1}$ in a neighborhood of l,

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0(\{X_{T_{U_{l',b,L}}} \cdot l' < 0\}) < 0, \tag{1.3}$$

for all $b > 0$, where $U_{l',b,L} = \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$ is a slab and $T_{U_{l',b,L}} = \inf\{n \geq 0 : X_n \notin U_{l',b,L}\}$ is the first exit time of this slab. On the other hand, one says that condition (T') holds relative to l if condition $(T)_\gamma$ holds relative to l for every $\gamma \in (0, 1)$. It is known that whenever the environment is elliptic and i.i.d., for each $\gamma \in (0, 1)$ condition $(T)_\gamma$ relative to l implies transience in direction l and that a.s. an asymptotic direction exists which is deterministic. Also, whenever the environment is uniformly elliptic and i.i.d., Sznitman proved [9] that for each $\gamma \in (1/2, 1)$, condition $(T)_\gamma$ relative to l implies condition (T'), which in turn implies ballisticity. One of the open problems related to condition $(T)_\gamma$ is the following.

Open Problem 1.6. *Consider a RWRE in an elliptic i.i.d. environment. Does* (1.3) *for some $l' \in \mathbb{S}^{d-1}$ already imply $(T)_\gamma$ relative to l'?*

Recently in [5], Simenhaus established the following theorem which gives equivalent conditions for the existence of an a.s. asymptotic direction and showing that transience in a neighborhood of a given direction implies that an a.s. asymptotic direction exists.

Theorem 1.7 (Simenhaus). *Consider a RWRE in an elliptic i.i.d. environment. Then the following are equivalent*:

(a) *There exists a nonempty open set $O \subset \mathbb{S}^{d-1}$ such that*

$$P_0(A_l) = 1 \qquad \forall l \in O. \tag{1.4}$$

(b) *There exists $\nu \in \mathbb{S}^{d-1}$ such that P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \nu.$$

(c) *There exists $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for all $l \in \mathbb{S}^{d-1}$ with $l \cdot \nu > 0$.*

The example of Sabot and Tournier [4,12] of a RWRE in an elliptic i.i.d. environment which is transient but not ballistic in a given direction, shows that the above thoerem does apply in nontrivial situations. On the other hand, it is natural to wonder if there exists a statement analogous to Theorem 1.5, but related only to the existence of a possibly nondeterministic asymptotic direction. Here we answer affirmatively this question proving the following generalization of Theorem 1.7.

Theorem 1.8. *Consider a RWRE in an elliptic i.i.d. environment. Then the following are equivalent*:

(a) *There exists a nonempty open set $O \subset \mathbb{S}^{d-1}$ such that*

$$P_0(A_l \cup A_{-l}) = 1 \qquad \forall l \in O. \tag{1.5}$$

(b) *There exist d linearly independent unit vectors $l_1, \dots, l_d \in \mathbb{R}^d$ such that*

$$P_0(A_{l_k} \cup A_{-l_k}) = 1 \qquad \forall k \in \{1, \dots, d\}. \tag{1.6}$$

(c) *There exists $\nu \in \mathbb{S}^{d-1}$ with $P_0(A_\nu \cup A_{-\nu}) = 1$ such that P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \mathbb{1}_{A_\nu}\nu - \mathbb{1}_{A_{-\nu}}\nu. \tag{1.7}$$

(d) *There exists $\nu \in \mathbb{S}^{d-1}$ such that*

$$P_0(A_l \cup A_{-l}) = 1$$

if and only if $l \in \mathbb{S}^{d-1}$ is such that $l \cdot \nu \neq 0$. In this case, $P_0(A_l \Delta A_\nu) = 0$ and $P_0(A_{-l} \Delta A_{-\nu}) = 0$ for all l such that $l \cdot \nu > 0$.

If condition (1.5) is fulfilled but (1.4) is not, then if asymptotic directions exist we have to expect at least (and as it turns out at most, see also Proposition 1 in [5]) two of them. However, in dimensions $d \geq 3$, it is not known whether condition (1.5) can be fulfilled while (1.4) is not. In fact, if the statement of the Conjecture 1.4 holds, which is true in dimensions $d = 2$ [15], then the two conditions are equivalent. Note that due to Kalikow's zero–one law, condition (d) of Theorem 1.8 yields a complete characterisation of $P_0(A_l \cup A_{-l})$ for all $l \in \mathbb{S}^{d-1}$. As a consequence of this result, we obtain an a priori sharper version of (c) in Theorem 1.7:

(c′) *There exists $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for all $l \in \mathbb{S}^{d-1}$ with $l \cdot \nu > 0$ and $P_0(A_l) = 0$ if $l \cdot \nu \leq 0$.*

This observation and Theorem 1.3 imply that in dimension $d = 2$ there are at most three possibilities for the values of the set of probabilities $\{P_0(A_l) : l \in \mathbb{S}^{d-1}\}$.

Corollary 1.9. *Consider a RWRE in an elliptic i.i.d. environment in dimension $d = 2$. Then necessarily, only one of the following is satisfied*:

(a) *For all l, $P_0(A_l) = 0$.*
(b) *There exists a $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_\nu) = 1$ while $P_0(A_l) = 0$ for $l \neq \nu$.*
(c) *There exists a $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for l such that $l \cdot \nu > 0$ while $P_0(A_l) = 0$ for l such that $l \cdot \nu \leq 0$.*

The following corollary, which can be deduced from Theorem 1.8, shows that knowing that there is an l^* such that $P_0(A_{l^*}) = 1$ and $P_0(A_l) > 0$ for all l in a neighborhood of l^*, determines the value of $P_0(A_l)$ for all directions l.

Corollary 1.10. *Consider a RWRE in an elliptic i.i.d. environment. The following are equivalent*:

(a) *There exists $l^* \in \mathbb{S}^{d-1}$ and some neighborhood $\mathcal{U}(l^*)$ such that $P_0(A_{l^*}) = 1$ and $P_0(A_l) > 0$ for all $l \in \mathcal{U}(l^*)$.*

(b) *There exists* $\nu \in \mathbb{R}^d$ *such that* $P_0(A_l) = 1$ *for* l *such that* $l \cdot \nu > 0$, *while* $P_0(A_l) = 0$ *for* l *such that* $l \cdot \nu \leq 0$.

In particular, this shows that in Theorem 1.7, condition (a) can be replaced by the a priori weaker condition (a) of this corollary.

In the rest of this paper we prove Theorem 1.8 and Corollary 1.10. In Section 2 we prove some preliminary results needed for the proofs and in Section 3 we apply them to prove the theorem and the corollary.

2 Preliminary results

The implications (d) ⇒ (a) ⇒ (b) of Theorem 1.8 are obvious, so here we introduce the renewal structure and prove some preliminary results needed to show that (b) ⇒ (c) ⇒ (d). For $l \in \mathbb{R}^d$ set

$$D_l := \inf\{n \in \mathbb{N} : X_n \cdot l < X_0 \cdot l\}$$

and for $B \subset \mathbb{R}^d$ define the first-exit time

$$D_B := \inf\{n \in \mathbb{N} : X_n \notin B\};$$

as usual, we set $\inf \varnothing := \infty$. We also define for $l \in \mathbb{R}^d$ and $s \in [0, \infty)$,

$$T_s^l := \inf\{n \in \mathbb{N} : X_n \cdot l > s\}.$$

Due to their linear independence, the vectors $l_1, \ldots, l_d$ of Theorem 1.8(b) give rise to the following 2^d cones:

$$C_\sigma := \bigcap_{k=1}^{d} \{x \in \mathbb{R}^d : \sigma_k(l_k \cdot x) \geq 0\}, \qquad \sigma \in \{-1, 1\}^d.$$

Furthermore, for $\lambda \in (0, 1]$ and $l \in \mathbb{R}^d \backslash \{0\}$ we will employ the notation

$$C_\sigma(\lambda, l) := \bigcap_{k=1}^{d} \{x \in \mathbb{R}^d : \big(\lambda\sigma_k l_k + (1-\lambda)l\big) \cdot x \geq 0\}, \tag{2.1}$$

where the vectors defining the cone are now interpolations of the $\sigma_k l_k$ with l. Note that $C_\sigma(\lambda, l)$ is a nondegenerate cone with base of finite area if and only if the vectors $\lambda\sigma_k l_k + (1-\lambda)l$, $k = 1, \ldots, d$, are linearly independent. In particular, $C_\sigma(1, l) = C_\sigma$ for all $\sigma \in \{-1, 1\}^d$ and l.

We will often choose σ such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$, which under (1.6) is possible since we then have

$$1 = P_0\Bigg(\bigcap_{k=1}^{d} A_{l_k} \cup A_{l_{-k}}\Bigg) = P_0\Bigg(\bigcup_{\sigma} \bigcap_{k=1}^{d} A_{\sigma_k l_k}\Bigg) = \sum_{\sigma} P_0\Bigg(\bigcap_{k=1}^{d} A_{\sigma_k l_k}\Bigg). \tag{2.2}$$

For a given $\sigma \in \{-1, 1\}^d$ which will usually be clear from the context, we will frequently consider vectors $l \in \mathbb{R}^d$ satisfying the condition

$$\inf_{x \in C_\sigma \cap \mathbb{S}^{d-1}} l \cdot x > 0. \tag{2.3}$$

Note here that for σ such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$ and l satisfying (2.3), the inequality $P_0(A_l) \geq P_0(\bigcap_{k=1}^d A_{\sigma_k l_k})$ implies that the measure $P_0(\cdot | A_l)$ is well defined. For such l we will then show the existence of a $P_0(\cdot | A_l)$-a.s. asymptotic direction. The strategy of our proof is based to a significant part on that of Theorem 1.7.

We start with the following lemma which ensures that if with positive probability the random walk finally ends up in a cone, then the probability that it does so and never exits a half-space containing this cone is positive as well.

Lemma 2.1. *Let $\sigma \in \{-1, 1\}^d$ and $l \in \mathbb{S}^{d-1}$ be such that (2.3) holds. Then*

$$P_0\left(\bigcap_{k=1}^d A_{\sigma_k l_k}\right) > 0 \quad \Longrightarrow \quad P_0\left(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}\right) > 0.$$

Proof. Assume $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) = 0$. Then $\mathbb{P}$-a.s.

$$P_{0,\omega}\left(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}\right) = 0. \tag{2.4}$$

For $y \in \mathbb{R}^d$ with $l \cdot y \geq 0$ this implies

$$P_y\left(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x : l \cdot x \geq 0\}} = \infty\}\right) = 0. \tag{2.5}$$

Indeed, if there existed such y with $\mathbb{P}(\{\omega \in \Omega | P_{y,\omega}(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x \geq 0\}} = \infty\}) > 0\}) > 0$ then for ω such that $P_{y,\omega}(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq 0\}} = \infty\}) > 0$, a random walker starting in 0 would, with positive probability with respect to $P_{0,\omega}$, hit y before hitting $\{x : l \cdot x < 0\}$ (due to ellipticity) and from there on finally end up in C_σ without hitting $\{x : l \cdot x < 0\}$; this is a contradiction to (2.4), hence (2.5) holds.

Choosing a sequence $(y_n) \subset C_\sigma$ such that $l \cdot y_n \to \infty$ as $n \to \infty$ we therefore get

$$\begin{aligned} 0 &= P_{y_n}\left(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq 0\}} = \infty\}\right) \\ &\geq P_0\left(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x \geq -l\cdot y_n\}} = \infty\}\right) \to P_0\left(\bigcap_{k=1}^d A_{\sigma_k l_k}\right) \end{aligned}$$

as $n \to \infty$. To obtain the inequality we employed the translation invariance of $\mathbb{P}$ as well as the monotonicity of events. $\square$

The following lemma will be employed to set up a renewal structure; it can in some way be seen as an analog to Lemma 1 of [5].

Lemma 2.2. *Let $\sigma \in \{-1,1\}^d$ be such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$. Then for each l such that (2.3) holds, one has*

$$P_0\big(D_{C_\sigma(\lambda,l)} = \infty\big) > 0 \tag{2.6}$$

for $\lambda > 0$ small enough.

Proof. Lemma 2.1 implies $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) > 0$. Due to the ellipticity of the walk and the independence of the environment we therefore obtain

$$P_0\left(\{X_1 \cdot l > 0\} \cap \bigcap_{k=1}^d A_{\sigma_k l_k}(X_{1+\cdot} - X_1) \cap \{D_l(X_{1+\cdot} - X_1) = \infty\}\right) > 0, \tag{2.7}$$

where we name explicitly the path $X_{1+\cdot} - X_1$ to which the corresponding events $A_{\sigma_k l_k}$ and D_l refer. Each path of the event in (2.7) is fully contained in $C_\sigma(\lambda,l)$ for $\lambda > 0$ small enough. Thus, the continuity from above of P_0 yields

$$\begin{aligned} P_0\Bigg(&\big\{D_{C_\sigma(\lambda,l)} = \infty\big\} \cap \{X_1 \cdot l > 0\} \\ &\cap \bigcap_{k=1}^d A_{\sigma_k l_k}(X_{1+\cdot} - X_1) \cap \{D_l(X_{1+\cdot} - X_1) = \infty\}\Bigg) > 0 \end{aligned} \tag{2.8}$$

for all $\lambda > 0$ small enough. $\square$

Employing Lemma 2.2, for $\sigma \in \{-1,1\}^d$ with $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$ as in [5] we can introduce a cone renewal structure, where we choose $l \in \mathbb{S}^{d-1}$ such that (2.3) is fulfilled and the cone to work with is $C_l := C_\sigma(\lambda,l)$, where we fixed $\lambda > 0$ small enough as in the statement of Lemma 2.2. Note that for fixed l the set $C_\sigma(\lambda,l)$ is indeed a cone as long as $\lambda > 0$ is chosen small enough [since the defining vectors in (2.1) are linearly independent].

For $k \in \mathbb{N}$ let $\theta_k : (\mathbb{Z}^d)^{\mathbb{N}_0} \ni (x_n) \mapsto (x_{n+k}) \in (\mathbb{Z}^d)^{\mathbb{N}_0}$ be the k-fold time shift. We define

$$S_0^l := T_0^l, \qquad R_0^l := D_{X_{S_0^l} + C_l} \circ \theta_{S_0^l} + S_0^l, \qquad M_0^l := \max\{X_n \cdot l : 0 \le n \le R_0^l\}$$

and inductively for $k \ge 1$:

$$\begin{aligned} &S_k^l := T_{M_{k-1}^l}^l, \qquad R_k^l := D_{X_{S_k^l} + C_l} \circ \theta_{S_{k-1}^l} + S_k^l, \\ &M_k^l := \max\{X_n \cdot l : 0 \le n \le R_k^l\}, \end{aligned}$$

where for $x \in \mathbb{Z}^d$ by $x + C_l$ we denote the cone C_l shifted such that its apex lies at x. Furthermore, set

$$K^l := \inf\{k \in \mathbb{N} : S_k^l < \infty, R_k^l = \infty\}$$

as well as

$$\tau_1^l := S_{K^l}^l,$$

that is, τ_1^l is the first time at which the walk reaches a new maximum in direction l and never exits the cone C_l shifted to $X_{\tau_1^l}$. We define inductively the sequence of cone renewal times with respect to C_l by

$$\tau_k^l := \tau_1^l(X_{\cdot+\tau_{k-1}^l} - X_{\tau_{k-1}^l}) + \tau_{k-1}^l$$

for $k \geq 2$.

The following lemma shows that under the conditions of Lemma 2.2 the sequence τ_k^l is well defined on A_l. It can be seen as an analog to Proposition 2 of [5].

Lemma 2.3. *Let $\sigma \in \{-1, 1\}^d$ be such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose l and λ such that (2.3) and (2.6) hold. Then $P_0(\cdot|A_l)$-a.s. one has $K^l < \infty$.*

Proof. Employing Lemma 2.2, the proof takes advantage of the fact that each time the walk hits a new maximum in direction l, the event that from there on it never exits the cone centred at that point is independent from the past and has the same positive probability. This then gives rise to a geometrically distributed renewal structure. For further details on these standard renewal arguments see the proofs of Proposition 2 in [5] or Proposition 1.2 in [10] which proceed in an analogous way. □

Lemma 2.4. *Let $\sigma \in \{-1, 1\}^d$ be such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose l and λ such that (2.3) and (2.6) hold. Then $((X_{\tau_1^l \wedge \cdot}, \tau_1^l), \ldots, (X_{(\tau_k^l+\cdot)\wedge\tau_{k+1}^l} - X_{\tau_k^l}, \tau_{k+1}^l - \tau_k^l)), \ldots$ are independent under $P_0(\cdot|A_l)$ and for $k \geq 1$, $((X_{(\tau_k^l+\cdot)\wedge\tau_{k+1}^l} - X_{\tau_k}^l), \tau_{k+1}^l - \tau_k^l)$ under $P_0(\cdot|A_l)$ is distributed like $(X_{\tau_1^l \wedge \cdot}, \tau_1^l)$ under $P_0(\cdot|\{D_{C_l} = \infty\})$.*

Proof. This result is intrinsic to the i.i.d. property of the environment and the proof is analogous to the proof of Corollary 1.5 in [10]. □

The following lemma has been derived in Simenhaus' thesis [6] (Lemma 2 in there). Here we state it and prove it under a slightly weaker assumption.

Lemma 2.5. *Let $\sigma \in \{-1, 1\}^d$ be such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose $l \in \mathbb{Z}^d$ and λ such that (2.3) and (2.6) hold and the g.c.d. of the coordinates of l is 1. Then*

$$E_0(X_{\tau_1^l} \cdot l | D_{C_l} = \infty) = \frac{1}{P_0(D_{C_l} = \infty|A_l) \lim_{i\to\infty} P_0(T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i)}$$
$$< \infty$$

and

$$E_0(X_{\tau_1^l}|D_{C_l}=\infty) \tag{2.9}$$

is well defined.

Remark 2.6. A fundamental consequence of working with the cone renewal structure instead of working with slabs is the existence of (2.9); see also Proposition 2.7.

Proof. The proof leans on the proof of Lemma 3.2.5 in [11] which is due to Zerner. Due to the strong Markov property and the independence and translation invariance of the environment we have for $i>0$:

$$\begin{aligned}
&P_0(\{\exists k\geq 1: X_{\tau_k^l}\cdot l=i\}\cap A_l)\\
&\quad=\sum_{x\in\mathbb{Z}^d,l\cdot x=i}\mathbb{E}P_{0,\omega}(T_{i-1}^l<\infty, X_{T_{i-1}^l}=x, D_{C_l+X_{T_{i-1}^l}}\circ\theta_{T_{i-1}^l}=\infty)\\
&\quad=\sum_{x\in\mathbb{Z}^d,l\cdot x=i}\mathbb{E}P_{0,\omega}(T_{i-1}^l<\infty, X_{T_{i-1}^l}=x)P_{x,\omega}(D_{C_l+x}=\infty)\\
&\quad=P_0(T_{i-1}^l<\infty, X_{T_{i-1}^l}\cdot l=i)P_0(D_{C_l}=\infty).
\end{aligned} \tag{2.10}$$

At the same time using $\{\tau_1^l<\infty\}=A_l$, a fact which is proven similarly to Proposition 1.2 of [10], we compute

$$\begin{aligned}
&\lim_{i\to\infty}P_0(\{\exists k\geq 1: X_{\tau_k^l}\cdot l=i\}|A_l)\\
&\quad=\lim_{i\to\infty}P_0(\{\exists k\geq 2: X_{\tau_k^l}\cdot l=i\}|A_l)\\
&\quad=\lim_{i\to\infty}\sum_{n\geq 1}P_0(\{\exists k\geq 2: X_{\tau_k^l}\cdot l=i\}\cap\{X_{\tau_1^l}\cdot l=n\}|A_l)\\
&\quad=\lim_{i\to\infty}\sum_{n\geq 1}P_0\big(\{\exists k\geq 2: (X_{\tau_k^l}-X_{\tau_1^l})\cdot l=i-n\}\cap\{X_{\tau_1^l}\cdot l=n\}|A_l\big)\\
&\quad=\lim_{i\to\infty}\sum_{n\geq 1}P_0\big(\{\exists k\geq 2: (X_{\tau_k^l}-X_{\tau_1^l})\cdot l=i-n\}|A_l\big)P_0(X_{\tau_1^l}\cdot l=n|A_l),
\end{aligned} \tag{2.11}$$

where to obtain the last equality we took advantage of Lemma 2.4. Blackwell's renewal theorem in combination with Lemma 2.4 now yields

$$\lim_{i\to\infty}P_0\big(\{\exists k\geq 2: (X_{\tau_k^l}-X_{\tau_1^l})\cdot l=i-n\}|A_l\big)=\frac{1}{E_0(X_{\tau_1^l}\cdot l|D_{C_l}=\infty)}$$

and thus (2.11) implies

$$\lim_{i\to\infty}P_0(\exists k\geq 1: X_{\tau_k^l}\cdot l=i|A_l)=\frac{1}{E_0(X_{\tau_1^l}\cdot l|D_{C_l}=\infty)}.$$

Therefore, taking into consideration (2.10) we infer

$$\begin{aligned} &E_0(X_{\tau_1^l}\cdot l|D_{C_l}=\infty)\\ &\quad=\frac{1}{P_0(D_{C_l}=\infty|A_l)\lim_{i\to\infty}P_0(T_{i-1}^l<\infty, X_{T_{i-1}^l}\cdot l=i)}. \end{aligned} \tag{2.12}$$

It remains to show that the right-hand side of (2.12) is finite. Writing $l_{\max} := \max\{|l_1|,\dots,|l_d|\}$ for the maximum of the absolute values of the coordinates of l we have

$$\begin{aligned} &\sum_{i=k}^{k+l_{\max}-1} P_0(\{T_{i-1}^l<\infty, X_{T_{i-1}^l}\cdot l=i\})\\ &\quad\geq \sum_{i=k}^{k+l_{\max}-1} P_0(\{T_{i-1}^l<\infty, X_{T_{k-1}^l}\cdot l=i\})\geq P_0(A_l) \qquad \forall k\in\mathbb{N}, \end{aligned}$$

where the first inequality follows since $\{X_{T_{k-1}^l}\cdot l=i\}\subseteq\{X_{T_{i-1}^l}\cdot l=i\}$ for all $k\in\mathbb{N}$ and $i\in\{k,\dots,k+l_{\max}-1\}$. This now yields $\lim_{i\to\infty}P_0(\{T_{i-1}^l<\infty, X_{T_{i-1}^l}\cdot l = i\})\geq l_{\max}^{-1}P_0(A_l)>0$, whence due to (2.12) we obtain

$$E_0(X_{\tau_1^l}\cdot l|\{D_{C_l}=\infty\})<\infty. \tag{2.13}$$

Since on $\{D_{C_l}=\infty\}$ there exists a constant $C>0$ such that $|X_{\tau_1^l}|\leq CX_{\tau_1^l}\cdot l$, we infer as a direct consequence of (2.13) that (2.9) is well-defined. □

We can now employ the above renewal structure to obtain an a.s. constant asymptotic direction on A_l.

Proposition 2.7. *Let $\sigma\in\{-1,1\}^d$ be such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k})>0$ and choose $l\in\mathbb{Z}^d$ and λ such that* (2.3) *and* (2.6) *hold and the g.c.d. of the coordinates of l is* 1. *Then $P_0(\cdot|A_l)$-a.s.*

$$\lim_{n\to\infty}\frac{X_n}{|X_n|}=\frac{E_0(X_{\tau_1^l}|\{D_{C_l}=\infty\})}{|E_0(X_{\tau_1^l}|\{D_{C_l}=\infty\})|}.$$

Remark 2.8. In particular, this proposition implies that the limit does not depend on the particular choice of l nor λ (for λ sufficiently small). Note that the independence of l stems from the fact that if l_1, l_2 satisfy (2.3) we have $P_0(A_{l_1}\cap A_{l_2})>0$.

Proof. Due to Lemmas 2.2 to 2.5 we may apply the law of large numbers to the sequence $(X_{\tau_k^l})_{k\in\mathbb{N}}$ yielding

$$\frac{X_{\tau_k^l}}{k}\to E_0(X_{\tau_1^l}|D_{C_l}=\infty) \qquad P_0(\cdot|A_l)\text{-a.s.}, \qquad k\to\infty,$$

and hence

$$\frac{X_{\tau_k^l}}{|X_{\tau_k^l}|} \to \frac{E_0(X_{\tau_1^l}|D_{C_l}=\infty)}{|E_0(X_{\tau_1^l}|D_{C_l}=\infty)|} \qquad P_0(\cdot|A_l)\text{-a.s.}, \qquad k\to\infty.$$

Using standard methods to estimate the intermediate terms (cf. page 9 in [5]) one obtains

$$\lim_{n\to\infty}\frac{X_n}{|X_n|} = \frac{E_0(X_{\tau_1^l}|D_{C_l}=\infty)}{|E_0(X_{\tau_1^l}|D_{C_l}=\infty)|} \qquad P_0(\cdot|A_l)\text{-a.s.}$$

□

The following two results will be needed to obtain results about transience in directions orthogonal to the asymptotic direction.

Lemma 2.9. *Let $(Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence on some probability space $(\mathcal{X},\mathcal{F},P)$ with expectation $EY_1=0$ and variance $EY_1^2\in(0,\infty]$. Then, for $S_n := \sum_{k=1}^n Y_k$ we have $P(\liminf_{n\to\infty} S_n=-\infty) = P(\limsup_{n\to\infty} S_n=\infty)=1$.*

Proof. We only prove $P(\liminf_{n\to\infty} S_n=-\infty)=1$, the remaining equality is proved in a similar way. Setting $\varepsilon := (-\operatorname{ess\,inf} Y_1/2)\wedge 1$ one can show for all $x\in\mathbb{R}$, using the strong Markov property at the entrance times of S_n to the interval $[x, x+\varepsilon]$, that $P(\liminf_{n\to\infty} S_n\in[x,x+\varepsilon])=0$. This then implies $P(\liminf_{n\to\infty} S_n=\pm\infty)=1$. But Kesten's result in [3] yields $\liminf_{n\to\infty} S_n/n>0$ $P(\cdot\cap\{\liminf_{n\to\infty} S_n=\infty\})$-a.s., while by the strong law of large numbers we have $\lim_{n\to\infty} S_n/n=0$ P-a.s. This yields $P(\liminf_{n\to\infty} S_n = \infty)=0$ and hence finishes the proof. □

Lemma 2.10. *Let $l\in\mathbb{R}^d$ be such that*

$$P_0\Big(\lim_{n\to\infty} X_n/|X_n| = l\Big) > 0. \tag{2.14}$$

Then, for $l^*\in\mathbb{R}^d$ such that $l^*\cdot l=0$ one has $P_0((A_{l^*}\cup A_{-l^*})\cap A_l)=0$.

Proof. We choose a basis $l_1,\ldots,l_d$ of $\mathbb{R}^d$ and σ such that l is contained in the interior of the cone C_σ corresponding to $l_1,\ldots,l_d$ and (2.3) is satisfied. Furthermore, by (2.14) and Lemma 2.2 we may choose λ such that condition (2.6) is satisfied for the corresponding cone $C_\sigma(\lambda,l)$. Lemma 2.3 yields that the sequence $(\tau_k^l)_{k\in\mathbb{N}}$ is well defined and Lemmas 2.4 and 2.5 yield that under $P_0(\cdot|A_l)$ the sequence $((X_{\tau_2^l}-X_{\tau_1^l})\cdot l^*, (X_{\tau_3^l}-X_{\tau_2^l})\cdot l^*,\ldots)$ is i.i.d. with expectation 0, the latter being due to the validity of Lemma 2.5 as well as (1.7) and $l^*\cdot l=0$. Indeed, Proposition 2.7 yields

$$E_0\big(X_{\tau_1^l}\cdot l^*|D_{C_\sigma(\lambda,l)}=\infty\big) = \big|E_0\big(X_{\tau_1^l}|D_{C_\sigma(\lambda,l)}=\infty\big)\big| \underbrace{\lim_{k\to\infty}\frac{X_{\tau_k^l}}{|X_{\tau_k^l}|}}_{=l}\cdot l^* = 0$$

$P_0(\cdot|A_l)$-a.s. Applying Lemma 2.9 to the sequence $((X_{\tau_2^l} - X_{\tau_1^l}) \cdot l^*, (X_{\tau_3^l} - X_{\tau_2^l}) \cdot l^*, \dots)$ yields $P_0((A_{l^*} \cup A_{-l^*}) \cap A_l) = 0$. □

3 Proof of Theorem 1.8 and Corollary 1.10

3.1 Proof of Theorem 1.8

We first prove that condition (b) implies (c). For this purpose consider σ such that $P_0(\bigcap_{k=1}^d A_{\sigma_k l_k}) > 0$ and $l \in \mathbb{Z}^d$ that satisfies (2.3) and for which the g.c.d. of the coordinates of l is 1. Then, since $P(\bigcap_{k=1}^d A_{\sigma_k l_k} \backslash A_l) = 0$, Proposition 2.7 yields that $P_0(\cdot | \bigcap_{k=1}^d A_{\sigma_k l_k})$-a.s.

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \frac{E_0(X_{\tau_1^l} | D_{C_l} = \infty)}{|E_0(X_{\tau_1^l} | D_{C_l} = \infty)|} =: \nu, \tag{3.1}$$

which due to Remark 2.8 is independent of the respective l chosen.

In combination with (2.2) this implies that

$$\lim_{n\to\infty} X_n/|X_n| \qquad \text{exists a.s.} \tag{3.2}$$

Now Proposition 1 of [5] states that if two elements $\nu \neq \nu'$ of $\mathbb{S}^{d-1}$ occur with positive probability each with respect to P_0 as asymptotic directions, then $\nu = -\nu'$. Thus, (3.2) already implies (c).

Now with respect to the implication (c) $\Rightarrow$ (d) note that the only thing that is not obvious at a first glance is that $l \cdot \nu = 0$ implies $P_0(A_l \cup A_{-l}) = 0$. However, Lemma 2.10 yields $P_0((A_l \cup A_{-l}) \cap (A_\nu \cup A_{-\nu})) = 0$ which due to $P_0(A_\nu \cup A_{-\nu}) = 1$ yields the desired result.

3.2 Proof of Corollary 1.10

We only have to prove (a) $\Rightarrow$ (b). Given (a), Theorem 1.8 yields the existence of $\nu \in \mathbb{S}^{d-1}$ such that

$$P_0(A_\nu \cup A_{-\nu}) = 1 \tag{3.3}$$

and (1.7) holds.

Now if $l^* \cdot \nu \neq 0$ then $P_0(A_\nu \cap A_{l^*}) = 1$ or $P_0(A_{-\nu} \cap A_{l^*}) = 1$, respectively, and hence $P_0(A_\nu) = 1$ or $P_0(A_{-\nu}) = 1$, which due to Theorem 1.7 finishes the proof. Thus, assume

$$l^* \cdot \nu = 0 \tag{3.4}$$

from now on. Then Lemma 2.10 yields $P_0((A_{l^*} \cup A_{-l^*}) \cap (A_\nu \cup A_{-\nu})) = 0$ which due to (3.3) implies $P_0(A_{l^*} \cup A_{-l^*}) = 0$, a contradiction to assumption (a).

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