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An expansion for the maximum likelihood estimator of location and its distribution function

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Abstract. In this paper stochastic expansions of the maximum likelihood estimator for location and for the asymptotic expansion of the distribution function of this estimator are derived. A Cornish–Fisher expansion is also given for the quantile function of the distribution function. All expansions are given in explicit expressions.

1 Introduction

In many papers we may find results about the asymptotic expansion of the maximum likelihood estimator (MLE) and its distribution function. We mention Pfanzagl (1973), who considers asymptotic expansions related to minimum contrast estimators. Pfanzagl (1979) discovered the phenomenon "first-order efficiency implies second-order efficiency." Klaassen and Venetiaan (1994) have given an alternative proof for this phenomenon. Furthermore, we mention Bickel and van Zwet (1978), Pfanzagl and Wefelmeyer (1985) and Bickel, Götze and van Zwet (1985). Chibisov (1973) considers expansions for minimum contrast estimators, which include the maximum likelihood estimator. We will be using his approach. Hall (1992) presents expansions for estimators admitting a stochastic expansion and relates these to the bootstrap. Akahira (1996) proved a conjecture of Ghosh (1994), namely that "third-order efficiency implies fourth-order efficiency." He does so by considering estimators with the most concentration probability. Kano (1998) also considers the concentration probability of estimators and shows that proper bias adjustment of the estimators leads to "third-order efficiency implies fourth-order efficiency" a necessary and sufficient condition for fifth-order efficiency and a proof that the MLE is fifth-order efficient.

In this paper, the author explores the location model and gives explicit expressions for the expansions. Akahira (1996) gives explicit expressions as well, but in the $1/n^{(3/2)}$ term of his expansion, he leaves some parameters for the reader to calculate.

In this paper, the reader will also find an expansion for the quantiles of the distribution function of the MLE for location. These expansions are particularly useful when one is interested in optimizing the length of a confidence interval.

Key words and phrases. Asymptotic expansion, maximum likelihood estimator, Cornish–Fisher expansions.

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Now we introduce the theoretical framework of this paper. Consider estimation of the location parameter θ in the one-dimensional location model of i.i.d. random variables $X_1, \dots, X_n$. Assume that the common distribution of the random variables has finite Fisher information for location. This means that this distribution is absolutely continuous with an absolutely continuous density $f(\cdot - \theta)$ with derivative $f'(\cdot - \theta)$ such that

$$I(f) = \int \left(\frac{f'}{f}\right)^2 f < \infty$$

holds. We have a regular parametric model then, and asymptotically efficient estimation is possible. Let $\hat{\theta}_n$ denote the maximum likelihood estimator for location (MLE) and

$$G_n(y) = P_{f(\cdot - \theta)}\left(\sqrt{nI(f)}(\hat{\theta}_n - \theta) \le y\right), \qquad y \in \mathbf{R}. \tag{1.1}$$

Asymptotic efficiency of $\hat{\theta}_n$ means that G_n converges weakly to the standard normal distribution function Φ as $n \to \infty$ or equivalently

$$G_n^{-1}(u) \to \Phi^{-1}(u), \qquad 0 < u < 1. \tag{1.2}$$

In Section 2 we will present an stochastic expansion for the MLE for location, by using and adjusting the results of Chibisov (1973) about expansions for minimum contrast estimators. Furthermore, we will note that our approach fits in the theoretical framework of Hall (1992) and so we arrive at an asymptotic expansion for the distribution function of the MLE. The last result is a Cornish–Fisher expansion for the quantiles of the distribution function of the MLE. Proofs of the above-mentioned results may be found in Section 3.

1.1 Notation

We will be using the following notation:

$$\begin{aligned}
&\eta_2 = E\psi_2^2(X_1)/I^2(f), \qquad \eta_3 = E\psi_1^3(X_1)/I^{3/2}(f),\\
&\eta_4 = E\psi_1^4(X_1)/I^2(f), \qquad \eta_5 = E\psi_1^5(X_1)/I^{(5/2)}(f),\\
&\eta_6 = E(\psi_2(X_1)\psi_3(X_1))/I^{(5/2)}(f)\\
&\qquad \text{with } \psi_i(x) = \frac{f^{(i)}}{f}(x).
\end{aligned}$$

2 Maximum likelihood estimator and expansions

In this section we will state a result about the maximum likelihood estimator, by viewing it as a special case of a minimum contrast estimator.

Definition. We say $\hat{\theta}_n$ is a minimum contrast estimator if $\hat{\theta}_n$ satisfies

$$L_n(\hat{\theta}_n) = \inf_{\theta \in \mathbf{R}} L_n(\theta), \tag{2.1}$$

where $L_n(\theta) = n^{-1} \sum_{i=1}^{n} \rho(X_i - \theta)$, with $\rho : \mathbf{R} \to \mathbf{R}$ measurable. The MLE is obtained when we take $\rho(\cdot) = -\log f(\cdot)$.

Theorem 1. *Let $X, X_1, \ldots, X_n$ be i.i.d. with common density $f(\cdot - \theta_0)$. Let $\hat{\theta}_n$ be the MLE defined by (2.1) with $\rho(\cdot) = -\log f(\cdot)$ and let $\rho(\cdot)$ satisfy the following conditions.*

(1) *For all $K \subset \mathbf{R}$ compact, $\sup_{\theta \in K} E_\theta \rho^2(X) = A < \infty$.*
(2) *$\rho(\cdot)$ is five times differentiable.*
(3) *There exists a finite function $R(\cdot)$ and a $\delta > 0$ such that for every $y \in \mathbf{R}$, $|\theta| < \delta$:*

$$\left|\rho^{(5)}(y) - \rho^{(5)}(y - \theta)\right| \leq R(y)|\theta| \quad \textit{and} \quad E_0 R^{5/2}(X) < \infty.$$

(4) *$E_0|\rho^{(\alpha)}(X)|^5 < \infty$ for $\alpha = 1, \ldots, 5$.*

Then $\hat{\theta}_n$ admits a stochastic expansion

$$\begin{aligned}
\sqrt{n}(\hat{\theta}_n - \theta_0) = {} & \frac{\xi_1}{a_2} + \frac{1}{\sqrt{n}}\left(\frac{-\xi_1\xi_2}{a_2^2} + \frac{a_3\xi_1^2}{2a_2^3}\right) \\
& + \frac{1}{n}\left(\frac{\xi_1\xi_2^2}{a_2^3} - \frac{3a_3\xi_1^2\xi_2}{2a_2^4} + \frac{\xi_1^2\xi_3}{2a_2^3} + \frac{a_3^2\xi_1^3}{2a_2^5} - \frac{a_4\xi_1^3}{6a_2^4}\right) \\
& + \frac{1}{n^{3/2}}\left(\frac{3\xi_1^2 a_3 \xi_2^2}{a_2^5} + \frac{5\xi_1^4 a_3^3}{8a_2^7} - \frac{5\xi_1^4 a_4 a_3}{12a_2^6} - \frac{3\xi_1^2\xi_3\xi_2}{2a_2^4}\right. \\
& \quad - \frac{5\xi_1^3 a_3^2 \xi_2}{2a_2^6} + \frac{\xi_1^4 a_5}{24a_2^5} + \frac{\xi_1^3\xi_3 a_3}{a_2^5} + \frac{2\xi_1^3 a_4 \xi_2}{3a_2^5} \\
& \quad \left. - \frac{\xi_1^3\xi_4}{6a_2^4} - \frac{\xi_1\xi_2^3}{a_2^4}\right) + \gamma_n,
\end{aligned} \tag{2.2}$$

where ξ_j denotes the normalized sum of the independent random variables $\rho^{(j)}(X_i)$.

$$\xi_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\rho^{(j)}(X_i) - a_j), \qquad a_j = E_0 \rho^{(j)}(X) \textit{ for } j = 1, \ldots, 5.$$

Furthermore, for any sequence of positive constants $\{\varepsilon_n\}$, with

$$\varepsilon_n \sqrt{n} (\log n)^{-2} \to \infty,$$

we have

$$P_0\left(|\gamma_n| \geq \frac{\varepsilon_n}{n^{3/2}}\right) = o\left(\frac{1}{n^{3/2}}\right). \tag{2.3}$$

We now present the expansion for the distribution function.

Theorem 2. *Let the conditions of Theorem 1 hold, then the distribution function of the MLE admits an Edgeworth expansion to order $n^{-3/2}$, namely*

$$\begin{aligned}
G_n(x) = \Phi(x) - \phi(x)\Bigg\{ & \frac{\eta_3(x^2+2)}{12\sqrt{n}} \\
& + \frac{1}{n}\Bigg(\frac{\eta_3^2}{288}x^5 - \Big(\frac{1}{8} - \frac{\eta_2}{6} + \frac{5\eta_4}{72} + \frac{\eta_3^2}{72}\Big)x^3 \\
& \quad + \Big(\frac{\eta_4}{24} - \frac{\eta 3^2}{24} - \frac{1}{8}\Big)x\Bigg) \\
& + \frac{1}{n^{3/2}}\Bigg(\frac{\eta_3^3}{10368}x^8 - \Big(\frac{\eta_3}{96} - \frac{\eta_2\eta_3}{72} + \frac{19\eta_3^3}{10368} + \frac{5\eta_3\eta_4}{864}\Big)x^6 \\
& \quad + \Big(\frac{\eta_4\eta_3}{72} + \frac{\eta_5}{30} - \frac{19\eta_3^3}{1728} - \frac{\eta_6}{8}\Big)x^4 \\
& \quad - \Big(\frac{35\eta_3^3}{864} + \frac{\eta_3}{32} + \frac{\eta_5}{80} - \frac{5\eta_4\eta_3}{96}\Big)x^2 \\
& \quad + \frac{5\eta_4\eta_3}{48} - \frac{35\eta_3^3}{432} - \frac{\eta_3}{16} - \frac{\eta_5}{40}\Bigg)\Bigg\} \\
& + o\Big(\frac{1}{n^{3/2}}\Big).
\end{aligned}$$

The next result is about a Cornish–Fisher expansion for the inverse of the distribution function of the MLE.

Theorem 3. *If the conditions of Theorem 1 are fulfilled, then the inverse of the distribution function of the MLE admits a Cornish–Fisher expansion*

$$\begin{aligned}
G_n^{-1}(v) = \Phi^{-1}(v) & + \frac{\eta_3}{12\sqrt{n}}((\Phi^{-1}(v))^2 + 2) \\
& + \frac{1}{n}\Bigg(\Big(-\frac{5\eta_4}{72} - \frac{1}{8} + \frac{\eta_2}{6} - \frac{\eta_3^2}{72}\Big)(\Phi^{-1}(v))^3 \\
& \quad + \Big(-\frac{\eta_3^2}{36} - \frac{1}{8} + \frac{\eta_4}{24}\Big)\Phi^{-1}(v)\Bigg) \\
& + \frac{1}{n^{3/2}}\Bigg(\Big(\frac{\eta_2\eta_3}{24} - \frac{\eta_4\eta_3}{144} - \frac{\eta_3}{48} - \frac{\eta_6}{8} + \frac{\eta_5}{30} - \frac{19\eta_3^3}{1728}\Big)(\Phi^{-1}(v))^4 \qquad (2.4) \\
& \quad + \Big(\frac{\eta_4\eta_3}{48} - \frac{67\eta_3^3}{1296} - \frac{5\eta_3}{48} + \frac{\eta_2\eta_3}{12} - \frac{\eta_5}{80}\Big)(\Phi^{-1}(v))^2
\end{aligned}$$

$$-\frac{\eta_5}{40}+\frac{\eta_3\eta_4}{9}-\frac{113\eta_3^3}{1296}-\frac{\eta_3}{12}\Big)$$

$$+o\left(\frac{1}{n^{3/2}}\right).$$

3 Proofs

3.1 Proof of Theorem 1

The proof presented here is a reorganization of the proof of Theorem 5 of Chibisov (1973). First we prove that $\hat{\theta}_n$ is a solution of the equation

$$L_n'(\theta)=0, \tag{3.1}$$

with probability $1+o(n^{-1})$. Because $\rho(\cdot)$ is differentiable, the infimum of $L_n(\theta)$ is attained either at a stationary point or at $\pm\infty$. We will show that the probability that the infimum is taken at the boundary is sufficiently small. To do so we will apply Lemma 4 of Michel and Pfanzagl (1971). With regular techniques it may be shown that their conditions are satisfied, among others by using the L_1 continuity theorem and by using the remark Lemma 2 of Pfanzagl (1973) to obtain a similar but stronger result than the Chebyshev inequality. With the above we come to the following consistency result. For all $\delta>0$,

$$P(|\hat{\theta}_n-\theta_0|>\delta)=o\left(\frac{1}{n}\right). \tag{3.2}$$

Consequently, with probability $1+o(1/n)$ the infimum is attained at $(-\delta,\delta)$ and hence at a stationary point; in other words $\hat{\theta}_n$ satisfies (3.1).

Assume without loss of generality that $\theta_0=0$. We expand $L_n'(\theta)$ by using condition (2).

$$\begin{aligned}L_n'(\theta)&=\frac{1}{n}\sum_{i=1}^n\rho'(X_i-\theta)\\&=\frac{1}{n}\sum_{i=1}^n\rho'(X_i)-\frac{\theta}{n}\sum_{i=1}^n\rho''(X_i)+\cdots+\frac{\theta^4}{24n}\sum_{i=1}^n\rho^{(5)}(X_i-\theta'),\end{aligned} \tag{3.3}$$

where $|\theta'|\le|\theta|$. We write (3.3) as

$$\begin{aligned}L_n'(\theta)=\frac{\xi_{1n}}{\sqrt{n}}-\theta\left(\frac{\xi_{2n}}{\sqrt{n}}+a_2\right)+\frac{\theta^2}{2}\left(\frac{\xi_{3n}}{\sqrt{n}}+a_3\right)\\-\frac{\theta^3}{6}\left(\frac{\xi_{4n}}{\sqrt{n}}+a_4\right)+\frac{\theta^4}{24}\left(\frac{\xi_{5n}}{\sqrt{n}}+a_5\right)+\lambda_n(\theta),\end{aligned}$$

where $\lambda_n(\theta) = \frac{\theta^4}{24n}\sum_{i=1}^n[\rho^{(5)}(X_i - \theta') - \rho^{(5)}(X_i)]$. Since $\hat{\theta}_n$ solves $L_n'(\theta) = 0$ an approximation $\theta_n^{(4)}$ of $\hat{\theta}_n$ may be obtained by solving $L_n'(\theta) - \lambda_n(\theta) = 0$ and writing the solution as $\theta_n^{(4)} + O_p(n^{-5/2})$ where $\sqrt{n}\theta_n^{(4)}$ equals the right-hand side of (2.2) minus γ_n. In Venetiaan (2007) a thorough description is given of the way the expansions were obtained.

Note that

$$L_n'(\hat{\theta}_n) - L_n(\theta_n^{(4)}) = -(\hat{\theta}_n - \theta_n^{(4)})\left(\frac{\xi_{2n}}{\sqrt{n}} + a_2\right) + \cdots + \frac{[\hat{\theta}_n^4 - (\theta_n^{(4)})^4]}{24}\left(\frac{\xi_{5n}}{\sqrt{n}} + a_5\right) + \lambda_n(\hat{\theta}_n) - \lambda_n(\theta_n^{(4)}).$$

From this equation and noting that $L_n'(\hat{\theta}_n) = 0$, we obtain the following inequality

$$\begin{aligned} |\hat{\theta}_n - \theta_n^{(4)}|\left|\frac{\xi_{2n}}{\sqrt{n}} + a_2\right| \le{}& |L_n'(\theta_n^{(4)}) - \lambda_n(\theta_n^{(4)})| \\ &+ \left|\frac{\hat{\theta}_n^2 - (\theta_n^{(4)})^2}{2}\right|\left|\frac{\xi_{3n}}{\sqrt{n}} + a_3\right| \\ &+ \left|\frac{\hat{\theta}_n^3 - (\theta_n^{(4)})^3}{6}\right|\left|\frac{\xi_{4n}}{\sqrt{n}} + a_4\right| \\ &+ \left|\frac{\hat{\theta}_n^4 - (\theta_n^{(4)})^4}{24}\right|\left|\frac{\xi_{5n}}{\sqrt{n}} + a_5\right| + |\lambda_n(\hat{\theta}_n)|. \end{aligned} \tag{3.4}$$

For C sufficiently large, we have

$$\begin{aligned} P\left(\left|\frac{\xi_{jn}}{\sqrt{\log n}}\right| > C\right) &= P\left(\frac{1}{\sqrt{n}}\left|\sum_{i=1}^n(\rho^{(j)}(X_i) - a_j)\right| > C\sqrt{\log n}\right) \\ &= o\left(\frac{1}{n^{3/2}}\right), \qquad j = 1, \ldots, 5. \end{aligned} \tag{3.5}$$

Here we used condition (4) and we applied Theorem 1 of Nagaev (1965) and the proof of Theorem 3.2 of Chibisov (1972) with minor adjustments. With (3.5) we see that $P(|\theta_n^{(4)}| > \delta) = o(1/n^{3/2})$. Together with (3.2) it follows that $|\hat{\theta}_n^2 - (\theta_n^{(4)})^2|$, $|\hat{\theta}_n^3 - (\theta_n^{(4)})^3|$ and $|\hat{\theta}_n^4 - (\theta_n^{(4)})^4|$ are of smaller order than $|\hat{\theta}_n - \theta_n^{(4)}|$. This means, again in view of (3.5) and because the conditions imply $a_2 \neq 0$, that the second, third and fourth term at the right-hand side of (3.4) are of smaller order than the left-hand side of (3.4). Consequently, since $\gamma_n = \sqrt{n}(\hat{\theta}_n - \theta_n^{(4)})$ and since a_2 does not vanish, claim (2.3) and hence the theorem have been proved once it has been shown that

$$P\left(|L_n'(\theta_n^{(4)}) - \lambda_n(\theta_n^{(4)})| > \frac{\varepsilon_n}{n^2}\right) = o\left(\frac{1}{n^{3/2}}\right), \tag{3.6}$$

and

$$P\left(|\lambda_n(\hat{\theta}_n)| > \frac{\varepsilon_n}{n^2}\right) = o\left(\frac{1}{n^{3/2}}\right). \tag{3.7}$$

Note that $L_n'(\theta_n^{(4)}) - \lambda_n(\theta_n^{(4)}) = -(\theta_n^{(4)})^4 \xi_{5n}/24\sqrt{n} + R_n$, with

$$R_n = \sum_{l=5}^{16} n^{-l/2} \sum_{0 \le i,j,k,m < l, i+j+k+m=l} c_{i,j,k,m} \xi_{1n}^i \xi_{2n}^j \xi_{3n}^k \xi_{4n}^m$$

for some constants $c_{i,j,k,m}$. (3.6) holds if

$$P\left(|\theta_n^{(4)}|^4 \frac{\xi_{5n}}{24\sqrt{n}} > \frac{\varepsilon_n}{n^2}\right) + P\left(|R_n| > \frac{\varepsilon_n}{n^2}\right) = o\left(\frac{1}{n^{3/2}}\right). \tag{3.8}$$

With (3.5) and the conditions on ε_n, we see that the first term at the left-hand side of (3.8) is bounded by

$$\begin{aligned} P\left(|\theta_n^{(4)}|^4 \frac{\xi_{5n}}{24\sqrt{n}} > \frac{\varepsilon_n}{n^2}\right) &\le P\left(|\theta_n^{(4)}|^4 > \left(\frac{\varepsilon_n}{n^2}\right)^{4/5}\right) \\ &\quad + P\left(\left|\frac{\xi_{5n}}{24\sqrt{n}}\right| > \left(\frac{\varepsilon_n}{n^2}\right)^{1/5}\right) \\ &\le P(|\xi_{1n}| > D_1 \varepsilon_n^{1/5} n^{1/10}) \\ &\quad + P(|\xi_{5n}| > D_2 \varepsilon_n^{1/5} n^{1/10}) \\ &= o\left(\frac{1}{n^{3/2}}\right), \end{aligned} \tag{3.9}$$

where D_1 and D_2 are certain constants. For the second term at the left-hand side of (3.8) we note, with the help of arguments as in (3.9), that it is of the order

$$\begin{aligned} O\left(\sum_{l=5}^{16} \sum_{j=1}^{4} P(|\xi_{jn}| > \varepsilon_n^{1/l} n^{1/2-2/l})\right) &= O\left(\sum_{j=1}^{4} P(|\xi_{jn}| > \varepsilon^{1/5} n^{1/10})\right) \\ &= o\left(\frac{1}{n^{3/2}}\right). \end{aligned}$$

We have proved (3.6). With the use of condition (3) we bound $|\lambda_n(\hat{\theta}_n)|$ which equals $\frac{|\hat{\theta}_n|^4}{24n} |\sum_{i=1}^n [\rho^{(5)}(X_i - \hat{\theta}_n') - \rho^{(5)}(X_i)]|$, by

$$\frac{|\hat{\theta}_n|^4}{24n} |\hat{\theta}_n'| \left|\sum_{i=1}^n R(X_i)\right| \le \frac{|\hat{\theta}_n|^5}{24n} \left|\sum_{i=1}^n R(X_i)\right|.$$

Note that because $E_0 R^{5/2}(X) < \infty$, it may be shown that

$$P\left(\frac{1}{n}\left|\sum_{i=1}^n R(X_i)\right| > C\right) = o\left(\frac{1}{n^{3/2}}\right).$$

Then

$$P\big(|\lambda_n(\hat{\theta}_n)| \le c_0\hat{\theta}_n^5\big) = 1 + o\Big(\frac{1}{n}\Big).$$

We note that

$$\hat{\theta}_n^5 \le 31\big(\hat{\theta}_n - \theta_n^{(4)}\big)^5 + 31\big(\theta_n^{(4)}\big)^5. \tag{3.10}$$

With arguments similar to those immediately after (3.5) we find that the first term at the right-hand side of (3.10) is of smaller order than $|\hat{\theta}_n - \theta_n^{(4)}|$. This means that we may focus on

$$P\bigg(\big(\theta_n^{(4)}\big)^5 > \frac{c_1\varepsilon_n}{n^2}\bigg) = O\big(P(|\xi_{1n}| > \varepsilon_n^{1/5}n^{-1/10})\big) = o\bigg(\frac{1}{n^{3/2}}\bigg),$$

which follows from (3.9). This proves (3.7) and hence the theorem (end of proof of Theorem 1).

3.2 Proof of Theorem 2

Let S_n be $\sqrt{n}\hat{\theta}_n^{(4)}$, then

$$\sqrt{n}(\theta_n - \theta_0) = S_n + \gamma_n. \tag{3.11}$$

Because the sub terms introduced in (2.2) are polynomials in normalized sums, S_n fits in the model described by Hall (1992), Section 2.4. With his Theorem 2.2 we see that the distribution of S_n admits an Edgeworth expansion. We take $\mathbf{Y}_i = (\rho'(X_i), \rho''(X_i) - a_2, \rho^{(3)}(X_i) - a_3, \rho^{(4)}(X_i) - a_4)^T$ and put $\bar{\mathbf{Y}} = \frac{1}{n}\sum \mathbf{Y}_i$. Then $S_n = \sqrt{n}A(\bar{\mathbf{Y}})$ with $A(y_1, y_2, y_3, y_4) = g_1(y_1) + g_2(y_1, y_2) + g_3(y_1, y_2, y_3) + g_4(y_1, y_2, y_3, y_4)$; cf. (2.2). Let

$$\sigma = \{E(\rho'(X_1))^2\}^{1/2}(E\rho''(X_1))^{-1}.$$

Note that $\sigma = (I(f))^{-1/2}$ with our choice for $\rho(\cdot)$. Also put $\|t\| = [\sum_{j=1}^4 (t^{(j)})^2]^{1/2}$ and let $\chi(t)$ denote the characteristic function of $\mathbf{Y}_1$. In view of (3.11) we have

$$\begin{aligned}
&\sup_x\bigg|P\bigg(\frac{\sqrt{n}(\hat{\theta}_n - \theta_0)}{\sigma} \le x\bigg) - P\bigg(\frac{S_n}{\sigma} \le x\bigg)\bigg| \\
&\quad \le P\bigg(|\gamma_n| \ge \frac{\sigma\varepsilon_n}{n^{3/2}}\bigg) + \sup_x\bigg|P\bigg(\frac{S_n}{\sigma} \le x\bigg) - P\bigg(\frac{S_n}{\sigma} \le x - \frac{\varepsilon_n}{n^{3/2}}\bigg)\bigg| \qquad (3.12)\\
&\qquad + \sup_x\bigg|P\bigg(\frac{S_n}{\sigma} \le x\bigg) - P\bigg(\frac{S_n}{\sigma} \le x + \frac{\varepsilon_n}{n^{3/2}}\bigg)\bigg|.
\end{aligned}$$

It may be shown with techniques adapted from Chibisov and van Zwet (1984a) that

$$\limsup_{\|t\|\to\infty} |\chi(t)| < 1. \tag{3.13}$$

Note that the conditions of Theorem 2.2 with $j=3$ of Hall (1992), page 56, are fulfilled for S_n. So, it suffices to prove that the right-hand side of (3.12) is of the order $o(1/n^{3/2})$ for an appropriate choice of $\{\varepsilon_n\}$. Indeed, with $\{\varepsilon_n\}$ as in (2.3) the first term at the right-hand side of (3.12) is $o(1/n^{3/2})$. Taking $\varepsilon_n \downarrow 0$ and using the existence of an Edgeworth expansion for S_n, uniformly in x, we see that the last two terms of (3.12) are of the order $o(1/n^{3/2})$ too. The fact that S_n fits Hall's (1992) model for a stochastic expansion ensures that the cumulants of S_n will determine the polynomials in the expansion for the distribution function G_n. In Venetiaan (2007) one may find the calculations of the cumulants and verify that indeed they are of the structure indicated by Hall (1992). In that paper it is also described how the polynomials are obtained from the cumulants, and one will find that these equal the polynomials in (2.4). The proof is complete (end of proof of Theorem 2).

3.3 Proof of Theorem 3

Theorem 3 follows in a straightforward way from Theorem 2. Directions for the computations may be found in Venetiaan (2007).

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