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Comment: A Selective Overview of Nonparametric Methods in Financial Econometrics

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Abstract. These comments concentrate on two issues arising from Fan's overview. The first concerns the importance of finite sample estimation bias relative to the specification and discretization biases that are emphasized in Fan's discussion. Past research and simulations given here both reveal that finite sample effects can be more important than the other two effects when judged from either statistical or economic viewpoints. Second, we draw attention to a very different nonparametric technique that is based on computing an empirical version of the quadratic variation process. This technique is not mentioned by Fan but has many advantages and has accordingly attracted much recent attention in financial econometrics and empirical applications.

Key words and phrases: Nonparametric method, continuous time models, financial time series, jackknife, realized volatility.

1. INTRODUCTION

In recent years there has been increased interest in using nonparametric methods to deal with various aspects of financial data. The paper by Fan gives an overview of some nonparametric techniques that have been used in the financial econometric literature, focusing on estimation and inference for diffusion models in continuous time and estimation of state price and transition density functions.

Continuous time specifications have been heavily used in recent work, partly because of the analytic convenience of stochastic calculus in mathematical finance and partly because of the availability of high-frequency data sets for many financial series. While the early work in continuous-time finance began in the 1970s with the work of Merton [29] and Black and Scholes [16], economists have been looking at the econometric problems of fitting continuous time systems for much longer. The idea of statistically fitting diffusion models and continuously distributed lagged dependencies with discretely observed data has a long history dating back to some original work in econometrics by Koopmans [27] and subsequent work by Phillips [31], Bergstrom [14], Sims [35], Phillips [32] and Sargan [34]. Bartlett and Rajalakshman [13] and Bartlett [12] are two references in the early statistical literature on fitting linear diffusions. Bergstrom [15] provides a short history of some of this early work. Also, the history of mathematical finance and stochastic integration prior to 1970 has recently been overviewed in an interesting historical review by Jarrow and Protter [24].

Our comments on Fan's paper will concentrate on two issues that relate in important ways to the paper's focus on misspecification and discretization bias and the role of nonparametric methods in empirical finance. The first issue deals with the finite sample effects of various estimation methods and their implications for asset pricing. A good deal of recent attention in the econometric literature has focused on the benefits of full maximum likelihood (ML) estimation of diffusions and mechanisms for avoiding discretization bias in the construction of the likelihood. However, many of the

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problems of estimating dynamic models that are well known in discrete time series, such as the bias in ML estimation, also manifest in the estimation of continuous time systems and affect subsequent use of these estimates, for instance in derivative pricing. In consequence, a relevant concern is the relative importance of the estimation and discretization biases. As we will show below, the former often dominates the latter even when the sample size is large (at least 500 monthly observations, say). Moreover, it turns out that correction for the finite sample estimation bias continues to be more important when the diffusion component of the model is itself misspecified. Such corrections appear to be particularly important in models that are nonstationary or nearly nonstationary.

The second issue we discuss deals with a very different nonparametric technique, which is not discussed by Fan, but which has recently attracted much attention in financial econometrics and empirical applications. This method involves the use of quadratic variation measures of realized volatility using ultra high frequency financial data. Like other nonparametric methods, empirical quadratic variation techniques also have to deal with statistical bias, which in the present case arises from the presence of microstructure noise. The field of research on this topic in econometrics is now very active.

2. FINITE SAMPLE EFFECTS

In his overview of diffusion equation estimation, Fan discusses two sources of bias, one arising from the discretization process and the second from misspecification. We review these two bias effects and then discuss the bias that comes from finite sample estimation effects.

The attractions of Itô calculus have made it particularly easy to work with stochastic differential equations driven by Brownian motion. Diffusion processes in particular have been used widely in finance to model asset prices, including stock prices, interest rates and exchange rates. Despite their mathematical attractability, diffusion processes present some formidable challenges for econometric estimation. The primary reason for the difficulty is that sample data, even very high-frequency data, are always discrete and for many popular nonlinear diffusion models the transition density of the discrete sample does not have a closed form expression, as noted by Fan. The problem is specific to nonlinear diffusions, as consistent methods for estimating exact discrete models corresponding to linear systems of diffusions have been available since Phillips [32]. A simple approach discussed in the paper is to use the Euler approximation scheme to discretize the model, a process which naturally creates some discretization bias. This discretization bias can lead to erroneous financial pricing and investment decisions. In consequence, the issue of discretization has attracted a lot of attention in the literature and many methods have been proposed to reduce the bias that it causes. Examples are Pedersen [30], Kessler [26], Durham and Gallant [18], Aït-Sahalia [2, 3] and Elerian, Chib and Shephard [19], among many others.

Next, many diffusion models in practical use are specified in a way that makes them mathematically convenient. These specifications are typically not derived from any underlying economic theory and are therefore likely to be misspecified. Potential misspecifications, like discretization, can lead to erroneous financial decisions. Accordingly, specification bias has attracted a great deal of attention in the literature and has helped to motivate the use of functional estimation techniques that treat the drift and diffusion coefficients nonparametrically. Important contributions include Aït-Sahalia [1], Stanton [36], Bandi and Phillips [5] and Hong and Li [21].

While we agree that both discretization and specification bias are important issues, finite sample estimation bias can be of equal or even greater importance for financial decision making, as noted by Phillips and Yu [33] in the context of pricing bonds and bond options. The strong effect of the finite sample estimation bias in this context can be explained as follows. In continuous time specifications, the prices of bonds and bond options depend crucially on the mean reversion parameter in the associated interest rate diffusion equation. This parameter is well known to be subject to estimation bias when standard methods like ML are used. The bias is comparable to, but generally has larger magnitude than, the usual bias that appears in time series autoregression. As the parameter is often very close to zero in empirical applications (corresponding to near martingale behavior and an autoregressive root near unity in discrete time), the estimation bias can be substantial even in very large samples.

To reduce the finite sample estimation bias in parameter estimation as well as the consequential bias that arises in asset pricing, Phillips and Yu [33] proposed the use of jackknife techniques. Suppose a sample of n observations is available and that this sample is decomposed into m consecutive sub-samples each with ℓ observations ($n = m \times \ell$). The jackknife estimator of a

parameter θ in the model is defined by

$$\hat{\theta}_{\text{jack}} = \frac{m}{m-1}\hat{\theta}_n - \frac{\sum_{i=1}^{m}\hat{\theta}_{\ell i}}{m^2-m}, \tag{2.1}$$

where $\hat{\theta}_n$ and $\hat{\theta}_{\ell i}$ are the extreme estimates of θ based on the entire sample and the i'th sub-sample, respectively. The parameter θ can be a coefficient in the diffusion process, such as the mean reversion parameter, or a much more complex function of the parameters of the diffusion process and the data, such as an asset price or derivative price. Typically, the full sample extreme estimator has bias of order $O(n^{-1})$, whereas under mild conditions the bias in the jackknife estimate is of order $O(n^{-2})$.

The following simulation illustrates these various bias effects and compares their magnitudes. In the experiment, the true generating process is assumed to be the following commonly used model (CIR hereafter) of short term interest rates due to Cox, Ingersoll and Ross [17]:

$$dr(t) = \kappa\big(\mu - r(t)\big)\,dt + \sigma r^{1/2}(t)\,dB(t). \tag{2.2}$$

The transition density of the CIR model is known to be $ce^{-u-v}(v/u)^{q/2}I_q(2(uv)^{1/2})$ and the marginal density is $w_1^{w_2}r^{w_2-1}e^{-w_1 r}/\Gamma(w_2)$, where $c = 2\kappa/(\sigma^2(1-e^{-\kappa\Delta}))$, $u = cr(t)e^{-\kappa\Delta}$, $v = cr(t+\Delta)$, $q = 2\kappa\mu/\sigma^2 - 1$, $w_1 = 2\kappa/\sigma^2$, $w_2 = 2\kappa\mu/\sigma^2$, Δ is the sampling frequency, and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The transition density together with the marginal density can be used for simulation purposes as well as to obtain the exact ML estimator of θ $(= (\kappa, \mu, \sigma)')$. In the simulation, we use this model to price a discount bond, which is a three-year bond with a face value of \$1 and initial interest rate of 5%, and a one-year European call option on a three-year discount bond which has a face value of \$100 and a strike price of \$87. The reader is referred to [33] for further details.

In addition to exact ML estimation, we may discretize the CIR model via the Euler method and estimate the discretized model using (quasi-) ML. The Euler scheme leads to the discretization

$$\begin{aligned} r(t+\Delta) &= \kappa\mu\Delta + (1-\kappa\Delta)r(t) \\ &\quad + \sigma N\big(0, \Delta r(t)\big). \end{aligned} \tag{2.3}$$

One thousand samples, each with 600 monthly observations (i.e., $\Delta = 1/12$), are simulated from the true model (2.2) with $(\kappa, \mu, \sigma)'$ being set at $(0.1, 0.08, 0.02)'$, which are settings that are realistic in many financial applications. To investigate the effects of discretization bias, we estimate model (2.3) by the (quasi-) ML approach. To investigate the finite sample estimation bias effects, we estimate model (2.2) based on the true transition density. To examine the effects of bias reduction in estimation, we apply the jackknife method (with $m = 3$) to the mean reversion parameter κ, the bond price and the bond option price.

To examine the effects of specification bias, we fit each simulated sequence from the true model to the misspecified Vasicek model [37] to obtain the exact ML estimates of κ, the bond price and the option price from this misspecified model. The Vasicek model is given by the simple linear diffusion

$$dr(t) = \kappa\big(\mu - r(t)\big)\,dt + \sigma\,dB(t). \tag{2.4}$$

We use this model to price the same bond and bond option. Vasicek [37] derived the expression for bond prices and Jamshidian [23] gave the corresponding formula for bond option prices. The transition density for the Vasicek model is

$$\begin{aligned} &r(t+\Delta)|r(t) \\ &\sim N\big(\mu(1-e^{-\kappa\Delta}) \\ &\qquad + e^{-\kappa\Delta}r_t, \sigma^2(1-e^{-2\kappa\Delta})/(2\kappa)\big). \end{aligned} \tag{2.5}$$

This transition density is utilized to obtain the exact ML estimates of κ, the bond price and the bond option price, all under the mistaken presumption that the misspecified model (2.4) is correctly specified.

Table 1 reports the means and root mean square errors (RMSEs) for all these cases. It is clear that the finite sample estimation bias is more substantial than

TABLE 1

Finite sample properties of ML and jackknife estimates of κ, bond price and option price for the (true) CIR model using a (correctly specified) fitted CIR model and a (misspecified) fitted Vasicek model (sample size $n = 600$)

Parameter		κ	**Bond price**	**Option price**
True value		**0.1**	**0.8503**	**2.3920**
Exact ML	Mean	0.1845	0.8438	1.8085
of CIR	RMSE	0.1319	0.0103	0.9052
Euler ML	Mean	0.1905	0.8433	1.7693
of CIR	RMSE	0.1397	0.0111	0.9668
Jackknife ($m = 3$)	Mean	0.0911	0.8488	2.1473
of CIR	RMSE	0.1205	0.0094	0.8704
ML of Vasicek	Mean	0.1746	0.8444	1.8837
(misspecified)	RMSE	0.1175	0.0088	0.7637
Jackknife ($m = 2$) of	Mean	0.0977	0.8488	2.2483
Vasicek (misspecified)	RMSE	0.1628	0.0120	1.0289

the discretization bias and the specification bias for all three quantities, at least in this experiment. In particular, κ is estimated by the exact ML method with 84.5% upward bias, which contributes toward the -0.76% bias in the bond price and the -24.39% bias in the option price. Relative to the finite sample bias, the bias in κ due to the discretization is almost negligible since the total bias in κ changes from 84.5% to 90.5%. (The increase in the total bias indicates that the discretization bias effect is in the same direction as that of the estimation bias.) The total bias changes from -0.76% to -0.82% in the bond price and from -24.39% to -26.03% in the option price. These changes are marginal. Similarly, relative to the finite sample bias, the bias in κ due to misspecification of the drift function is almost negligible since the total bias changes from 84.5% to 74.6%. (The decrease in the total bias indicates that the misspecification bias effect is in the opposite direction to that of the estimation bias.) The total bias changes from -0.76% to -0.69% in the bond price and from -24.39% to -21.25% in the option price. Once again, these changes are marginal. When the jackknife method is applied to the correctly specified model, the estimation bias is greatly reduced in all cases (from 84.5% to -8.9% for κ; from -0.76% to -0.18% for the bond price; and from -24.39% to -10.23% for the option price).

Even more remarkably, when the jackknife method is applied to the incorrectly specified model (see the final row of Table 1), the estimation bias is also greatly reduced in all cases (from 84.5% to -2.3% for κ; from -0.76% to -0.18% for the bond price; and from -24.39% to -6.01% for the option price). These figures reveal that dealing with estimation bias can be much more important than ensuring correct specification in diffusion equation estimation, suggesting that general econometric treatment of the diffusion through nonparametric methods may not address the major source of bias effects on financial decision making.

Although the estimation bias is not completely removed by the jackknife method, the bias reduction is clearly substantial and the RMSE of the jackknife estimate is smaller in all cases than that of exact ML. In sum, it is apparent from Table 1 that the finite sample estimation bias is larger in magnitude than either of the biases due to discretization and misspecification and correcting this bias is therefore a matter of importance in empirical work on which financial decisions depend.

Although this demonstration of the relative importance of finite sample estimation bias in relation to discretization bias and specification bias is conducted in a parametric context, similar results can be expected for some nonparametric models. For example, in the semiparametric model examined in [1], the diffusion function is nonparametrically specified and the drift function is linear, so that the mean reversion parameter is estimated parametrically as in the above example. In such cases, we can expect substantial finite sample estimation bias to persist and to have important practical implications in financial pricing applications.

3. REALIZED VOLATILITY

As noted in Fan's overview, many models used in financial econometrics for modeling asset prices and interest rates have the fully functional scalar differential form

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t, \tag{3.1}$$

where both drift and diffusion functions are nonparametric and where the equation is driven by Brownian motion increments dB_t. For models such as (3.1), we have $(dX_t)^2 = \sigma^2(X_t)\,dt$ a.s. and hence the quadratic variation of X_t is

$$[X]_T = \int_0^T (dX_t)^2\,dt = \int_0^T \sigma^2(X_t)\,dt, \tag{3.2}$$

where $\int_0^T \sigma^2(X_t)\,dt$ is the accumulated or integrated volatility of X. Were X_t observed continuously, $[X]_T$ and, hence, integrated volatility, would also be observed. For discretely recorded data, estimation of (3.2) is an important practical problem. This can be accomplished by direct nonparametric methods using an empirical estimate of the quadratic variation that is called realized volatility. The idea has been discussed for some time, an early reference being Maheswaran and Sims [28], and it has recently attracted a good deal of attention in the econometric literature now that very high frequency data has become available for empirical use. Recent contributions to the subject are reviewed in [4] and [8].

Suppose X_t is recorded discretely at equispaced points $(\Delta, 2\Delta, \ldots, n_\Delta\Delta(\equiv T))$ over the time interval $[0, T]$. Then, $[X]_T$ can be consistently estimated by the realized volatility of X_t defined by

$$[X_\Delta]_T = \sum_{i=2}^{n_\Delta} \big(X_{i\Delta} - X_{(i-1)\Delta}\big)^2, \tag{3.3}$$

as $\Delta \to 0$, as is well known. In fact, any construction of realized volatility based on an empirical grid of observations where the maximum grid size tends to zero will produce a consistent estimate. It follows that the

integrated volatility can be consistently estimated by this nonparametric approach, regardless of the form of $\mu(X_t)$ and $\sigma(X_t)$. The approach has received a great deal of attention in the recent volatility literature and serves as a powerful alternative to the methods discussed by Fan, especially when ultra-high frequency data are available.

While this approach is seemingly straightforward, it is not without difficulties. First, in order for the approach to be useful in empirical research, it is necessary to estimate the precision of the realized volatility estimates. Important contributions on the central limit theory of these empirical quadratic variation estimates by Jacod [22] and Barndorff-Nielson and Shephard [10, 11] has facilitated the construction of suitable methods of inference. Second, in practical applications, realized volatility measures such as (3.3) are usually contaminated by microstructure noise bias, especially at ultra high frequencies and tick-by-tick data. Noise sources arise from various market frictions and discontinuities in trading behavior that prevent the full operation of efficient financial markets. Recent work on this subject (e.g., [8, 9, 21, 38]) has developed various methods, including nonparametric kernel techniques, for reducing the effects of microstructure noise bias.

4. ADDITIONAL ISSUES

Given efficient market theory, there is good reason to expect that diffusion models like (3.1) may have nonstationary characteristics. Similar comments apply to term structure models and yield curves. In such cases, nonparametric estimation methods lead to the estimation of the local time (or sojourn time) of the corresponding stochastic process and functionals of this quantity, rather than a stationary probability density. Moreover, rates of convergence in such cases become path dependent and the limit theory for nonparametric estimates of the drift and diffusion functions in (3.1) is mixed normal. Asymptotics of this type require an enlarging time span of data as well as increasing in-fill within each discrete interval as $n \to \infty$. An overview of this literature and its implications for financial data applications is given in [6]. Nonparametric estimates of yield curves in multifactor term structure models are studied in [25].

Not all models in finance are driven by Brownian motion. In some cases, one can expect noise to have to have some memory and, accordingly, models such as (3.1) have now been extended to accommodate fractional Brownian motion increments. The stochastic calculus of fractional Brownian motion, which is not a semi-martingale, is not as friendly as that of Brownian motion and requires new constructs, involving Wick products and versions of the Stratonovich integral. Moreover, certain quantities, such as quadratic variation, that have proved useful in the recent empirical literature may no longer exist and must be replaced by different forms of variation, although the idea of volatility is still present. Developing a statistical theory of inference to address these issues in financial econometric models is presenting new challenges.

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