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REPLY TO "ON SOME PROBLEMS IN THE ARTICLE *EFFICIENT LIKELIHOOD ESTIMATION IN STATE SPACE MODELS*[1]" BY CHENG-DER FUH [*Ann. Statist.* 34 (2006) 2026–2068]

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The author is grateful for the comments by Dr. Jensen. This note is in reply to his comments.

Problem 2.1. Definition of iterated function system.

$$\mathbf{P}_\theta(\xi_j)h(x) = \int_{y\in\mathcal{X}} p_\theta(y,x) f(\xi_j;\theta|x,\xi_{j-1})h(y)m(dy). \tag{2.6}$$

Define the composition of two random functions as

$$\begin{aligned}
&\mathbf{P}_\theta(\xi_{j+1})\circ\mathbf{P}_\theta(\xi_j)h(x)\\
&\quad= \int_{z\in\mathcal{X}} p_\theta(z,x) f(\xi_{j+1};\theta|x,\xi_j)\\
&\qquad\times\left(\int_{y\in\mathcal{X}} p_\theta(y,z) f(\xi_j;\theta|z,\xi_{j-1})h(y)m(dy)\right)m(dz).
\end{aligned} \tag{2.7}$$

Page 2042. C1. ...for all $s_0, s_1 \in \mathbf{R}^d$, and $\sup_{x\in\mathcal{X}}\int p_\theta(y,x)m(dy) < \infty$. Since m is σ-finite, there exist pairwise disjoint $\mathcal{X}_n$ such that $\mathcal{X} = \bigcup_{n=1}^\infty \mathcal{X}_n$, and $0 < m(\mathcal{X}_n) < \infty$. Assume $E[\sum_{n=1}^\infty \frac{1}{2^n}\sup_{x\in\mathcal{X}_n} f(\xi_1;\theta|x,s_0)] < \infty$ for all $s_0 \in \mathbf{R}^d$. Denote $g_\theta(\xi_0,\xi_1) = \sup_{x\in\mathcal{X}}\int p_\theta(y,x) f(\xi_1;\theta|x,\xi_0)m(dy)$. Furthermore, we assume that there exists $p \geq 1$ as in K2 such that

$$\sup_{(x_0,s_0)\in\mathcal{X}\times\mathbf{R}^d} E^\theta_{(x_0,s_0)}\left\{\log\left(g_\theta(s_0,\xi_1)\cdots g_\theta(\xi_{p-1},\xi_p)\frac{w(X_p,\xi_p)}{w(x_0,s_0)}\right)\right\} < 0. \tag{5.2}$$

The example on Page 2044, L12, holds if $\alpha \neq 0$. The original (5.6) was wrong; it should be

$$M_n := \mathbf{P}_\theta(\xi_n)\circ\cdots\circ\mathbf{P}_\theta(\xi_1)\circ\mathbf{P}_\theta(\xi_0)\pi \qquad \text{(page 2045)}. \tag{5.6}$$

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[1]Editors' note: The standard *Annals of Statistics* policy is that brief author responses and corrigenda are only reviewed for style and appropriateness; authors themselves are responsible for their correctness.

Page 2046. LEMMA 3. ...*Furthermore*, *under conditions* C1, C6–C9, *the function g defined in* (5.7) *belongs to* $\mathcal{L}(Q \times Q)$.

PROOF OF LEMMA 3. We consider only the case of $\mathbf{P}(\xi_1)$, since the case of $\mathbf{P}(\xi_0)$ and $\mathbf{P}(\xi_j)$, for $j = 2, \ldots, n$, is a straightforward consequence. For any two elements $h_1, h_2 \in \mathbf{M}$, and two fixed elements $s_0, s_1 \in \mathbf{R}^d$, by (5.8) we have

$$\begin{aligned}
&d(\mathbf{P}(s_1)h_1, \mathbf{P}(s_1)h_2)\\
&\quad = \sup_{x\in\mathcal{X}}\biggl|\int p_\theta(y,x) f(s_1;\theta|x,s_0)h_1(y)m(dy)\\
&\qquad\qquad - \int p_\theta(y,x) f(s_1;\theta|x,s_0)h_2(y)m(dy)\biggr|\\
&\quad \le d(h_1,h_2)\sup_{x\in\mathcal{X}}\int p_\theta(y,x) f(s_1;\theta|x,s_0)m(dy)\\
&\quad \le C\biggl(\sup_{x\in\mathcal{X}}\int p_\theta(y,x)m(dy)\biggr)d(h_1,h_2),
\end{aligned}$$

where $0 < C = \sup_{x\in\mathcal{X}} f(s_1;\theta|x,s_0) < \infty$, and by assumption C1, is a constant. Note that $\sup_{x\in\mathcal{X}}\int p_\theta(y,x)m(dy) < \infty$ by assumption C1. The equality holds only if $h_1 = h_2$ m-almost surely. This proves the condition of Lipschitz continuity in the second argument.

Note that C1 implies that K1 holds. Recall that $M_n = \mathbf{P}(\xi_n)\circ\cdots\circ\mathbf{P}(\xi_1)\circ\mathbf{P}(\xi_0)\pi$ for $\pi \in \mathbf{M}$ in (5.6). To prove the weighted mean contraction property K2, we observe that for $p \ge 1$,

$$\begin{aligned}
&\sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\biggl\{\log\biggl(L_p\frac{w(X_p,\xi_p)}{w(x_0,s_0)}\biggr)\biggr\}\\
&\quad = \sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\biggl\{\log\biggl(\sup_{h_1\ne h_2}\frac{d(M_ph_1,M_ph_2)}{d(h_1,h_2)}\frac{w(X_p,\xi_p)}{w(x_0,s_0)}\biggr)\biggr\}\\
&\quad < \sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\biggl\{\log\biggl(\prod_{j=1}^{p}\biggl[\sup_{x_j\in\mathcal{X}}\int p_\theta(x_{j-1},x_j)\\
&\qquad\qquad \times f(\xi_j;\theta|x_j,s_{j-1})\\
&\qquad\qquad \times m(dx_{j-1})\biggr]\frac{w(X_p,\xi_p)}{w(x_0,s_0)}\biggr)\biggr\} < 0.
\end{aligned}\tag{7.1}$$

The last inequality follows from (5.2) in condition C1.

To verify that assumption K3 holds, as m is σ-finite, we have $\mathcal{X} = \bigcup_{n=1}^{\infty}\mathcal{X}_n$ where the $\mathcal{X}_n$ are pairwise disjoint and $0 < m(\mathcal{X}_n) < \infty$. Set

$$h(x) = \sum_{n=1}^{\infty}\frac{I_{\mathcal{X}_n}(x)}{2^n m(\mathcal{X}_n)}.\tag{7.2}$$

It is easy to see that $\int_{x\in\mathcal{X}} h(x)m(dx) = 1$ and hence belongs to $\mathbf{M}$. Observe that

$$\begin{aligned}
&\mathbf{E}d^2(\mathbf{P}(\xi_1)h, h)\\
&\quad= \mathbf{E}\left[\sup_{x_1\in\mathcal{X}}\left|\int p_\theta(x_0,x_1)f(\xi_1;\theta|x_1,s_0)h(x_0)m(dx_0) - h(x_1)\right|\right]\\
&\quad\le \mathbf{E}\left[\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{x_1\in\mathcal{X}_n} f(\xi_1;\theta|x_1,s_0)\right]\left[\sup_{x_1\in\mathcal{X}}\int p_\theta(x_0,x_1)m(dx_0)\right]\\
&\qquad+\sup_{x_1\in\mathcal{X}}|h(x_1)|.
\end{aligned}\tag{7.3}$$

Note that $h(x)$ is piecewise constant by definition (7.2), $E[\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{x\in\mathcal{X}_n} f(\xi_1;\theta|x,s_0)] < \infty$ for all $s_0 \in \mathbf{R}^n$ by assumption C1 and $p_\theta(x_0,x_1)$ is integrable of x_0 over the subset $\mathcal{X}_n$ by assumption C1. These imply that (7.3) is finite.

Finally, we observe

$$\begin{aligned}
&\sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\left\{L_1\frac{w(X_1,\xi_1)}{w(x_0,s_0)}\right\}\\
&\quad= \sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\left\{\sup_{h_1\neq h_2}\frac{d(\mathbf{P}(\xi_1)h_1,\mathbf{P}(\xi_1)h_2)}{d(h_1,h_2)}\frac{w(X_1,\xi_1)}{w(x_0,s_0)}\right\}\\
&\quad< \sup_{x_0,s_0}\mathbf{E}_{(x_0,s_0)}\left\{\left(\sup_{x_1\in\mathcal{X}}\int p_\theta(x_0,x_1)f(\xi_1;\theta|x_1,s_0)m(dx_0)\right)\frac{w(X_1,\xi_1)}{w(x_0,s_0)}\right\}\\
&\quad< \infty.
\end{aligned}$$

The last inequality follows from (5.3) in condition C1.

Note that C8 and C9 imply that $g \in \mathcal{L}(Q\times Q)$. Hence, the proof is complete. □

Problem 2.2. Harris recurrence of iterated function. This paper is an extension of Fuh (2003) for finite state space in which the likelihood function can be expressed as the L_1-norm of products of Markovian random matrices. Note that M_n defined in (5.6) is an iterated random functions system governed by a Markov chain Y_n. And $Y_n = (X_n, \xi_n)$ in the state space models case. In Theorem 1 I only assume $Y_n = (X_n, \xi_n)$ is Harris recurrent. The purpose of the statement, "*Note that under* K1–K3, … *a Markovian iterated random functions system in Theorem* 2," is to relate Theorems 1 and 2, to which I can apply limiting theorems in Markov chains to the law of large numbers and central limit theorem (and Edgeworth expansion) for (Y_n, M_n).

In Lemma 4 I want to prove $Z_n = ((X_n, \xi_n), M_n)$ is Harris recurrent (Z_n is defined in lines 1 and 2 on page 2056). In the proof, I can use the results in Theorem 1 since only $Y_n = (X_n, \xi_n)$ is assumed to be Harris recurrent in Theorem 1. It is known that C1 implies that $Y_n = (X_n, \xi_n)$ is Harris recurrent. A new proof of Lemma 3 was given on pages 1 and 2.

Problem 2.3. Asymptotic properties of score function and observed information. Page 2060, L12. In the proof of Lemma 6, (7.9) defined a new iterated functions system; therefore Corollary 1 cannot be used directly. The same situation happens for Theorems 5 and 7. The rigorous proofs of these results will be given in a separate paper.

Problem 2.4. Generality of conditions. C5. For $\theta \in N_\delta(\theta_0)$,

$$E_x^\theta \left(\frac{\partial \log \int_{y \in \mathcal{X}} \pi(x) p(x, y) f(s_0; \theta | x) f(\xi_1; \theta | y, s_0) m(dy)}{\partial \theta_i} \right)^2 < w(x, s_0)$$

for all $i = 1, \ldots, q$.

Change C5 accordingly. It is straightforward to check that C5 holds for the examples considered in Section 6. The proof of Lemma 5 can be done under C5.

Other typos and mistakes. Page 2032, L1. $\cdots\ p_\theta(y, x) f(\xi_j; \theta | x, \xi_{j-1}) \cdots$ $X_{j-1} = y$ and $X_j \in dx, \ldots$

$$(3.7) \quad \pi(y)\mathbf{P}(Y_n \in dz, M_n \in \cdot | Y_0 = y) = \pi(z)\tilde{\mathbf{P}}(\tilde{Y}_n \in dy, \tilde{M}_n \in \cdot | \tilde{Y}_0 = z).$$

Page 2028, L5. $(1 - \alpha^2)$. Page 2043, C7, $\theta \to \varphi_x(\theta)$ was a typo; delete it. Page 2047, L1, then, “each component of” the Fisher information matrix. L5, replace “positive definite” by “finite.” Page 2048, Theorem 5, assume $\mathbf{I}(\theta_0)$ is invertible. Page 2057, L3, the notation $m \times Q \times Q$ may be confusing; change it to $m \times Q \times \bar{Q}$.

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