Electronic Journal of Statistics
Vol. 3 (2009) 546–556
ISSN: 1935-7524
DOI: 10.1214/08-EJS303

Almost sure convergence of extreme order statistics

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Abstract: Let $M_n^{(k)}$ denote the kth largest maximum of a sample $(X_1, X_2, \ldots, X_n)$ from parent X with continuous distribution. Assume there exist normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and a nondegenerate distribution G such that $a_n^{-1}(M_n^{(1)} - b_n) \xrightarrow{w} G$. Then for fixed $k \in \mathbb{N}$, the almost sure convergence of

$$\frac{1}{D_N}\sum_{n=k}^{N} d_n \mathbb{I}\{M_n^{(1)} \le a_n x_1 + b_n, M_n^{(2)} \le a_n x_2 + b_n, \ldots, M_n^{(k)} \le a_n x_k + b_n\}$$

is derived if the positive weight sequence (d_n) with $D_N = \sum_{n=1}^{N} d_n$ satisfies conditions provided by Hörmann. Some practical issues of this result are also discussed.

AMS 2000 subject classifications: Primary 62F15; secondary 60G70, 60F15.
Keywords and phrases: Almost sure convergence, order statistics.

Received October 2008.

Contents

1. Introduction

The concept of almost sure central limit theorems (ASCLTs) is relatively new compared to that of classical central limit theorems. The original papers on this concept are those by Brosamler [4], Schatte [20] and Lacey and Philipp [17]. The concept has already started to receive applications in many areas. For example, Brosamler [5, 6] has shown applications of ASCLTs for occupation measures of Brownian-motion on a compact Riemannian manifold and for diffusions and its application to path energy and eigenvalues of the Laplacian. His work has been followed up in many other applied areas, including condensed matter physics, statistical mechanics, ergodic theory and dynamical systems, occupational health psychology, control and information sciences and rehabilitation counseling.

More recently, Thangavelu [23] has studied applications of ASCLTs for quanitle estimation, one-sample hypothesis testing, two-sample hypothesis testing, random intervals, the Behrens-Fisher Problem, rank statistics, quality control and decision making. One of the key findings is that in hypothesis testing methods using ASCLTs one does not need to estimate or use the variance of the observations. For other advantages, see Chapters 2 and 3 in Thangavelu [23].

Most recently, Bercu et al. [3] have shown applications of ASCLTs for the estimation and prediction in linear autoregressive models and branching processes with immigration.

The first ASCLTs were reported in the papers of Brosamler [4], Schatte [20] and Lacey and Philipp [17]. For an independent and identically distributed (i.i.d.) sequence $\{X_n, n \geq 1\}$ with mean 0, variance 1 and partial sum $S_n = \sum_{k=1}^{n} X_k$, the simplest version of the ASCLT says

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbb{I}\{S_n \leq \sqrt{n}x\} \to \Phi(x), \quad a.s. \forall x \in \mathbb{R},$$

where $a.s.$ means almost surely, $\mathbb{I}_A$ denotes the indicator function and $\Phi(x)$ is the standard normal distribution function. For unbounded functional ASCLTs, Ibragimov and Lifshits [16] and Berkes et al. [2] obtained the following ASCLT under different restrictions on the continuous function f:

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f(S_n/\sqrt{n}) \to \int_{-\infty}^{\infty} f(x) d\Phi(x), \quad a.s.$$

The universal version of the ASCLT discussed by Berkes and Csáki [1] includes the case of the ASCLT of extremes of i.i.d random sequences which was first studied by Fahrner and Stadtmüller [11] and Cheng et al. [7]. Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence, and let $M_n = \max_{1 \leq k \leq n} X_k$ denote the partial maximum. If there exist normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and a nondegenerate distribution $G(x)$ such that nondegenerate distribution $G(x)$ such that

$$P\left(M_n \leq a_n x + b_n\right) \to G(x) =: G_\gamma(x) = \exp\left\{-(1+\gamma x)^{-1/\gamma}\right\}, \tag{1.1}$$

where γ is the so-called extreme value index, then

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbb{I}\{M_n \leq a_n x + b_n\} \to G(x), \quad a.s. \forall x \in \mathbb{R}. \tag{1.2}$$

Fahrner [10] extended (1.2) to unbounded continuous functions. The general result is

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}f(a_n^{-1}(M_n-b_n))\to\int_{-\infty}^{\infty}f(x)dG(x),\quad a.s.$$

See Theorem 1 of Fahrner [10]. Stadtmüller [21] considered the ASCLT of the kth maximum as $k=k_n$ satisfies $\log k_n = O((\log n)^{1-\varepsilon})$ for some $\varepsilon>0$ or $(n-k_n)/n = p+O(1/\sqrt{n\log^{\varepsilon} n})$ for some $0<p<1$. Especially for fixed k he showed

$$\frac{1}{\log N}\sum_{n=k}^{N}\frac{1}{n}\mathbb{I}\{M_n^{(k)}\le a_nx+b_n\}\to G(x)\sum_{j=0}^{k-1}\frac{(-\log G(x))^j}{j!},\quad a.s.\forall x\in\mathbb{R},$$

where $M_n^{(k)}$ denotes the kth maximum of $X_1,X_2,\dots,X_n$ and $M_n := M_n^{(1)}$. Peng and Qi [19] proved the ASCLT of central order statistics, see also Hörmann [12].

In this note, we consider the following averages:

$$\frac{1}{D_N}\sum_{n=k}^{N}d_n\mathbb{I}\{M_n^{(1)}\le a_nx_1+b_n, M_n^{(2)}\le a_nx_2+b_n,\dots,M_n^{(k)}\le a_nx_k+b_n\}\quad(1.3)$$

provided the positive weights d_n, $n\ge 1$ satisfy the following conditions:

$$\liminf_{n\to\infty} nd_n>0,\qquad(1.4)$$

$$n^{\alpha}d_n\ \text{ is eventually nonincreasing for some } 0<\alpha<1,\qquad(1.5)$$

and

$$\limsup_{n\to\infty} nd_n\left(\log D_n\right)^{\rho}/D_n<\infty\qquad(1.6)$$

for some $\rho>0$, where $D_n=\sum_{k=1}^{n}d_k$. Under conditions (1.4)–(1.6) it follows from the results in Hörmann [13, 14, 15] that

$$\frac{1}{D_N}\sum_{n=1}^{N}d_n\mathbb{I}\{S_n\le\sqrt{n}x\}\to\Phi(x),\quad a.s.,$$

and

$$\frac{1}{D_N}\sum_{n=1}^{N}d_n\mathbb{I}\{M_n\le a_nx+b_n\}\to G(x),\quad a.s.$$

The main results on the convergence of (1.3) are provided in Section 2. The proofs are deferred to Section 3. Some practical implications of the main results are discussed in Section 4.

As discussed by Berkes and Csáki [1] and Hömann [12, 13, 14, 15], the larger the D_n, the stronger the ASCLT. If $d_n < 1/n$ such that $D_n \to \infty$, then the ASCLT holds. If $d_n = 1$, there is no ASCLT on the partial sum and partial maxima. Conditions (1.4), (1.5) and (1.6) tell us that there exists a large class of sequences $1/n < d_n < 1$ such that the ASCLT holds. For example, we may assume $D_n \to \infty$ with Karamata representation

$$D_n = \exp\left(\int_a^n \frac{\theta(u)}{u} du\right), \quad n > a,$$

where $\theta(x)$ is a slowly varying function such that

$$\liminf_{n\to\infty} \theta(n) D_n > 0, \quad \text{and} \quad \limsup_{n\to\infty} \theta(n) \left(\log D_n\right)^{\rho} < \infty$$

for some $\rho > 0$, which guarantees that (1.4), (1.5) and (1.6) hold. By the mean value theorem, we may choose $d_n \sim \theta(n) D_n/n$. This implies that (d_n) is a regularly varying function with index -1. We mention the following examples:

(a) $D_n = (\log n)^{\kappa}$ with $d_n \sim \kappa(\log n)^{\kappa-1}/n$ for $\kappa > 1$, $\rho > 0$;
(b) $D_n = \exp((\log n)^{\kappa})$ with $d_n \sim \kappa \exp((\log n)^{\kappa})(\log n)^{\kappa-1}/n$ for $0 < \kappa < 1$, $0 < \rho < (1-\kappa)/\kappa$;
(c) $D_n = (\log n)^{1-\kappa} \exp((\log n)^{\kappa})$ with $d_n \sim \kappa \exp((\log n)^{\kappa})/n$ for $0 \le \kappa < 1/2$, $0 < \rho < 1/\kappa - 1$.

Throughout this note we assume that $F(x)$, the univariate marginal distribution of X_n, $n \ge 1$ is continuous. This assumption implies that the order statistics are a.s. uniquely defined. Before providing the main results, recall the joint limit distribution of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)})$ for fixed k if (1.1) holds. Define levels $u_n(x_j) = a_n x_j + b_n$, $j = 1, 2, \ldots, k$, $x_1 > x_2 > \cdots > x_k$ and define the point process χ_n of exceedances of levels $u_n(x_j), j = 1, 2, \ldots, k$ by i.i.d random variables $X_1, X_2, \ldots, X_n$. Then χ_n converges in distribution to a Poisson process on $(0, 1] \times \mathbb{R}$, for more details see Chapter 5 of Leadbetter et al. [18], which states the joint limit distribution of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)})$ and

$$P\big(M_n^{(j)} \le a_n x + b_n\big) \to G(x) \sum_{i=0}^{j-1} \frac{(-\log G(x))^i}{i!} =: H_j(x), \quad j = 1, 2, \ldots, k \tag{1.7}$$

as $n \to \infty$. The joint limit distribution of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)})$ is so complicated that we express it as $H(x_1, x_2, \ldots, x_k)$ with the marginal distribution $H_j(x)$ defined in (1.7), $j = 1, 2, \ldots, k$, i.e.

$$\begin{aligned} &\lim_{n\to\infty} \quad P\big(M_n^{(1)} \le a_n x_1 + b_n, M_n^{(2)} \le a_n x_2 + b_n, \ldots, M_n^{(k)} \le a_n x_k + b_n\big) \\ &= \quad \begin{cases} H\left(x_1, x_2, \ldots, x_k\right), & x_1 > x_2 > \cdots > x_k; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{1.8}$$

2. Main results

In this section, we provide the main results. The proofs are deferred to the next section.

Theorem 1. *Suppose (1.1) holds for an i.i.d. random sequence* $(X_n, n \geq 1)$. *Further assume (1.4)–(1.6) hold for positive weights* $d_n, n \geq 1$. *Then for fixed* $k \in \mathbb{N}$ *and real numbers* $x_1 > x_2 > \cdots > x_k$, *we have*

$$\begin{aligned} & \frac{1}{D_N}\sum_{n=k}^{N} d_n \mathbb{I}\{M_n^{(1)} \leq u_n(x_1), M_n^{(2)} \leq u_n(x_2), \ldots, M_n^{(k)} \leq u_n(x_k)\} \\ \to \quad & H(x_1, x_2, \ldots, x_k), \quad a.s., \end{aligned} \tag{2.1}$$

where $u_n(x_j), j = 1, 2, \ldots, k$ *and* $H(x_1, x_2, \ldots, x_k)$ *are defined as before.*

Corollary 1. *Under the conditions of Theorem 1, for real numbers* $x_{k_1} > x_{k_2} > \cdots > x_{k_l}$ *with* $1 \leq k_1 < k_2 < \cdots < k_l \leq k$, *we have*

$$\begin{aligned} & \frac{1}{D_N}\sum_{n=k_l}^{N} d_n \mathbb{I}\{M_n^{(k_1)} \leq u_n(x_{k_1}), M_n^{(k_2)} \leq u_n(x_{k_2}), \ldots, M_n^{(k_l)} \leq u_n(x_{k_l})\} \\ \to \quad & H^*(x_{k_1}, x_{k_2}, \ldots, x_{k_l}), \quad a.s., \end{aligned}$$

where $H^*(x_{k_1}, x_{k_2}, \ldots, x_{k_l})$ *is the marginal distribution of* $H(x_1, x_2, \ldots, x_k)$. *Especially, for fixed* $k \in \mathbb{N}$,

$$\frac{1}{D_N}\sum_{n=k}^{N} d_n \mathbb{I}\{M_n^{(k)} \leq u_n(x)\} \to H_k(x) = G(x)\sum_{j=0}^{k-1}\frac{(-\log G(x))^j}{j!}, \quad a.s.$$

For bounded Lipschitz 1 functions, we have the following ASCLT of order statistics.

Corollary 2. *Under the conditions of Theorem 1, for fixed* $k \in \mathbb{N}$ *and bounded Lipschitz* 1 *function* f, *we have*

$$\frac{1}{D_N}\sum_{n=k}^{N} d_n f\left(a_n^{-1}\big(M_n^{(k)} - b_n\big)\right) \to \int_{-\infty}^{\infty} f(x) dH_k(x), \quad a.s.$$

3. Proofs

As mentioned above, denote levels $u_n(x) = a_n x + b_n, x \in \mathbb{R}, n \geq 1$ and real numbers $x_1 > x_2 > \cdots > x_k$ for fixed k. For convenience, let $M_{m,n}^{(j)}$ denote the jth maxima of $X_{m+1}, X_{m+2}, \ldots, X_n, 0 \leq m < n$ and $M_n^{(j)} := M_{0,n}^{(j)}$. Set

$$\begin{aligned} \eta_{m,n} \quad = \quad & \mathbb{I}\big\{M_{m,n}^{(1)} \leq u_n(x_1), M_{m,n}^{(2)} \leq u_n(x_2), \ldots, M_{m,n}^{(k)} \leq u_n(x_k)\big\} \\ & - P\big(M_{m,n}^{(1)} \leq u_n(x_1), M_{m,n}^{(2)} \leq u_n(x_2), \ldots, M_{m,n}^{(k)} \leq u_n(x_k)\big) \end{aligned}$$

for $0 \leq m < n$ and $\eta_n = \eta_{0,n}$. Before proving the main results, we need some lemmas. Our first lemma provides a bound on the expectation of the difference of the indicator functions for the jth maxima of the whole sequence and the jth maxima of the subsequence for $j = 1, 2, \ldots, k$.

Lemma 1. *Assume (1.1) holds and $m \geq k, n - m \geq k$. Then*

$$\mathbb{E}\left|\mathbb{I}\{M_n^{(j)} \leq u_n(x)\} - \mathbb{I}\{M_{m,n}^{(j)} \leq u_n(x)\}\right| \leq k\frac{m}{n}$$

uniformly for $j = 1, 2, \ldots, k$ and $x \in \mathbb{R}$.

Proof. First note $\mathbb{I}\{M_n^{(j)} \leq u_n(x)\} - \mathbb{I}\{M_{m,n}^{(j)} \leq u_n(x)\} \neq 0$ if and only if $M_n^{(j)} > M_{m,n}^{(j)}$. The latter implies that $M_m^{(1)} > M_{m,n}^{(j)}$. It is known that the distribution of the general order statistic $M_n^{(j)}$ is

$$P\big(M_n^{(j)} \leq x\big) = \sum_{i=0}^{j-1} \binom{n}{i} (F(x))^{n-i} (1 - F(x))^i,$$

where $\binom{n}{i} = n!/\{i!(n-i)!\}$. Hence,

$$\begin{aligned}
& \mathbb{E}\left|\mathbb{I}\{M_n^{(j)} \leq u_n(x)\} - \mathbb{I}\{M_{m,n}^{(j)} \leq u_n(x)\}\right| \\
\leq\ & P\big(M_n^{(j)} > M_{m,n}^{(j)}\big) \leq P\big(M_m^{(1)} > M_{m,n}^{(j)}\big) \\
=\ & \sum_{i=0}^{j-1} m\binom{n}{i} \int_{-\infty}^{\infty} (F(x))^{n+m-i-1} (1 - F(x))^i \, dF(x) \\
\leq\ & \sum_{i=0}^{j-1} m\binom{n}{i} \int_0^1 x^{n+m-i-1}(1-x)^i dx \\
=\ & \sum_{i=0}^{j-1} m\binom{n}{i} \frac{(n+m-i-1)!i!}{(n+m)!} \\
\leq\ & j\frac{m}{n} \leq k\frac{m}{n}
\end{aligned}$$

uniformly for $1 \leq j \leq k$ and $x \in \mathbb{R}$. □

Our next lemma provides a bound for the covariance of η_m and η_n, which will be used later for estimating the moment of the weighted sum of η_n.

Lemma 2. *Assume (1.1) holds. Then*

$$|\text{Cov}\,(\eta_m, \eta_n)| \leq 2k^2\frac{m}{n} \tag{3.1}$$

for $m \geq k, n - m \geq k$.

Proof. The desired result follows by Lemma 1 and noting

$$\begin{aligned}|\mathrm{Cov}(\eta_n,\eta_m)| \quad \leq \quad & 2\left(\mathbb{E}\left|\mathbb{I}\{M_n^{(k)}\leq u_n(x_k)\}-\mathbb{I}\{M_{m,n}^{(k)}\leq u_n(x_k)\}\right|\right.\\ & +\mathbb{E}\left|\mathbb{I}\{M_n^{(k-1)}\leq u_n(x_{k-1})\}-\mathbb{I}\{M_{m,n}^{(k-1)}\leq u_n(x_{k-1})\}\right|\\ & +\cdots\\ & \left.+\mathbb{E}\left|\mathbb{I}\{M_n^{(1)}\leq u_n(x_1)\}-\mathbb{I}\{M_{m,n}^{(1)}\leq u_n(x_1)\}\right|\right).\end{aligned}$$

□

The following lemma is useful to estimate the moment of the weighted sum of $\eta_n-\eta_{m,n}$.

Lemma 3. *Assume (1.1) holds. For $m\geq k, n-m\geq k$, we have*

$$\mathbb{E}\left|\eta_n-\eta_{m,n}\right|\leq 2k^2\frac{m}{n}. \tag{3.2}$$

Proof. Note

$$\begin{aligned}& \mathbb{E}\left|\eta_n-\eta_{m,n}\right|\\ = \quad & 2\mathbb{E}\left(\prod_{j=1}^{k}\mathbb{I}\{M_{m,n}^{(j)}\leq u_n(x_j)\}-\prod_{j=1}^{k}\mathbb{I}\{M_n^{(j)}\leq u_n(x_j)\}\right)\\ \leq \quad & 2\sum_{j=1}^{k}\mathbb{E}\left(\mathbb{I}\{M_{m,n}^{(j)}\leq u_n(x_j)\}-\mathbb{I}\{M_n^{(j)}\leq u_n(x_j)\}\right)\end{aligned}$$

by the elementary inequality $|\prod_{j=1}^{l}y_j-\prod_{j=1}^{l}z_j|\leq\sum_{j=1}^{l}|y_j-z_j|$ for all $|y_j|\leq 1,|z_j|\leq 1, j=1,2,\ldots,l$. By using Lemma 1, the proof is complete. □

Lemma 4. *Under the conditions of Theorem 1, for any ω with $k\leq m\leq\omega\leq n$ and $p\in\mathbb{N}$,*

$$\mathbb{E}\left|\sum_{l=\omega}^{n}d_l\left(\eta_l-\eta_{m,l}\right)\right|^p\leq 2^{2p-1}k\left(2+kp^{\frac{p}{2}}\right)\left(\sum_{l=\omega}^{n}ld_l^2\right)^{\frac{p}{2}}.$$

Proof. Note $|\eta_l-\eta_{k,l}|\leq 4$ and, for $m\geq k$, $l-m\geq k$, by using Lemma 3, we have

$$\mathbb{E}\left|\eta_l-\eta_{m,l}\right|^p\leq 2\cdot 4^{p-1}\mathbb{E}\left|\eta_l-\eta_{m,l}\right|\leq 4^pk^2\left(\frac{m}{l}\right).$$

Then by Hölder inequality and (1.5), similar to the arguments in Lemma 3 of Hörmann [13], we have

$$\mathbb{E}\left|\sum_{l=\omega+k}^{n}d_l\left(\eta_l-\eta_{m,l}\right)\right|^p\leq 4^pk^2p^{\frac{p}{2}}\left(\sum_{l=\omega}^{n}ld_l^2\right)^{\frac{p}{2}}.$$

By using the C_r inequality,

$$
\begin{aligned}
& \mathbb{E}\left|\sum_{l=\omega}^{n} d_l\left(\eta_l-\eta_{m,l}\right)\right|^p \\
\leq\ & 2^{p-1}\left(\mathbb{E}\left|\sum_{l=\omega}^{\omega+k-1} d_l\left(\eta_l-\eta_{m,l}\right)\right|^p+\mathbb{E}\left|\sum_{l=\omega+k}^{n} d_l\left(\eta_l-\eta_{m,l}\right)\right|^p\right) \\
\leq\ & 2^{p-1}\left(2k\cdot 4^p \max_{\omega\leq l\leq \omega+k-1} d_l^p+\mathbb{E}\left|\sum_{l=\omega+k}^{n} d_l\left(\eta_l-\eta_{m,l}\right)\right|^p\right) \\
\leq\ & 2^{2p-1}k\left(2+kp^{\frac{p}{2}}\right)\left(\sum_{l=\omega}^{n} l d_l^2\right)^{\frac{p}{2}}.
\end{aligned}
$$

The proof is complete. □

The following is the result of Lemma 2, Lemma 4 and slight changes to the proof of Lemma 4 of Hörmann [13] (or Lemma 2 of Hörmann [14]).

Lemma 5. *Under the conditions of Theorem 1, for every $p\in\mathbb{N}$, there exists a constant $C_p>0$ such that*

$$
\mathbb{E}\left|\sum_{n=k}^{N} d_n\eta_n\right|^p \leq C_p\left(\sum_{k\leq m\leq n\leq N} d_m d_n\left(\frac{m}{n}\right)^{\alpha}\right)^{\frac{p}{2}}.
$$

The following is the result of Hörmann [13, 14].

Lemma 6. *Assume (1.6) holds. For any $\alpha>0$ and $\eta<\rho$, we have*

$$
\sum_{k\leq m\leq n\leq N} d_m d_n\left(\frac{m}{n}\right)^{\alpha}=O\left(\frac{D_N^2}{(\log D_N)^{\eta}}\right).
$$

Proof of Theorem 1. By Lemmas 4 and 5, using Markov inequality and the subsequence procedure, we obtain the desired results, cf. Hörmann [13, 14, 15]. □

4. Discussion

The main result given by Theorem 1 can be of practical use in many different ways. Here, we discuss four problems.

Firstly, we should note that (2.1) provides a "time-average" version of (1.8). So, a statistical model based on (2.1) for a fixed N should be more accurate and efficient than one based on (1.8) for fixed n (see Tawn [22] for an example of the latter). This is because for a fixed N a model based on (2.1) will consider the data values $\{M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)}\}$ for $n=k,k+1,\ldots,N$ while for a fixed n a model based on (1.8) will only consider the data values $\{M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)}\}$.

Secondly, Theorem 1 can be used to construct tests of hypotheses about the extreme value index γ: for example,

$$H_0 : \gamma = 0; \quad H_1 : \gamma \neq 0. \tag{4.1}$$

There has been much research on developing procedures for tests of this kind. One approach is to derive asymptotic distributions of known estimators of the extreme value index (such as the moment-type estimator due to Dekkers et al. [9]) and then deduce the corresponding asymptotic rejection regions. However, this approach has challenging problems: the choice of optimal threshold, size and power are open for debate.

We now show how Theorem 1 can be used to construct a test for (4.1). With D_N as defined in Section 1, set

$$\begin{aligned} & H_n\left(x_1, x_2, \ldots, x_k\right) \\ = \quad & \frac{1}{D_N} \sum_{n=k}^{N} d_n \mathbb{I}\left\{ M_n^{(1)} \leq u_n(x_1), M_n^{(2)} \leq u_n(x_2), \ldots, M_n^{(k)} \leq u_n(x_k) \right\}. \end{aligned}$$

By arguments similar to those of Lemma 3 and Theorem 4 in Thangavelu [23], one can see that $H_n(x_1, x_2, \ldots, x_k)$ is an empirical distribution and that the following Glivenko-Cantelli Theorem

$$\lim_{N\to\infty} \sup_{x_k \leq x_{k-1} \leq \cdots \leq x_1} \left| H_n\left(x_1, x_2, \ldots, x_k\right) - H\left(x_1, x_2, \ldots, x_k\right) \right|, \quad a.s.$$

holds by (2.1). So, an almost sure rejection region can be established as in page 1807 of Dekkers and de Haan [8]. Note that the parameters, a_n and b_n, should be estimated. It is not clear, however, how one can test for $\gamma < 0$ or $\gamma > 0$ if the hypothesis $\gamma = 0$ is rejected. These are interesting and challenging problems for the future.

Thirdly, as pointed by Brosamler [4], one might consider using (2.1) to test a random number generator. One would only have to check one (typical) sequence, rather than many sequences as in tests based on (1.8).

Finally, the result in (2.1) is of interest for mathematical statistics as it shows that assertions are possible for almost every realization of the random variables.

Acknowledgments

The authors would like to thank the Editor and the Associate Editor for careful reading and for their comments which greatly improved this note.

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