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On a one-dimensional model of infection spreading

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Abstract. We study sequential infection spreading models on $\mathbb{Z}$, where healthy particles stay inactive and infected particles have positive displacement rate. When one infected particle reaches a healthy site, all particles contained there become infected. The main focus of our study is the speed of infection spreading.

1 Introduction

The model that we are considering here is similar to the frog model, see Alves, Machado and Popov (2001, 2002), Fontes, Machado and Sarkar (2004), Lebensztayn, Machado and Popov (2005) and Popov (2001). In the basic version of the frog model only the origin contains one infected particle, with positive rate of displacement. In the other sites the particles are healthy and do not move. When an infected particle reaches a site only with healthy particles, all particles contained there become infected and start moving. Moreover, particles of the same kind do not interact with each other. In our model particles are allocated on $\mathbb{Z}$ and can only move to the right, and besides the conditions above, we admit the existence of more than one site with infected particles in the initial configuration, that is, at time 0, particles at $x \leq 0$ are considered infected. The model described above is called the *general model*.

The number of particles at site $x \in \mathbb{Z}$, at moment t, will be denoted by $\eta_t(x)$. Thus, for the initial moment, $\{\eta_0(x) : x \in \mathbb{Z}\}$ is called the initial configuration. In our model the configuration in the origin will be $\eta_0(0) = 1 + \nu$, $\nu \sim Pois(\lambda)$, where $Pois(\lambda)$ stands for the Poisson distribution with parameter $\lambda > 0$. Note that the origin necessarily contains at least one particle at time 0. In the other sites $\eta_0(x) \sim Pois(\lambda)$, and moreover $\{\eta_0(x) : x \in \mathbb{Z}\}$ is a sequence of independent random variables.

By convention, in the beginning the negative sites, and also the origin, will be infected, while the positive ones are healthy. For each infected particle we associate a homogeneous Poisson process of rate 1, whose we call the clock of the process. The clocks are mutually independent. Thus, infected particles move to the right, and the intervals between jumps are independent random variables with

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exponential distribution of parameter 1, denoted by *Expon*(1). The healthy particles stay sleeping (displacement rate equal to zero), until awakened by an infected particle.

The model that we deal in this paper is a discrete approach to one-dimensional diffusion process, see Itô and McKean (1974). In a diffusion process particles move on a continuous state space $I \subset \mathbb{R}$, moreover they have a lifetime $m \leq \infty$ (also called killing time), but here we will not assume this.

Our main result is about the speed of infection spreading, that is, we have:

Theorem 1. *Let* $v := \lim_{t\to\infty} \frac{Z_t}{t}$ *be the speed of infection spreading, being* Z_t *the current boundary, that is, the rightmost site at time* t *which contains infected particles. Then,* $v > 1$.

It is worth noting that there is a substantial difference between this paper and the previous work on the frog model: in Alves, Machado and Popov (2001, 2002) the coupling used to prove the subadditivity is particle-oriented (particles are distinguishable and the sequence of particles' jumps is fixed). In this paper, the coupling is site-oriented (particles are indistinguishable and the sequences of jump moments are attached to the sites).

2 Preliminaries

In this section we present some preliminaries, allowing prove the crucial Theorem 1 later.

The following theorem guarantees the existence of v and also supplies a lower bound. We use the version of Liggett (1985).

Theorem 2 (Subadditive Ergodic Theorem). *Let* $\{X(m,n) : 0 \leq m \leq n;\ m,n \in \mathbb{N}\}$ *be a sequence of positive random variables such that*:

(i) $X(0,n) \leq X(0,m) + X(m,n)$ *for all* $0 \leq m < n$;

(ii) *the joint distribution* $\{X(m+1, m+k+1) : k \geq 1\}$ *is the same that* $\{X(m, m+k) : k \geq 1\}$, $\forall m \geq 0$;

(iii) *for each* $k \geq 1$ *the sequence* $\{X(nk, (n+1)k) : n \geq 1\}$ *is a ergodic stationary process*;

(iv) $E(X(0,1)) < \infty$.

Then:

$$\lim_{n\to\infty} \frac{X(0,n)}{n} = \gamma \quad \textit{almost surely},$$

where $\gamma := \inf_{n\geq 0} \frac{E(X(0,n))}{n}$.

Now, our goal is apply the subadditive ergodic theorem, so let us restrict to a simplified model, which we call *truncated model*. This model has almost surely the same speed of infection spreading than general model, as we will see in the following section. In truncated model the sites are $\{0, 1, \ldots\}$ and at time 0 only the origin is infected. For the positive sites the configuration and the dynamics is the same that in the general model. Through truncated model we obtain results that will be applied to the general model. To apply Theorem 2 we construct a certain two-parameter family of random variables, which correspond to certain hitting times with respect to the truncated model, and prove that the subadditivity property holds for this construction. So we introduce the process A which has no particles in $\{0, 1, \ldots, x-1\}$, and begins with $\eta_0(x)+1$ infected particles at x, $\eta_0(x) \sim Pois(\lambda)$. Note that the truncated model can contain particles at $\{0, 1, \ldots, x-1\}$, but at x should contain a greater or equal quantity of particles than the process A. Now, for the process A, let $T(x, y)$ be the time it takes for an infected particle from site x to reach y, with $0 \le x < y < \infty$, that is, the time so that the infection reach y, leaving from a particle at x. The assumption of not having particles in $\{0, 1, \ldots, x-1\}$ is to guarantee the subadditivity, that is, $T(0, x) + T(x, y) \ge T(0, y)$, allowing to apply Theorem 2.

We need to define N_u^A and N_u as being the number of particles at site u, in the process A and truncated model, respectively. Note that $N_u \ge N_u^A$, for any $x \le u < y$. Moreover, let $\mathcal{T}_n^A(u)$ and $\mathcal{T}_n(u)$ the instant when the nth particle left site u.

Lemma 1 and 2 below are for the proof of Lemma 3, which ensure that $0 \le \gamma \le 1$, when applied to the sequence $\{T(x, y) : x, y \in \mathbb{Z}\}$. Later, through Lemma 3, we will prove that $v > 1$ for the general model.

Lemma 1. *Let $x \le u < y$ and $n \le N_u^A$. Then, $\mathcal{T}_n^A(u) \ge \mathcal{T}_n(u)$.*

Proof. We will use induction in n. First define $\{\mathcal{E}_1^x, \mathcal{E}_2^x, \ldots\}$ to be an i.i.d. sequence of random variables with law $Expon(1)$ and assume $x = u$. Then

$$\mathcal{T}_1^A(x) = \min\{\mathcal{E}_j^x : j = 1, \ldots, N_x^A\} \quad \text{and} \quad \mathcal{T}_1(x) = \min\{\mathcal{E}_j^x : j = 1, \ldots, N_x\}.$$

It follows that $\mathcal{T}_1(x) \le \mathcal{T}_1^A(x)$. Now, we concentrate on the process A. For the second particle at site x, $\mathcal{T}_2^A(x) = \min\{\{\mathcal{E}_j^x : j = 1, \ldots, N_x^A\} \setminus \{\mathcal{T}_1^A(x)\}\}$ and we proceed in this way until obtaining $\mathcal{T}_{N_x^A}^A(x)$. Note that through the site $x+1$ will pass $m + N_x^A$ particles, being m the initial number of particles which were present in $x+1$. Here we introduce the notation

$$\begin{aligned}\mathcal{C}^A(x+1) = \{&\mathcal{T}_1^A(x) + \mathcal{E}_1^{x+1}, \mathcal{T}_1^A(x) + \mathcal{E}_2^{x+1}, \ldots,\\ &\mathcal{T}_1^A(x) + \mathcal{E}_m^{x+1}, \mathcal{T}_1^A(x) + \mathcal{E}_{m+1}^{x+1},\\ &\mathcal{T}_2^A(x) + \mathcal{E}_{m+2}^{x+1}, \mathcal{T}_3^A(x) + \mathcal{E}_{m+3}^{x+1}, \ldots, \mathcal{T}_{N_x^A}^A(x) + \mathcal{E}_{m+N_x^A}^{x+1}\}.\end{aligned}$$

In the Process A, the instant when the first particle jumps from $x+1$ to $x+2$ will be $\mathcal{T}_1^A(x+1) = \min\{\mathcal{C}^A(x+1)\}$. For the second particle, in $\mathcal{C}^A(x+1)$ we exclude $\mathcal{T}_1^A(x+1)$ and we take the next minor value, that is, $\mathcal{T}_2^A(x+1) = \min\{\mathcal{C}^A(x+1) \setminus \{\mathcal{T}_1^A(x+1)\}\}$. In this way we proceed until obtaining $\mathcal{T}^A_{m+N_x^A}(x+1)$. Assuming that $N_x = N_x^A$, similarly, in the truncated model we will have the set

$$\mathcal{C}(x+1) = \{\mathcal{T}_1(x) + \mathcal{E}_1^{x+1}, \mathcal{T}_1(x) + \mathcal{E}_2^{x+1}, \dots, \mathcal{T}_1(x) + \mathcal{E}_m^{x+1}, \mathcal{T}_1(x) + \mathcal{E}_{m+1}^{x+1},$$
$$\mathcal{T}_2(x) + \mathcal{E}_{m+2}^{x+1}, \mathcal{T}_3(x) + \mathcal{E}_{m+3}^{x+1}, \dots, \mathcal{T}_{N_x^A}(x) + \mathcal{E}_{m+N_x^A}^{x+1}\}.$$

Proceeding as before we will obtain the instants

$$\mathcal{T}_1(x+1), \mathcal{T}_2(x+1), \dots, \mathcal{T}_{m+N_x^A}(x+1).$$

If $N_x > N_x^A$, we will have also

$$\mathcal{T}_{m+N_x^A+1}(x+1), \dots, \mathcal{T}_{m+N_x}(x+1).$$

Using the fact that $\mathcal{T}_1(x) \le \mathcal{T}_1^A(x)$ and the hypothesis $\mathcal{T}_n(x) \le \mathcal{T}_n^A(x)$ we have

$$\mathcal{T}_1^A(x) + \mathcal{E}_1^{x+1} \ge \mathcal{T}_1(x) + \mathcal{E}_1^{x+1}, \qquad \mathcal{T}_1^A(x) + \mathcal{E}_2^{x+1} \ge \mathcal{T}_1(x) + \mathcal{E}_2^{x+1}, \dots,$$
$$\mathcal{T}^A_{N_x^A}(x) + \mathcal{E}^{x+1}_{m+N_x^A} \ge \mathcal{T}_{N_x^A}(x) + \mathcal{E}^{x+1}_{m+N_x^A},$$

hence $\mathcal{T}_n^A(x+1) \ge \mathcal{T}_n(x+1)$, for $n \in \{1, \dots, m+N_x^A\}$. In the same way one obtains that $\mathcal{T}_n^A(u) \ge \mathcal{T}_n(u)$ for $x < u < y$ and $n \le N_u^A$. □

Lemma 2. *The sequence* $\{T(x,y) : x, y \in \mathbb{Z} \text{ and } 0 \le x < y\}$ *satisfies the hypothesis of Theorem 2.*

Proof. (i) In Lemma 1 let $n = 1$ and $u = y - 1$. Then, $T(x,y) = \mathcal{T}_1^A(y-1)$. Also $T(0,y) - T(0,x) = \mathcal{T}_1(y-1)$. Thus, $T(x,y) \ge T(0,y) - T(0,x)$, that is, $T(0,y) \le T(0,x) + T(x,y)$.

(ii) We set E_η and P_η as being the expectation and probability, respectively, for $T(0,x)$ with fixed initial configuration η. In other words, E_η and P_η are the expectation and probability conditioned on configuration η, the so-called quenched probability and expectation. Moreover, we set $\mathbb{E} = \mathbf{E}E_\eta$ and $\mathbb{P} = \mathbf{P}P_\eta$, that is, the annealed expectation and probability, respectively. Here, $\mathbf{E}$ and $\mathbf{P}$ are expectation and probability with respect to the initial configuration η.

Suppose that $0 \le x < y$. Then, for $m > 0$,

$$\begin{aligned}\mathbb{P}\big(T(x,x+1) > t\big) &= \sum_{k=m}^{\infty} \mathbf{P}\big(\eta(x) = k\big) P_\eta\big(T(x,x+1) > t \,|\, \eta(x) = k\big) \\ &= \sum_{k=0}^{\infty} \left(e^{-\lambda} \frac{\lambda^k}{k!}\right) e^{-kt} \\ &= \exp\{\lambda(e^{-t} - 1)\}.\end{aligned}$$

In the same way we conclude that

$$\mathbb{P}\big(T(y, y+1) > t\big) = \exp\{\lambda(e^{-t} - 1)\}.$$

Thus, we conclude that $T(x, x+1)$ and $T(y, y+1)$ have the same probability law with respect to measure $\mathbb{P}$. By the same kind of argument, $\{T(x, x+k) : k \geq 1\}$ and $\{T(x+1, x+k+1) : k \geq 1\}$ also have the same annealed law, that is, $\{T(x, x+k) : k \geq 1\}$ is stationary.

(iii) This is proved in the same way as in the item (ii). The ergodicity follows from the fact that the sequence $\{T(nk, (n+1)k : n \geq 1\}$, for k positive integer, is i.i.d.

(iv) The initial configuration at site zero is $\eta(0) + 1$, with $\eta(0) \sim Pois(\lambda)$. Since each infected particle jumps with rate 1, $T(0, 1) \sim Expon(\eta(0) + 1)$. Thus,

$$E_\eta T(0, 1) = \frac{1}{\eta(0) + 1}$$

and

$$\mathbb{E}T(0, 1) = \mathbf{E}E_\eta T(0, 1) = \mathbf{E}\Big(\frac{1}{\eta(0) + 1}\Big) = \sum_{j=0}^{\infty} \frac{1}{j+1} e^{-\lambda}\Big(\frac{\lambda^j}{j!}\Big)$$

$$= \frac{1 - e^{-\lambda}}{\lambda} < 1. \qquad \square$$

Lemma 3. *The sequence* $\{T(0, x) : x, y \in \mathbb{Z}$ *and* $0 \leq x < y\}$ *is such that*

$$0 < \inf_{x \geq 0} \frac{E(T(0, x))}{x} < 1.$$

Proof. Let us denote by N_t^r the total number of infected particles up to time t. Set $H(t) := EN_t^r$. As the initial boundary is in the origin,

$$H(0) = EN_0^r = E\big(1 + \eta_0(0)\big) = 1 + \lambda.$$

For a small h,

$$H(t + h) = E\Big(\sum_{i=1}^{N_t^r} V_i\Big), \tag{2.1}$$

where

$$V_i := \begin{cases} 1, & \text{if the } i\text{th particle at } Z_t \text{ does not jump in the interval } [0, h], \\ 1 + \lambda, & \text{otherwise.} \end{cases}$$

To explain the definition of V_i, in the case when the ith particle from Z_t jumps, the site $Z_t + 1$ will have, in average, $1 + \lambda$ particles (remember that before jumping, in average, $Z_t + 1$ was with λ normal particles). For small h, it is likely that on

interval $[h, t+h]$ only one particle will jump. Since each particle has a Poisson clock, the jump probability is proportional to $1 \times h + o(h)$. It follows that

$$EV_i = 1 \times (1-h) + (1+\lambda) \times h + \bar{o}(h) \approx 1 + \lambda \times h.$$

Let us come back to the equation (2.1). Since N_t^r is independent of $\{V_i\}_{i \geq 1}$, and the latter is an i.i.d. sequence of random variables, from Wald formula we obtain

$$H(t+h) = EN_t^r EV_1 = H(t)(1+h\lambda) = H(t) + \lambda h H(t).$$

Hence,

$$\begin{aligned} H'(t) &= \lim_{h \downarrow 0} \frac{H(t+h) - H(t)}{h} \\ &= \lim_{h \downarrow 0} \frac{H(t) + \lambda h H(t) - H(t)}{h} \\ &= \lambda H(t). \end{aligned} \tag{2.2}$$

With the initial condition $H(0) = 1 + \lambda$, the solution of this differential equation is $H(t) = (1+\lambda)e^{\lambda t}$. By Chebyshev inequality,

$$P\big(N_t^r > 2(1+\lambda)e^{\lambda t}\big) \leq \frac{H(t)}{2(1+\lambda)e^{\lambda t}} = \frac{1}{2},$$

therefore,

$$P\big(N_t^r \leq 2(1+\lambda)e^{\lambda t}\big) \geq \frac{1}{2}. \tag{2.3}$$

Now, we come back to the truncated model and define

$N_t :=$ "total number of infected particles in the original model at time t".

Thus,

$$P\big(N_t \leq 2(1+\lambda)e^{\lambda t}\big) \geq P\big(N_t^r \leq 2(1+\lambda)e^{\lambda t}\big),$$

hence, by (2.3)

$$P\big(N_t \leq 2(1+\lambda)e^{\lambda t}\big) \geq \frac{1}{2}. \tag{2.4}$$

Now, fix a particle and define

$Y_t :=$ "number of jumps that this particle performes up to time t".

Note that $Y_t \sim Pois(t)$. For $a, s > 0$, from Chernoff upper bound,

$$\begin{aligned} P(Y_t > at) &= P(e^{sY_t} > e^{sat}) \leq \frac{M_{Y_t}(s)}{e^{sat}} \\ &= \left(\frac{\exp\{e^s - 1\}}{\exp\{sa\}}\right)^t \\ &= \exp\{-t(sa - e^s + 1)\}. \end{aligned} \tag{2.5}$$

A direct computation shows that the minimum of $(-t(sa - e^s + 1))$ occurs for $s = \ln(a)$. Thus, from (2.5) we obtain

$$P(Y_t > at) \leq \exp\{-t(a\ln(a) - a + 1)\}. \tag{2.6}$$

Let $\{U_i\}_{i\geq 1}$ be i.i.d. random variables with distribution $Pois(t)$. The interpretation of U_i is the position that ith particle occupies, at time t. Also we set $W_t := \max\{U_i : i = 1, \ldots, N_t\}$. For $b > 0$, from the independence of $\{U_i\}_{i\geq 1}$, and using the fact that $\{U_i\}_{i\geq 1}$ are independent from N_t for all i,

$$\begin{aligned} P(W_t \leq bt | N_t \leq 2(1+\lambda)e^{\lambda t}) &= P\Big(\bigcap_{1\leq i\leq N_t} \{U_i \leq bt\} \big| N_t \leq 2(1+\lambda)e^{\lambda t}\Big) \\ &\geq \big(P(U_1 \leq bt)\big)^{2(1+\lambda)e^{\lambda t}}. \end{aligned}$$

By (2.6),

$$\begin{aligned} \big(P(U_1 \leq bt)\big)^{2(1+\lambda)e^{\lambda t}} &\geq \big(1 - \exp\{-t(b\ln(b) - b + 1)\}\big)^{2(1+\lambda)e^{\lambda t}} \\ &= \exp\{2(1+\lambda)\exp\{\lambda t\} \\ &\qquad \times \ln\big(1 - \exp\{-t(b\ln(b) - b + 1)\}\big)\}. \end{aligned} \tag{2.7}$$

Let us remember the fact $\ln(1-\alpha) \sim -\alpha$, for α small enough. Then,

$$\alpha = \exp\{-t(b\ln(b) - b + 1)\}$$

is small if b is large enough. Thus, for b large enough, (2.7) is close to

$$\exp\{-2(1+\lambda)\exp\{-t(b\ln(b) - b + 1)\}\} \geq c, \quad 0 < c < 1. \tag{2.8}$$

By (2.4) and (2.8),

$$P(W_t \leq bt) \geq P(W_t \leq bt | N_t \leq 2(1+\lambda)e^{\lambda t}) P(N_t \leq 2(1+\lambda)e^{\lambda t}) \geq \frac{c}{2}.$$

But this means that, with probability at least $\frac{c}{2}$, the process does not reach bt, having N_t infected particles at time t. Since the event $\{W_t \leq bt\}$ is equivalent to $\{T(0, bt) > t\}$, we have

$$P(T(0, bt) > t) \geq \frac{c}{2},$$

which implies that

$$E(T(0, bt)) \geq \frac{c}{2}t.$$

Hence, for t large enough,

$$\frac{E(T(0, bt))}{bt} \geq \frac{c}{2b},$$

and therefore, $\gamma > 0$.

In the following few lines we prove that $\gamma < 1$. By (i) and (ii) in Lemma 2, $\mathbb{E}T(0,x) \le \mathbb{E}T(0,1) + \mathbb{E}T(1,x)$ and $\mathbb{E}T(1,x) = \mathbb{E}T(0,x-1)$, therefore,

$$\begin{aligned}\mathbb{E}T(0,x) &\le \mathbb{E}T(0,1) + \mathbb{E}T(1,x)\\ &= \mathbb{E}T(0,1) + \mathbb{E}T(0,x-1)\\ &\le \mathbb{E}T(0,1) + \mathbb{E}T(0,1) + \mathbb{E}T(1,x-1)\\ &= 2\mathbb{E}T(0,1) + \mathbb{E}T(0,x-2)\\ &\vdots\\ &= x\mathbb{E}T(0,1), \end{aligned} \tag{2.9}$$

where the last equality was obtained by successive applications of Lemma 2. By (2.9),

$$\frac{\mathbb{E}T(0,x)}{x} \le \mathbb{E}T(0,1). \tag{2.10}$$

But, in the proof (iv) of Lemma 2 we saw that $\mathbb{E}T(0,1) = \frac{1-e^{-\lambda}}{\lambda}$. Since $\frac{1-e^{-\lambda}}{\lambda}$ is monotonously decreasing function and $\lim_{\lambda\downarrow 0}\frac{1-e^{-\lambda}}{\lambda} = 1$, $\mathbb{E}T(0,1) < 1$. Returning to (2.10), by Theorem 2,

$$\gamma = \inf_{x\ge 0}\frac{\mathbb{E}T(0,x)}{x} < 1,$$

hence $\gamma < 1$. □

3 Existence of a lower bound for the speed of infection spreading

Initially we establish a lower bound for the speed of infection spreading in the truncated model and then extend it to the general model.

Lemma 4. *The speed of infection spreading in truncated model is greater than* 1.

Proof. In Lemma 2 we saw that $\lim_{x\to\infty}\frac{T(0,x)}{x} = \gamma$, almost surely. This means that for any $\delta > 0$, there exists random x_0, such that for any $x \ge x_0$,

$$(\gamma - \delta)x < T(0,x) < (\gamma + \delta)x. \tag{3.1}$$

Fix $\varepsilon > 0$. It follows that there exists $t > 0$ such that $x = (\frac{1}{\gamma} + \varepsilon)t$. From (3.1),

$$T\left(0, \left(\frac{1}{\gamma} + \varepsilon\right)t\right) > (\gamma - \delta)\left(\frac{1}{\gamma} + \varepsilon\right)t.$$

Since $\delta > 0$ is arbitrary, taking it small enough,

$$(\gamma - \delta)\left(\frac{1}{\gamma} + \varepsilon\right) > t,$$

therefore, for t large enough,

$$T\left(0, \left(\frac{1}{\gamma} + \varepsilon\right)t\right) > t.$$

But, the event $\{T(0, y) > t\}$ implies that $\{Z_t < y\}$, therefore we conclude that $Z_t < (\frac{1}{\gamma} + \varepsilon)t$. In a similar way, $Z_t > (\frac{1}{\gamma} - \varepsilon)t$, hence,

$$\left(\frac{1}{\gamma} - \varepsilon\right)t < Z_t < \left(\frac{1}{\gamma} + \varepsilon\right)t.$$

Since ε is arbitrary, $\lim_{t\to\infty} \frac{Z_t}{t} = \frac{1}{\gamma} > 1$, hence in accordance with Lemma 3, $\gamma < 1$. □

Now, we show that the speed in the general model is almost surely equal to the speed in the truncated model. Empirically, although the general model have negative sites, and therefore more particles, the speed of any individual particle is equal to 1, and in the truncated model the speed of infection spreading is greater than 1. Thus, it is unlikely that particles from negative sites reach Z_t.

Proposition 1. *In the general model, with positive probability, particles from negative sites do not reach the current boundary.*

Proof. For the initial configuration,

$$\begin{aligned} &P_\eta(\text{particles from negative sites do not reach the current boundary}) \\ &\quad = \prod_{k<0} P_\eta(C_k), \end{aligned} \tag{3.2}$$

where $C_k :=$ "particles from site k do not reach the boundary". Note that the product in (3.2) appears because of independence between particles originating in different sites.

Remember that the initial configuration is such that $\eta(k) \sim Pois(\lambda)$, thus

$$P(\eta(k) \geq |k|) = \sum_{j\geq|k|} e^{-\lambda}\left(\frac{\lambda^j}{j!}\right) \leq e^{-\lambda}\frac{\lambda^{|k|}}{|k|!}\sum_{j\geq|k|}\frac{\lambda^{j-|k|}}{(j-|k|)!} = \frac{\lambda^{|k|}}{|k|!}$$

and

$$\sum_{|k|>0} P(\eta(k) \geq |k|) \leq \sum_{|k|\geq 0}\frac{\lambda^{|k|}}{|k|!} = e^\lambda < \infty.$$

From Borel–Cantelli lemma, the event $\{\eta(k) \geq |k|$, infinitely often$\}$ has probability zero. We set the event $A_k :=$ "a particle from site k do not meets the boundary ". The occurrence of the C_k implies that A_k occurs for each particle at k. Moreover, particles on the same site are independent, and also have equal probability for the event A_k, then

$$P_\eta(C_k) = (P_\eta(A_k))^{\eta(k)} \geq (P_\eta(A_k))^{|k|}, \tag{3.3}$$

since $\{\eta(k) < |k|\}$ with probability 1.

It follows from Lemma 2(iv), without loss of generality, that $\frac{Z_t}{t} > a > 1$, for any $t > 0$. Then,

$$A_k = \{n \leq |k| + at, \forall n \geq 1\},$$

where $t = \sum_{i=1}^n T_i$, $\{T_i\}_{i=1}^n$ i.i.d. random variables with law $Expon(1)$. Now, set

$$A_k^n := \left\{ \frac{n - |k|}{a} < \sum_{i=1}^n T_i \right\}.$$

Therefore, $A_k = \bigcap_{n\geq 1} A_k^n$. For $s > 0$, from Chernoff upper bound,

$$\begin{aligned} P_\eta((A_k^n)^c) &= P_\eta\left(\sum_{i=1}^n T_i \leq \frac{n - |k|}{a}\right) \\ &= P_\eta\left(\exp\left\{-s\sum_{i=1}^n T_i\right\} \geq \exp\left\{-s\left(\frac{n-|k|}{a}\right)\right\}\right) \\ &\leq \frac{(E(\exp\{-sT_1\}))^n}{\exp\{-s(n-|k|)/a\}} \\ &= \exp\left\{-\left(\frac{s}{a}\right)|k|\right\}\left(\exp\left\{-\ln(1+s) + \frac{s}{a}\right\}\right)^n. \end{aligned} \tag{3.4}$$

Minimizing $(-\ln(1+s) + \frac{s}{a})$ with respect to s in (3.4), we obtain $s = a - 1 > 0$. Returning to (3.4),

$$P_\eta((A_k^n)^c) \leq \exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}\left(\exp\left\{-\ln a + \left(\frac{a-1}{a}\right)\right\}\right)^n. \tag{3.5}$$

By (3.5),

$$\begin{aligned} P_\eta(A_k^c) &\leq \sum_{n=1}^\infty P((A_k^n)^c) = \sum_{n=1}^\infty \exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}\left(\exp\left\{-\ln a + \left(\frac{a-1}{a}\right)\right\}\right)^n \\ &= \exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}\alpha, \end{aligned}$$

therefore

$$P_\eta(A_k) \geq 1 - \exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}\alpha. \tag{3.6}$$

By (3.2), (3.3) and (3.6)

$$P_\eta(\text{particles from negative sites do not reach the current boundary}) \geq \prod_{k<0}\left(1-\exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}\alpha\right)^{|k|}. \tag{3.7}$$

We claim that the product in (3.7) is positive. In fact, from integral criterion,

$$\sum_{|k|\geq 0}\alpha|k|\exp\left\{-\left(\frac{a-1}{a}\right)|k|\right\}<\infty,$$

allowing us to conclude that

$$P_\eta(\text{particles from negative sites do not reach the current boundary})>0,$$

and therefore the proposition is proved. □

Lemma 5. *Let v_g be the speed of infection spreading in the general model. Then, $v = v_g$ almost surely.*

Proof. It is trivial that $v_g \geq v$. To show that $v_g \leq v$ we introduce couples of the general model, that is, processes $\{Y_t^{(m)} : t \geq 0, m = 1, 2, \ldots\}$ with the same configuration that general model, but with the difference that particles until site m are infected and above m are healthy. For each fixed m we set

$$B_m := \text{“no particle starting below } m \text{ meets the boundary”}.$$

The respective speed of infection is denoted by $v_g^{(m)}$. Note that $v_g^{(m)} \geq v_g^{(0)} = v_g$. Now, according to Proposition 1,

$$E\big(I_{\{B_m\}}\big) = E\big(I_{\{B_0\}}\big) = P(B_0) > 0,$$

being $I_{\{\cdot\}}$ indicator function. Applying the Birkhoff ergodic theorem,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m} I_{\{B_j\}} = P(B_0) \qquad \text{almost surely},$$

implying the existence of m_0 such that B_{m_0} occurs, so $v_g^{(m_0)} = v$. Thus,

$$v = v_g^{(m_0)} \geq v_g^{(0)} = v_g,$$

and we complete the proof of the fact that $v = v_g$, therefore the proof of the Theorem 1. □

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