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Annealed vs quenched critical points for a random walk pinning model

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Abstract. We study a random walk pinning model, where conditioned on a simple random walk Y on $\mathbb{Z}^d$ acting as a random medium, the path measure of a second independent simple random walk X up to time t is Gibbs transformed with Hamiltonian $-L_t(X,Y)$, where $L_t(X,Y)$ is the collision local time between X and Y up to time t. This model arises naturally in various contexts, including the study of the parabolic Anderson model with moving catalysts, the parabolic Anderson model with Brownian noise, and the directed polymer model. It falls in the same framework as the pinning and copolymer models, and exhibits a localization-delocalization transition as the inverse temperature β varies. We show that in dimensions $d=1,2$, the annealed and quenched critical values of β are both 0, while in dimensions $d\geq 4$, the quenched critical value of β is strictly larger than the annealed critical value (which is positive). This implies the existence of certain intermediate regimes for the parabolic Anderson model with Brownian noise and the directed polymer model. For $d\geq 5$, the same result has recently been established by Birkner, Greven and den Hollander [Quenched LDP for words in a letter sequence (2008)] via a quenched large deviation principle. Our proof is based on a fractional moment method used recently by Derrida et al. [*Comm. Math. Phys.* **287** (2009) 867–887] to establish the non-coincidence of annealed and quenched critical points for the pinning model in the disorder-relevant regime. The critical case $d=3$ remains open.

Résumé. Nous considérons le modèle de marche aléatoire avec *pinning* suivant : étant donné une marche aléatoire simple Y sur $\mathbb{Z}^d$ qui sert d'environnement aléatoire, on se donne une mesure de Gibbs sur les trajectoires d'une marche aléatoire X jusqu'au temps t de Hamiltonien $-L_t(X,Y)$ où $L_t(X,Y)$ est le temps local d'intersection entre X et Y jusqu'au temps t. Ce modèle apparaît naturellement dans des contextes variés tels que l'étude du modèle parabolique d'Anderson avec catalyseurs mouvants, l'étude du modèle parabolique d'Anderson avec bruit Brownien ainsi que dans le cadre de l'étude de polymères dirigés. Ce modèle appartient à la même classe que les modèles de *pinning* et copolymères et présente une transition localisation / délocalisation quand la température inverse β varie. Nous montrons qu'en dimension $d=1,2$ les valeurs critiques *annealed* et *quenched* de β sont toutes deux 0 mais que en dimension $d\geq 4$ la valeur critique *quenched* de β est strictement supérieure à la valeur *annealed* (qui est positive). Ceci entraine l'existence de certains régimes intermédiaires pour le modèle parabolique de Anderson avec bruit Brownien et pour les polymères dirigés. Pour $d\geq 5$ des résultats similaires ont été récemment établis par Birkner, Greven et den Hollander [Quenched LDP for words in a letter sequence (2008)] *via* un principe de grandes déviations *quenched*. Notre preuve se fonde sur la méthode des moments fractionnaires utilisée récemment par Derrida, Giacomin, Lacoin et Toninelli [*Comm. Math. Phys.* **287** (2009) 867–887] pour établir la non-coïncidence des valeurs critiques *quenched* et *annealed* du modèle de pinning dans le régime lié au désordre. Le cas de la dimension critique $d=3$ reste ouvert.

MSC: 60K35; 82B44

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1. Introduction and main result

1.1. *The model and main results*

We first define the continuous time version of the *random walk pinning model*, which more precisely, could be called the *random walk pinned to random walk model*. Let X and Y be two independent continuous time simple random walks on $\mathbb{Z}^d$ with jump rates 1 and $\rho \geq 0$ respectively. Let μ_t denote the law of $(X_s)_{0\leq s\leq t}$. For $\beta \in \mathbb{R}$, which plays the role of the inverse temperature (if $\beta > 0$), and for a fixed realization of Y acting as a random medium, we define a Gibbs transformation of the path measure μ_t. Namely, we define a new path measure $\mu^{\beta}_{t,Y}$ on $(X_s)_{0\leq s\leq t}$ which is absolutely continuous w.r.t. μ_t with Radon–Nikodym derivative

$$\frac{\mathrm{d}\mu^{\beta}_{t,Y}}{\mathrm{d}\mu_t}(X) = \frac{\mathrm{e}^{\beta L_t(X,Y)}}{Z^{\beta}_{t,Y}}, \tag{1.1}$$

where $L_t(X,Y) = \int_0^t 1_{\{X_s=Y_s\}}\,\mathrm{d}s$ is the collision local time between X and Y up to time t, and

$$Z^{\beta}_{t,Y} = \mathbb{E}^X_0\bigl[\mathrm{e}^{\beta L_t(X,Y)}\bigr] \tag{1.2}$$

is the *quenched partition function* which makes $\mu^{\beta}_{t,Y}$ a probability measure, where $\mathbb{E}^X_x[\cdot]$ denotes expectation w.r.t. X starting from $x \in \mathbb{Z}^d$. The *quenched free energy* of the model is defined by

$$F(\beta,\rho) = \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta}_{t,Y}. \tag{1.3}$$

We will show below that the limit exists and is non-random. As a disordered system, it is also natural to consider the *annealed partition function* $\mathbb{E}^Y_0[Z^{\beta}_{t,Y}]$ and the *annealed free energy*

$$F_{\mathrm{ann}}(\beta,\rho) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^Y_0\bigl[Z^{\beta}_{t,Y}\bigr]. \tag{1.4}$$

Note that $\mathbb{E}^Y_0[Z^{\beta}_{t,Y}] = \mathbb{E}^{X-Y}_0[\mathrm{e}^{\beta L_t(X-Y,0)}]$ is also the partition function of a homogeneous pinning model (see e.g. Giacomin [14]), namely a random walk pinning model where the random walk $X-Y$ (with jump rate $1+\rho$) is pinned to the site 0 instead of to a random trajectory.

To define the discrete time version of the random walk pinning model, let X, Y be discrete time simple random walks on $\mathbb{Z}^d$. The Gibbs transformed path measure $\hat{\mu}^{\beta}_{N,Y}$, $N \in \mathbb{N}$, can be defined similarly as in (1.1), where we replace $L_t(X,Y)$ by $L_N(X,Y) = \sum_{i=1}^N 1_{\{X_i=Y_i\}}$. We then define $\hat{Z}^{\beta}_{N,Y}$, $\hat{F}(\beta)$, $\hat{\mu}^{\beta}_{N,\mathrm{ann}}$, $\hat{F}_{\mathrm{ann}}(\beta)$ similarly for the discrete time model as for the continuous time model. Note that the free energies $\hat{F}(\beta)$ and $\hat{F}_{\mathrm{ann}}(\beta)$ now only depend on β since there are no more jump rates to adjust. To keep things simple, we focus only on X and Y being simple random walks in this paper. However, we expect much of the same results to hold and the proofs to be adaptable for general random walks, and we will comment on possible adaptations when appropriate.

Our first result is the existence of the quenched free energies $F(\beta,\rho)$ and $\hat{F}(\beta)$. Existence of the annealed free energies $F_{\mathrm{ann}}(\beta,\rho)$ and $F_{\mathrm{ann}}(\beta)$ is well known (see e.g. Chapter 2 in [14]). Before stating the result, we first introduce a two-parameter family of constrained partition functions for the random walk pinning model, where apart from a shift in time for the disorder Y, the random walk X is subject to the constraint $X_t = Y_t$ in (1.1). In continuous time setting, for $0 < s < t < \infty$, define

$$Z^{\beta,\mathrm{pin}}_{[s,t],Y} = \mathbb{E}^X_{Y_s}\left[\exp\left\{\beta\int_0^{t-s} 1_{\{X_u=Y_{s+u}\}}\,\mathrm{d}u\right\}1_{\{X_{t-s}=Y_t\}}\right]. \tag{1.5}$$

For $0 \leq m < n < \infty$ with $m,n \in \mathbb{N}_0$, we define $\hat{Z}^{\beta,\mathrm{pin}}_{[m,n],Y}$ analogously for the discrete time model. For simplicity, we will denote $Z^{\beta,\mathrm{pin}}_{[0,t],Y}$ by $Z^{\beta,\mathrm{pin}}_{t,Y}$, and $\hat{Z}^{\beta,\mathrm{pin}}_{[0,N],Y}$ by $\hat{Z}^{\beta,\mathrm{pin}}_{N,Y}$.

Theorem 1.1 (Existence of quenched free energy). *For any $\beta \in \mathbb{R}$ and $\rho \geq 0$, there exists a non-random constant $F(\beta, \rho)$ such that*

$$F(\beta, \rho) = \lim_{t\to\infty} \frac{1}{t} \log Z_{t,Y}^{\beta} = \lim_{t\to\infty} \frac{1}{t} \log Z_{t,Y}^{\beta,\mathrm{pin}}, \tag{1.6}$$

where the convergence are a.s. and in L^1 w.r.t. Y. Furthermore, we have the representation

$$F(\beta, \rho) = \sup_{t>0} \frac{1}{t} \mathbb{E}_0^Y \bigl[\log Z_{t,Y}^{\beta,\mathrm{pin}}\bigr]. \tag{1.7}$$

Analogous statements hold for the discrete time model.

Corollary 1.1 (Existence of critical points). *There exist $0 \leq \beta_{\mathrm{c}}^{\mathrm{ann}} \leq \beta_{\mathrm{c}} < \infty$ depending on $\rho \geq 0$ such that: $F_{\mathrm{ann}}(\beta, \rho) = 0$ if $\beta < \beta_{\mathrm{c}}^{\mathrm{ann}}$ and $F_{\mathrm{ann}}(\beta, \rho) > 0$ if $\beta > \beta_{\mathrm{c}}^{\mathrm{ann}}$; $F(\beta, \rho) = 0$ if $\beta < \beta_{\mathrm{c}}$ and $F(\beta, \rho) > 0$ if $\beta > \beta_{\mathrm{c}}$. Analogous statements hold for the discrete time model with annealed and quenched critical points $\hat{\beta}_{\mathrm{c}}^{\mathrm{ann}}$ and $\hat{\beta}_{\mathrm{c}}$ respectively.*

Remark. *See (5.5) and (4.4) for the exact values of $\beta_{\mathrm{c}}^{\mathrm{ann}}$ and $\hat{\beta}_{\mathrm{c}}^{\mathrm{ann}}$.*

Remark. *As in the pinning model (see e.g. [14]), β_{c} marks the transition between a localized and a delocalized phase: when $\beta < \beta_{\mathrm{c}}$ and $F(\beta, \rho) = 0$, $L_t(X, Y)$ is typically of order $\mathrm{o}(t)$ w.r.t. $\mu_{t,Y}^{\beta}$ for t large; when $\beta > \beta_{\mathrm{c}}$ and $F(\beta, \rho) > 0$, $L_t(X, Y)$ is typically of order t w.r.t. $\mu_{t,Y}^{\beta}$ for t large. Similarly, $\beta_{\mathrm{c}}^{\mathrm{ann}}$ marks the transition between the localized and delocalized phase for the annealed homogeneous pinning model.*

One question of fundamental interest in the study of disordered systems is to determine when is the disorder strong enough to shift the critical point of the model, i.e., when is $\beta_{\mathrm{c}}^{\mathrm{ann}} < \beta_{\mathrm{c}}$? For the pinning model, this question has recently been essentially fully resolved independently by Derrida et al. [8], and Alexander and Zygouras [1]. For the random walk pinning model, our main result is the following.

Theorem 1.2 (Annealed vs quenched critical points). *In dimensions $d = 1$ and 2, we have $\beta_{\mathrm{c}}^{\mathrm{ann}} = \beta_{\mathrm{c}} = \hat{\beta}_{\mathrm{c}}^{\mathrm{ann}} = \hat{\beta}_{\mathrm{c}} = 0$. In dimensions $d \geq 4$, we have $0 < \beta_{\mathrm{c}}^{\mathrm{ann}} < \beta_{\mathrm{c}}$ for each $\rho > 0$ and $0 < \hat{\beta}_{\mathrm{c}}^{\mathrm{ann}} < \hat{\beta}_{\mathrm{c}}$. For $d \geq 5$, there exists $a > 0$ s.t. $\beta_{\mathrm{c}} - \beta_{\mathrm{c}}^{\mathrm{ann}} \geq a\rho$ for all $\rho \in [0, 1]$. For $d = 4$ and for each $\delta > 0$, there exists $a_\delta > 0$ s.t. $\beta_{\mathrm{c}} - \beta_{\mathrm{c}}^{\mathrm{ann}} \geq a_\delta \rho^{1+\delta}$ for all $\rho \in [0, 1]$.*

For purposes relevant to applications for the parabolic Anderson model with Brownian noise and the directed polymer model, in $d \geq 4$, we prove instead a stronger version of Theorem 1.2. Define

$$\beta_{\mathrm{c}}^{*} = \sup\Bigl\{\beta \in \mathbb{R}\colon \sup_{t>0} Z_{t,Y}^{\beta} < \infty \text{ a.s. w.r.t. } Y\Bigr\}. \tag{1.8}$$

Define $\hat{\beta}_{\mathrm{c}}^{*}$ for the discrete time model analogously. Clearly $\beta_{\mathrm{c}}^{*} \leq \beta_{\mathrm{c}}$ and $\hat{\beta}_{\mathrm{c}}^{*} \leq \hat{\beta}_{\mathrm{c}}$. We have

Theorem 1.3 (Non-coincidence of critical points strengthened). *For $d \geq 4$, we have $\beta_{\mathrm{c}}^{\mathrm{ann}} < \beta_{\mathrm{c}}^{*}$ for each $\rho > 0$ and $\hat{\beta}_{\mathrm{c}}^{\mathrm{ann}} < \hat{\beta}_{\mathrm{c}}^{*}$. For $d \geq 5$, there exists $a > 0$ s.t. $\beta_{\mathrm{c}}^{*} - \beta_{\mathrm{c}}^{\mathrm{ann}} \geq a\rho$ for all $\rho \in [0, 1]$. For $d = 4$ and for each $\delta > 0$, there exists $a_\delta > 0$ s.t. $\beta_{\mathrm{c}}^{*} - \beta_{\mathrm{c}}^{\mathrm{ann}} \geq a_\delta \rho^{1+\delta}$ for all $\rho \in [0, 1]$.*

Remark. *Theorem 1.3 for $d \geq 5$ (without bounds on the gap) has recently been established by Birkner, Greven and den Hollander [3] as an application of a quenched large deviation principle for renewal processes in random scenery. Our aim here is to give an alternative proof based on adaptations of the fractional moment method used recently by Derrida et al. [8] in the pinning model context, and to extend to the $d = 4$ case. Loosely speaking, because $\mathbb{P}(X_n = Y_n) \sim Cn^{-d/2} = Cn^{-1-\alpha}$ by the local central limit theorem, $d \geq 5$ corresponds to the case $\alpha > 1$ in [8]; $d = 4$ corresponds to the case $\alpha = 1$, which was not covered in [8], but included in [1]; while $d = 3$ corresponds to the marginal case $\alpha = 1/2$, which for the pinning model with Gaussian disorder was recently shown by Giacomin et al. [15] to be disorder relevant. For the random walk pinning model, $d = 3$ remains open.*

Remark. *It is an interesting open question whether $\beta_c^* = \beta_c$, i.e., whether the quenched partition function $Z_{t,Y}^{\beta}$ is uniformly bounded in t a.s. w.r.t. Y in the entire delocalized phase. As communicated to us by F. L. Toninelli, this question also remains open for the pinning and the copolymer models.*

Theorem 1.3 for the continuous time model confirms Conjecture 1.8 of Greven and den Hollander [16] (for $d \geq 4$) that the parabolic Anderson model with Brownian noise could admit an equilibrium measure with an infinite second moment. Theorem 1.3 for the discrete time model can be used to disprove a conjecture of Garel and Monthus [10] that for the directed polymer model in random environment, the transition from weak to strong disorder occurs at β_c^{ann}. See Section 1.4 for more details. For some special environments in special dimensions, this conjecture has already been disproved by Camanes and Carmona [5]. In Section 1.4, we will show that the results of Derrida et al. [8] on the pinning model can also be used to disprove the Garel–Monthus conjecture in $d \geq 4$. The reader can also consult Section 1.5 of Birkner et al. [3] for more detailed expositions on the implication of Theorem 1.3 for the various models mentioned above.

In the remainder of the introduction, we point out a connection between the random walk pinning model and the parabolic Anderson model with a single moving catalyst, and how does the random walk pinning model fit in the same framework as the pinning and copolymer models. Lastly, we will introduce an *inhomogeneous random walk pinning model* which generalizes both the pinning and the random walk pinning model.

1.2. *Parabolic Anderson model with a single moving catalyst*

As for the continuous time random walk pinning model, let Y be a continuous time simple random walk on $\mathbb{Z}^d$ with jump rate $\rho \geq 0$. The parabolic Anderson model with a single moving catalyst is the solution of the following Cauchy problem for the heat equation in a time-dependent random potential

$$\begin{aligned} &\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) + \beta \delta_{Y_t}(x) u(t,x), \qquad x \in \mathbb{Z}^d, t \geq 0, \\ &u(0,x) = 1, \end{aligned} \tag{1.9}$$

where $\beta \in \mathbb{R}$ and $\Delta f(x) = \frac{1}{2d} \sum_{\|y-x\|=1} (f(y) - f(x))$ is the discrete Laplacian on $\mathbb{Z}^d$. Heuristically, the time-dependent potential $\beta \delta_{Y_t}(x)$ can be interpreted as a single catalyst with strength β moving as Y, $u(t,x)$ is then simply the expected number of particles alive at position x at time t for a branching particle system, where initially one particle starts from each site of $\mathbb{Z}^d$, and independently, each particle moves on $\mathbb{Z}^d$ as a simple random walk, and whenever the particle is at the same location as the catalyst Y, it splits into two particles with rate β if $\beta > 0$ and is killed with rate $-\beta$ if $\beta < 0$. For further motivations and a survey on the parabolic Anderson model, see e.g. Gärtner and König [11].

Quantities of special interest in the study of the parabolic Anderson model are the quenched and annealed pth moment Lyapunov exponents.

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(t,0), \qquad \lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0^Y\big[u(t,0)^p\big]. \tag{1.10}$$

The annealed pth moment Lyapunov exponents for $p \in \mathbb{N}$ have been studied by Gärtner and Heydenreich in [12]. Here we show that

Theorem 1.4 (Existence of quenched Lyapunov exponent). *For any $\beta \in \mathbb{R}$ and $\rho \geq 0$, there exists a non-random constant $\lambda_0 = \lambda_0(\beta,\rho)$ such that for all $x \in \mathbb{Z}^d$,*

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(t,x) \quad \text{a.s. and in } L^1 \text{ w.r.t. } Y. \tag{1.11}$$

Furthermore, $\lambda_0(\beta,\rho) = F(\beta,\rho)$, where $F(\beta,\rho)$ is as in (1.6).

Indeed, the solution of (1.9) admits the Feynman–Kac representation

$$u(t,x)=\mathbb{E}_x^X\left[\exp\left\{\beta\int_0^t 1_{\{X_{t-s}=Y_s\}}\,\mathrm{d}s\right\}\right], \tag{1.12}$$

where X is a simple random walk on $\mathbb{Z}^d$ with jump rate 1 and $X_0=x$. Except for the time reversal of X in (1.12), $u(t,x)$ has the same representation as that for $Z_{t,Y}^{\beta}$. The same proof as for Theorem 1.1 then applies, which gives rise to the same representation for λ_0 as for $F(\beta,\rho)$ in (1.7) due to the fact that the variational expression in (1.7) is invariant w.r.t. time reversal for X.

1.3. *Relation to pinning and copolymer models*

We now explain in what sense does the random walk pinning model belong to the same framework as the pinning and the copolymer models. For simplicity, we will examine the discrete time random walk pinning model with a path measure associated with the partition function $\hat{Z}_{[0,N],Y}^{\beta,\mathrm{pin}}$, c.f. (1.5).

The pinning and copolymer models are both Gibbs transformation of a renewal process. More precisely, let $\sigma=(\sigma_0=0,\sigma_1,\sigma_2,\ldots)$ be a renewal process on $\mathbb{N}_0$, where the inter-arrival times $(\sigma_i-\sigma_{i-1})_{i\in\mathbb{N}}$ are i.i.d. $\mathbb{N}\cup\{\infty\}$-valued random variables with distribution $\mathbb{P}(\sigma_1=i)=K(i)$ for some probability kernel K on $\mathbb{N}\cup\{\infty\}$. Let $(\omega_i)_{i\in\mathbb{N}}$ be i.i.d. real-valued random variables with $\mathbb{E}[\omega_1]=0$ and $\mathbb{E}[\mathrm{e}^{\lambda\omega_1}]<\infty$ for all $\lambda\in\mathbb{R}$. Let $h\in\mathbb{R}$ and $\beta\ge 0$. Then for a fixed $N\in\mathbb{N}$, the finite volume Gibbs weight for a given realization of the renewal sequence σ for both models are of the form

$$W(\sigma)=\begin{cases}\prod_{i=1}^m w\big(\beta,h,(\omega_j)_{\sigma_{i-1}<j\le\sigma_i}\big) & \text{if } N=\sigma_m \text{ for some } m\ge 1,\\ 0 & \text{otherwise,}\end{cases} \tag{1.13}$$

where

$$w\big(\beta,h,(\omega_j)_{0<j\le n}\big)=\begin{cases}\mathrm{e}^{\beta\omega_n+h} & \text{pinning model,}\\ \frac{\mathrm{e}^{\beta\sum_{j=1}^n(\omega_j+h)}+\mathrm{e}^{-\beta\sum_{j=1}^n(\omega_j+h)}}{2} & \text{copolymer model.}\end{cases} \tag{1.14}$$

See [14] for more on the pinning and copolymer models. For the discrete time random walk pinning model, we can write

$$\begin{aligned}\hat{Z}_{N,Y}^{\beta,\mathrm{pin}}&=\mathbb{E}_0^X\big[\mathrm{e}^{\beta L_N(X,Y)}1_{\{X_N=Y_N\}}\big]\\&=\sum_{m=1}^N\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=N}\prod_{i=1}^m\big(\mathrm{e}^{\beta}\mathbb{P}_0^X(\tau_{\theta_{\sigma_{i-1}}Y}=\sigma_i-\sigma_{i-1})\big),\end{aligned} \tag{1.15}$$

where $\theta_nY=(Y_{n+i}-Y_n)_{i\in\mathbb{N}_0}$ denotes a shift in Y, and $\tau_Y=\tau_Y(X)=\min\{i\ge 1:\ X_i=Y_i\}$. Let us denote $K(i)=\mathbb{E}_0^Y[\mathbb{P}_0^X(\tau_Y=i)]=\mathbb{P}_0^{X-Y}(\tau_0=i)$, then K with $K(\infty)=\mathbb{P}_0^{X-Y}(\tau_0=\infty)$ is the return time distribution of a renewal process on $\mathbb{N}_0$. Let $\Delta_i=Y_i-Y_{i-1}$. We can then rewrite (1.15) as

$$\hat{Z}_{N,Y}^{\beta,\mathrm{pin}}=\sum_{m=1}^N\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=N}\prod_{i=1}^m\big(K(\sigma_i-\sigma_{i-1})w\big(\beta,(\Delta_j)_{\sigma_{i-1}<j\le\sigma_i}\big)\big), \tag{1.16}$$

where

$$w\big(\beta,(\Delta_i)_{0<i\le n}\big)=\frac{\mathrm{e}^{\beta}\mathbb{P}_0^X(\tau_Y=n)}{K(n)},\qquad Y_i=\sum_{j=1}^i\Delta_j. \tag{1.17}$$

In view of (1.16) and (1.17), we see that the random walk pinning model associated with $\hat{Z}_{[0,N],Y}^{\beta,\mathrm{pin}}$ is also a Gibbs transformation of a renewal process with inter-arrival law K, except that the disorder $(\Delta_i)_{i\in\mathbb{N}}$ take values in $\mathbb{Z}^d$

and the Gibbs weight factor $w(\cdot)$ for each renewal gap has a more complicated dependence on the disorder than for the pinning and copolymer models. Nevertheless, this simple observation motivates us to try to adapt the fractional moment method from the pinning model to our context. In the actual proof, we will use an alternative representation for $\hat{Z}^{\beta,\mathrm{pin}}_{[0,N],Y}$, as well as for $Z^{\beta,\mathrm{pin}}_{[0,t],Y}$, which admits a simpler form for the weight factor $w(\cdot)$ than (1.17). See (4.3) and (5.3). We will see later on that despite the entirely different nature of the disorder, the random walk pinning model turns out to be a close analogue of the pinning model. Lastly we note that the fractional moment method has recently been successfully applied also to the copolymer model, see Bodineau et al. [4] and Toninelli [17].

1.4. *An inhomogeneous random walk pinning model*

Another common feature between the pinning and the random walk pinning model is that, for both models, the annealed partition function is that of a homogeneous pinning model. A further intriguing interplay between the two models is that we can define an *inhomogeneous random walk pinning model*, from which both models can be obtained by partial annealing. More precisely, let X and Y be discrete time simple random walks on $\mathbb{Z}^d$, let $(\omega_i)_{i\in\mathbb{N}}$ be i.i.d. real-valued random variables with $\mathbb{E}[\omega_1]=0$, and $M(\lambda)=\log\mathbb{E}[\mathrm{e}^{\lambda\omega_1}]$ is well-defined for all $\lambda\geq 0$. Let $h\in\mathbb{R}$ and $\beta\geq 0$. Then the discrete time inhomogeneous random walk pinning model is the Gibbs transformation of the path measure μ_N of X up to time N with Radon–Nikodym derivative

$$\frac{\mathrm{d}\mu^{\beta,h}_{N,Y,\omega}}{\mathrm{d}\mu_N}(X)=\frac{\exp\{\sum_{i=1}^N(\beta\omega_i+h)1_{\{X_i=Y_i\}}\}}{Z^{\beta,h}_{N,Y,\omega}},\tag{1.18}$$

where $Z^{\beta,h}_{N,Y,\omega}=\mathbb{E}^X_0[\exp\{\sum_{i=1}^N(\beta\omega_i+h)1_{\{X_i=Y_i\}}\}]$ is the partition function, and we now have two sources of disorder: the location of pinning as given by Y, and the strength of pinning as given by $\beta\omega_i+h$. Note that under annealing w.r.t. Y,

$$\mathbb{E}^Y_0\left[Z^{\beta,h}_{N,Y,\omega}\right]=\mathbb{E}^{X-Y}_0\left[\exp\left\{\sum_{i=1}^N(\beta\omega_i+h)1_{\{(X-Y)_i=0\}}\right\}\right]\tag{1.19}$$

is the partition function of a pinning model (without boundary constraint $(X-Y)_N=0$), where the underlying renewal process is given by the return times of $X-Y$ to 0. On the other hand, under annealing w.r.t. ω,

$$\mathbb{E}^\omega\left[Z^{\beta,h}_{N,Y,\omega}\right]=\mathbb{E}^X_0\left[\mathrm{e}^{(M(\beta)+h)L_N(X,Y)}\right]$$

is the partition function of a random walk pinning model with parameter $M(\beta)+h$.

The continuous time version of the inhomogeneous random walk pinning model can be defined similarly with partition function

$$Z^{\beta,h}_{t,Y,B}=\mathbb{E}^X_0\left[\exp\left\{\beta\int_0^t 1_{\{X_s=Y_s\}}(\mathrm{d}B_s+h\,\mathrm{d}s)\right\}\right],$$

where B_s is a standard Brownian motion.

The discrete time inhomogeneous random walk pinning model first appeared implicitly in Birkner [2] in the study of the directed polymer model (the continuous time analogue can be found in Greven and den Hollander [16]). Given a simple random walk X on $\mathbb{Z}^d$, $\lambda\geq 0$, i.i.d. real-valued random variables $(\omega(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^d}$ with $M(\lambda')=\log\mathbb{E}[\mathrm{e}^{\lambda'\omega(1,1)}]$ well-defined for all $\lambda'\geq 0$, the (normalized) partition function of the directed polymer model is given by

$$Z^\lambda_{N,\omega}=\mathbb{E}^X_0\left[\mathrm{e}^{\sum_{i=1}^N\{\lambda\omega(i,X_i)-M(\lambda)\}}\right].$$

Note that $(Z^\lambda_{N,\omega})_{N\in\mathbb{N}}$ is a positive martingale. The critical point of the model can be defined by

$$\lambda_c=\sup\left\{\lambda\geq 0\colon\ \left(Z^\lambda_{N,\omega}\right)_{N\in\mathbb{N}}\text{ is uniformly integrable}\right\}=\sup\left\{\lambda\geq 0\colon\ \lim_{N\to\infty}Z^\lambda_{N,\omega}>0\text{ a.s.}\right\}.$$

In the literature, $[0,\lambda_c)$ and (λ_c,∞) are called respectively the weak and strong disorder regimes, characterized respectively by the uniform integrability (or the lack of u.i.) of $(Z^\lambda_{N,\omega})_{N\in\mathbb{N}}$. See [7] for an overview of the directed polymer model, and see [6], Theorem 1.1 and Proposition 3.1, for the existence of λ_c. The Garel–Monthus conjecture [10] asserts that $\lambda_c = \lambda_2 := \sup\{\lambda \ge 0\colon \sup_{N\in\mathbb{N}} \mathbb{E}[(Z^\lambda_{N,\omega})^2] < \infty\}$. On the other hand, Birkner [2], Lemma 1, showed that if Y is an independent copy of X, and $(\tilde\omega(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^d}$ is an i.i.d. field with a tilted law $\mathbb{P}(\tilde\omega(n,x)\in d\zeta) = e^{\lambda\zeta - M(\lambda)}\mathbb{P}(\omega(n,x)\in d\zeta)$, independent of X, Y and ω, then the size-biased law of $Z^\lambda_{N,\omega}$ is the same as the law of

$$\tilde Z^\lambda_{N,\omega,\tilde\omega,Y} = \mathbb{E}^X_0\left[\exp\left\{\sum_{i=1}^N \bigl(1_{\{X_i\neq Y_i\}}\lambda\omega(i,X_i) + 1_{\{X_i=Y_i\}}\lambda\tilde\omega(i,X_i) - M(\lambda)\bigr)\right\}\right]. \tag{1.20}$$

Namely, $\mathbb{E}[f(\tilde Z^\lambda_{N,\omega,\tilde\omega,Y})] = \mathbb{E}[Z^\lambda_{N,\omega} f(Z^\lambda_{N,\omega})]$ for all bounded $f:\mathbb{R}_+\to\mathbb{R}$. The uniform integrability of $(Z^\lambda_{N,\omega})_{N\in\mathbb{N}}$ is then equivalent to the uniform tightness of the laws of $(Z^\lambda_{N,\omega,\tilde\omega,Y})_{N\in\mathbb{N}}$. If we integrate out the disorder ω in (1.20), then

$$\mathbb{E}\bigl[\tilde Z^\lambda_{N,\omega,\tilde\omega,Y}|\tilde\omega, Y\bigr] = \mathbb{E}^X_0\bigl[e^{\sum_{i=1}^N(\lambda\tilde\omega(i,X_i)-M(\lambda))1_{\{X_i=Y_i\}}}\bigr] \tag{1.21}$$

is precisely the partition function of the inhomogeneous random walk pinning model. Further integrating out $\tilde\omega$ gives the partition function of a random walk pinning model with parameter $\hat\beta(\lambda) = M(2\lambda) - 2M(\lambda)$,

$$\mathbb{E}\bigl[\tilde Z^\lambda_{N,\omega,\tilde\omega,Y}|Y\bigr] = \mathbb{E}^X_0\bigl[e^{\sum_{i=1}^N(M(2\lambda)-2M(\lambda))1_{\{X_i=Y_i\}}}\bigr].$$

Since $\mathbb{E}[(Z^\lambda_{N,\omega})^2] = \mathbb{E}[\tilde Z^\lambda_{N,\omega,\tilde\omega,Y}]$, $\hat\beta(\lambda_2) = \hat\beta^{\text{ann}}_c$ with $\hat\beta^{\text{ann}}_c$ being the annealed critical point as in Theorem 1.3. Since for non-degenerate ω, $\hat\beta(\lambda)$ is strictly increasing in λ, Theorem 1.3 implies that in $d\ge 4$, there exists $\lambda' > \lambda_2$ such that $\mathbb{E}[\tilde Z^{\lambda'}_{N,\omega,\tilde\omega,Y}|Y]$ is uniformly bounded in N a.s. w.r.t. Y. Therefore the law of $(\tilde Z^{\lambda'}_{N,\omega,\tilde\omega,Y})_{N\in\mathbb{N}}$ is uniformly tight, and hence $\lambda_c \ge \lambda' > \lambda_2$, which disproves the conjecture of Garel and Monthus [10]. Since our proof is based on bounding fractional moments, we will in fact exhibit a $\lambda' > \lambda_2$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\bigl[\mathbb{E}\bigl[\tilde Z^{\lambda'}_{N,\omega,\tilde\omega,Y}|Y\bigr]^\gamma\bigr] < \infty \quad \text{for some } \gamma\in(0,1).$$

See (4.5). Since $\tilde Z^\lambda_{N,\omega,\tilde\omega,Y}$ is the size-biased version of the partition function $Z^{\lambda'}_{N,\omega}$ of the directed polymer model,

$$\mathbb{E}\bigl[\bigl(Z^{\lambda'}_{N,\omega}\bigr)^{1+\gamma}\bigr] = \mathbb{E}\bigl[\bigl(\tilde Z^{\lambda'}_{N,\omega,\tilde\omega,Y}\bigr)^\gamma\bigr] \le \mathbb{E}\bigl[\mathbb{E}\bigl[\tilde Z^{\lambda'}_{N,\omega,\tilde\omega,Y}|Y\bigr]^\gamma\bigr].$$

Therefore, beyond the regime of λ where $Z^\lambda_{N,\omega}$ is a L_2 bounded martingale, there is a regime where $Z^\lambda_{N,\omega}$ has uniformly bounded $(1+\gamma)$th moment for some $\gamma\in(0,1)$.

Finally, we point out that based on (1.20), the results of Derrida et al. [8] for the pinning model can also be used to disprove the Garel–Monthus conjecture in $d\ge 4$: In (1.21), conditioned on Y, $(\tilde\omega(i,Y_i))_{1\le i\le N}$ are i.i.d. Therefore if we fix an i.i.d. sequence $(\bar\omega_i)_{i\in\mathbb{N}}$ equally distributed with $\tilde\omega(1,1)$, then $\mathbb{E}[\tilde Z^\lambda_{N,\omega,\tilde\omega,Y}|\tilde\omega, Y]$ is equally distributed with

$$\mathbb{E}^X_0\bigl[e^{\sum_{i=1}^N(\lambda\bar\omega_i - M(\lambda))1_{\{X_i=Y_i\}}}\bigr].$$

Integrating out Y then gives the partition of a pinning model,

$$Z^{\beta,h}_{N,\bar\omega} = \mathbb{E}^{X-Y}_0\bigl[e^{\sum_{i=1}^N(\lambda\bar\omega_i - M(\lambda))1_{\{(X-Y)_i=0\}}}\bigr] \tag{1.22}$$

with parameters $\beta(\lambda)=\lambda$, $h(\lambda) = -M(\lambda)$ (c.f. (1.19)), and underlying renewal process $K(n) = \mathbb{P}^{X-Y}_0(\tau_0 = n)$ where τ_0 is the first return time of $X-Y$ to 0. It is easy to check that the critical curve for the annealed pinning model is given by $h^{\text{ann}}_c(\beta) = M(\lambda) - M(\lambda+\beta) - \log \mathbb{P}^{X-Y}_0(\tau_0<\infty)$. By the definition of λ_2, $(\beta(\lambda_2), h(\lambda_2))$ lies on this annealed critical curve. Since in $d\ge 4$, $K(n)\sim cn^{-d/2}$ has tail exponent $\alpha = \frac{d}{2} - 1 \ge 1$, it follows from Derrida et al. [8] that there exists a continuous curve $h^*(\beta)$ strictly above $h^{\text{ann}}_c(\beta)$, such that for all $h\le h^*(\beta)$, $\mathbb{E}[(Z^{\beta,h}_{N,\bar\omega})^\gamma]$ is uniformly bounded in N for some $\gamma \in (0,1)$. Therefore we can choose $\lambda' > \lambda_2$ such that $-M(\lambda') \le h^*(\lambda')$, and hence $\mathbb{E}[(Z_{N,\tilde{\omega}}^{\lambda',-M(\lambda')})^\gamma]$ is uniformly bounded in N for some $\gamma \in (0,1)$. By the same reasoning as before, this implies the uniform tightness of $(\tilde{Z}_{N,\omega,\tilde{\omega},Y}^{\lambda'})_{N\in\mathbb{N}}$, and hence $\lambda_c \ge \lambda' > \lambda_2$. We remark that in [8], only the constrained version of the partition function $Z_{N,\tilde{\omega}}^{\beta,h}$ is considered, i.e., the constraint $1_{\{X_N=Y_N\}}$ is inserted in (1.22). However, the proof there can be easily adapted to the non-constrained version, as can be seen later in our analysis of the random walk pinning model. Most recently, Giacomin, Lacoin and Toninelli [15] extended their fractional moment technique to the pinning model with Gaussian disorder in the critical dimension, i.e., $K(n) \sim cn^{-3/2}$, which corresponds to $d=3$ for the random walk pinning model considered here. Except for the technical point that [15] only considered the constrained pinning model, their result would imply $\lambda_c > \lambda_2$ for the directed polymer model in Gaussian environment in $d=3$, since in (1.22), the exponentially tilted law of a Gaussian is a shifted Gaussian.

1.5. *Outline*

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1, Corollary 1.1, and Theorem 1.4. In Section 3, we prove Theorem 1.2 for $d=1,2$. In Section 4, we prove Theorem 1.3 in the discrete time case. Lastly in Section 5, we prove Theorem 1.3 in the continuous time case. The proof of Theorem 1.3 does not rely on the existence of the quenched free energies. Readers interested in how the fractional moment method is applied in this context can go directly to Sections 4 and 5.

2. Existence of the quenched free energy

In this section, we prove Theorems 1.1, 1.4 and Corollary 1.1.

Proof of Theorem 1.1. We consider first the constrained partition functions $\hat{Z}_{N,Y}^{\beta,\mathrm{pin}}$ and $Z_{t,Y}^{\beta,\mathrm{pin}}$. For the discrete time model, by the super-additive ergodic theorem (see e.g. Section 6.6 in Durrett [9]) applied to $(\log \hat{Z}_{[m,n],Y}^{\beta,\mathrm{pin}})_{0\le m<n}$, we have

$$\hat{F}(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \hat{Z}_{n,Y}^{\beta,\mathrm{pin}} = \sup_{n\in\mathbb{N}} \mathbb{E}_0^Y\big[\log \hat{Z}_{n,Y}^{\beta,\mathrm{pin}}\big],$$

where the convergence is a.s. and in L^1. For the continuous time model, we have to apply the super-additive ergodic theorem first along the integer times, and then extend the convergence along all real times. Clearly $(\log Z_{[m,n],Y}^{\beta,\mathrm{pin}})_{0\le m<n}$ satisfies all the conditions of the super-additive ergodic theorem, therefore

$$F^{\mathrm{pin}}(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,Y}^{\beta,\mathrm{pin}} = \sup_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E}_0^Y\big[\log Z_{n,Y}^{\beta,\mathrm{pin}}\big] \quad \text{a.s. and in } L^1. \tag{2.1}$$

To extend the a.s. convergence to real $t\to\infty$, we need the following crude estimates.

Proposition 2.1. *Let $(X_t)_{t\ge 0}$ be a continuous time random walk on $\mathbb{Z}^d$ with jump rate 1. Let $\|\cdot\|_1$ denote L^1 norm in $\mathbb{Z}^d$. Then*

(i) *There exists $C>0$ such that a.s. $\|X_t\|_1 < C\sqrt{t\log\log t}$ for all t sufficiently large.*
(ii) *$\mathbb{P}_0^X(X_t = x) \ge C(1+t)^{-d/2}(2d)^{-\|x\|_1}$ uniformly for all $t>0$ and $x\in\mathbb{Z}^d$ with $\|x\|_1 \le t/2$.*

Proof. Part (i) is a consequence of the law of the iterated logarithm. Part (ii) follows by forcing X to visit x after exactly $\|x\|_1$ number of jumps, and then return to x at time t. The factor $(1+t)^{-d/2}$ arises from the local central limit theorem. □

Note that for $t\ge 1$, by super-additivity, we have

$$\frac{1}{t}\big(\log Z_{\lfloor t-t^{2/3}\rfloor,Y}^{\beta,\mathrm{pin}} + \log Z_{[\lfloor t-t^{2/3}\rfloor,t],Y}^{\beta,\mathrm{pin}}\big) \le \frac{1}{t}\log Z_{t,Y}^{\beta,\mathrm{pin}} \le \frac{1}{t}\big(\log Z_{\lfloor t+t^{2/3}\rfloor,Y}^{\beta,\mathrm{pin}} - \log Z_{[t,\lfloor t+t^{2/3}\rfloor],Y}^{\beta,\mathrm{pin}}\big). \tag{2.2}$$

By (2.1), a.s. $F^{\mathrm{pin}} = \lim_{t\to\infty} t^{-1} \log Z^{\beta,\mathrm{pin}}_{\lfloor t-t^{2/3}\rfloor, Y} = \lim_{t\to\infty} t^{-1} \log Z^{\beta,\mathrm{pin}}_{\lfloor t+t^{2/3}\rfloor, Y}$. On the other hand,

$$Z^{\beta,\mathrm{pin}}_{[\lfloor t-t^{2/3}\rfloor, t], Y} \le \mathrm{e}^{|\beta|(t-\lfloor t-t^{2/3}\rfloor)} \mathbb{P}^X_0\big(X_{t-\lfloor t-t^{2/3}\rfloor} = Y_t - Y_{\lfloor t-t^{2/3}\rfloor}\big),$$
$$Z^{\beta,\mathrm{pin}}_{[\lfloor t-t^{2/3}\rfloor, t], Y} \ge \mathrm{e}^{-|\beta|(t-\lfloor t-t^{2/3}\rfloor)} \mathbb{P}^X_0\big(X_{t-\lfloor t-t^{2/3}\rfloor} = Y_t - Y_{\lfloor t-t^{2/3}\rfloor}\big). \tag{2.3}$$

By Proposition 2.1, for t sufficiently large, $\|Y_t - Y_{\lfloor t-t^{2/3}\rfloor}\|_1 \le 2C\sqrt{t\log\log t} < \frac{(t-\lfloor t-t^{2/3}\rfloor)}{2}$, and hence

$$\mathbb{P}^X_0\big(X_{t-\lfloor t-t^{2/3}\rfloor} = Y_t - Y_{\lfloor t-t^{2/3}\rfloor}\big) \ge C\big(1 + t - \lfloor t - t^{2/3}\rfloor\big)^{-d/2} (2d)^{-2C\sqrt{t\log\log t}},$$

from which we obtain $\lim_{t\to\infty} t^{-1} |\log Z^{\beta,\mathrm{pin}}_{[\lfloor t-t^{2/3}\rfloor, t], Y}| = 0$. Similarly, $\lim_{t\to\infty} t^{-1}|\log Z^{\beta,\mathrm{pin}}_{[t, \lfloor t+t^{2/3}\rfloor], Y}| = 0$. This establishes the a.s. convergence in (2.1) for $t\to\infty$ in place of $n\to\infty$ for $n\in\mathbb{N}$. To obtain L^1 convergence, it remains to verify the uniform integrability of $(t^{-1}\log Z^{\beta,\mathrm{pin}}_{t,Y})_{t\ge 1}$. Note that

$$t^{-1}\log Z^{\beta,\mathrm{pin}}_{t,Y} \le t^{-1}\log Z^{\beta}_{t,Y} \le \beta \quad \text{a.s. w.r.t. } Y,$$

while

$$t^{-1}\log Z^{\beta,\mathrm{pin}}_{t,Y} \ge t^{-1}\log p_t(Y_t),$$

where p_t denotes the transition kernel of X. Using estimates (3.5)–(3.7) below, it is easy to see that $|t^{-1}\log p_t(Y_t)|_{t\ge 1}$ is uniformly integrable, hence $(t^{-1}\log Z^{\beta,\mathrm{pin}}_{t,Y})_{t\ge 1}$ is also uniformly integrable. Note that because $\log Z^{\beta}_{t,Y} \ge 0$, the unconstrained partition function $(t^{-1}\log Z^{\beta}_{t,Y})_{t\ge 0}$ is also uniformly integrable.

We now consider the unconstrained partition functions $\hat{Z}^{\beta}_{N,Y}$ and $Z^{\beta}_{t,Y}$. The argument is the same for discrete and continuous times, so we only consider the latter. Clearly $Z^{\beta}_{t,Y} > Z^{\beta,\mathrm{pin}}_{t,Y}$. To upper bound $Z^{\beta}_{t,Y}$ in terms of $Z^{\beta,\mathrm{pin}}_{t,Y}$, we can let X run freely until time $t - t^{3/4}$ (3/4 is somewhat ad hoc), which gives a contribution of order $Z^{\beta}_{t-t^{3/4},Y}$, and then force X to go to Y_t at time t. If $X_{t-t^{3/4}}$ is not too far from $Y_{t-t^{3/4}}$, then we expect the cost of forcing $X_t = Y_t$ to be negligible, and if such X gives the dominant contribution in $Z^{\beta}_{t-t^{3/4},Y}$, then we are essentially done.

We now make the above heuristics precise. Note that

$$Z^{\beta}_{t,Y} \le \mathrm{e}^{|\beta| t^{3/4}} Z^{\beta}_{t-t^{3/4},Y}. \tag{2.4}$$

We claim that for t sufficiently large,

$$\mathbb{E}^X_0\big[\mathrm{e}^{\beta L_{t-t^{3/4}}(X,Y)} 1_{\{\|X_{t-t^{3/4}}\|_1 \le t^{2/3}\}}\big] \ge \mathbb{E}^X_0\big[\mathrm{e}^{\beta L_{t-t^{3/4}}(X,Y)} 1_{\{\|X_{t-t^{3/4}}\|_1 > t^{2/3}\}}\big]. \tag{2.5}$$

By Proposition 2.1, for t sufficiently large, we have $\sup_{0\le s\le t}\|Y_s\|_1 \le C\sqrt{t\log\log t}$. Define recursively stopping times $\sigma_1 = 0$, and for $n\in\mathbb{N}$,

$$\tau_n = \inf\big\{s \in \big(\sigma_n, t - t^{3/4}\big]\colon \|X_s\|_1 \ge t^{2/3}/2\big\},$$
$$\sigma_{n+1} = \inf\big\{s \in \big(\tau_n, t - t^{3/4}\big]\colon \|X_s\|_1 \le C\sqrt{t\log\log t}\big\}, \tag{2.6}$$

where we set σ_n, τ_n to $t - t^{3/4}$ if the infimum is taken over an empty set. Then

$$\mathbb{E}^X_0\big[\mathrm{e}^{\beta L_{t-t^{3/4}}(X,Y)} 1_{\{\|X_{t-t^{3/4}}\|_1 > t^{2/3}\}}\big]$$
$$= \sum_{n=1}^{\infty} \mathbb{E}^X_0 2\big[\mathrm{e}^{\beta L_{\tau_n}(X,Y)} 1_{\{\tau_n < \sigma_{n+1} = t-t^{3/4}, \|X_{t-t^{3/4}}\|_1 > t^{2/3}\}}\big]$$

$$= \sum_{n=1}^{\infty} \mathbb{E}_0^X \big[\mathrm{e}^{\beta L_{\tau_n}(X,Y)} 1_{\{\tau_n < t - t^{3/4}\}} \mathbb{P}_0^X \big(\sigma_{n+1} = t - t^{3/4}, \|X_{t-t^{3/4}}\|_1 > t^{2/3} | X_{\tau_n} \big) \big]$$

$$\le \sum_{n=1}^{\infty} \mathbb{E}_0^X \big[\mathrm{e}^{\beta L_{\tau_n}(X,Y)} 1_{\{\tau_n < t - t^{3/4}\}} \mathbb{P}_0^X \big(\sigma_{n+1} = t - t^{3/4}, \|X_{t-t^{3/4}}\|_1 \le t^{2/3} | X_{\tau_n} \big) \big]$$

$$\le \mathbb{E}_0^X \big[\mathrm{e}^{\beta L_{t-t^{3/4}}(X,Y)} 1_{\{\|X_{t-t^{3/4}}\|_1 \le t^{2/3}\}} \big], \tag{2.7}$$

where in the first inequality we used the fact that $t^{2/3}/2 \gg \sqrt{t \log\log t} \gg \sqrt{t}$ for t large. This proves the claim (2.5). By Proposition 2.1, we have $\mathbb{P}_0^X(X_t = Y_t | X_{t-t^{3/4}} = x) \ge C(1+t^{3/4})^{-d/2}(2d)^{-2t^{2/3}}$ uniformly for $\|x\|_1 \le t^{2/3}$. Hence

$$Z_{t,Y}^{\beta,\mathrm{pin}} \ge C\big(1+t^{3/4}\big)^{-d/2}(2d)^{-2t^{2/3}} \mathrm{e}^{-|\beta| t^{3/4}} \mathbb{E}_0^X \big[\mathrm{e}^{\beta L_{t-t^{3/4}}(X,Y)} 1_{\{\|X_{t-t^{3/4}}\|_1 \le t^{2/3}\}} \big].$$

Combined with (2.4) and (2.5), we find

$$Z_{t,Y}^{\beta} \le 2C^{-1}\big(1+t^{3/4}\big)^{d/2}(2d)^{2t^{2/3}} \mathrm{e}^{2|\beta| t^{3/4}} Z_{t,Y}^{\beta,\mathrm{pin}}.$$

Since $Z_{t,Y}^{\beta} > Z_{t,Y}^{\beta,\mathrm{pin}}$, (1.6) follows with $F(\beta,\rho) = F^{\mathrm{pin}}$.

Lastly, (1.7) holds because (2.1) is valid with $F^{\mathrm{pin}} = F(\beta,\rho)$ if we take the limit in (2.1) along nt, $n \in \mathbb{N}$, for any fixed $t > 0$. □

Proof of Corollary 1.1. From the theory for homogeneous pinning models (see e.g. Chapter 2 of [14]), it is known that β_c^{ann} exists, and $\beta_c^{\mathrm{ann}} = 0$ if the renewal process underlying the pinning model is recurrent (i.e., the random walk $X - Y$ is recurrent), and $\beta_c^{\mathrm{ann}} > 0$ if the random walk $X - Y$ is transient. The statement $\beta_c^{\mathrm{ann}} \le \beta_c$ follows from

$$F(\beta,\rho) = \lim_{t\to\infty} t^{-1} \mathbb{E}_0^Y \big[\log Z_{t,Y}^{\beta} \big] \le \lim_{t\to\infty} t^{-1} \log \mathbb{E}_0^Y \big[Z_{t,Y}^{\beta} \big] = F_{\mathrm{ann}}(\beta,\rho)$$

by the L^1 convergence in Theorem 1.1 and Jensen's inequality. The statement $\beta_c \ge 0$ follows from the fact that for $\beta < 0$, $F(\beta,\rho) = 0$. Indeed, for $\beta < 0$, $Z_{t,Y}^{\beta} \le 1$, while

$$\log Z_{t,Y}^{\beta} = \log \mathbb{E}_0^X \big[\mathrm{e}^{\beta L_t(X,Y)} \big] \ge \beta \mathbb{E}_0^X \big[L_t(X,Y) \big] \ge \beta \int_0^t \frac{C}{(1+s)^{d/2}} \, \mathrm{d}s = \mathrm{o}(t),$$

where we used the local central limit theorem that $\mathbb{P}_0^X(X_t = x) \le C(1+t)^{-d/2}$ uniformly in $t > 0$ and $x \in \mathbb{Z}^d$. The existence and finiteness of β_c then follows from (1.7) and the monotonicity of $F(\beta,\rho)$ in β. The proof for the discrete time model is identical. □

Proof of Theorem 1.4. The difference between the Feynman–Kac representation of $u(t,x)$ in (1.12) and the representation for $Z_{t,Y}^{\beta}$ in (1.2) is: (1) time-reversal for X; (2) in (1.12), X starts at x instead of on Y. The same proof as for Theorem 1.1 shows that $\lim_{t\to\infty} t^{-1} u(t, Y_t) = F(\beta,\rho)$ a.s. w.r.t. Y where $F(\beta,\rho)$ is as in (1.7). To compare $u(t,x)$ with $u(t,Y_t)$, note that

$$u(t,x) \ge \mathbb{P}_0^X(X_{t^{2/3}} = Y_{t-t^{2/3}} - x) \mathrm{e}^{-|\beta| t^{2/3}} u\big(t - t^{2/3}, Y_{t-t^{2/3}}\big), \tag{2.8}$$

which a.s. gives the correct lower bound on the exponential scale as $t \to \infty$. For the upper bound, note that if $\beta \le 0$, then $u(t,x) \le 1$, which suffices by Corollary 1.1. If $\beta > 0$, then for any $\varepsilon > 0$, a.s. we can find $T_{\varepsilon,Y}$ sufficiently large s.t. $t^{-1} \log u(t,Y_t) \le F(\beta,\rho) + \varepsilon$ for all $t \ge T_{\varepsilon,Y}$. In (1.12), let $\tau = \inf\{s \in [0,t]\colon X_s = Y_{t-s}\}$ with $\tau = t$ if the set is empty. Then for all $t > T_{\varepsilon,Y}$ and $x \in \mathbb{Z}^d$, we have

$$u(t,x) \le \mathbb{P}_x^X(\tau \ge t - T_{\varepsilon,Y}) \mathrm{e}^{\beta T_{\varepsilon,Y}} + \mathbb{E}_x^X \big[u(t-\tau, Y_{t-\tau}) 1_{\{\tau < t - T_{\varepsilon,Y}\}} \big] \le \mathrm{e}^{\beta T_{\varepsilon,Y}} + \mathrm{e}^{(F(\beta,\rho)+\varepsilon)t}. \tag{2.9}$$

Since $\varepsilon > 0$ can be arbitrarily small, a.s. this provides the correct upper bound for $u(t,x)$ on the exponential scale as $t \to \infty$. The L^1 convergence in (1.11) follows from the uniform boundedness of $|\log u(t,x)|$ in t, x and Y. □

3. Coincidence of critical points in $d = 1$ and 2

Proof of Theorem 1.2 for $d = 1$ and 2. The proof for the discrete and continuous time cases are essentially the same, except that the estimates for the continuous time random walk transition kernel is slightly more involved. So we will only consider the continuous time case. As pointed out in the proof of Corollary 1.1, because the random walk $X - Y$ is recurrent in $d = 1$ and 2, $\beta_c^{\mathrm{ann}} = 0$. By (1.7), to show $\beta_c = 0$, it suffices to show that for any $\beta > 0$, there exists $t > 0$ such that $\mathbb{E}_0^Y[\log Z_{t,Y}^{\beta,\mathrm{pin}}] > 0$. We can write

$$\mathbb{E}_0^Y\big[\log Z_{t,Y}^{\beta,\mathrm{pin}}\big] = \mathbb{E}_0^Y\big[\log \mathbb{P}_0^X(X_t = Y_t)\big] + \mathbb{E}_0^Y\big[\log \mathbb{E}_0^X\big[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t\big]\big]. \tag{3.1}$$

We first we estimate $\mathbb{E}_0^Y[\log \mathbb{P}_0^X(X_t = Y_t)] = \sum_{x\in\mathbb{Z}^d} p_{\rho t}(x)\log p_t(x)$, where $p_t(x)$ denotes the transition probability of a jump rate 1 continuous time simple random walk on $\mathbb{Z}^d$. We then find lower bounds for the second term in (3.1) for $d = 1$ and $d = 2$.

Lemma 3.1. *For all $\rho \geq 0$, we have*

$$\lim_{t\to\infty} \frac{\sum_{x\in\mathbb{Z}^d} p_{\rho t}(x)\log p_t(x)}{\log t} = -\frac{d}{2}. \tag{3.2}$$

Proof. By the local central limit theorem, $p_t(x) \leq C(1+t)^{-d/2}$ uniformly for $t > 0$ and $x \in \mathbb{Z}^d$. Hence

$$\limsup_{t\to\infty} \frac{\sum_{x\in\mathbb{Z}} p_{\rho t}(x)\log p_t(x)}{\log t} \leq -\frac{d}{2}. \tag{3.3}$$

For a matching lower bound, we need lower bounds for $p_t(x)$ for all $x \in \mathbb{Z}^d$. Note that if $p_t^{(1)}(\cdot)$ denotes the transition probability kernel of a rate 1 simple random walk on $\mathbb{Z}$, then $p_t(x) = \Pi_{i=1}^d p_{t/d}^{(1)}(x_i)$, and $\sum_{x\in\mathbb{Z}^d} p_{\rho t}(x)\log p_t(x) = d\sum_{x\in\mathbb{Z}} p_{\rho t/d}^{(1)}(x)\log p_{t/d}^{(1)}(x)$. Hence it suffices to show

$$\liminf_{t\to\infty} \frac{\sum_{x\in\mathbb{Z}} p_{\rho t}^{(1)}(x)\log p_t^{(1)}(x)}{\log t} \geq -\frac{1}{2}. \tag{3.4}$$

For $0 < \varepsilon \ll 1 \ll A < \infty$, we have the following estimates. There exist $C_1, C_2, C_3, T > 0$ depending on ε and A, such that

$$p_t^{(1)}(x) \geq C_1 t^{-1/2}\mathrm{e}^{-C_2x^2/t} \quad \forall t \geq T, |x| \leq \varepsilon t, \tag{3.5}$$

$$p_t^{(1)}(x) \geq \mathrm{e}^{-C_3t} \quad \forall t \geq T, \varepsilon t < |x| < At, \tag{3.6}$$

$$p_t^{(1)}(x) \geq \mathrm{e}^{-2|x|\log|x|} \quad \forall t \geq T, At \leq |x|. \tag{3.7}$$

To derive (3.4) from (3.5)–(3.7), we partition the sum $\sum_{x\in\mathbb{Z}}$ into $\sum_{|x|\leq\varepsilon t}$, $\sum_{\varepsilon t<|x|<At}$, and $\sum_{|x|\geq At}$ with $\varepsilon \ll \rho \ll A$. By (3.5),

$$\begin{aligned}\sum_{|x|\leq\varepsilon t} p_{\rho t}^{(1)}(x)\log p_t^{(1)}(x) &\geq \sum_{|x|\leq\varepsilon t} p_{\rho t}^{(1)}(x)\log\big(C_1t^{-1/2}\mathrm{e}^{-C_2x^2/t}\big)\\ &\geq -\frac{\log t}{2} - |\log C_1| - \frac{C_2}{t}\sum_{x\in\mathbb{Z}} x^2 p_{\rho t}^{(1)}(x) = -\frac{\log t}{2} - |\log C_1| - C_2\rho.\end{aligned} \tag{3.8}$$

By (3.6) and the Markov inequality,

$$\sum_{\varepsilon t<|x|<At} p_{\rho t}^{(1)}(x)\log p_t^{(1)}(x) \geq -C_3t\sum_{|x|>\varepsilon t} p_{\rho t}^{(1)}(x) \geq -C_3t\frac{\sum_{x\in\mathbb{Z}} x^2 p_{\rho t}^{(1)}(x)}{\varepsilon^2t^2} = -\frac{C_3\rho}{\varepsilon^2}. \tag{3.9}$$

And by (3.7), for t sufficiently large, we have

$$\sum_{|x|\ge At} p^{(1)}_{\rho t}(x)\log p^{(1)}_t(x) \ge -2\sum_{|x|\ge At} p^{(1)}_{\rho t}(x)|x|\log|x| \ge -2\sum_{|x|\ge At} p^{(1)}_{\rho t}(x)\frac{|x|^2}{At/\log(At)}$$
$$\ge -\frac{2\rho}{A}\log(At). \tag{3.10}$$

Combining (3.8)–(3.10), we obtain the lower bound

$$\liminf_{t\to\infty}\frac{\sum_{x\in\mathbb{Z}} p^{(1)}_{\rho t}(x)\log p^{(1)}_t(x)}{\log t} \ge -\frac{1}{2}-\frac{2\rho}{A}. \tag{3.11}$$

Since A can be chosen arbitrarily large, (3.4) follows.

We now verify (3.5)–(3.7). Let $P_n(x)$ denote the probability that a discrete time simple random walk starting from 0 visits x at time n. Then for x and n having the same parity, by Stirling's formula,

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n}\frac{n!}{((n+x)/2)!((n-x)/2)!}\\
&= \frac{(1+\mathrm{o}(1))\sqrt{2\pi n}(n/\mathrm{e})^n}{2^n\sqrt{2\pi((n+x)/2)}((n+x)/(2\mathrm{e}))^{(n+x)/2}\sqrt{2\pi((n-x)/2)}((n-x)/(2\mathrm{e}))^{(n-x)/2}}\\
&= \big(1+\mathrm{o}(1)\big)\sqrt{\frac{2n}{\pi(n^2-x^2)}}\mathrm{e}^{-((n+x)/2)\log(1+x/n)-((n-x)/2)\log(1-x/n)}\\
&= \big(1+\mathrm{o}(1)\big)\sqrt{\frac{2n}{\pi(n^2-x^2)}}\mathrm{e}^{-x^2/(2n)+\mathrm{o}(x^2/n^2)n}.
\end{aligned} \tag{3.12}$$

Hence for n sufficiently large and $|x|/n$ sufficiently small, we have

$$P_n(x) \ge Cn^{-1/2}\mathrm{e}^{-x^2/n}. \tag{3.13}$$

If N_t denotes a Poisson random variable with mean t, then (3.5) follows from (3.13) and the observation that $N_t/t\to 1$ in probability with $|\mathbb{P}(N_t \text{ is odd})-\mathbb{P}(N_t \text{ is even})|\to 0$ as $t\to\infty$.

For (3.6), note that for $|x|<At$, by (3.13),

$$\begin{aligned}
p^{(1)}_t(x) &\ge \sum_{\substack{At/\varepsilon\le n\le 2At/\varepsilon\\ n\equiv x \bmod 2}} \mathbb{P}(N_t=n)P_n(x)\\
&\ge C\sqrt{\frac{\varepsilon}{2At}}\mathrm{e}^{-\varepsilon x^2/(At)}\mathbb{P}(At/\varepsilon\le N_t\le 2At/\varepsilon,\, N_t\equiv x \bmod 2)\\
&\ge C\sqrt{\frac{\varepsilon}{2At}}\mathrm{e}^{-\varepsilon At}\mathrm{e}^{-C't} \ge \mathrm{e}^{-C_3 t},
\end{aligned} \tag{3.14}$$

where we used the fact that N_t/t satisfies a large deviation principle with a finite rate function on $[0,\infty)$.

For $|x|\ge At$, we can bound $p^{(1)}_t(x)$ from below by requiring that the random walk makes exactly $|x|$ jumps in the time interval $[0,1]$ so that the random walk is at x at time 1, and at time t the random walk returns to x. Thus, by the local central limit theorem, for t large,

$$p^{(1)}_t(x) \ge \frac{1}{\mathrm{e}|x|!}2^{-|x|}\frac{C}{t} = \big(1+\mathrm{o}(1)\big)\frac{\mathrm{e}^{-1+|x|-|x|\log|x|}}{\sqrt{2\pi|x|}}2^{-|x|}\frac{C}{t}. \tag{3.15}$$

It is then clear that (3.7) holds. □

Remark. *We point out that, for general mean zero finite variance random walks, the estimates (3.5)–(3.7) can still be established by adapting the proof here and decomposing the random walk transition kernel to extract a simple random walk part.*

Remark. *The analogue of Lemma 3.1 also holds for discrete time simple random walks. The proof is similar and omitted.*

Lower bound for $\mathbb{E}_0^Y[\log \mathbb{E}_0^X[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t]]$ *for* $d = 1$

By Jensen's inequality,

$$\mathbb{E}_0^Y\bigl[\log \mathbb{E}_0^X\bigl[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t\bigr]\bigr] \geq \mathbb{E}_0^Y\bigl[\mathbb{E}_0^X\bigl[\beta L_t(X,Y)|X_t = Y_t\bigr]\bigr] = \beta \int_0^t \mathbb{E}_0^Y\left[\frac{p_s(Y_s)p_{(t-s)}(Y_t - Y_s)}{p_t(Y_t)}\right]\mathrm{d}s.$$

By Donsker's invariance principle, there exists $\alpha > 0$ s.t. $\mathbb{P}_0^Y(\sup_{s\in[0,t]}|Y_s| \leq \sqrt{t}) \geq \alpha$ for all $t > 0$. On the other hand, if $\sup_{s\in[0,t]}|Y_s| \leq \sqrt{t}$, then by the local central limit theorem, $p_s(Y_s) \wedge p_{t-s}(Y_t - Y_s) \geq C/\sqrt{t}$ for all $s \in [t/3, 2t/3]$ for some C independent of Y and $t > 1$, while $p_t(Y_t) \leq C'/\sqrt{t}$. Therefore

$$\mathbb{E}_0^Y\bigl[\log \mathbb{E}_0^X\bigl[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t\bigr]\bigr] \geq \alpha\beta \int_{t/3}^{2t/3} \frac{(C/\sqrt{t})(C/\sqrt{t})}{C'/\sqrt{t}}\,\mathrm{d}s = C'\sqrt{t} \tag{3.16}$$

for some $C' > 0$ independent of t. In view of (3.1) and Lemma 3.1, this proves that $\mathbb{E}_0^Y[\log Z_{t,Y}^{\beta,\mathrm{pin}}] > 0$ for t large, and hence $\beta_{\mathrm{c}} = 0$ for $d = 1$.

Lower bound for $\mathbb{E}_0^Y[\log \mathbb{E}_0^X[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t]]$ *for* $d = 2$

Since in $d = 2$, $L_t(X,Y)$ is typically of order $\log t$, the argument above for $d = 1$ fails for $d = 2$. Instead, we apply an a.s. limit theorem for $L_t(X,Y)/\log t$ conditioned on Y. More precisely, by Theorem 1.2 of Gärtner and Sun [13], a.s. w.r.t. Y, $L_t(X,Y)/\log t$ conditioned on Y converges in distribution to an exponential random variable with mean $1/\pi(1+\rho)$. We only need to bypass the conditioning on $X_t = Y_t$.

Let $\mu_{t/\log t}$ denote the law of $(X_s)_{0\leq s\leq t/\log t}$, and let $\mu_{t/\log t}^{(t,y)}$ denote the law of $(X_s)_{0\leq s\leq t/\log t}$ conditioned on $X_t = y$. Then $\mu_{t/\log t}$ and $\mu_{t/\log t}^{(t,y)}$ are equivalent with density

$$\frac{\mathrm{d}\mu_{t/\log t}^{(t,y)}}{\mathrm{d}\mu_{t/\log t}}(X) = \frac{p_{t-t/\log t}(y - X_{t/\log t})}{p_t(y)} = \frac{t}{t - t/\log t}\,\frac{\mathrm{e}^{-\|y-X_{t/\log t}\|^2/(t-t/\log t)} + \mathrm{o}(1)}{\mathrm{e}^{-\|y\|^2/t} + \mathrm{o}(1)}, \tag{3.17}$$

where we applied the local central limit theorem. Since $\|X_{t/\log t}\|/\sqrt{t} \to 0$ in probability as $t \to \infty$, it is clear that in total variational distance,

$$\sup_{\|y\|\leq\sqrt{t}} \bigl\|\mu_{t/\log t}^{(t,y)} - \mu_{t/\log t}\bigr\|_{\mathrm{TV}} \underset{t\to\infty}{\longrightarrow} 0. \tag{3.18}$$

We can thus remove the conditioning at the cost of reducing the time interval from t to $t/\log t$.

Fix $A > 0$. Let

$$G_{t/\log t}^A = \bigl\{Y\colon\ \mu_{t/\log t}\bigl(L_{t/\log t}(X,Y) \geq A\log t\bigr) \geq \mathrm{e}^{-\alpha A}\bigr\}. \tag{3.19}$$

By Theorem 1.2 of [13], if we choose $\alpha > \pi(1+\rho)$, then $\mathbb{P}_0^Y(G_{t/\log t}^A) \to 1$ as $t \to \infty$. We now write

$$\begin{aligned}
&\mathbb{E}_0^Y\bigl[\log \mathbb{E}_0^X\bigl[\mathrm{e}^{\beta L_t(X,Y)}|X_t = Y_t\bigr]\bigr] \\
&\quad\geq \mathbb{E}_0^Y\bigl[1_{\{\|Y_t\|\leq\sqrt{t},\,Y\in G_{t/\log t}^A\}} \log \mathbb{E}_0^X\bigl[\mathrm{e}^{\beta L_{t/\log t}(X,Y)}|X_t = Y_t\bigr]\bigr]
\end{aligned}$$

$$\geq \mathbb{E}_0^Y\big[1_{\{\|Y_t\|\leq\sqrt{t}, Y\in G^A_{t/\log t}\}}\big(\beta A\log t+\log\mu^{(t,Y_t)}_{t/\log t}\big(L_{t/\log t}(X,Y)\geq A\log t\big)\big)\big]$$

$$\geq \beta A\mathbb{P}_0^Y\big(\|Y_t\|\leq\sqrt{t}, Y\in G^A_{t/\log t}\big)\log t$$

$$+\mathbb{E}_0^Y\big[1_{\{\|Y_t\|\leq\sqrt{t}, Y\in G^A_{t/\log t}\}}\log\big(\mu_{t/\log t}\big(L_{t/\log t}(X,Y)\geq A\log t\big)+\mathrm{o}(1)\big)\big].$$

$$\geq \big(C-\mathrm{o}(1)\big)\big(\beta A\log t+\log\big(\mathrm{e}^{-\alpha A}+\mathrm{o}(1)\big)\big), \tag{3.20}$$

where $C=\inf_{t>0}\mathbb{P}_0^Y(\|Y_t\|\leq\sqrt{t})$ is positive and independent of A. Since A can be chosen arbitrarily large, in view of (3.1) and Lemma 3.1, this proves that $\mathbb{E}_0^Y[\log Z^{\beta,\mathrm{pin}}_{t,Y}]>0$ for t large, and hence $\beta_c=0$ for $d=2$. □

4. Gap between critical points: discrete time

4.1. *Proof of Theorem 1.3 in discrete time*: $d\geq 5$

Our proof is based on adaptations of the fractional moment method used recently by Derrida et al. [8] to show the non-coincidence of annealed and quenched critical points for the pinning model in the disorder-relevant regime. Two ingredients are needed for the adaptation. First, a suitable representation for the partition function $\hat{Z}^{\beta}_{N,Y}$ and its constrained counterpart $\hat{Z}^{\beta,\mathrm{pin}}_{N,Y}$ in a similar form as in (1.16), except with a Gibbs weight factor $w(\cdot)$ that has a simpler dependence on the disorder $(\Delta_i)_{i\in\mathbb{N}}=(Y_{i+1}-Y_i)_{i\in\mathbb{N}}$ than in (1.17). Second, a suitable change of measure for the disorder Y when estimating fractional moments $\mathbb{E}_0^Y[(\hat{Z}^{\beta,\mathrm{pin}}_{N,Y})^{\gamma}]$ for N on the order of the correlation length of the annealed model.

We split the proof into three parts: representation for $\hat{Z}^{\beta}_{N,Y}$ and $\hat{Z}^{\beta,\mathrm{pin}}_{N,Y}$; fractional moment method; change of measure. To simplify notation, C, C_1, C', etc., will denote generic constants whose precise values may change from place to place.

Representation for $\hat{Z}^{\beta}_{N,Y}$ and $\hat{Z}^{\beta,\mathrm{pin}}_{N,Y}$

The representation we now derive was already used in [3]. It is based on binomial expansion for $(1+\mathrm{e}^{\beta}-1)^{L_N(X,Y)}$. Let $p^X_n(\cdot)$, resp. $p^{X-Y}_n(\cdot)$, be the n-step transition probability kernel of X, resp. $X-Y$. Let $G^{X-Y}=\sum_{n=1}^{\infty}p^{X-Y}_n(0)$, $K(n)=p^{X-Y}_n(0)/G^{X-Y}$, $z'=\mathrm{e}^{\beta}-1$, $z=z'G^{X-Y}$, and $\check{Z}^{z}_{N,Y}=\hat{Z}^{\beta}_{N,Y}$. Then

$$\check{Z}^{z}_{N,Y}=\mathbb{E}_0^X\big[\big(1+z'\big)^{L_N(X,Y)}\big]=\mathbb{E}_0^X\left[1+\sum_{m=1}^{N}\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m\leq N}(z')^m\prod_{i=1}^{m}1_{\{X_{\sigma_i}=Y_{\sigma_i}\}}\right]$$

$$=1+\sum_{m=1}^{N}\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m\leq N}\big(z'\big)^m\prod_{i=1}^{m}p^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})$$

$$=1+\sum_{m=1}^{N}\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m\leq N}\prod_{i=1}^{m}K(\sigma_i-\sigma_{i-1})w(z,\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}}), \tag{4.1}$$

where

$$w(z,\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}})=zp^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})/p^{X-Y}_{\sigma_i-\sigma_{i-1}}(0). \tag{4.2}$$

If we denote $\check{Z}^{z,\mathrm{pin}}_{N,Y}=\frac{z'}{1+z'}\hat{Z}^{\beta,\mathrm{pin}}_{N,Y}$, then similarly,

$$\check{Z}^{z,\mathrm{pin}}_{N,Y}=\mathbb{E}_0^X\big[\big(1+z'\big)^{L_{N-1}(X,Y)}z'1_{\{X_N=Y_N\}}\big]$$

$$=\sum_{m=1}^{N}\sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=N}\prod_{i=1}^{m}K(\sigma_i-\sigma_{i-1})w(z,\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}}). \tag{4.3}$$

Note that (4.3) casts $\check{Z}^{z,\mathrm{pin}}_{N,Y}$ in the same form as (1.16), except now $K(n)$ equals $p_n^{X-Y}(0)/G^{X-Y}$ instead of $\mathbb{P}^{X-Y}(\tau_0 = n)$. This mapping from one underlying renewal process to another defined in terms of the Green function decomposition of the original renewal process applies to any pinning model with an underlying transient renewal distribution. Of course the disorder also changes and the terms in (4.3) may not be positive in general. This is not the case here, and the key point for us is that the weight factor w now has a much simpler dependence on the disorder $(\Delta_j)_{\sigma_{i-1}<j\le\sigma_i}$ (i.e. only on $\sigma_i - \sigma_{i-1}$ and $\sum_{j=\sigma_{i-1}+1}^{\sigma_i}\Delta_j$) than in (1.17). We note that if $\tilde{K}(n) \sim \frac{c}{n^{1+\alpha}}$ for some $\alpha > 0$ is the first return time distribution of a transient renewal process, then the corresponding return probability at time n satisfies $p(n) \sim \frac{c'}{n^{1+\alpha}}$. See [14], Theorem A.4.

Because K is the return time distribution of a recurrent renewal process σ on $\mathbb{N}_0$, and $\mathbb{E}_0^Y[w(z, \sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}})] = z$, the critical point for the annealed model associated with $\check{Z}^{z,\mathrm{pin}}_{N,Y}$ is $z_c^{\mathrm{ann}} = 1$, or equivalently, $1 = z_c^{\mathrm{ann}} = (e^{\hat{\beta}_c^{\mathrm{ann}}} - 1)G^{X-Y}$ so that

$$\hat{\beta}_c^{\mathrm{ann}} = \log\left(1 + \frac{1}{G^{X-Y}}\right). \tag{4.4}$$

Fractional moment method

We now recall the fractional moment method used by Derrida et al. in [8]. Due to the common framework between pinning models and the random walk pinning model as pointed out in Section 1.3, the basic strategy carries over without change. The only model dependent part of the argument lies in estimating $\mathbb{E}_0^Y[(\check{Z}^{z,\mathrm{pin}}_{N,Y})^\gamma]$, $\gamma \in (0,1)$, for N on the order of the correlation length of the annealed model, where a change of measure argument for the disorder needs to be adapted.

In terms of the new variables $z = (e^\beta - 1)G^{X-Y}$ and $\check{Z}^z_{N,Y}$, Theorem 1.3 reduces to showing that for some $z > z_c^{\mathrm{ann}} = 1$, $\sup_{N\in\mathbb{N}_0}\check{Z}^z_{N,Y} < \infty$ a.s. w.r.t. Y. Since for $z > 1$, $\check{Z}^z_{N,Y}$ is a.s. increasing in N, it suffices to show that for some $z > 1$ and $\gamma \in (0,1)$,

$$\sup_{N\in\mathbb{N}_0} \mathbb{E}_0^Y\big[\big(\check{Z}^z_{N,Y}\big)^\gamma\big] < \infty. \tag{4.5}$$

The basic idea is to suitably group terms in the expansion for $\check{Z}^z_{N,Y}$ in (4.1) and then apply the fractional moment inequality

$$\left(\sum_{i=1}^n |a_i|\right)^\gamma \le \sum_{i=1}^n |a_i|^\gamma, \quad \gamma \in (0,1). \tag{4.6}$$

However, the effectiveness of (4.6) depends crucially on how $\check{Z}^z_{N,Y}$ is decomposed. In [8], Derrida et al., studied analogues of the constrained partition function $\check{Z}^{z,\mathrm{pin}}_{N,Y}$, and their clever choice is to group terms in (4.3) according to the starting and the ending position of the gap in the renewal sequence σ straddling a fixed position $L \in \mathbb{N}$. Namely,

$$\check{Z}^{z,\mathrm{pin}}_{N,Y} = \sum_{i=0}^{L-1}\sum_{j=0}^{N-L} \check{Z}^{z,\mathrm{pin}}_{i,Y} K(N-j-i) w(z, N-j-i, Y_{N-j} - Y_i)\check{Z}^{z,\mathrm{pin}}_{j,\theta_{N-j}Y},$$

where $\theta_n Y = (Y_{n+i} - Y_n)_{i\in\mathbb{N}_0}$ denotes a shift in Y. For $\check{Z}^z_{N,Y}$, we can perform a similar grouping of terms in (4.1) and get

$$\check{Z}^z_{N,Y} = \check{Z}^z_{L-1,Y} + \sum_{i=0}^{L-1}\sum_{j=0}^{N-L} \check{Z}^{z,\mathrm{pin}}_{i,Y} K(N-j-i) w(z, N-j-i, Y_{N-j} - Y_i)\check{Z}^z_{j,\theta_{N-j}Y}. \tag{4.7}$$

Fix $\gamma \in (0,1)$. Denote $\check{A}^z_N = \mathbb{E}_0^Y[(\check{Z}^z_{N,Y})^\gamma]$ and $\check{A}^{z,\mathrm{pin}}_N = \mathbb{E}_0^Y[(\check{Z}^{z,\mathrm{pin}}_{N,Y})^\gamma]$. Since

$$K(N-j-i)w(z, N-j-i, Y_{N-j} - Y_i) = \frac{z p^X_{N-j-i}(Y_{N-j} - Y_i)}{G^{X-Y}} \le C(N-j-i)^{-d/2}$$

for some $C>0$ independent of i, j, N, Y and $z\in[1,2]$ by the local central limit theorem, applying (4.6) to (4.7) and taking expectation w.r.t. Y gives

$$\begin{aligned}\check{A}^z_N &\le \check{A}^z_{L-1} + C\sum_{i=0}^{L-1}\check{A}^{z,\mathrm{pin}}_i\sum_{j=0}^{N-L}(N-j-i)^{-d\gamma/2}\check{A}^z_j\\ &\le \check{A}^z_{L-1} + C\left(\sum_{i=0}^{L-1}\frac{\check{A}^{z,\mathrm{pin}}_i}{(L-i)^{d\gamma/2-1}}\right)\max_{0\le j\le N-L}\check{A}^z_j.\end{aligned}\tag{4.8}$$

If for some choice of $z>1$ and $L\in\mathbb{N}$,

$$\check{\varrho} = C\left(\sum_{i=0}^{L-1}\frac{\check{A}^{z,\mathrm{pin}}_i}{(L-i)^{d\gamma/2-1}}\right) < 1,\tag{4.9}$$

then iterating (4.8) clearly implies that $\check{A}^z_N$ is uniformly bounded in N, and hence (4.5).

By Jensen's inequality, $\check{A}^{z,\mathrm{pin}}_N \le \mathbb{E}^Y_0[\check{Z}^{z,\mathrm{pin}}_{N,Y}]^\gamma$. It is clear from (4.3) and (4.2) that $\mathbb{E}^Y_0[\check{Z}^{z,\mathrm{pin}}_{N,Y}]$ is the partition function of a homogeneous pinning model with critical point $z^{\mathrm{ann}}_{\mathrm{c}}=1$. Hence $\check{F}_{\mathrm{ann}}(z)=\lim_{N\to\infty}N^{-1}\log\mathbb{E}^Y_0[\check{Z}^{z,\mathrm{pin}}_{N,Y}]$ exists, and $\check{F}_{\mathrm{ann}}(z)=\hat{F}_{\mathrm{ann}}(\beta)$ with $z=(\mathrm{e}^\beta-1)G^{X-Y}$. Since $d\ge5$, $K(\cdot)$ has finite first moment, and hence by Theorem 2.1 of [14], $\check{F}_{\mathrm{ann}}(z)\sim C(z-1)$ for some $C>0$ as $z\downarrow1$. Since $(\mathbb{E}^Y_0[\check{Z}^{z,\mathrm{pin}}_{n,Y}])_{n\in\mathbb{N}}$ is super-multiplicative, $\mathbb{E}^Y_0[\check{Z}^{z,\mathrm{pin}}_{N,Y}]\le \mathrm{e}^{N\check{F}_{\mathrm{ann}}(z)}\le\mathrm{e}^{CN(z-1)}$ for all $N\in\mathbb{N}$. So if we choose

$$L=L(z)=\frac{1}{z-1},\tag{4.10}$$

where we abused notation and assumed L to be an integer for simplicity, then $\sup_{1\le i\le L}\check{A}^{z,\mathrm{pin}}_i\le C$ for some $C>0$ independent of z. Therefore

$$\check{\varrho}\le\sum_{i=0}^{L-R}\frac{C}{(L-i)^{d\gamma/2-1}}+\sum_{i=L-R+1}^{L-1}\frac{C\check{A}^{z,\mathrm{pin}}_i}{(L-i)^{d\gamma/2-1}}\le CR^{2-d\gamma/2}+C\max_{L-R\le i\le L}\check{A}^{z,\mathrm{pin}}_i.\tag{4.11}$$

For $d\ge5$, we can choose $\gamma<1$ close to 1 such that the first term on the RHS of (4.11) can be made arbitrarily small (uniformly in z) by choosing R large. To show $\check{\varrho}<1$ for some $z>1$, it then suffices to show that

$$\lim_{z\downarrow1}\max_{L-R\le N\le L}\check{A}^{z,\mathrm{pin}}_N=0,\tag{4.12}$$

where $R\in\mathbb{N}$ is large and fixed, and $L=\frac{1}{z-1}$. This summarizes the model independent part of the fractional moment method as used in [8].

Change of measure

The basic idea in [8] to prove (4.12) is to apply a change of measure to the disorder so that the cost of changing the measure is small, yet under the new disorder, the annealed partition function for a system of size L is small. For the pinning model, the choice of changing the measure in [8] is to make the disorder more repulsive, i.e., tilt the measure of ω_i in (1.14) by a factor $\mathrm{e}^{-\lambda\omega_i}$ for some $\lambda>0$. In our setting, it turns out that for the continuous time model, the appropriate change of measure is to increase the jump rate of the random walk Y. For the discrete time model, the analogue is to increase the variance of the random walk increment each step without changing the support of the random walk transition kernel. However, among nearest-neighbor random walks on $\mathbb{Z}^d$, the variance of simple random walk is already maximal. To overcome this difficulty, we change measure for Y two steps at a time. More precisely, for $h \in (0, \frac{1}{2d})$, let $(Y_n^h)_{n\in\mathbb{N}_0}$ be a process on $\mathbb{Z}^d$ with $Y_0 = 0$ and transition probabilities

$$\mathbb{P}\big(Y_{n+1}^h - Y_n^h = e_i \big| (Y_k^h)_{0\le k\le n}\big) = \begin{cases} \frac{1}{2d} & \text{if } n \text{ is even, or } n \text{ is odd and } e_i \neq \pm\big(Y_n^h - Y_{n-1}^h\big), \\ \frac{1+h}{2d} & \text{if } n \text{ is odd, and } e_i = Y_n^h - Y_{n-1}^h, \\ \frac{1-h}{2d} & \text{if } n \text{ is odd, and } e_i = -\big(Y_n^h - Y_{n-1}^h\big) \end{cases} \tag{4.13}$$

for each of the $2d$ unit vectors $e_i \in \mathbb{Z}^d$. Note that $\mathbb{P}(Y_2^h = 2e_i) = \mathbb{P}(Y_2 = 2e_i) + \frac{h}{4d^2}$ for each unit vector $e_i \in \mathbb{Z}^d$, $\mathbb{P}(Y_2^h = 0) = \mathbb{P}(Y_2 = 0) - \frac{h}{2d}$, and $\mathbb{P}(Y_2^h = x) = \mathbb{P}(Y_2 = x)$ for all other $x \in \mathbb{Z}^d$. Thus Y_2^h has larger variances than Y_2. Clearly up to any time $N \in \mathbb{N}$, the distribution of Y and Y^h are equivalent. Let $f(N, Y)$ denote the Radon–Nikodym derivative of the law of $(Y_i^h)_{0\le i\le N}$ w.r.t. $(Y_i)_{0\le i\le N}$. Then

$$\begin{aligned} \check{A}_N^{z,\text{pin}} &= \mathbb{E}_0^{Y^h}\big[f\big(N, Y^h\big)^{-1}\big(\check{Z}_{N,Y^h}^{z,\text{pin}}\big)^\gamma\big] \le \mathbb{E}_0^{Y^h}\big[f\big(N, Y^h\big)^{-1/(1-\gamma)}\big]^{1-\gamma}\mathbb{E}_0^{Y^h}\big[\check{Z}_{N,Y^h}^{z,\text{pin}}\big]^\gamma \\ &= \mathbb{E}_0^{Y}\big[f(N, Y)^{-\gamma/(1-\gamma)}\big]^{1-\gamma}\mathbb{E}_0^{Y^h}\big[\check{Z}_{N,Y^h}^{z,\text{pin}}\big]^\gamma. \end{aligned} \tag{4.14}$$

Since $(Y_{2n+1} - Y_{2n}, Y_{2n+2} - Y_{2n})_{n\in\mathbb{N}_0}$ are i.i.d. and the distribution of $Y_{2n+1}^h - Y_{2n}^h$ conditioned on Y_{2n}^h is the same as a simple random walk, we have

$$\begin{aligned} \mathbb{E}_0^{Y}\big[f(N, Y)^{-\gamma/(1-\gamma)}\big] &= \mathbb{E}_0^{Y}\big[f(2, Y)^{-\gamma/(1-\gamma)}\big]^{\lfloor N/2\rfloor} \\ &= \left(1 - \frac{1}{d} + \frac{(1+h)^{-\gamma/(1-\gamma)}}{2d} + \frac{(1-h)^{-\gamma/(1-\gamma)}}{2d}\right)^{\lfloor N/2\rfloor} \\ &\le \mathrm{e}^{\gamma h^2 N/(2d(1-\gamma)^2)} \end{aligned}$$

for h sufficiently small. Therefore if we choose $h = \frac{1}{\sqrt{L}}$, then the first factor in (4.14) is uniformly bounded for $L - R \le N \le L$, and to prove (4.12), it only remains to estimate $\mathbb{E}_0^{Y^h}[\check{Z}_{N,Y^h}^{z,\text{pin}}]$ for $h = \frac{1}{\sqrt{L}} = \sqrt{z-1}$.

By (4.3), we have

$$\mathbb{E}_0^{Y^h}\big[\check{Z}_{N,Y^h}^{z,\text{pin}}\big] = \sum_{m=1}^N \left(\frac{z}{G^{X-Y}}\right)^m \sum_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=N} \mathbb{E}_0^{Y^h}\left[\prod_{i=1}^m p_{\sigma_i-\sigma_{i-1}}^X\big(Y_{\sigma_i}^h - Y_{\sigma_{i-1}}^h\big)\right]. \tag{4.15}$$

Note that when σ_{i-1} is even, by the properties of Y^h, we have

$$\mathbb{E}_0^{Y^h}\big[p_{\sigma_i-\sigma_{i-1}}^X\big(Y_{\sigma_i}^h - Y_{\sigma_{i-1}}^h\big)\big|\big(Y_j^h\big)_{0\le j\le\sigma_{i-1}}\big] = \mathbb{E}_0^{Y^h}\big[p_{\sigma_i-\sigma_{i-1}}^X\big(Y_{\sigma_i-\sigma_{i-1}}^h\big)\big].$$

Similarly when σ_{i-1} is odd, by symmetry and translation invariance, we have

$$\mathbb{E}_0^{Y^h}\big[p_{\sigma_i-\sigma_{i-1}}^X\big(Y_{\sigma_i}^h - Y_{\sigma_{i-1}}^h\big)\big|\big(Y_j^h\big)_{0\le j\le\sigma_{i-1}}\big] = \mathbb{E}_0^{Y^h}\big[p_{\sigma_i-\sigma_{i-1}}^X\big(Y_{\sigma_i-\sigma_{i-1}+1}^h - Y_1^h\big)\big|Y_1^h = e_1\big],$$

which is a constant independent of $(Y_j^h)_{0\le j\le\sigma_{i-1}}$. Thus in (4.15), we can successively condition w.r.t. $(Y_j^h)_{0\le j\le\sigma_n}$, $(Y_j^h)_{0\le j\le\sigma_{n-1}}, \ldots, (Y_j^h)_{0\le j\le\sigma_1}$. To write the result in a more compact form, let us denote

$$K_{h,\text{even}}(n) = \frac{\mathbb{E}_0^{Y^h}[p_n^X(Y_n^h)]}{G_{h,\text{even}}}, \quad \text{where } G_{h,\text{even}} = \sum_{n=1}^\infty \mathbb{E}_0^{Y^h}\big[p_n^X\big(Y_n^h\big)\big],$$

$$K_{h,\text{odd}}(n) = \frac{\mathbb{E}_0^{Y^h}[p_n^X(Y_{n+1}^h - Y_1^h)|Y_1^h = e_1]}{G_{h,\text{odd}}}, \quad \text{where } G_{h,\text{odd}} = \sum_{n=1}^\infty \mathbb{E}_0^{Y^h}\big[p_n^X\big(Y_{n+1}^h - Y_1^h\big)\big|Y_1^h = e_1\big].$$

Let $K_h(i,j) = K_{h,\mathrm{even}}(j-i)$ when i is even, and $K_h(i,j) = K_{h,\mathrm{odd}}(j-i)$ when i is odd. Let $\iota = \{0, \iota_1, \iota_2, \ldots\}$ be a renewal process on $\mathbb{N}_0$ with parity-dependent inter-arrival law $K_h(\cdot,\cdot)$, and denote expectation w.r.t. ι by $\mathbb{E}^{K_h}[\cdot]$. Then (4.15) reduces to

$$\mathbb{E}_0^{Y^h}\big[\check{Z}^{z,\mathrm{pin}}_{N,Y^h}\big] = \mathbb{E}^{K_h}\bigg[\bigg(\frac{z}{G^{X-Y}}\bigg)^{|\iota\cap[1,N]|} G_{h,\mathrm{even}}^{|\iota_{\mathrm{e}}\cap[1,N]|} G_{h,\mathrm{odd}}^{|\iota_{\mathrm{o}}\cap[1,N]|} 1_{\{N\in\iota\}}\bigg]$$
$$\le \mathbb{E}^{K_h}\bigg[\bigg(\frac{z(G_{h,\mathrm{even}}\vee G_{h,\mathrm{odd}})}{G^{X-Y}}\bigg)^{|\iota\cap[1,N]|}\bigg],$$

where ι_{e} and ι_{o} denote respectively the even and odd subsets of ι. In $d \ge 5$, by the local central limit theorem, it is easy to see that there exists an inter-arrival probability distribution $K_*(\cdot)$ on $\mathbb{N}$ with finite first moment, such that K_* stochastically dominates both $K_{h,\mathrm{even}}(\cdot)$ and $K_{h,\mathrm{odd}}(\cdot)$ for h sufficiently small, i.e., $\sum_{i\ge n} K_*(i) \ge \sum_{i\ge n} K_{h,\mathrm{even}}(i)$ and $\sum_{i\ge n} K_*(i) \ge \sum_{i\ge n} K_{h,\mathrm{odd}}(i)$ for all $n\in\mathbb{N}$ and $h\in[0,\frac{1}{2}]$. Recall our choice $h = \frac{1}{\sqrt{L}} = \sqrt{z-1}$. We will show that

$$\frac{z(G_{h,\mathrm{even}}\vee G_{h,\mathrm{odd}})}{G^{X-Y}} = 1 - c\sqrt{z-1} + o\big(\sqrt{z-1}\big) \tag{4.16}$$

for some $c>0$. Then for all $z>1$ sufficiently close to 1,

$$\mathbb{E}_0^{Y^h}\big[\check{Z}^{z,\mathrm{pin}}_{N,Y^h}\big] \le \mathbb{E}^{K_*}\big[\big(1 - c\sqrt{z-1} + o\big(\sqrt{z-1}\big)\big)^{|\iota^*\cap[1,N]|}\big], \tag{4.17}$$

where ι^* is a renewal process with inter-arrival law K_* and is independent of z. By the law of large numbers, a.s. w.r.t. ι^*,

$$\lim_{n\to\infty} N^{-1}\big|\iota^*\cap[1,N]\big| = \frac{1}{\sum_{i\in\mathbb{N}} i K_*(i)} > 0,$$

and hence

$$\lim_{z\downarrow 1} \max_{(z-1)^{-1}-R\le N\le (z-1)^{-1}} \big(1 - c\sqrt{z-1} + o\big(\sqrt{z-1}\big)\big)^{|\iota^*\cap[1,N]|} = 0.$$

Thus

$$\lim_{z\downarrow 1} \max_{L-R\le N\le L} \mathbb{E}_0^{Y^h}\big[\check{Z}^{z,\mathrm{pin}}_{N,Y^h}\big] = 0, \quad L = \frac{1}{z-1}, h = \sqrt{z-1}, \tag{4.18}$$

which together with (4.14) implies (4.12).

It only remains to verify (4.16). For $k = (k_1,\ldots,k_d)\in\mathbb{R}^d$, we have

$$\phi(k) := \mathbb{E}_0^X\big[\mathrm{e}^{\mathrm{i}k\cdot X_1}\big] = \frac{1}{d}\sum_{i=1}^d \cos k_i,$$
$$\psi(k) := \mathbb{E}_0^{Y^h}\big[\mathrm{e}^{\mathrm{i}k\cdot Y_2^h}\big] = \phi(k)^2 - \frac{h}{d^2}\sum_{i=1}^d \sin^2 k_i, \tag{4.19}$$
$$\varphi(k) := \mathbb{E}_0^{Y^h}\big[\mathrm{e}^{\mathrm{i}k\cdot(Y_2^h - Y_1^h)}\big| Y_1^h = e_1\big] = \phi(k) + \mathrm{i}\frac{h}{d}\sin k_1.$$

Since X and Y^h are independent, $(Y_{2n}^h - Y_{2n-2}^h)_{n\in\mathbb{N}}$ are i.i.d., $Y_{2n+1}^h - Y_{2n}^h$ is independent of $(Y_j^h)_{0\le j\le 2n}$ and is distributed as X_1, while conditioned on $Y_1^h = e_1$, $Y_2^h - Y_1^h$ is independent of $(Y_j^h - Y_2^h)_{j\ge 2}$, we obtain by Fourier inversion

$$G^{X-Y} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\big(\phi(k)^2+\phi(k)^4+\cdots\big)\,\mathrm{d}k = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\phi(k)^2}{1-\phi(k)^2}\,\mathrm{d}k, \tag{4.20}$$

$$G_{h,\mathrm{even}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\big(\phi(k)^2+\phi(k)^2\psi(k)+\phi(k)^4\psi(k)+\phi(k)^4\psi(k)^2+\cdots\big)\,\mathrm{d}k$$

$$= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\phi(k)^2(1+\psi(k))}{1-\phi(k)^2\psi(k)}\,\mathrm{d}k, \tag{4.21}$$

$$G_{h,\mathrm{odd}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\big(\varphi(k)\phi(k)+\varphi(k)\phi(k)^3+\varphi(k)\phi(k)^3\psi(k)+\varphi(k)\phi(k)^5\psi(k)+\cdots\big)\,\mathrm{d}k$$

$$= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\varphi(k)\phi(k)(1+\phi(k)^2)}{1-\phi(k)^2\psi(k)}\,\mathrm{d}k = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\phi(k)^2(1+\phi(k)^2)}{1-\phi(k)^2\psi(k)}\,\mathrm{d}k, \tag{4.22}$$

where in (4.22) we have used the formula for $\varphi(k)$ and the fact that $\phi(k)$ and $\psi(k)$ are even functions while $\sin k_1$ is odd. Since $\psi(k) < \phi(k)^2$ and $\phi(k), \psi(k) \in [-1,1]$, we have $G_{h,\mathrm{even}} < G_{h,\mathrm{odd}}$, while

$$G^{X-Y} - G_{h,\mathrm{odd}} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left(\frac{\phi(k)^2}{1-\phi(k)^2} - \frac{\phi(k)^2(1+\phi(k)^2)}{1-\phi(k)^2\psi(k)}\right)\mathrm{d}k$$

$$= \frac{h}{(2\pi)^d d^2}\int_{[-\pi,\pi]^d}\frac{\phi(k)^4\sum_{i=1}^d \sin^2 k_i}{(1-\phi(k)^2)(1-\phi(k)^2\psi(k))}\,\mathrm{d}k, \tag{4.23}$$

which implies (4.16) since $h=\sqrt{z-1}$.

Remark. *Equation (4.16) reveals the close resemblance between the random walk pinning model and the pinning model (compare (4.17) here with (4.12) in [8]). In both cases, after changing the measure, we end up comparing with a homogeneous pinning model of size N with weight factor $\mathrm{e}^{-c/\sqrt{N}}$ for each renewal return. The factor $c/\sqrt{N}$ partly explains why $\alpha = 1/2$, resp. $d=3$, is the critical case for the pinning, resp. random walk pinning model.*

Remark. *For general random walks, we can try to change measure for Y one-step at a time. More precisely, let $S = \{y\in\mathbb{Z}^d\colon\ p_1^Y(y)>0\}$. Then for any $A, B\subset S$ and for any transition probability kernels $p_1^A(\cdot)$ and $p_1^B(\cdot)$ with support resp. A and B, and for $h\in\mathbb{R}$ sufficiently close to 0, we can change measure for Y by replacing $p_1^Y(\cdot)$ with $p_1^{Y^h}(x) = p_1^Y(x) + h(p_1^A(x) - p_1^B(x))$. In (4.14), the estimate involving the density $f(N,Y)$ is similar, while the estimate for $\mathbb{E}_0^{Y^h}[\check{Z}^{z,\mathrm{pin}}_{N,Y^h}]$ reduces to estimating*

$$G^{X-Y} - G^{X-Y^h} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left(\frac{1}{1-\phi_X(k)\overline{\phi}_Y(k)} - \frac{1}{1-\phi_X(k)\overline{\phi}_{Y^h}(k)}\right)\mathrm{d}k$$

$$= \frac{h}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{\phi_X(\overline{\phi}_B-\overline{\phi}_A)}{(1-\phi_X\overline{\phi}_Y)(1-\phi_X\overline{\phi}_{Y^h})}\,\mathrm{d}k,$$

where $\phi_X(k) = \sum_x \mathrm{e}^{\mathrm{i}k\cdot x}p_1^X(x)$, $\overline{\phi}_X(k) = \phi_X(-k)$, and $\overline{\phi}_Y(k)$, $\overline{\phi}_A(k)$ and $\overline{\phi}_B(k)$ are defined similarly. Note that in $d\ge 4$, $\int|\frac{\phi_X(\overline{\phi}_B-\overline{\phi}_A)}{(1-\phi_X\overline{\phi}_Y)^2}|\,\mathrm{d}k<\infty$. Therefore based on Taylor expansion in h, all calculations carry through as long as

$$Q := \int\frac{\phi_X(\overline{\phi}_B-\overline{\phi}_A)}{(1-\phi_X\overline{\phi}_Y)^2}\,\mathrm{d}k \neq 0 \tag{4.24}$$

and h is chosen to have the same sign. When X and Y are simple random walks, we have $Q=0$ for any choice of A, B, p_1^A and p_1^B due to symmetry. In particular, changing the drift for the simple random walk fails. On the other *hand, if S contains enough points so as to break symmetry, then it is reasonable to expect the existence of A, B, p_1^A and p_1^B which give $Q \neq 0$. When such A, B, p_1^A and p_1^B exist, we may even take A and B to be singletons in S. We were not able to verify (4.24) for some $A, B \subset S$ for general random walks, such as for all walks with zero mean and finite variance and whose support S contains at least two points which are not related by reflections or permutations of coordinates. However, when X and Y are i.i.d. so that $\phi_X = \phi_Y$, $\phi_X \geq 0$, and $0 \in S$, it is easily seen that $Q > 0$ for $B = \{0\}$ and $p_1^A = p_1^X$. This includes random walks X which are symmetric with $p_1^X(0) \geq \frac{1}{2}$, as well as walks X that can be expressed as the difference of two i.i.d. random walks.*

4.2. *Proof of Theorem 1.3 in discrete time*: $d = 4$

For $d = 4$, in the representation (4.1), we have $K(n) = p_n^{X-Y}(0)/G^{X-Y} \sim Cn^{-2}$ which has infinite first moment. Thus $d = 4$ corresponds to the case $\alpha = 1$ in [8] for the pinning model. In [8], the case $\alpha = 1$ was left out. However, as we will show below, there is no difficulty in extending the fractional moment method to the $d = 4$ case, and we expect the same to be true for the $\alpha = 1$ case for the pinning model.

As in $d \geq 5$, it suffices to verify (4.9). What differs in $d = 4$ is that $\sum_{i=R}^{\infty} i^{1-d\gamma/2} = \sum_{i=R}^{\infty} i^{1-2\gamma} = \infty$ for any $\gamma \in (0, 1)$ and $R \in \mathbb{N}$. Hence a more careful estimate of $\check{\varrho}$ than in (4.11) is needed. By Theorem 2.1 of [14] and super-multiplicativity of $(\mathbb{E}_0^Y[\check{Z}_{n,Y}^{z,\mathrm{pin}}])_{n\in\mathbb{N}}$, we have $\mathbb{E}_0^Y[\check{Z}_{N,Y}^{z,\mathrm{pin}}] \leq \mathrm{e}^{CN(z-1)}$ for some $C > 0$ uniformly in $z > 1$ sufficiently close to 1 and $N \in \mathbb{N}$. Therefore the same choice $L = (z-1)^{-1}$ as in $d \geq 5$ ensures that $\sup_{1\leq i\leq L} \check{A}_i^{z,\mathrm{pin}} \leq C < \infty$ uniformly for $z > 1$ close to 1. Fix $\varepsilon > 0$ small, then let $\gamma \in (0, 1)$ such that $2\gamma - 1 > 1 - \varepsilon$. Analogous to (4.11), we have

$$\check{\varrho} \leq \sum_{i=0}^{L^{1-\varepsilon}} \frac{C}{(L-i)^{2\gamma-1}} + \sum_{i=L^{1-\varepsilon}}^{L-1} \frac{C\check{A}_i^{z,\mathrm{pin}}}{(L-i)^{2\gamma-1}} \leq \frac{CL^{1-\varepsilon}}{(L-L^{1-\varepsilon})^{2\gamma-1}} + CL^{2-2\gamma} \max_{L^{1-\varepsilon}\leq i\leq L} \check{A}_i^{z,\mathrm{pin}}. \tag{4.25}$$

Therefore to show $\check{\varrho} < 1$ for some $z > 1$, it suffices to show that with $L = (z-1)^{-1}$,

$$\lim_{z\downarrow 1} L^{2-2\gamma} \max_{L^{1-\varepsilon}\leq N\leq L} \check{A}_N^{z,\mathrm{pin}} = 0. \tag{4.26}$$

Tracing through the arguments for $d \geq 5$, we see that analogous to (4.17), for $h = 1/\sqrt{L} = \sqrt{z-1}$, uniformly for $L^{1-\varepsilon} \leq N \leq L$ and $z > 1$ sufficiently close to 1, we have

$$\check{A}_N^{z,\mathrm{pin}} \leq C\mathbb{E}_0^{Y^h}\big[\check{Z}_{N,Y^h}^{z,\mathrm{pin}}\big]^{\gamma} \leq C\mathbb{E}^{K_*}\big[\exp\big\{-c\sqrt{z-1}\big|\iota^* \cap \big[1, (z-1)^{\varepsilon-1}\big]\big|\big\}\big]^{\gamma}, \tag{4.27}$$

where ι^* is a renewal process on $\mathbb{N}_0$ with inter-arrival probability distribution K_* satisfying the property that $K_*(n) \sim Cn^{-2}$ for some $C > 0$. Set $M = (z-1)^{\varepsilon-1}$. Then

$$0 \leq \lim_{z\downarrow 1} L^{2-2\gamma} \max_{L^{1-\varepsilon}\leq N\leq L} \check{A}_N^{z,\mathrm{pin}} \leq \lim_{M\to\infty} CM^{(2-2\gamma)/(1-\varepsilon)}\mathbb{E}^{K_*}\big[\exp\big\{-cM^{-1/(2(1-\varepsilon))}\big|\iota^* \cap [1, M]\big|\big\}\big]^{\gamma} = 0,$$

where we applied Proposition A.1 with $\delta_1 = \frac{1}{2(1-\varepsilon)}$ and $1 - \delta_2 = \frac{2-2\gamma}{\gamma(1-\varepsilon)}$, which satisfy the condition $0 < \delta_1 < \delta_2 < 1$ if $\varepsilon > 0$ is small, and $\gamma \in (0, 1)$ is then chosen sufficiently close to 1.

5. Gap between critical points: continuous time

5.1. *Proof of Theorem 1.3 in continuous time*: $d \geq 5$

As in discrete time, we split the proof into three parts: representation for $Z_{t,Y}^{\beta}$ and $Z_{t,Y}^{\beta,\mathrm{pin}}$; fractional moment method; change of measure. Compared to the discrete time case, the main complication here is to suitably discretize time so that the fractional moment inequality (4.6) can be applied. The change of measure argument however becomes much simpler.

Representation for $Z^{\beta}_{t,Y}$ and $Z^{\beta,\mathrm{pin}}_{t,Y}$

We now Taylor expand $e^{\beta L_t(X,Y)}$. Let $p_s(\cdot)$ be the transition probability kernel of a rate 1 continuous time simple random walk on $\mathbb{Z}^d$. Let $G_{1+\rho} = \int_0^\infty p_{(1+\rho)s}(0)\,\mathrm{d}s$, $K_{1+\rho}(s) = p_{(1+\rho)s}(0)/G_{1+\rho}$, $\bar\beta = \beta G_{1+\rho}$, and $\bar Z^{\bar\beta}_{t,Y} = Z^{\beta}_{t,Y}$. Then

$$
\begin{aligned}
\bar Z^{\bar\beta}_{t,Y} &= \mathbb{E}^X_0\left[1 + \sum_{m=1}^\infty \frac{\beta^m}{m!}\left(\int_0^t 1_{\{X_s=Y_s\}}\,\mathrm{d}s\right)^m\right] \\
&= \mathbb{E}^X_0\left[1 + \sum_{m=1}^\infty \beta^m \underset{0<s_1\cdots<s_m<t}{\int\cdots\int} 1_{\{X_{s_1}=Y_{s_1},\dots,X_{s_m}=Y_{s_m}\}}\,\mathrm{d}s_1\cdots\mathrm{d}s_m\right] \\
&= 1 + \sum_{m=1}^\infty \beta^m \underset{0<s_1\cdots<s_m<t}{\int\cdots\int} p_{s_1}(Y_{s_1})p_{s_2-s_1}(Y_{s_2}-Y_{s_1})\cdots p_{s_m-s_{m-1}}(Y_{s_m}-Y_{s_{m-1}})\,\mathrm{d}s_1\cdots\mathrm{d}s_m \\
&= 1 + \sum_{m=1}^\infty \underset{s_0=0<s_1\cdots<s_m<t}{\int\cdots\int} \prod_{i=1}^m \bigl(K_{1+\rho}(s_i-s_{i-1})w(\bar\beta, s_i-s_{i-1}, Y_{s_i}-Y_{s_{i-1}})\bigr)\,\mathrm{d}s_1\cdots\mathrm{d}s_m,
\end{aligned}
\tag{5.1}
$$

where

$$
w(\bar\beta, s_i-s_{i-1}, Y_{s_i}-Y_{s_{i-1}}) = \frac{\bar\beta p_{s_i-s_{i-1}}(Y_{s_i}-Y_{s_{i-1}})}{p_{(1+\rho)(s_i-s_{i-1})}(0)}. \tag{5.2}
$$

If we denote $\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y} = \beta Z^{\beta,\mathrm{pin}}_{t,Y}$, then similarly,

$$
\begin{aligned}
\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y} &= K_{1+\rho}(t)w(\bar\beta,t,Y_t) \\
&\quad + \sum_{m=1}^\infty \underset{s_0=0<s_1\cdots<s_m<s_{m+1}=t}{\int\cdots\int} \prod_{i=1}^{m+1} K_{1+\rho}(s_i-s_{i-1})w(\bar\beta, s_i-s_{i-1}, Y_{s_i}-Y_{s_{i-1}})\,\mathrm{d}s_1\cdots\mathrm{d}s_m.
\end{aligned}
\tag{5.3}
$$

Note that (5.3) casts $\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y}$ in the same form as (4.3), except that the underlying renewal process is in continuous time with return time distribution $K_{1+\rho}(s)\,\mathrm{d}s$. Since

$$
\mathbb{E}^Y_0\bigl[w(\bar\beta, s_i-s_{i-1}, Y_{s_i}-Y_{s_{i-1}})\bigr] = \bar\beta, \tag{5.4}
$$

and $K_{1+\rho}(\cdot)$ defines a recurrent renewal process on $[0,\infty)$, $\mathbb{E}^Y_0[\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y}]$ is the partition function of a homogeneous pinning model (in continuous time) with critical point $\bar\beta^{\mathrm{ann}}_{\mathrm{c}} = 1$, or equivalently,

$$
\beta^{\mathrm{ann}}_{\mathrm{c}} = \frac{\bar\beta^{\mathrm{ann}}_{\mathrm{c}}}{G_{1+\rho}} = \frac{1}{G_{1+\rho}}. \tag{5.5}
$$

Fractional moment method

Analogous to (4.7), for fixed $L \in \mathbb{N}$, we have the decomposition

$$
\bar Z^{\bar\beta}_{t,Y} = \bar Z^{\bar\beta}_{L,Y} + \iint_{0\le u<L<v\le t} K_{1+\rho}(v-u)w(\bar\beta, v-u, Y_v-Y_u)\bar Z^{\bar\beta,\mathrm{pin}}_{u,Y}\bar Z^{\bar\beta}_{t-v,\theta_v Y}\bigl(1+\delta_0(u)\bigr)\,\mathrm{d}u\,\mathrm{d}v, \tag{5.6}
$$

where $\theta_v Y = (Y_{v+s}-Y_v)_{s\ge 0}$ denotes a shift in Y, $\delta_0(u)$ is the delta function at 0, and $\bar Z^{\bar\beta,\mathrm{pin}}_{0,Y} = 1$. In the continuous setting, the analogue of (4.6), $(\int |a(x)|\,\mathrm{d}x)^\gamma \le \int |a(x)|^\gamma\,\mathrm{d}x$ for $\gamma \in (0,1)$, is false in general. Therefore, we need to discretize the integrals in (5.6). In order to obtain uniform control for the integrand in (5.6) on intervals, it turns out to be more suitable to study the following quantities in place of $\bar{Z}^{\bar{\beta}}_{t,Y}$ and $\bar{Z}^{\bar{\beta},\mathrm{pin}}_{t,Y}$.

$$\begin{aligned}
\bar{Z}^{\bar{\beta},1}_{t,Y} &= 1+\sum_{m=1}^{\infty}\int\cdots\int_{s_0=0<s_1\cdots<s_m<t}\prod_{i=1}^{m}K_{1+\rho}(s_i-s_{i-1})\prod_{i=2}^{m}w(\bar{\beta},s_i-s_{i-1},Y_{s_i}-Y_{s_{i-1}})\,\mathrm{d}s_1\cdots\mathrm{d}s_m,\\
\bar{Z}^{\bar{\beta},\mathrm{pin}1}_{t,Y} &= K_{1+\rho}(t)\\
&\quad+\sum_{m=1}^{\infty}\int\cdots\int_{s_0=0<s_1\cdots<s_m<s_{m+1}=t}\prod_{i=1}^{m+1}K_{1+\rho}(s_i-s_{i-1})\prod_{i=2}^{m+1}w(\bar{\beta},s_i-s_{i-1},Y_{s_i}-Y_{s_{i-1}})\,\mathrm{d}s_1\cdots\mathrm{d}s_m,\\
\bar{Z}^{\bar{\beta},\mathrm{pin}2}_{t,Y} &= K_{1+\rho}(t)+\sum_{m=1}^{\infty}\int\cdots\int_{s_0=0<s_1\cdots<s_m<s_{m+1}=t}\prod_{i=1}^{m+1}K_{1+\rho}(s_i-s_{i-1})\prod_{i=2}^{m}w(\bar{\beta},s_i-s_{i-1},Y_{s_i}-Y_{s_{i-1}})\,\mathrm{d}s_1\cdots\mathrm{d}s_m,
\end{aligned}\tag{5.7}$$

where $\prod_{i=2}^{m}w=1$ if $m=1$. Note that $\bar{Z}^{\bar{\beta},1}_{t,Y}$ differs from $\bar{Z}^{\bar{\beta}}_{t,Y}$ in that the factor $w(\bar{\beta},s_1,Y_{s_1})$ in (5.1) has been omitted, while $\bar{Z}^{\bar{\beta},\mathrm{pin}1}_{t,Y}$ (resp. $\bar{Z}^{\bar{\beta},\mathrm{pin}2}_{t,Y}$) differs from $\bar{Z}^{\bar{\beta},\mathrm{pin}}_{t,Y}$ in that the factors $w(\bar{\beta},t,Y_t)$ and $w(\bar{\beta},s_1,Y_{s_1})$ (resp. as well as $w(\bar{\beta},t-s_m,Y_t-Y_{s_m})$) in (5.3) have been omitted. Omitting these random factors will provide flexibility in adjusting the lengths of the renewal gaps $(s_i-s_{i-1})_{i\in\mathbb{N}}$.

Note that

$$w(\bar{\beta},v-u,Y_v-Y_u)=\frac{\bar{\beta}p_{v-u}(Y_v-Y_u)}{p_{(1+\rho)(v-u)}(0)}\le\frac{\bar{\beta}p_{v-u}(0)}{p_{(1+\rho)(v-u)}(0)}\le C \tag{5.8}$$

for some $C\in(1,\infty)$ independent of $v-u\ge 0$ and $\bar{\beta}\in[1,2]$, which is furthermore uniformly bounded for $\rho\in[0,1]$. Therefore,

$$\bar{Z}^{\bar{\beta}}_{t,Y}\le C\bar{Z}^{\bar{\beta},1}_{t,Y}. \tag{5.9}$$

By the monotonicity of $Z^{\beta}_{t,Y}=\bar{Z}^{\bar{\beta}}_{t,Y}$ in t, to show $\beta<\beta_c^*$ (i.e., $\sup_{t\ge 0}Z^{\beta}_{t,Y}<\infty$ a.s. w.r.t. Y), it suffices to show that for $\bar{\beta}=\beta G_{1+\rho}$, there exists $\gamma\in(0,1)$ such that

$$\sup_{t\ge 0}\mathbb{E}^Y_0\big[\big(\bar{Z}^{\bar{\beta},1}_{t,Y}\big)^{\gamma}\big]<\infty. \tag{5.10}$$

Note that $\bar{Z}^{\bar{\beta},1}_{t,Y}$ is increasing in t for every Y, therefore we may assume $t\in\mathbb{N}$. Similar to (5.6), we have

$$\begin{aligned}
\bar{Z}^{\bar{\beta},1}_{t,Y} &= \bar{Z}^{\bar{\beta},1}_{L,Y}+\int_L^t K_{1+\rho}(v)\bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y}\,\mathrm{d}v\\
&\quad+\iint_{0<u<L<v<t}K_{1+\rho}(v-u)w(\bar{\beta},v-u,Y_v-Y_u)\bar{Z}^{\bar{\beta},\mathrm{pin}1}_{u,Y}\bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y}\,\mathrm{d}u\,\mathrm{d}v\\
&= \bar{Z}^{\bar{\beta},1}_{L,Y}+\sum_{j=L}^{t-1}\int_j^{j+1}K_{1+\rho}(v)\bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y}\,\mathrm{d}v\\
&\quad+\sum_{i=0}^{L-1}\sum_{j=L}^{t-1}\iint_{\substack{i<u<i+1\\ j<v<j+1}}K_{1+\rho}(v-u)w(\bar{\beta},v-u,Y_v-Y_u)\bar{Z}^{\bar{\beta},\mathrm{pin}1}_{u,Y}\bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}\tag{5.11}$$

We will establish uniform estimates on the integrand for each integral in (5.11) by bounding $\bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y}$ in terms of $\bar{Z}^{\bar{\beta},1}_{t-j-1,\theta_{j+1}Y}$ and bounding $\bar{Z}^{\bar{\beta},\mathrm{pin1}}_{u,Y}$ in terms of $\bar{Z}^{\bar{\beta},\mathrm{pin2}}_{i,Y}$.

We first make a few observations which will come in handy. Note that for all $s \in [0,1]$ and all realizations of Y,

$$\begin{aligned} \bar{Z}^{\bar{\beta}}_{s,Y} &= Z^{\beta}_{s,Y} = \mathbb{E}^X_0\left[\mathrm{e}^{\beta L_s(X,Y)}\right] \le \mathrm{e}^{\beta}, \\ \bar{Z}^{\bar{\beta},\mathrm{pin}}_{s,Y} &= \beta Z^{\beta,\mathrm{pin}}_{s,Y} = \beta \mathbb{E}^X_0\left[\mathrm{e}^{\beta L_s(X,Y)} 1_{\{X_s=Y_s\}}\right] \le \beta \mathrm{e}^{\beta}. \end{aligned} \tag{5.12}$$

Next note that

$$C_\rho = \sup_{\substack{u\ge 0\\ 0\le s\le 1}} \frac{K_{1+\rho}(u)}{K_{1+\rho}(u+s)} < \infty, \tag{5.13}$$

which is uniformly bounded for $\rho \in [0,1]$.

If $v \in (j, j+1)$ for some $L \le j \le t-1$, then by the same decomposition as (5.6) with $s_1, s_2, j+1$ now playing the roles of u, v, L and by the observations above, we have

$$\begin{aligned} \bar{Z}^{\bar{\beta}}_{t-v,\theta_v Y} &= \bar{Z}^{\bar{\beta}}_{j+1-v,\theta_v Y} \\ &\quad + \iint\limits_{\substack{v\le s_1<j+1\\ j+1<s_2<t}} K_{1+\rho}(s_2-s_1) w(\bar{\beta}, s_2-s_1, Y_{s_2}-Y_{s_1}) \bar{Z}^{\bar{\beta},\mathrm{pin}}_{s_1-v,\theta_v Y} \bar{Z}^{\bar{\beta}}_{t-s_2,\theta_{s_2}Y}\big(1+\delta_v(s_1)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &\le C + C\int_{j+1}^{t} K_{1+\rho}(s_2-j-1)\bar{Z}^{\bar{\beta}}_{t-s_2,\theta_{s_2}Y}\,\mathrm{d}s_2 = C\bar{Z}^{\bar{\beta},1}_{t-j-1,\theta_{j+1}Y}, \end{aligned} \tag{5.14}$$

where $C<\infty$ is independent of $t, v, Y, \bar{\beta} \in [1,2]$, and furthermore is uniformly bounded for $\rho \in [0,1]$.

If $u \in (i, i+1)$ for some $0 \le i \le L-1$, then by a similar decomposition as above, we have

$$\begin{aligned} \bar{Z}^{\bar{\beta},\mathrm{pin1}}_{u,Y} &= \int_{i<s_2\le u} K_{1+\rho}(s_2)\bar{Z}^{\bar{\beta},\mathrm{pin}}_{u-s_2,\theta_{s_2}Y}\big(1+\delta_u(s_2)\big)\,\mathrm{d}s_2 \\ &\quad + \iint\limits_{0<s_1<i<s_2\le u} K_{1+\rho}(s_2-s_1) w(\bar{\beta}, s_2-s_1, Y_{s_2}-Y_{s_1}) \\ &\quad \times \bar{Z}^{\bar{\beta},\mathrm{pin1}}_{s_1,Y}\bar{Z}^{\bar{\beta},\mathrm{pin}}_{u-s_2,\theta_{s_2}Y}\big(1+\delta_u(s_2)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &\le CK_{1+\rho}(i) + C\int_{0<s_1<i} K_{1+\rho}(i-s_1)\bar{Z}^{\bar{\beta},\mathrm{pin1}}_{s_1,Y}\,\mathrm{d}s_1 = C\bar{Z}^{\bar{\beta},\mathrm{pin2}}_{i,Y}. \end{aligned} \tag{5.15}$$

Substituting the bounds (5.8), (5.13)–(5.15) into (5.11) gives

$$\begin{aligned} \bar{Z}^{\bar{\beta},1}_{t,Y} &\le \bar{Z}^{\bar{\beta},1}_{L,Y} + C'\sum_{j=L}^{t-1} K_{1+\rho}(j+1)\bar{Z}^{\bar{\beta},1}_{t-j-1,\theta_{j+1}Y} \\ &\quad + C'\sum_{i=0}^{L-1}\sum_{j=L}^{t-1} K_{1+\rho}(j+1-i)\bar{Z}^{\bar{\beta},\mathrm{pin2}}_{i,Y}\bar{Z}^{\bar{\beta},1}_{t-j-1,\theta_{j+1}Y} \\ &\le \bar{Z}^{\bar{\beta},1}_{L,Y} + C\sum_{i=0}^{L-1}\sum_{j=L}^{t-1} K_{1+\rho}(j+1-i)\bar{Z}^{\bar{\beta},\mathrm{pin2}}_{i,Y}\bar{Z}^{\bar{\beta},1}_{t-j-1,\theta_{j+1}Y}, \end{aligned} \tag{5.16}$$

where $C<\infty$ is independent of $t, Y, \bar{\beta} \in [1,2]$, and can be chosen uniformly for $\rho \in [0,1]$.

Fix $\gamma \in (0,1)$ such that $\frac{d\gamma}{2} > 2$ for $d \geq 5$. Denote $\bar{A}_t^{\bar{\beta},1} = \mathbb{E}_0^Y[(\bar{Z}_{t,Y}^{\bar{\beta},1})^\gamma]$ and $\bar{A}_t^{\bar{\beta},\text{pin2}} = \mathbb{E}_0^Y[(\bar{Z}_{t,Y}^{\bar{\beta},\text{pin2}})^\gamma]$. Then the same calculations as those leading to (4.8) yields

$$\bar{A}_t^{\bar{\beta},1} \leq \bar{A}_L^{\bar{\beta},1} + \varrho \sup_{0\leq j\leq t-L} \bar{A}_j^{\bar{\beta},1} \quad \text{with } \varrho = C\left(\sum_{i=0}^{L-1} \frac{\bar{A}_i^{\bar{\beta},\text{pin2}}}{(L-i)^{d\gamma/2-1}}\right), \tag{5.17}$$

where $C < \infty$ is independent of t and $\bar{\beta} \in [1,2]$, and can be chosen uniformly for $\rho \in [0,1]$. As in the discrete time case, we aim to show $\varrho < 1$.

Note that $\bar{A}_s^{\bar{\beta},\text{pin2}} \leq \mathbb{E}_0^Y[\bar{Z}_{s,Y}^{\bar{\beta},\text{pin2}}]^\gamma \leq \mathbb{E}_0^Y[\bar{Z}_{s,Y}^{\bar{\beta},\text{pin}}]^\gamma \leq \mathbb{E}_0^Y[\bar{Z}_{s,Y}^{\bar{\beta}}]^\gamma$ by Jensen and (5.4), where we see from (5.1) that $\mathbb{E}_0^Y[\bar{Z}_{s,Y}^{\bar{\beta}}]$ is the partition function of a continuous time homogeneous pinning model with return time distribution $K_{1+\rho}(\cdot)$ and critical point $\bar{\beta}_c^{\text{ann}} = 1$. For $d \geq 5$, it is easy to verify (by law of large numbers and elementary large deviation estimates for the number of returns of the renewal process before time s) that

$$\mathbb{E}_0^Y\big[\bar{Z}_{s,Y}^{\bar{\beta}}\big] \leq C\mathrm{e}^{C(\bar{\beta}-1)s} \tag{5.18}$$

for some $C \in (0,\infty)$ independent of $s \geq 0$ and $\bar{\beta} \in [1,2]$, and is furthermore uniformly bounded for $\rho \in [0,1]$. As in the discrete time case, we choose

$$L = (\bar{\beta}-1)^{-1}. \tag{5.19}$$

In view of (5.10) and (5.17), and by the same arguments as those leading to (4.12) in the discrete time case, to show $\beta_c^* > \beta_c^{\text{ann}}$ for any $\rho > 0$, it suffices to show that

$$\lim_{\bar{\beta}\downarrow 1} \sup_{L-R\leq t\leq L} \bar{A}_t^{\bar{\beta},\text{pin2}} = 0, \tag{5.20}$$

where $R \in \mathbb{N}$ is large and fixed and can be chosen uniformly for $\rho \in [0,1]$. On the other hand, showing

$$\beta_c^* - \beta_c^{\text{ann}} \geq a\rho \tag{5.21}$$

for some $a > 0$ and all $\rho \in [0,1]$ reduces to showing that: (1) the convergence in (5.20) is in fact uniform for $\rho \in [\rho_0, 1]$ for any $0 < \rho_0 \leq 1$, which implies that $\inf_{\rho\in[\rho_0,1]}(\bar{\beta}_c^* - 1) > 0$ where $\bar{\beta}_c^* = G_{1+\rho}\beta_c^*$, and hence $\inf_{\rho\in[\rho_0,1]}(\beta_c^* - \beta_c^{\text{ann}}) > 0$; (2) for $\bar{\beta} = 1 + a\rho$ with $a > 0$ sufficiently small, $L = (\bar{\beta}-1)^{-1}$, and $R \in \mathbb{N}$ large and independent of $\rho \in [0,1]$,

$$\limsup_{\rho\downarrow 0} \sup_{L-R\leq t\leq L} \bar{A}_t^{\bar{\beta},\text{pin2}} < 1, \tag{5.22}$$

which implies that for some $\rho_0 \in (0,1]$, $\bar{\beta}_c^* - 1 = G_{1+\rho}(\beta_c^* - \beta_c^{\text{ann}}) \geq a\rho$ for all $\rho \in [0,\rho_0]$.

Change of measure

We now prove (5.20) and (5.22), where the convergence in (5.20) will be shown to be uniform in $\rho \in [\rho_0, 1]$ for any $0 < \rho_0 \leq 1$. Here, the appropriate change of measure for the disorder Y is simply to increase the jump rate of the random walk Y. Let $Y^{\rho+h}$ be a simple random walk on $\mathbb{Z}^d$ with jump rate $\rho + h$ for some $h > 0$, then the path measures $(Y_s)_{0\leq s\leq t}$ and $(Y_s^{\rho+h})_{0\leq s\leq t}$ are equivalent, and the Radon–Nikodym derivative of the law of $(Y_s^{\rho+h})_{0\leq s\leq t}$ w.r.t. that of $(Y_s)_{0\leq s\leq t}$ is given by

$$f(t,Y) = \mathrm{e}^{-ht}\big(1 + h\rho^{-1}\big)^{N_t(Y)},$$

where $N_t(Y)$ is the number of jumps of Y in $[0,t]$. Then as in (4.14),

$$\bar{A}_t^{\bar{\beta},\text{pin2}} = \mathbb{E}_0^{Y^{\rho+h}}\big[f\big(t,Y^{\rho+h}\big)^{-1}\big(\bar{Z}_{t,Y^{\rho+h}}^{\bar{\beta},\text{pin2}}\big)^\gamma\big] \leq \mathbb{E}_0^Y\big[f(t,Y)^{-\gamma/(1-\gamma)}\big]^{1-\gamma}\mathbb{E}_0^{Y^{\rho+h}}\big[\bar{Z}_{t,Y^{\rho+h}}^{\bar{\beta},\text{pin2}}\big]^\gamma. \tag{5.23}$$

Note that

$$\begin{aligned}
&\mathbb{E}_0^Y\big[f(t,Y)^{-\gamma/(1-\gamma)}\big]\\
&\quad= \mathrm{e}^{\gamma ht/(1-\gamma)}\mathbb{E}_0^Y\big[\big(1+h\rho^{-1}\big)^{-\gamma N_t/(1-\gamma)}\big] = \mathrm{e}^{\gamma ht/(1-\gamma)}\sum_{n=0}^{\infty}\mathrm{e}^{-\rho t}\frac{(\rho t)^n}{n!}\big(1+h\rho^{-1}\big)^{-\gamma n/(1-\gamma)}\\
&\quad= \exp\bigg\{\bigg(\rho\big(1+h\rho^{-1}\big)^{-\gamma/(1-\gamma)}-\rho+\frac{\gamma h}{1-\gamma}\bigg)t\bigg\} \le \exp\bigg\{\frac{\gamma h^2 t}{2\rho(1-\gamma)^2}\bigg\},
\end{aligned} \tag{5.24}$$

where second order Taylor expansion in h in the exponent provides a true upper bound. For $L-R\le t\le L$, if we choose $h=\frac{\sqrt{\rho}}{\sqrt{L}}$, then the first term in (5.23) is bounded and independent of ρ, $\bar\beta$ and t. Thus it only remains to estimate $\mathbb{E}_0^{Y^{\rho+h}}[\bar Z^{\bar\beta,\mathrm{pin2}}_{t,Y^{\rho+h}}]$.

First note that $\mathbb{E}_0^{Y^{\rho+h}}[\bar Z^{\bar\beta,\mathrm{pin2}}_{t,Y^{\rho+h}}]\le C\mathbb{E}_0^{Y^{\rho+h}}[\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y^{\rho+h}}]$ for some $C>0$ independent of $\rho\ge 0$, $\bar\beta\in[1,2]$ and $t\ge 0$, because each term in the expansion for $\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y}$ in (5.3) differs from the corresponding term in (5.7) for $\bar Z^{\bar\beta,\mathrm{pin2}}_{t,Y}$ by at most two factors of w, and $\mathbb{E}_0^{Y^{\rho+h}}[w(\bar\beta,v-u,Y_v-Y_u)]=\frac{\bar\beta p_{(1+\rho+h)(v-u)}(0)}{p_{(1+\rho)(v-u)}(0)}\ge C$ for some $C>0$ independent of $\rho\ge 0$, $h\in[0,1]$, $\bar\beta\in[1,2]$ and $v-u\ge 0$. Recall $G_{1+\rho}=\int_0^\infty p_{(1+\rho)s}(0)\,\mathrm{d}s$,

$$\begin{aligned}
\mathbb{E}_0^{Y^{\rho+h}}\big[\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y^{\rho+h}}\big] &= \bigg(\frac{\bar\beta}{G_{1+\rho}}\bigg)p_{(1+\rho+h)t}(0)\\
&\quad+\sum_{m=1}^{\infty}\int\cdots\int_{0=s_0<s_1\cdots<s_m<s_{m+1}=t}\bigg(\frac{\bar\beta}{G_{1+\rho}}\bigg)^{m+1}\prod_{i=1}^{m+1}p_{(1+\rho+h)(s_i-s_{i-1})}(0)\,\mathrm{d}s_1\cdots\mathrm{d}s_m\\
&= \frac{(1+\rho)\bar\beta}{1+\rho+h}K_{1+\rho+h}(t)\\
&\quad+\int\cdots\int_{0=s_0<s_1\cdots<s_m<s_{m+1}=t}\bigg(\frac{(1+\rho)\bar\beta}{1+\rho+h}\bigg)^{m+1}\prod_{i=1}^{m+1}K_{1+\rho+h}(s_i-s_{i-1})\,\mathrm{d}s_1\cdots\mathrm{d}s_m,
\end{aligned} \tag{5.25}$$

where $K_{1+\rho+h}(s)=p_{(1+\rho+h)s}(0)/G_{1+\rho+h}$ with $G_{1+\rho+h}=\int_0^\infty p_{(1+\rho+h)s}(0)\,\mathrm{d}s=\frac{(1+\rho)G_{1+\rho}}{1+\rho+h}$.

Denote $\bar\beta'=\frac{(1+\rho)\bar\beta}{1+\rho+h}$. Let $\sigma^{\rho+h}=(0,\sigma_1^{\rho+h},\sigma_2^{\rho+h},\ldots)$ be a renewal sequence on $[0,\infty)$ with inter-arrival law $K_{1+\rho+h}(\cdot)$, and let $\mathbb{E}^{K_{1+\rho+h}}[\cdot]$ denote expectation w.r.t. $\sigma^{\rho+h}$. Then in view of (5.25),

$$\mathbb{E}^{K_{1+\rho+h}}\big[\big(\bar\beta'\big)^{1+|\sigma^{\rho+h}\cap[0,t]|}1_{\{\sigma^{\rho+h}\cap[t,t+1]\neq\varnothing\}}\big]\ge \inf_{\substack{u\ge 0,\\ 0\le s\le 1}}\frac{K_{1+\rho+h}(u+s)}{K_{1+\rho+h}(u)}\mathbb{E}_0^{Y^{\rho+h}}\big[\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y^{\rho+h}}\big].$$

Recall the definition of $C_{1+\rho}$ from (5.13), we then have

$$\mathbb{E}_0^{Y^{\rho+h}}\big[\bar Z^{\bar\beta,\mathrm{pin}}_{t,Y^{\rho+h}}\big]\le C_{\rho+h}\mathbb{E}^{K_{1+\rho+h}}\big[\big(\bar\beta'\big)^{1+|\sigma^{\rho+h}\cap[0,t]|}\big]. \tag{5.26}$$

Now to prove (5.22), we recall that $L=(\bar\beta-1)^{-1}$ and hence $h=\frac{\sqrt{\rho}}{\sqrt{L}}=\sqrt{\rho(\bar\beta-1)}$. Therefore there exists $\bar\beta_0>1$ sufficiently small such that for all $\rho>0$ and $\bar\beta\in[1,\bar\beta_0]$,

$$\bar\beta'=\frac{(1+\rho)\bar\beta}{1+\rho+h}\le(1+\bar\beta-1)\bigg(1-\frac{\sqrt{\rho(\bar\beta-1)}}{2(1+\rho)}\bigg). \tag{5.27}$$

First note that by our choice $\bar\beta=1+a\rho$, we have $\bar\beta'\le 1-\rho\sqrt{a}/8$ for all $\rho\in[0,1]$ if $0<a<1/64$. Next note that $C_{\rho+h}$ is uniformly bounded for $\rho\in[0,1]$ and $\bar\beta\in[1,2]$. For $d\ge 5$, by the local central limit theorem, there exists an inter-arrival probability distribution K_* on $(0,\infty)$ with finite first moment $m=\int_0^\infty sK_*(s)\,\mathrm{d}s$, such that K_* stochastically dominates $K_{1+\rho+h}$ for all $h\in[0,1]$ and $\rho\in[0,1]$. Namely, $\int_t^\infty K_*(s)\,\mathrm{d}s\ge\int_t^\infty K_{1+\rho+h}(s)\,\mathrm{d}s$ for all $t\ge 0$, $h\in[0,1]$ and $\rho\in[0,1]$. Combining the above observations, we have

$$\begin{aligned}\limsup_{\rho\downarrow 0}\sup_{L-R\le t\le L}\bar{A}_t^{\bar\beta,\mathrm{pin2}}&\le C\limsup_{\rho\downarrow 0}\sup_{L-R\le t\le L}\mathbb{E}_0^{Y^{\rho+h}}\big[\bar{Z}^{\bar\beta,\mathrm{pin}}_{t,Y^{\rho+h}}\big]\\&\le C\limsup_{\rho\downarrow 0}\mathbb{E}^{K_*}\big[\big(1-\rho\sqrt{a}/8\big)^{|\iota^*\cap[0,L-R]|}\big],\end{aligned}\tag{5.28}$$

where ι^* is a renewal process on $[0,\infty)$ with return time distribution K_*. By the law of large numbers, a.s. w.r.t. ι^*,

$$\lim_{\rho\downarrow 0}\big(1-\rho\sqrt{a}/8\big)^{|\iota^*\cap[0,L-R]|}=\lim_{\rho\downarrow 0}\exp\left\{-\frac{\rho\sqrt{a}}{8}\cdot\frac{(a\rho)^{-1}-R}{m}\right\}=\exp\left\{-\frac{1}{8m\sqrt{a}}\right\},$$

which can be made arbitrarily small if $a>0$ is chosen sufficiently small. Inequality (5.22) then follows by applying the dominated convergence theorem in (5.28).

The proof of (5.20) for any $\rho>0$ and the uniform convergence in (5.20) for $\rho\in[\rho_0,1]$ for any $\rho_0\in(0,1]$ follows by similar arguments. It suffices to observe that $\bar\beta'\le 1-C\sqrt{\bar\beta-1}$ for some $C>0$ uniformly in $\rho\in[\rho_0,1]$ and $\bar\beta>1$ sufficiently small. This concludes the proof of Theorem 1.3.

Remark. *Note that the change of measure argument here applies equally well to any random walks X and Y with an identical symmetric transition kernel.*

5.2. *Proof of Theorem* 1.3 *in continuous time*: $d=4$

As in $d\ge 5$, proving Theorem 1.3 reduces to proving $\varrho<1$ (see (5.17)) for appropriate choices of $\bar\beta$ and L depending on the diffusion constant ρ. Since $\mathbb{E}_0^Y[\bar{Z}^{\bar\beta}_{t,Y}]$ is the partition function of a homogeneous pinning model with parameter $\bar\beta\ge 1$ and return time distribution $K_{1+\rho}(t)\sim Ct^{-2}$, by comparing $K_{1+\rho}$ with a return time distribution K' which is stochastically smaller than $K_{1+\rho}$ and has finite first moment, we see that (5.18) also holds in $d=4$. Therefore, setting $L=(\bar\beta-1)^{-1}$ as in $d\ge 5$, we have $\sup_{0\le t\le L}\bar{A}_t^{\bar\beta,\mathrm{pin2}}\le C<\infty$, and analogous to (4.25), we have

$$\varrho\le\sum_{i=0}^{L^{1-\varepsilon}}\frac{C}{(L-i)^{2\gamma-1}}+\sum_{i=L^{1-\varepsilon}}^{L-1}\frac{C\bar{A}_i^{\bar\beta,\mathrm{pin2}}}{(L-i)^{2\gamma-1}}\le\frac{CL^{1-\varepsilon}}{(L-L^{1-\varepsilon})^{2\gamma-1}}+CL^{2-2\gamma}\sup_{L^{1-\varepsilon}\le t\le L}\bar{A}_t^{\bar\beta,\mathrm{pin2}},\tag{5.29}$$

where $\varepsilon>0$, $\gamma\in(0,1)$ is chosen so that $2\gamma-1>1-\varepsilon$, and $C\in(0,\infty)$ is independent of $\bar\beta\in[1,2]$ and is furthermore uniformly bounded for $\rho\in[0,1]$. Therefore, to show $\beta_c^*>\beta_c^{\mathrm{ann}}$ for any $\rho>0$, it suffices to show

$$\lim_{\bar\beta\downarrow 1}L^{2-2\gamma}\sup_{L^{1-\varepsilon}\le t\le L}\bar{A}_t^{\bar\beta,\mathrm{pin2}}=0.\tag{5.30}$$

On the other hand, to show that for any $\delta>0$, there exists $a_\delta>0$ such that

$$\beta_c^*-\beta_c^{\mathrm{ann}}\ge a_\delta\rho^{1+\delta}\quad\forall\rho\in[0,1],\tag{5.31}$$

it suffices to show that: (1) the convergence in (5.30) is uniform for $\rho\in[\rho_0,1]$ for any $0<\rho_0\le 1$, which implies that $\inf_{\rho\in[\rho_0,1]}(\beta_c^*-\beta_c^{\mathrm{ann}})>0$; (2) for $\bar\beta=1+\rho^{1+\delta}$ and $L=(\bar\beta-1)^{-1}=\rho^{-1-\delta}$,

$$\lim_{\rho\downarrow 0}L^{2-2\gamma}\sup_{L^{1-\varepsilon}\le t\le L}\bar{A}_t^{\bar\beta,\mathrm{pin2}}=0,\tag{5.32}$$

which implies that for some $\rho_0\in(0,1]$, $\bar\beta_c^*-1=G_{1+\rho}(\beta_c^*-\beta_c^{\mathrm{ann}})\ge\rho^{1+\delta}$ for all $\rho\in[0,\rho_0]$.

Proceeding exactly as in the $d \geq 5$ case, we note that (5.26) still holds in $d = 4$. By the choice $h = \frac{\sqrt{\rho}}{\sqrt{L}} = \rho^{1+\delta/2}$, there exists $\rho_1 \in (0, 1)$ such that

$$\bar{\beta}' = \frac{(1+\rho)\bar{\beta}}{1+\rho+h} = \frac{(1+\rho)(1+\rho^{1+\delta})}{1+\rho+\rho^{1+\delta/2}} \leq 1 - \rho^{1+\delta/2}/2 \leq \mathrm{e}^{-\rho^{1+\delta/2}/2} \quad \forall \rho \in [0, \rho_1]. \tag{5.33}$$

If we choose K_* to be a return time distribution with $\int_0^\infty K_*(s)\,\mathrm{d}s = 1$ and $K_*(s) \sim Cs^{-2}$ such that K_* stochastically dominates $K_{1+\rho+h}$ for all $\rho, h \in [0, 1]$, and let ι^* be a renewal process on $[0, \infty)$ with return time distribution K_*, then

$$\begin{aligned}
0 &\leq \lim_{\rho \downarrow 0} L^{2-2\gamma} \sup_{L^{1-\varepsilon} \leq t \leq L} \bar{A}_t^{\bar{\beta},\mathrm{pin2}} \\
&\leq C \lim_{\rho \downarrow 0} \rho^{-(1+\delta)(2-2\gamma)} \mathbb{E}^{K_*}\Bigl[\exp\Bigl\{-\frac{1}{2}\rho^{1+\delta/2}\bigl|\iota^* \cap \bigl[0, \rho^{-(1+\delta)(1-\varepsilon)}\bigr]\bigr|\Bigr\}\Bigr]^\gamma \\
&= C \lim_{M \to \infty} M^{(2-2\gamma)/(1-\varepsilon)} \mathbb{E}^{K_*}\Bigl[\exp\Bigl\{-\frac{1}{2}M^{-(1+\delta/2)/((1+\delta)(1-\varepsilon))}\bigl|\iota^* \cap [0, M]\bigr|\Bigr\}\Bigr]^\gamma = 0,
\end{aligned}$$

where we applied Proposition A.1 with $\delta_1 = \frac{1+\delta/2}{(1+\delta)(1-\varepsilon)}$ and $1 - \delta_2 = \frac{2-2\gamma}{\gamma(1-\varepsilon)}$, which satisfy the condition $0 < \delta_1 < \delta_2 < 1$ if $\varepsilon > 0$ is small and γ is then chosen sufficiently close to 1. This proves (5.32).

The proof of (5.30) for any $\rho > 0$ and the uniform convergence therein for $\rho \in [\rho_0, 1]$ for any $\rho_0 \in (0, 1]$ follows by similar arguments. It suffices to note that for each $\rho > 0$, there exists $C > 0$ and $\bar{\beta}_0 > 1$ such that $\bar{\beta}' \leq 1 - C\sqrt{\bar{\beta} - 1}$ for all $\bar{\beta} \in [1, \bar{\beta}_0]$. Furthermore, C and $\bar{\beta}_0$ can be chosen uniformly for $\rho \in [\rho_0, 1]$ for any $\rho_0 > 0$. The rest of the proof proceeds exactly as for $d = 4$ in the discrete time case.

Appendix: A renewal process estimate

The following proposition complements Proposition A.2 in [8] for the case $\alpha = 1$.

Proposition A.1. *Let $\iota^* = \{\iota_0 = 0, \iota_1, \ldots\}$ be a renewal process on $\mathbb{N}_0$ with inter-arrival probability distribution K_* satisfying $\sum_{n \in \mathbb{N}} K_*(n) = 1$ and $K_*(n) \sim Cn^{-2}$ as $n \to \infty$. Then for any $c > 0$ and $0 < \delta_1 < \delta_2 < 1$, we have*

$$\lim_{N \to \infty} N^{1-\delta_2} \mathbb{E}^{K_*}\bigl[\exp\bigl\{-cN^{-\delta_1}\bigl|\iota^* \cap [0, N]\bigr|\bigr\}\bigr] = 0. \tag{A.1}$$

The same result holds if ι^* is a renewal process on $[0, \infty)$ with inter-arrival distribution K_* satisfying $\int_0^\infty K_*(s)\,\mathrm{d}s = 1$ and $K_*(s) \sim Cs^{-2}$ as $s \to \infty$.

Proof. Let $\delta_3 \in (\delta_1, \delta_2)$. Note that

$$\mathbb{E}^{K_*}\bigl[\exp\bigl\{-cN^{-\delta_1}\bigl|\iota^* \cap [0, N]\bigr|\bigr\}\bigr] \leq \mathbb{P}\bigl(0 \leq \bigl|\iota^* \cap [0, N]\bigr| < N^{\delta_3}\bigr) + \mathrm{e}^{-cN^{\delta_3 - \delta_1}}. \tag{A.2}$$

Let $(U_i)_{i \in \mathbb{N}}$ be i.i.d. random variables with distribution K_*. By our assumption on K_*, for each $\alpha \in (0, 1)$, we can find a constant $C_\alpha > 0$ and i.i.d. stable subordinators $(V_i)_{i \in \mathbb{N}}$ with exponent α, i.e., $\mathbb{P}(V_1 > 0) = 1$ and $V_1 \overset{\text{law}}{=} \sum_{i=1}^n V_i / n^{1/\alpha}$, such that $\mathbb{P}(U_1 > s) \leq \mathbb{P}(V_1 + C_\alpha > s)$ for all $s > 0$. Therefore, for $\alpha \in (\delta_3, 1)$,

$$\begin{aligned}
\mathbb{P}\bigl(0 \leq \bigl|\iota^* \cap [0, N]\bigr| < N^{\delta_3}\bigr) &= \mathbb{P}\Biggl(\sum_{n=1}^{N^{\delta_3}} U_n > N\Biggr) \leq \mathbb{P}\Biggl(\sum_{n=1}^{N^{\delta_3}} (V_n + C_\alpha) > N\Biggr) \\
&= \mathbb{P}\Biggl(\sum_{n=1}^{N^{\delta_3}} V_n > N - C_\alpha N^{\delta_3}\Biggr) = \mathbb{P}\bigl(V_1 > N^{1-\delta_3/\alpha} - C_\alpha N^{\delta_3(1-1/\alpha)}\bigr) \\
&\leq CN^{\delta_3 - \alpha},
\end{aligned} \tag{A.3}$$

where we used the fact that $\mathbb{P}(V_1 > x) \sim Cx^{-\alpha}$ as $x \to \infty$. It is easy to see that (A.1) follows from (A.2) and (A.3) if we choose $\alpha \in (0, 1)$ such that $1 - \delta_2 + \delta_3 - \alpha < 0$. The case when ι^* is a renewal process on $[0, \infty)$ can be treated identically. □

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