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Jump processes, $\mathscr{L}$-harmonic functions, continuity estimates and the Feller property

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Abstract. Given a family of Lévy measures $\nu = \{\nu(x, \cdot)\}_{x\in\mathbb{R}^d}$, the present work deals with the regularity of harmonic functions and the Feller property of corresponding jump processes. The main aim is to establish continuity estimates for harmonic functions under weak assumptions on the family ν. Different from previous contributions the method covers cases where lower bounds on the probability of hitting small sets degenerate.

Résumé. Soit $\nu = \{\nu(x, \cdot)\}_{x\in\mathbb{R}^d}$ une famille de mesures de Lévy, ce travail étudie la régularité de fonctions harmoniques et la propriété de Feller du processus de saut correspondant. Le but principal est d'établir des estimations de continuité pour les fonctions harmoniques sous des conditions faibles sur la famille ν. À la différence des contributions précédentes cette méthode couvre des cas où les bornes inférieures de la probabilité d'atteindre de petits ensembles dégénèrent.

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1. Introduction

Regularity of solutions to differential equations is closely related to qualitative properties of the corresponding Markov process. A good example is the modern theory of fully nonlinear partial differential equations of second-order which came to real life after Hölder a priori estimates for solutions to elliptic and parabolic second-order equations with irregular coefficients were established [21]. The derivation of these a priori estimates was first based on hitting time estimates for diffusion processes.

In the last years these regularity results which are by now classical for local diffusion operators have been investigated for nonlocal operators and related jump processes. In this article we discuss continuity a priori estimates for functions which are harmonic with respect to nonlocal integro-differential operators, respectively Markov jump processes. In comparison with existing results on Hölder a priori estimates we need to impose only weak conditions on the jump kernels.

The main tool used in previous proofs of Hölder regularity for functions harmonic with respect to Markov processes is to show that for all $r < 1/2$, $A \subset B(x_0, r)$ satisfying $|A| \geq \frac{1}{2}|B(x_0, r)|$ and for all $y \in B(x_0, \frac{r}{2})$,

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \geq c > 0. \tag{1.1}$$

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Here, T and τ denote entry and exit times, respectively, and $|A|$ the Lebesgue measure of the measurable set A. Estimate (1.1) is at the heart of [21] and is basically a probabilistic reformulation of what is known as growth lemmas, see [22]. In this work our main goal is to extend [3] and to prove a priori continuity estimates in situations where (1.1) fails, see Theorems 1.1 and 1.2.

With the help of Theorem 1.2 we are able to establish the Feller property for a certain class of Markov processes, see Theorem 1.3. It is interesting to compare this result to Theorem 1.9 of [1] where it is shown that the martingale problem may fail under slightly weaker conditions. One aim of the present work is to shed more light into this area of research.

Let us be more precise and present our results. Let $\nu = \{\nu(x,\cdot)\}_{x\in\mathbb{R}^d}$ be a family of Lévy measures satisfying

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\min\bigl(|h|^2,1\bigr)\nu(x,\mathrm{d}h)<\infty.$$

For $u\in C_b^2(\mathbb{R}^d)$ set

$$\mathscr{L}u(x)=\int_{\mathbb{R}^d\setminus\{0\}}\bigl(u(x+h)-u(x)-\mathbb{1}_{\{|h|<1\}}\langle h,\nabla u(x)\rangle\bigr)\nu(x,\mathrm{d}h). \tag{1.2}$$

Fix some $\delta<\frac{1}{2}$ and define

$$S(x,r)=\int_{|h|\ge r}\nu(x,\mathrm{d}h),$$

$$L(x,r)=S(x,r)+\frac{1}{r}\left|\int_{1\ge|h|\ge r}h\nu(x,\mathrm{d}h)\right|+\frac{1}{r^2}\int_{|h|<r}|h|^2\nu(x,\mathrm{d}h),$$

$$N(x,r)=\inf\bigl\{\nu(x,M):\ M\subset B(0,2r),\,|M|\ge\delta\bigl|B(0,r)\bigr|\bigr\}.$$

We will need the following assumptions:

(A) There is a strong Markov process $(X_t,\mathbb{P}^x)$ having right continuous paths with left limits such that $u(X_t)-u(x)-\int_0^t\mathscr{L}u(X_s)\,\mathrm{d}s$ is a $\mathbb{P}^x$-martingale for all $u\in C_b^2(\mathbb{R}^d)$ and any $x\in\mathbb{R}^d$.

(B1) $\sup_{x\in\mathbb{R}^d}L(x,1)<\infty$.

(B2) There exist $\kappa_1>0$ and $\sigma>0$ such that for all $x\in\mathbb{R}^d$, $r\in(0,1/2)$, $1<\lambda<\frac{1}{r}$,

$$S(x,\lambda r)\le\kappa_1\lambda^{-\sigma}S(x,r).$$

(B3) There exists $\kappa_2>0$ such that for all $x,y\in\mathbb{R}^d$, $r\in(0,1/2)$, $|x-y|<2r$,

$$N(x,r)\ge\frac{\kappa_2}{|\ln r|}L(y,r/2).$$

Assumptions (A), (B1) and (B2) are mild and also appear in [3]. Our central assumption is (B3) which differs significantly from Assumption 2.1(b) in [3]. It allows for a certain degeneracy which we focus on in the present work. At the end of this section we discuss some examples where (B1) through (B3) are satisfied.

Given an integro-differential operator $\mathscr{L}$ of type (1.2) we call functions $u:\mathbb{R}^d\to\mathbb{R}$ harmonic with respect to $\mathscr{L}$ or simply $\mathscr{L}$-harmonic in an open set $D\subset\mathbb{R}^d$ if for any open set $D'\Subset D$ the process $u(X_{s\wedge\tau_{D'}})$ is a $\mathbb{P}^x$-martingale. This definition of harmonicity ensures that functions $u\in C_b^2(\mathbb{R}^d)$ satisfying $\mathscr{L}u(x)=0$ for $x\in D$ are indeed $\mathscr{L}$-harmonic in D.

Define the local modulus of continuity of a function u on the ball $B(x_0,R)$ as follows:

$$\omega_u(t;x_0,R)=\sup_{\substack{x,y\in B(x_0,R)\\|x-y|<t}}\bigl|u(x)-u(y)\bigr|.$$

Let us introduce two kinds of a priori estimates:

(HC) The *Hölder continuity a priori estimate* (HC) holds if for every $R \in (0, 1)$ there exist $c > 0$ and $\gamma \in (0, 1)$ such that for any bounded function $u : \mathbb{R}^d \to \mathbb{R}$ which is $\mathscr{L}$-harmonic in a ball $B(x_0, R)$ we have

$$\omega_u(t; x_0, R/2) \le ct^\gamma \|u\|_\infty \quad \forall t > 0.$$

(C) The *continuity a priori estimate* (C) holds if for every $R \in (0, 1)$ there exists a function $\vartheta : (0, 1) \to \mathbb{R}_+$ with $\lim_{t\to 0} \vartheta(t) = 0$ such that for every bounded function $u : \mathbb{R}^d \to \mathbb{R}$ which is $\mathscr{L}$-harmonic in a ball $B(x_0, R)$ we have

$$\omega_u(t; x_0, R/2) \le \|u\|_\infty \vartheta(t) \quad \forall t \in (0, 1).$$

Clearly, (HC) implies (C) by the choice $\vartheta(t) = ct^\gamma$. (C) guarantees that any equibounded set of functions which are $\mathscr{L}$-harmonic in $B(x_0, R)$ is compact in $C(B(x_0, R/2))$. (C) is often the minimal condition that is needed, for example, when dealing with nonlinear elliptic operators satisfying so called natural growth conditions. (HC) was established by DeGiorgi [9] and Nash [23] for weak solutions to $\operatorname{div}(A(\cdot)\nabla u) = 0$ and later by Krylov–Safonov for diffusion equations in nondivergence form. We refer to the end of this section for a short discussion about known results in the case of jump processes.

As mentioned above we prove our main results under assumptions where uniform hitting time estimates as (1.1) do not hold necessarily. We illustrate this phenomenon for a fixed Lévy measure $\nu(x, \mathrm{d}h) = \nu(\mathrm{d}h)$. More precisely, we have the following result.

Theorem 1.1. *There exists a Lévy measure ν satisfying* (A), (B1)–(B3) *and sequences $r_n \to 0$, $A_n \subset B(0, r_n)$ satisfying $|A_n| \ge \frac{5}{8}|B(0, r_n)|$ such that*

$$\mathbb{P}^0(T_{A_n} < \tau_{B(0,r_n)}) \to 0 \quad \text{for } n \to \infty. \tag{1.3}$$

Note that (A) is automatically satisfied when considering a fixed Lévy measure. In light of (1.3) regularity of harmonic functions or resolvents under our assumptions is an interesting and subtle question. Our main result reads as follows.

Theorem 1.2. *Assume that ν satisfies assumptions* (A), (B1)–(B3). *Then for each $R \in (0, 1/2)$ there is $c > 0$ such that for all bounded functions $u : \mathbb{R}^d \to \mathbb{R}$ being $\mathscr{L}$-harmonic on $B(x_0, R)$ its modulus of continuity on $B(x_0, R/2)$ satisfies*

$$\omega_u(t; x_0, R/2) \le c\|u\|_\infty |\ln t|^{-\rho} \quad \forall t \in (0, 1/2). \tag{1.4}$$

The constant $\rho > 0$ depends only on the constants appearing in (B2) *and* (B3). *In particular, for each $p > 1/\rho$, u is p-Dini continuous, i.e.*

$$\lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{\omega_u(t; x_0, R/2)^p}{t} \,\mathrm{d}t < c.$$

Assumptions (B1)–(B3) are applicable to cases where the following two phenomena might appear simultaneously:

(i) For given $M \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ the mapping $x \mapsto \nu(x, M)$ might be discontinuous.
(ii) For given $x \in \mathbb{R}^d$ the measure $\nu(x, \cdot)$ might not be almost symmetric, i.e. the quantity $\inf_{M\subset B_r(0)\setminus\{0\}} \frac{\nu(x,M)}{\nu(x,-M)}$ might be zero for all $r > 0$.

Once Theorem 1.2 is established it is not too difficult to determine a Feller semigroup corresponding to $\nu(x, \mathrm{d}h)$. For this purpose it is not necessary to have (HC), see also [19]. Any uniform control over the modulus of continuity for the resolvents is good enough.

In the following result we apply our method in the framework of Dirichlet forms.

Theorem 1.3. *Define a regular Dirichlet-form* $(\mathscr{E}, D(\mathscr{E}))$ *by*

$$\begin{aligned} \mathscr{E}(u,v) &= \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)k(x,y)\,\mathrm{d}x\,\mathrm{d}y, \\ D(\mathscr{E}) &= \overline{C_c^{0,1}(\mathbb{R}^d)}^{\mathscr{E}_1}, \end{aligned} \tag{1.5}$$

where $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ *is measurable and satisfies* $k(x,y)=k(y,x)$ *and*

$$c_0|x-y|^{-d-\alpha} \le k(x,y) \le c_1 \ln\left(\frac{3}{|x-y|}\right)|x-y|^{-d-\alpha} \quad \text{for } |x-y|\le 1, \tag{1.6}$$

$$0 \le k(x,y) \le c_2|x-y|^{-d-\gamma} \quad \text{for } |x-y|>1, \tag{1.7}$$

with $\alpha\in(0,1)$, $c_0,c_1,c_2,\gamma>0$. *Then the restriction of the corresponding semi-group to* $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ *can be extended to a Feller semigroup* (T_t) *on* $C_\infty(\mathbb{R}^d)$.

Here $C_c^{0,1}(\mathbb{R}^d)$ is the space of all Lipschitz-continuous functions with compact support and $\overline{C_c^{0,1}(\mathbb{R}^d)}^{\mathscr{E}_1}$ denotes the closure of this space with respect to the norm $\mathscr{E}_1=\mathscr{E}(\cdot,\cdot)+\|\cdot\|_{L^2}^2$. The tuple $(\mathscr{E}, D(\mathscr{E}))$ is indeed a regular Dirichlet form as it can be proved like in Example 1.2.4 of [11]. For more information on Dirichlet forms, the corresponding Hunt process and other related objects we refer the reader to [11].

In light of Theorem 1.9 in [1] it is an interesting task to further weaken assumption (B3). An integrability test suggests that continuity estimates break down under an assumption of the type $N(x,r)\ge\frac{\kappa_2}{|\log r|^{1+\varepsilon}}L(y,r/2)$ for some $\varepsilon>0$. By our techniques we can get quite close to this if we replace (B3) by

(B3′) There exists $\kappa_2>0$, $r_0>0$ and $M\in\mathbb{N}$ such that for all $x,y\in\mathbb{R}^d$, $r\in(0,r_0)$, $|x-y|<2r$,

$$N(x,r)\ge\frac{\kappa_2}{\Psi(r)}L(y,r/2),$$

where $\Psi(r)=|\log r|\prod_{k=1}^M \log^k(|\log r|)$ and $\log^k=\log\circ\log\circ\cdots\circ\log$ denotes the $(k-1)$-times iterated logarithm.

Corollary 1.4. *Assume* (A), (B1), (B2) and (B3′). *Then* (C) *holds.*

We outline the proof of this result at the end of Section 4. Note that the logarithm in (1.6) can be replaced by the more general function Ψ without affecting Theorem 1.3.

Related results and examples

We close this section with a short overview on related results and some examples. Komatsu establishes a priori estimates in [19] and [20] in the case $\nu(x,\mathrm{d}h)=a(x)|h|^{-d-\alpha}\,\mathrm{d}h$ and $0<c_0\le a(x)\le c_1$. (HC) is proved by Bass and Levin [4] in the case where $\nu(x,\mathrm{d}h)$ is absolutely continuous with density $n(x,h)$ satisfying $n(x,h)=n(x,-h)$ and $c_0|h|^{-d-\alpha}\le n(x,h)\le c_1|h|^{-d-\alpha}$ with $\alpha\in(0,2)$, see also [27]. (HC) is also studied with probabilistic methods by Bass and one of us in [3] not assuming ν to have a density. In [25], Silvestre uses methods of partial differential equations to show (HC) in a similar context. Recently, the celebrated analytic methods of DeGiorgi, Nash and Moser were extended to nonlocal Dirichlet forms [18]. For symmetric jump processes corresponding to operators of type (5.1) with $\nu(x,\mathrm{d}h)=n(x,h)\,\mathrm{d}h$, $n(x,h)=n(x+h,-h)$, (HC) is established by Bass and Levin in [5] on the lattice and by Chen and Kumagai in [8] for quite general state spaces under the assumption $c_0|h|^{-d-\alpha}\le n(x,h)\le c_1|h|^{-d-\alpha}$, $\alpha\in(0,2)$. Schilling and Uemura [26] derive (HC) for such kernels allowing for certain mild perturbations for large h. [6] and [14] apply (HC) in order to prove convergence of approximation schemes for symmetric jump processes.[3]

[3]Note added in proof: While this work was under review and being modified, several new articles were published that discuss (HC). The reader is referred to [7] for a detailed discussion.

Concerning the Feller property, substantial work has been carried out using methods from the theory of partial differential and pseudo differential operators by Jacob [15,16], Hoh [12,13] and others, see [17] for references. Different from our context, the main assumption there is that given $M \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$, the mapping $x \mapsto \nu(x, M)$ is smooth.

Finally, let us give an example where assumptions (B1) through (B3) are satisfied. Let us mention that all examples of kernels $\nu(x, \mathrm{d}h)$ from the literature that satisfy (A) and lead to (HC) are covered by our assumptions.

Example 1.5. *Let $\alpha \in (0, 2)$, $2 > r_0 > 1$ and $\nu(x, \mathrm{d}h) = n(x, h)\,\mathrm{d}h$. Suppose*

$$c_0|h|^{-d-\alpha} \le n(x, h) \le c_1|h|^{-d-\alpha} \log\left(\frac{3}{|h|}\right) \quad \text{for } |h| \le r_0,$$

$$\inf_{x\in\mathbb{R}^d} S(x, 1) < \infty.$$

Then ν satisfies (B1)–(B3).

The example above indicates that large jumps have no substantial influence on our result. Note that (B1) through (B3) do not require $n(x, h)$ to be continuous neither in x nor h. Furthermore it includes cases which are not covered by earlier contributions since they all deal with what we call almost symmetric measures.

Definition 1.6. *Let μ be a Lévy-measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\mu(\mathbb{R}^d \setminus \{0\}) = \infty$. We say that μ is almost rotationally invariant at* 0 *if*

$$\exists c > 0 \quad \liminf_{r\to\infty} \inf_{M\subset B_r(0)\setminus\{0\}} \frac{\mu(M)}{\mu(\rho(M))} > c \tag{1.8}$$

for any rotation ρ about the origin. We say that μ is almost symmetric at 0 *if*

$$\liminf_{r\to\infty} \inf_{M\subset B_r(0)\setminus\{0\}} \frac{\mu(M)}{\mu(-M)} > 0. \tag{1.9}$$

It is clear that one can choose $\nu(x, \mathrm{d}h) = n(x, h)\,\mathrm{d}h$ as in Example 1.5 leading to measures $\nu(x, \cdot)$ which are neither almost symmetric nor almost rotationally invariant. Choose $d = 2$, $M = \{(x, y) \in \mathbb{R}^2;\, x > 0,\, y > 0\}$ and set

$$n(x, h) = \left\{|h|^{-3} + |h|^{-3} \ln\left(\frac{3}{|h|}\right)\mathbb{1}_{\{h\in M\}}(h)\right\}\mathbb{1}_{\{|h|\le 1\}}(h).$$

Then the measure $\nu(x, \cdot) = \nu(\cdot)$ is a Lévy-measure satisfying (B1) through (B3) but it is not almost symmetric. In Section 5 we discuss similar examples where $\nu(x, \cdot)$ depends on $x \in \mathbb{R}^d$ noncontinuously.

2. Preliminaries

We denote the open ball in $\mathbb{R}^d$ with center x and radius r by $B(x, r)$ or $B_r(x)$, the characteristic function of a set $A \subset \mathbb{R}^d$ by $\mathbb{1}_A$ and the Lebesgue measure of a Borel set A by $|A|$. Define the function spaces

$$C_\infty\big(\mathbb{R}^d\big) = \Big\{u \in C\big(\mathbb{R}^d\big)\colon \lim_{|x|\to\infty} u(x) = 0\Big\},$$

$$C_b^k\big(\mathbb{R}^d\big) = \big\{u \in C^k\big(\mathbb{R}^d\big)\colon \text{all derivatives up to order } k \text{ bounded}\big\},$$

$$C_c^k\big(\mathbb{R}^d\big) = \big\{u \in C^k\big(\mathbb{R}^d\big)\colon \operatorname{supp} u \text{ compact}\big\}.$$

The following lemma will be essential when proving properties of certain anisotropic Lévy processes. Let us define for $a, \rho \in (0,1)$ the following sets.

$$A = \{(x,y) \in \mathbb{R}^2; |y| \geq |x|^a, x^2 + y^2 < 1\},$$

$$E_\rho = A \cap \{(x,y) \in \mathbb{R}^2; \sqrt{x^2+y^2} \geq \rho\},$$

$$F_\rho = A \cap \{(x,y) \in \mathbb{R}^2; x \geq \rho\}.$$

Lemma 2.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be invariant under rotations, i.e. $g(x,y) = f(r)$ where $r = \sqrt{x^2+y^2}$ and $f : \mathbb{R}_+ \to \mathbb{R}$. Then*

$$\iint_{E_\rho} g(x,y)\,\mathrm{d}y\,\mathrm{d}x = \mathcal{O}\left(\int_\rho^1 r^{1/a} f(r)\,\mathrm{d}r\right) \quad \text{for } \rho \to 0. \tag{2.1}$$

Let $\beta \in (0,2)$, $\beta \neq 1/a - 1$. Asymptotically for $\rho \to 0$ we then obtain

$$\iint_{E_\rho} \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\sqrt{x^2+y^2}\right)^{-2-\beta}\mathrm{d}y\,\mathrm{d}x = \mathcal{O}\left(\ln\left(\frac{1}{\rho}\right)\rho^{1/a-1-\beta}\right), \tag{2.2}$$

$$\iint_{F_\rho} \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\sqrt{x^2+y^2}\right)^{-2-\beta}\mathrm{d}y\,\mathrm{d}x = \mathcal{O}\left(\ln\left(\frac{1}{\rho}\right)\rho^{1-a-a\beta}\right). \tag{2.3}$$

Proof. Let us prove (2.1) first. Using polar coordinates (r,θ) instead of Euclidean coordinates (x,y) we obtain

$$\iint_{E_\rho} g(x,y)\,\mathrm{d}y\,\mathrm{d}x = 4\int_\rho^1 \int_{\phi(r)}^{\pi/2} r f(r)\,\mathrm{d}\theta\,\mathrm{d}r = 4\int_\rho^1 \left(\frac{\pi}{2} - \phi(r)\right) r f(r)\,\mathrm{d}r, \tag{2.4}$$

where $\phi(r)$ is the unique angle satisfying

$$(1) \quad \frac{\pi}{4} \leq \phi(r) \leq \frac{\pi}{2} \quad \text{and} \quad (2) \quad r^a \cos^a\bigl(\phi(r)\bigr) = r \sin\bigl(\phi(r)\bigr).$$

Note that (2) is equivalent to

$$\frac{\pi}{2} - \phi(r) = r^{1/a-1} \sin^{1/a}\bigl(\phi(r)\bigr) \frac{\pi/2 - \phi(r)}{\cos(\phi(r))}.$$

Since both functions, $\sin^{1/a}(x)$ and $(\frac{\pi}{2} - x)/\cos(x)$ are bounded from above and below by positive constants for $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$ which is the range of $\phi(r)$ we obtain

$$\iint_{E_\rho} g(x,y)\,\mathrm{d}y\,\mathrm{d}x \approx \int_\rho^1 r^{1/a-1} r f(r)\,\mathrm{d}r, \tag{2.5}$$

which proves (2.1). Next, we prove (2.2). Using integration by parts we derive

$$\begin{aligned}\int_\rho^1 \ln\left(\frac{1}{r}\right) r^{1/a-2-\beta}\,\mathrm{d}r &= \frac{1}{1/a-1-\beta}\ln\left(\frac{1}{r}\right) r^{1/a-1-\beta}\bigg|_\rho^1 + \frac{1}{1/a-1-\beta}\int_\rho^1 r^{1/a-2-\beta}\,\mathrm{d}r \\ &= \frac{-1}{1/a-1-\beta}\left\{\ln\left(\frac{1}{\rho}\right)\rho^{1/a-1-\beta} - \frac{1}{1/a-1-\beta}\bigl(1-\rho^{1/a-1-\beta}\bigr)\right\},\end{aligned}$$

which proves (2.2). In order to prove (2.3) note that $F_\rho \subset E_{\sqrt{\rho^2+\rho^{2a}}}$. Together with (2.2) this implies

$$\iint_{F_\rho} \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\sqrt{x^2+y^2}\right)^{-2-\beta}\,\mathrm{d}y\,\mathrm{d}x \le c\ln\left(\frac{1}{\sqrt{\rho^2+\rho^{2a}}}\right)\left(\sqrt{\rho^2+\rho^{2a}}\right)^{1/a-1-\beta}$$
$$\le \mathcal{O}\left(\ln\left(\frac{1}{\rho}\right)\rho^{1-a-a\beta}\right) \quad \text{for } \rho\to 0.$$

Concerning the lower estimate in (2.2) we observe

$$\iint_{F_\rho} \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)\left(\sqrt{x^2+y^2}\right)^{-2-\beta}\,\mathrm{d}y\,\mathrm{d}x$$
$$\ge c\int_\rho^{2\rho}\int_{x^a}^{2(x^a)} \ln\left(\frac{1}{\sqrt{y^2+y^2}}\right)\left(\sqrt{y^2+y^2}\right)^{-2-\beta}\,\mathrm{d}y\,\mathrm{d}x$$
$$\ge c\int_\rho^{2\rho} \ln\left(\frac{1}{x^a}\right)\left(x^a\right)^{-1-\beta}\,\mathrm{d}x \ge ca\ln\left(\frac{1}{\rho}\right)\rho^{1-a-a\beta},$$

which proves (2.3). □

Let $(X_t,\mathbb{P}^x)$ be a strong Markov process according to assumption (A). Let $\Delta X_t = X_t - X_{t-}$ be the jump of X_t at time t and, for a Borel set A, let τ_A be the first exit time of A, T_A the first hitting time. Recall that a function $u:\mathbb{R}^d\to\mathbb{R}$ is said to be *harmonic with respect to* $\mathscr{L}$ in an open set $U\subset\mathbb{R}^d$, if for any open set $U'\Subset U$ the process $u(X_{s\wedge\tau_{U'}})$ is a $\mathbb{P}^x$-martingale.

Let us state and prove several technical lemmas. As in [4], Proposition 2.3, one can prove that $(\nu(x,x-\mathrm{d}h),\mathrm{d}t)$ is a Lévy system for X_t:

Lemma 2.2. *Suppose* (A) *holds. For disjoint Borel sets $A, B\subset\mathbb{R}^d$ and bounded stopping times S*,

$$\mathbb{E}^{x_0}\sum_{s\le S}\mathbb{1}_{\{X_{s-}\in A,X_s\in B\}} = \mathbb{E}^{x_0}\int_0^S \mathbb{1}_A(X_s)\nu(X_s,B-X_s)\,\mathrm{d}s.$$

We will now estimate some probabilities which play a crucial role in the proof of Theorem 1.2. Set

$$\overline{L}(x_0,r) = \sup_{x\in B(x_0,r)} L(x,r).$$

The proofs of the following results can be found in [3].

Lemma 2.3. *Assume that* (A), (B1) *hold. Then there exists a constant $\kappa_3>0$ such that for all $x_0\in\mathbb{R}^d$ and $r\in$* $(0,1/2)$,

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)}<t) \le \kappa_3 t\overline{L}(x_0,r).$$

Lemma 2.4. *Assume that* (A), (B1) *and* (B2) *hold. For a Borel set B and $r\in(0,1/2)$ let $U=\inf\{t\colon |\Delta X_t|\ge r\}$ be the time of the first jump greater than r. Then, for all $1<\lambda<\frac{1}{r}$ and $x\in\mathbb{R}^d$ we have*

$$\mathbb{P}^x\left(|\Delta X_{U\wedge\tau_B}|\ge\lambda r\right)\le\kappa_1\lambda^{-\sigma},$$

where κ_1 is the constant in (B2).

Lemma 2.5. *Assume that* (A), (B1) *and* (B3) *hold. Then there exists a constant* $\kappa_4 > 0$ *such that for* $r \in (0, 1/2)$, $A \subset B(x_0, r)$, $|A| \geq \delta |B(x_0, r)|$ *and* $y \in B(x_0, \frac{r}{2})$,

$$\mathbb{P}^y(T_A < \tau_{B(x_0,r)}) \geq \frac{\kappa_4}{|\ln r|}.$$

Due to the special form of (B3) the assertion of Lemma 2.5 is considerably weaker than the corresponding result in [3].

Lemma 2.6. *Assume* (A) *and* $\lim_{r\to 0} \inf_{x\in\mathbb{R}^d} S(x, r) = \infty$. *Then there exist a function* $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{r\to 0} \vartheta(r) = 0$ *and* $r_0 > 0$ *such that*

$$\mathbb{E}^y \tau_{B(x,r)} \leq \vartheta(r) \quad \forall x, y \in \mathbb{R}^d, r_0 > r > 0.$$

Proof. We follow the proof of Lemma 3.4 in [2]. Let U be the time of the first jump greater than $2r$. Note that $\tau_{B(x,r)} \leq U$ and that, because of the additional assumption on $S(x, r)$, there exists a function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ and $r_0 > 0$ with $\lim_{r\to 0} \vartheta(r) = 0$ and $S(x, 2r) \geq \frac{1}{2}\vartheta(r)^{-1}$ for all $r < r_0$. Assume $r < r_0$. If $\mathbb{P}^y(U \leq \vartheta(r)) \leq \frac{1}{2}$, then by the Lévy system identity

$$\begin{aligned}\mathbb{P}^y\big(U \leq \vartheta(r)\big) &= \mathbb{E}^y \sum_{s\leq U\wedge\vartheta(r)} \mathbb{1}_{\{|\Delta X_s|>2r\}} = \mathbb{E}^y \int_0^{U\wedge\vartheta(r)} S(X_s, 2r)\,\mathrm{d}s \\ &\geq \frac{1}{2}\vartheta(r)^{-1}\mathbb{E}^y\big(U \wedge \vartheta(r)\big) \geq \frac{1}{2}\mathbb{P}^y\big(U > \vartheta(r)\big) \geq \frac{1}{4}.\end{aligned}$$

Therefore in any case $\mathbb{P}^y(U \leq \vartheta(r)) \geq \frac{1}{4}$. Now let θ_t be the Markov shift operator. For $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}^y\big(U > (m+1)\vartheta(r)\big) &\leq \mathbb{P}^y\big(U > m\vartheta(r), U \circ \theta_{m\vartheta(r)} > \vartheta(r)\big) \\ &= \mathbb{E}^y\big(\mathbb{P}_{X_{m\vartheta(r)}}\big(U > \vartheta(r); U > m\vartheta(r)\big)\big) \\ &\leq \frac{1}{2}\mathbb{P}^y\big(U > m\vartheta(r)\big) \leq \cdots \leq 2^{-(m+1)}.\end{aligned}$$

Hence we see $\mathbb{E}^y U \leq 4\vartheta(r)$ completing the proof. □

3. Degeneration of hitting time estimates

The aim of this section is to prove Theorem 1.1. Thereby, we show that our assumptions made in Theorem 1.2 allows for cases where (1.1) does not hold. The construction below is similar to the class of examples given in Section 5 of [2] but not included in that class. However, a generalized Harnack inequality would fail for our example, too.

Proof of Theorem 1.1. Let $0 < \alpha < \beta < 1$ and $a = (1 + \beta - \alpha)^{-1}$. In particular we have $a \in (0, 1)$ and $\frac{1}{a} - 1 - \beta = -\alpha$. As in Section 2 set

$$A = \big\{(h_1, h_2) \in \mathbb{R}^2; |h_2| \geq |h_1|^a, (h_1)^2 + (h_2)^2 < 1\big\}.$$

Let $\nu(\mathrm{d}h) = n(h)\,\mathrm{d}h$ be the symmetric Lévy measure with density

$$n(h) = |h|^{-2-\alpha} + \mathbb{1}_A(h)\big|\ln|h|\big||h|^{-2-\beta}. \tag{3.1}$$

Observe that for a Lévy measure the martingale problem always has a unique solution, the Lévy process (X_t) with Lévy characteristic $(0, 0, \nu)$, see, for example, [24]. An application of (2.1) shows

$$S(r) = S(x, r) = \mathcal{O}\bigg(r^{-\alpha} \ln\frac{1}{r}\bigg) \quad \text{as } r \to 0.$$

Furthermore we have

$$\iint\limits_{A\cap\{|h|\le r\}} \big|\ln|h|\big||h|^{-\beta}\mathrm{d}h$$
$$\le 4\int_0^r\int_0^{r^{1/a}}\big(-\ln\sqrt{(h_1)^2+(h_2)^2}\big)\big(\sqrt{(h_1)^2+(h_2)^2}\big)^{-\beta}\,\mathrm{d}h_1\,\mathrm{d}h_2$$
$$\le 4\int_0^r\int_0^{r^{1/a}}\big(-(h_2)^{-\beta}\ln h_2\big)\,\mathrm{d}h_1\,\mathrm{d}h_2\le cr^{1+1/a-\beta}\ln\bigg(\frac{1}{r}\bigg).$$

Together with symmetry of the measure we obtain

$$L(r)=L(x,r)\le cr^{-\alpha}\ln\bigg(\frac{1}{r}\bigg).$$

Clearly, $|A\cap B(0,r)|/|B(0,r)|$ tends to 0 for $r\to 0$, hence

$$N(x,r)=N(r)=\mathcal{O}\big(r^{-\alpha}\big).$$

Altogether, ν satisfies the assumptions of Theorem 1.2. Set $B_r=B(0,r)$ and define $T(r)=r^{\alpha}/\sqrt{\ln(\frac{1}{r})}$. We will prove

$$\lim_{r\to 0}\mathbb{P}^0\Big(\sup_{s\le T(r)}|X_s|<r\Big)=0, \tag{3.2}$$

$$\lim_{r\to 0}\mathbb{P}^0\bigg(\sup_{s\le T(r)}|X_s^1|>\frac{r}{16}\bigg)=0, \tag{3.3}$$

where $X_t=(X_t^1,X_t^2)$ but X_t^1 and X_t^2 are not necessarily independent. Let r_n be an arbitrary sequence in $\mathbb{R}_+$ with $r_n\to 0$. Together, (3.2) and (3.3) mean that, in the limit $r_n\to 0$ the process has left $B(0,r_n)$ up to time $T(r_n)$ but has moved right or left not further than the distance $r/16$. (1.3) follows after choosing $A_n\subset B(0,r_n)\setminus\{(x,y)\in B(0,r_n),\frac{-r_n}{16}\le x\le\frac{r_n}{16}\}$ large enough.

Let us first prove (3.2). Note the following: Let μ be a Lévy measure, Y_t the associated pure-jump Lévy process and $B\subset\mathbb{R}^d$ a Borel set. Then, for any time T, the quantity $\sum_{s\le T}\mathbb{1}_{\{\Delta Y_s\in B\}}$, i.e. the number of jumps in B of the Lévy process before T, is Poisson distributed with parameter $T\mu(B)$. Using (2.1) and our choice of a show

$$I_1(r)=\nu\big(\{|h|>2r\}\big)=\mathcal{O}\bigg(r^{-\alpha}\ln\bigg(\frac{1}{r}\bigg)\bigg)\quad\text{for } r\to 0.$$

In particular $T(r)I_1(r)$ tends to ∞ for $r\to 0$. Thus the process exits a ball of radius r before time $T(r)$ with a probability that tends to 1 for $r\to 0$. This proves (3.2).

Next we write (X_t) as the sum of two independent Lévy processes (Y_t) and (Z_t) with Lévy measures $\nu_Y(\mathrm{d}h)=|h|^{-2-\alpha}\,\mathrm{d}h$ and $\nu_Z(\mathrm{d}h)=\mathbb{1}_A(h)|h|^{-2-\beta}\ln(\frac{1}{|h|})$. (Y_t) is a rotationally invariant α-stable process. Hence we get by scaling

$$\lim_{r\to 0}\mathbb{P}^0\Big(\sup_{s\le T(r)}|Y_s|>r/32\Big)=\lim_{r\to 0}\mathbb{P}^0\Big(\sup_{s\le(-\ln r)^{-1/2}}|Y_s|>1/32\Big)=0. \tag{3.4}$$

Lemma 2.1 implies for $r\to 0$,

$$I_2(r)=\nu_Z\big(\{|h_1|>r/32\}\big)=\mathcal{O}\bigg(r^{1-a-a\beta}\ln\bigg(\frac{1}{r}\bigg)\bigg).$$

Because of $1 - a - a\beta = -\alpha/(1+\beta-\alpha) > -\alpha$ we have $T(r)I_2(r) \to 0$ for $r \to 0$, which is the expected number of times Z_s^1 has jumps greater than $r/32$. In other words:

$$\lim_{r\to 0} \mathbb{P}^0\Big(\sup_{s\le T(r)} \big|\Delta Z_s^1\big| > r/32\Big) = 0. \tag{3.5}$$

It remains to handle the small jumps of (Z_t^1). For this we remove all jumps with $(\Delta Z_t) \in \{|h_1| > r/32\}$ and obtain a Lévy process (W_t) with Lévy measure

$$\nu_W(\mathrm{d}h) = \mathbb{1}_{\{|h_1|\le r/32\}} \nu_Z(\mathrm{d}h).$$

Note that (W_t) has bounded jumps and therefore moments of all orders. Hence (W_t^1) is a martingale. We apply Doob's inequality and estimate

$$\mathbb{P}^0\Big(\sup_{s\le T(r)} |W_s^1| > r/16\Big) \le \frac{4\mathbb{E}^0(W_{T(r)}^1)^2}{(r/16)^2}.$$

As a consequence of the Lévy–Itô decomposition $\mathbb{E}^0(W_t^1)^2 = tI_3(r)$, where

$$\begin{aligned} I_3(r) &= \int_{|h_1|\le r/16} (h_1)^2 \nu(\mathrm{d}h) = 4\int_0^{r/16}\int_{|h_1|^a}^1 (h_1)^2 |h|^{-2-\beta} \ln\Big(\frac{1}{|h|}\Big)\,\mathrm{d}h_2\,\mathrm{d}h_1 \\ &\le 4\int_0^{r/16} (h_1)^2 \int_{|h_1|^a}^1 |h_2|^{-2-\beta} \ln\Big(\frac{1}{|h_2|}\Big)\,\mathrm{d}h_2\,\mathrm{d}h_1 \le cr^{3-a-a\beta}. \end{aligned}$$

We obtain

$$\lim_{r\to 0} \mathbb{P}^0\Big(\sup_{s\le T(r)} \big|W_s^1\big| > r/16\Big) = 0. \tag{3.6}$$

Combining (3.4)–(3.6) we see that, starting in 0, the probability that (X_s^1) leaves the interval $(-r/8, r/8)$ before time $T(r)$ tends to 0 for $r \to 0$. Assertion (3.3) is proved. The proof of Theorem 1.1 is complete. □

4. Continuity of $\mathscr{L}$-harmonic functions

The aim of this section is to prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. We will use an alteration of the method worked out in [3] and prove the continuity of u in $z_1 \in B(z_0, \frac{R}{2})$ by an induction argument. The logarithmic degeneration in Lemma 2.5 requires a subtle change of the argument given in [3]. Set $K = \|u\|_\infty$ and define furthermore

$$r_n = \theta_2 4^{-n},$$

where we select $\theta_2 = R/32$, in particular $B(z_1, 2r_1) \subset B(z_0, \frac{3R}{4})$. We write $B_n = B(z_1, r_n)$, $\tau_n = \tau_{B_n}$ and

$$M_n = \sup_{x\in B_n} u(x), \qquad m_n = \inf_{x\in B_n} u(x).$$

We will show

$$M_n - m_n \le s_n \tag{4.1}$$

for all n where s_n is a series decreasing monotone to 0. In our case

$$s_n = \theta_1 n^{-\rho}$$

will do the job, where $\theta_1 > 2K$ and $1 > \rho > 0$ will be specified later. Here the role of the upper bound on ρ is only to keep notation simple.

Let us assume for a moment that (4.1) holds already for $1, \dots, n$. Choose arbitrary $y, z \in B_{n+1}$ and define

$$A_n = \left\{x \in B_n\colon\ u(x) \le \frac{M_n + m_n}{2}\right\}.$$

Without loss of generality suppose $|A_n| \ge \frac{1}{2}|B_n|$ (otherwise we look at the function $K - u$). Let $D \subset A_n$ compact with $|D| \ge \delta|B_n|$. By the $\mathscr{L}$-harmonicity of u in $B(x_0, R)$ we get

$$\begin{aligned}
u(z) - u(y) &= \mathbb{E}^z\big(u(X_{\tau_n \wedge T_D}) - u(y)\big)\\
&= \mathbb{E}^z\big(u(X_{\tau_n \wedge T_D}) - u(y); T_D < \tau_n, X_{\tau_n} \in B_{n-1} \setminus B_n\big)\\
&\quad + \mathbb{E}^z\big(u(X_{\tau_n \wedge T_D}) - u(y); T_D > \tau_n, X_{\tau_n} \in B_{n-1} \setminus B_n\big)\\
&\quad + \sum_{i=1}^{n-2} \mathbb{E}^z\big(u(X_{\tau_n \wedge T_D}) - u(y); X_{\tau_n} \in B_{n-i-1} \setminus B_{n-i}\big)\\
&\quad + \mathbb{E}^z\big(u(X_{\tau_n \wedge T_D}) - u(y); X_{\tau_n} \notin B_1\big)\\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Set

$$p_n = \mathbb{P}^z(T_D < \tau_n).$$

Then, by definition of A_n and (4.1), we derive the estimates

$$\begin{aligned}
I_1 &\le \left(\frac{M_n + m_n}{2} - m_n\right)\mathbb{P}^z(T_D < \tau_n) = \frac{1}{2} s_n p_n,\\
I_2 &\le s_{n-1}(1 - p_n).
\end{aligned}$$

To handle I_3 and I_4 we have to look at the probabilities

$$F_j = \mathbb{P}^z(X_{\tau_n} \notin B_{n-j}).$$

The event defining F_j can only take place, if the process (X_t) has no jumps larger than $2r_n$ for $t < \tau_n$ and jumps at least $r_{n-j} - r_n$ at time τ_n. So by Lemma 2.4 it follows:

$$\begin{aligned}
F_j &\le \mathbb{P}^z\Big(|\Delta X_{\tau_n}| \ge r_{n-j} - r_n, \sup_{s<\tau_n} |\Delta X_s| \le 2r_n\Big) \le \kappa_1 \left(\frac{2r_n}{r_{n-j} - r_n}\right)^{\sigma}\\
&\le \kappa_1 3^{\sigma} 4^{-j\sigma} = c_1 4^{-j\sigma}.
\end{aligned}$$

Again, we use our hypothesis (4.1) as well as summation by parts and obtain

$$\begin{aligned}
I_3 &\le \sum_{i=1}^{n-2} s_{n-i-1}(F_i - F_{i-1}) = s_1 F_{n-2} - s_{n-2} F_0 + \sum_{i=1}^{n-3} (s_{n-i-1} - s_{n-i}) F_i\\
&\le s_1 F_{n-2} + \sum_{i=1}^{n-3} (s_{n-i-1} - s_{n-i}) F_i \le \theta_1 c_1 4^{-\sigma(n-2)} + \sum_{i=1}^{n-3} (s_{n-i-1} - s_{n-i}) F_i.
\end{aligned}$$

Finally we estimate

$$I_4 \le 2K F_{n-1} \le \theta_1 c_1 4^{-\sigma(n-1)}.$$

In consequence, we have

$$
\begin{aligned}
u(z)-u(y) &\leq s_{n+1}\left[\frac{s_n}{s_{n+1}}\cdot\frac{p_n}{2}+\frac{s_{n-1}}{s_{n+1}}(1-p_n)+\frac{1}{s_{n+1}}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i+\frac{\theta_1 c_2}{s_{n+1}}4^{-n\sigma}\right]\\
&\leq s_{n+1}\left[-\frac{p_n}{2}\cdot\frac{s_{n-1}}{s_{n+1}}+\left(1+\frac{2}{n-1}\right)^{\rho}\right.\\
&\qquad\left.+\frac{(n+1)^{\rho}}{\theta_1}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i+c_2(n+1)^{\rho}4^{-n\sigma}\right]. \qquad (4.2)
\end{aligned}
$$

Lemma 2.5 implies

$$
p_n \geq \frac{\kappa_4}{|\ln r_n|}=\frac{\kappa_4}{|\ln\theta_2-n\ln 4|}\geq\frac{\kappa_4}{|\ln\theta_2|+n\ln 4}.
$$

s_{n-1}/s_{n+1} is bounded from below by 1, thus the first term in (4.2) is bounded from above by

$$
-\frac{c_4}{|\ln\theta_2|+n\ln 4}.
$$

Moreover, the second term (4.2) behaves for $n\to\infty$ as

$$
1+\frac{2\rho}{n-1}+\mathcal{O}\left(\frac{1}{(n-1)^2}\right).
$$

The most laborious part is estimating the sum in (4.2):

$$
\begin{aligned}
\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i &\leq \sum_{i=1}^{\lceil (n-3)/2\rceil}(s_{n-i-1}-s_{n-i})F_i+\sum_{i=\lceil (n-3)/2\rceil}^{n-3}(s_{n-i-1}-s_{n-i})F_i\\
&\leq (s_{\lceil n/2\rceil-1}-s_{\lceil n/2\rceil})\sum_{i=1}^{\infty}F_i+F_{\lceil (n-3)/2\rceil}\sum_{i=1}^{\infty}(s_i-s_{i+1}).
\end{aligned}
$$

Here both series converge. Finally an easy application of the mean value theorem yields

$$
s_k-s_{k+1}\leq\theta_1\rho k^{-\rho-1},
$$

and therefore there exist $c_5, c_6>0$ with

$$
\frac{1}{s_{n+1}}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i\leq\frac{c_5\rho}{n-1}+c_6(n+1)^{\rho}4^{-\sigma n/2}.
$$

Altogether we have

$$
u(z)-u(y)\leq s_{n+1}\left(1-\frac{c_4}{|\ln\theta_2|+n\ln 4}+\frac{c_5\rho}{n-1}+\frac{c_7}{(n-1)^2}+c_8 4^{-\sigma n/3}\right). \qquad (4.3)
$$

Note that the constants in (4.3) are independent of the choice of y, z, ρ, θ_1 and θ_2. Therefore the estimate in (4.3) gives us also an upper bound for $M_{n+1}-m_{n+1}$. Next, select ρ small enough and then n_0 large enough such that for all $n > n_0$

$$1 - \frac{1}{3} \cdot \frac{c_4}{|\ln\theta_2| + n\ln 4} + \frac{c_5\rho}{n-1} + \frac{c_7}{(n-1)^2} + c_8 4^{-\sigma n} < 1.$$

Finally, choose θ_1 in such a way that

$$M_n - m_n \le 2K \le s_n \quad \forall 1 \le n \le n_0.$$

Now, (4.1) holds for all n. Moreover, looking carefully over the preceding proof we see, that ρ and θ_2 only depend on R and not on u. Consequently we might choose θ_1 proportional to $\|u\|_\infty$. Therefore the modulus of continuity of u on $B(x_0, R/2)$ is bounded from above by $C\|u\|_\infty(-\ln t)^{-\rho}$. We now take into account that the integral

$$\int_0^{1/2} \frac{\mathrm{d}t}{t(-\ln t)^\eta}$$

exists for $\eta > 1$. Hence u is p-Dini continuous for every $p > 1/\overline{\rho}$, where $\overline{\rho}$ is the supremum over all $\rho > 0$ for which our induction works. □

We close this section by indicating how to prove Corollary 1.4. Let the notations be as in the proof of Theorem 1.2. Then Assumption (B3′) implies for n large enough

$$p_n = p(r_n) \ge c\Lambda_n, \quad \text{where } \Lambda_n = \left(n \prod_{k=1}^{M-1} \log^k(n)\right)^{-1}.$$

We now introduce the function $s(x) = (\log^M(x))^{-1}$ and choose, with a slight abuse of notation, $s_n = s(n)$. Proceeding as in (4.2) and using the mean value theorem to estimate differences of the type $s_k - s_{k-1}$ we obtain

$$\begin{aligned} u(z) - u(y) \le s_{n+1}\Bigg[& 1 - \frac{1}{2}\frac{s_{n-1}}{s_{n+1}} p_n + \frac{-2s'(n-1)}{s_{n+1}} + \frac{1}{s_{n+1}} \sum_{l=1}^{n-3} \bigl(-s'(n-l-1)\bigr) F_l \\ & + \frac{2K}{s_{n+1}} F_n + \frac{s_1}{s_{n+1}} F_{n-2} \Bigg]. \end{aligned}$$

The second summand on the right-hand side is the only negative one and bounded from above by $-c\Lambda_n$. Therefore it suffices to show that the positive summands converge faster to 0 than Λ_n for $n \to \infty$. For the last two terms this is trivial since they are of order $\mathcal{O}(\mathrm{e}^{-n\gamma})$ for $n \to \infty$. Moreover,

$$\frac{-2s'(n-1)}{s_{n+1}} = \frac{\log^M(n+1)}{(n-1)(\log^M(n-1))^2 \prod_{k=1}^{M-1} \log^k(n-1)} = \mathcal{O}\left(\frac{\Lambda_n}{\log^M(n)}\right).$$

Finally, the remaining term of the right-hand side can be treated as follows:

$$\begin{aligned} & \log^M(n+1) \sum_{l=1}^{n-3} \frac{4^{-l\sigma}}{(n-l-1)(\log^M(n-l-1))^2 \prod_{k=1}^{M-1} \log^k(n-l-1)} \\ & \quad = \log^M(n+1) 4^{-(n-1)\sigma} \sum_{l=2}^{n-2} \frac{4^{l\sigma}}{l(\log^M(l))^2 \prod_{k=1}^{M-1} \log^k(l)} \\ & \quad = \mathcal{O}\left(\log^M(n) 4^{-n\sigma} \int_2^n \frac{4^{x\sigma}\,\mathrm{d}x}{x(\log^M(x))^2 \prod_{k=1}^{M-1} \log^k(x)}\right). \end{aligned}$$

By applying L'Hôspital's rule we end up with

$$\lim_{n\to\infty}\frac{1}{\Lambda_n}\log^M(n)4^{-n\sigma}\int_2^n\frac{4^{x\sigma}\,\mathrm{d}x}{x(\log^M(x))^2\prod_{k=1}^{M-1}\log^k(x)}$$
$$=\lim_{n\to\infty}n\left[\prod_{k=1}^{M}\log^k(n)\right]4^{-\sigma n}\int_2^n\frac{4^{\sigma x}\,\mathrm{d}x}{x\log^M(x)\prod_{k=1}^{M}\log^k(x)}=0.$$

Thereby we have shown $u(z)-u(y)<s_{n+1}$ for n large.

5. The Feller property

In this section we prove and discuss Theorem 1.3. As mentioned in the introduction, one open problem in the area of jump processes is to understand when, given a jump kernel, one can construct a corresponding Feller process and (not less important) when one cannot. In [1] an example of a jump kernel is given for which the martingale problem fails to be unique. We recall this example. With the help of Theorem 1.3 we then construct an example which is similar to the one in [1] but results in a Feller process.

Proof of Theorem 1.3. Let us denote by $(\widetilde{\mathscr{A}},D(\widetilde{\mathscr{A}}))$ the L^2-generator of $(\mathscr{E},D(\mathscr{E}))$ and by $(\widetilde{T_t})$ the corresponding semigroup. Basic calculations imply $\widetilde{\mathscr{A}}u(x)=\mathscr{L}u(x)$ for all $x\in\mathbb{R}^d$ and all functions $u\in C_c^2(\mathbb{R}^d)$, where

$$\begin{aligned}(\mathscr{L}u)(x)&=\ \text{p.v.}\int_{\mathbb{R}^d}\big(u(x+h)-u(x)\big)k(x,x+h)\,\mathrm{d}h\\&:=\lim_{\varepsilon\to0}\int_{|h|>\varepsilon}\big(u(x+h)-u(x)\big)k(x,x+h)\,\mathrm{d}h.\end{aligned}\tag{5.1}$$

Note that the principal value integral exists for $u\in C_c^2(\mathbb{R}^d)$ because of $k(x,y)=k(y,x)$ and $\alpha<1$. Hence $(\widetilde{\mathscr{A}},D(\widetilde{\mathscr{A}}))$ is an extension of $(\mathscr{L},C_c^2(\mathbb{R}^d))$. Denote by X_t a Hunt process in $\mathbb{R}^d$ corresponding to $(\mathscr{E},D(\mathscr{E}))$ and by $\mathscr{N}$ the associated properly exceptional set $\mathscr{N}$. Note that any two such processes are equivalent and that the Lebesgue measure of $\mathscr{N}$ is zero, i.e. $|\mathscr{N}|=0$.

Following Theorem 5.2.2. in [11] one shows that for any starting point $x_0\in\mathbb{R}^d\setminus\mathscr{N}$ the process X_t solves the martingale problem for $(\widetilde{\mathscr{A}},D(\widetilde{\mathscr{A}}))$. For $\lambda>0$ and $f\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ denote by $\widetilde{R_\lambda}f$ the resolvent of $(\mathscr{E},D(\mathscr{E}))$. In the sense of Theorem 4.2.3 in [11], for any $f\in L^\infty(\mathbb{R}^d)$,

$$(\widetilde{R_\lambda}f)(x)=\int_0^\infty \mathrm{e}^{-\lambda t}(\widetilde{T_t}f)(x)\,\mathrm{d}t=\mathbb{E}^x\left(\int_0^\infty \mathrm{e}^{-\lambda t}f(X_t)\,\mathrm{d}t\right),\quad x\in\mathbb{R}^d\setminus\mathscr{N}.$$

From here, it needs only three more or less standard steps in order to complete the proof. The proof of Theorem 1.2 can be applied without changes in order to guarantee that bounded functions u which are $\mathscr{L}$-harmonic in $B(x_0,R)\setminus\mathscr{N}$ satisfy

$$\sup_{\substack{x,y\in B(x_0,R/2)\setminus\mathscr{N}\\|x-y|<t}}\big|u(x)-u(y)\big|\le c_0\|u\|_\infty|\ln t|^{-\rho},\tag{5.2}$$

where c_0 is independent of u but depends on R. Next, let us show that $\widetilde{R_\lambda}$ maps bounded functions into functions uniformly continuous on $\mathbb{R}^d\setminus\mathscr{N}$. We prove

$$\big|(\widetilde{R_\lambda}f)(x)-(\widetilde{R_\lambda}f)(y)\big|\le c_1(\lambda)\|f\|_\infty\vartheta\big(|x-y|\big),\tag{5.3}$$

with a function $\vartheta:(0,1)\to\mathbb{R}_+$ satisfying $\lim_{t\to0}\vartheta(t)=0$ and $c_1(\lambda)$ independent of $x,y\in\mathbb{R}^d\setminus\mathscr{N}$: $|x-y|<1/2$ and $f\in L^\infty(\mathbb{R}^d)$. (5.3) is only needed for x,y closeby. Choose $x_0\in\mathbb{R}^d$, $r>0$ such that $x,y\in B(x_0,r/2)$. Using the strong Markov property one obtains

$$(\widetilde{R_\lambda} f)(x) = \mathbb{E}^x\left(\int_0^{\tau_{B(x_0,r)}} \mathrm{e}^{-\lambda t} f(X_t)\,\mathrm{d}t\right) + \mathbb{E}^x\big((\widetilde{R_\lambda} f)(X_{\tau_{B(x_0,r)}})\big)$$
$$+ \mathbb{E}^x\big(\big(\mathrm{e}^{-\lambda\tau_{B(x_0,r)}} - 1\big)(\widetilde{R_\lambda} f)(X_{\tau_{B(x_0,r)}})\big),$$

and a similar expression for $(\widetilde{R_\lambda} f)(y)$. Note that the second term on the right-hand side is a $\mathscr{L}$-harmonic function in $B(x_0,r)$ as a function of x. Using the above representation we deduce

$$\big|(\widetilde{R_\lambda} f)(x) - (\widetilde{R_\lambda} f)(y)\big| \le \big(2\|f\|_\infty + 2\lambda\|\widetilde{R_\lambda} f\|_\infty\big) \sum_{z\in\{x,y\}} \mathbb{E}^z\tau_{B(x_0,r)} + c_0\|\widetilde{R_\lambda} f\|_\infty |\ln r|^{-\rho},$$

where we applied (5.2). Estimate (5.3) follows from an application of Lemma 2.6.

Therefore, there exists a modification R_λ of $\widetilde{R_\lambda}$ such that R_λ satisfies the strong Feller property which means that bounded functions are mapped into $C_b(\mathbb{R}^d)$. Furthermore, for any $v \in C_\infty(\mathbb{R}^d)$ one checks $\lim_{\lambda\to\infty} \|\lambda(\widetilde{R_\lambda} v - v)\|_\infty = 0$. From here, one concludes that there is a modification T_t of $\widetilde{T_t}$ such that (T_t) is a Feller semigroup on $C_\infty(\mathbb{R}^d)$, see Proposition 4.3 in [26]. Note that (T_t) might not be a strong Feller semigroup. □

Let us now review the counter-example of [1]. We construct a kernel $n_1 : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ as follows. Choose $a \in (0,1)$ and $0 < \varepsilon < 1 - a$. For $z \in \mathbb{R}^2$, $z_1 \neq z_2$:

$$m_1(z_1,z_2) = \begin{cases} \min\big(|z_1|^{-a-2}, |z_2|^{-(a+\varepsilon)-2}\big) & \text{if } |z_1| \vee |z_2| \le 1, \\ 0 & \text{if } |z_1| \vee |z_2| > 1. \end{cases} \tag{5.4}$$

Note that there are $0 < c_0 \le c_1$ such that for $|z_1| \vee |z_2| \le 1$ one has

$$c_0|z|^{-a-2} \le \min\big(|z_1|^{-a-2}, |z_2|^{-(a+\varepsilon)-2}\big) \le c_1|z|^{-(a+\varepsilon)-2}. \tag{5.5}$$

Now for $x, y \in \mathbb{R}^2$, $x \neq y$ define $k_1(x,y)$ as follows. Let $V = \{(x_1,x_2) \colon |x_1| < |x_2|\}$. Set

$$k_1(x,y) = \begin{cases} m_1\big(|x_1-y_1|, |x_2-y_2|\big)\mathbb{1}_{\{x-y\in B_1(0)\}}, & x, y \in V, \\ m_1\big(|x_2-y_2|, |x_1-y_1|\big)\mathbb{1}_{\{x-y\in B_1(0)\}}, & x, y \notin V, \\ \big(|x_1-y_1|^{-2-a} \wedge |x_2-y_2|^{-2-a}\big)\mathbb{1}_{\{x-y\in B_1(0)\}}, & \text{elsewhere.} \end{cases} \tag{5.6}$$

Theorem 5.1 [1]. *Let $\mathscr{L}$ be as in (5.1) with k replaced by k_1. Then the martingale problem for $(\mathscr{L}, C_c^2(\mathbb{R}^d))$ is not well-posed for the starting point $0 \in \mathbb{R}^d$.*

The proof of this result is far from being trivial but the main idea can be grasped easily. The above construction has the following effect on the corresponding process X_t. If X_t is started from V it moves in short time intervals rather up or down than left or right. Started in V^c the preferred directions are swapped. One obtains that the transition probability $p(t,x,y)$ is discontinuous for small t at $x = 0$. As a result we find a continuous function v such that $x \mapsto \mathbb{E}^x v(X_t)$ is not continuous at 0 when t is small. On the other hand, if uniqueness to the martingale problem for $\mathscr{L}$ started at 0 were to hold, one would have $\mathbb{P}^x \to \mathbb{P}^0$ as $x \to 0$. This is a contradiction.

Note that $\varepsilon > 0$ can be arbitrarily small in the construction of m_1 and k_1. The following example is a byproduct of Theorem 1.3. It shows that a replacement of an ε-power by a logarithmic term in the construction of k_1 above again leads to a nice Feller semigroup.

Assume $a \in (0,1)$. For $z \in \mathbb{R}^2$, $z_1 \neq z_2$, set

$$m_2(z_1,z_2) = \begin{cases} \min\big(|z_1|^{-a-2}, |z_2|^{-a-2}\ln\big(\frac{3}{|z_2|}\big)\big) & \text{if } |z_1| \vee |z_2| \le 1, \\ 0 & \text{if } |z_1| \vee |z_2| > 1. \end{cases} \tag{5.7}$$

Define $k_2(x,y)$ with the help of $m_2(z_1,z_2)$ in the same way as $k_1(x,y)$ is defined using $m_1(z_1,z_2)$ above. Further below we show that k_2 satisfies the assumption of Theorem 1.3. Next, let $(\mathscr{E}, D(\mathscr{E}))$ be as in (1.5) and $\mathscr{L}$ be as in (5.1) with k replaced by k_2. Then Theorem 1.3 applies. If $(\mathscr{A}, D(\mathscr{A}))$ denotes the C_∞-generator of (T_t), then well-posedness of the martingale problem for $(\mathscr{A}, D(\mathscr{A}))$ follows directly from Theorem 4.1, Chapter 4 in [10] and Dynkin's formula. This statement completes the presentation of the example. Note that the results obtained in [3,4,26,27] do not apply to this case.

We close this session with an auxiliary result which we have just used.

Lemma 5.2. *Set $M = \{(x, y) \in \mathbb{R}^2, x \neq y, \max(|x|, |y|) \leq 1\}$. Choose $\beta > 0$. For $(x, y) \in M$ set*

$$m_2(x, y) = \min\left(|x|^{-\beta}, |y|^{-\beta} \ln\left(\frac{3}{|y|}\right)\right). \tag{5.8}$$

There are positive constants c_0, c_1 such that for all $(x, y) \in M$

$$\frac{c_0}{\sqrt{x^2 + y^2}^\beta} \leq m_2(x, y) \leq c_1 \frac{\ln(3/\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}^\beta}. \tag{5.9}$$

Proof. Throughout the proof we assume $x > 0$, $y > 0$. The estimate of $m_2(x, y)$ from below is trivial since

$$m_2(x, y) \geq \frac{1}{x^\beta + y^\beta(\ln(3/y))^{-1}} \geq \frac{1}{x^\beta + y^\beta} \geq \frac{c_0}{\sqrt{x^2 + y^2}^\beta}.$$

For the estimate from above consider two cases. First we assume $x^{-\beta} \geq y^{-\beta} \ln(\frac{3}{y})$. Then $0 < x \leq y \leq 1$ and $y \geq \frac{1}{\sqrt{2}}\sqrt{x^2 + y^2}$. Therefore by monotony we get

$$y^{-\beta} \ln\left(\frac{3}{y}\right) \leq 2^{\beta/2} \frac{\ln(3\sqrt{2}/\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}^\beta} \leq c_1 \frac{\ln(3/\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}^\beta}. \tag{5.10}$$

Now assume $x^{-\beta} \leq y^{-\beta} \ln(\frac{3}{y})$. If one has $0 < x \leq y \leq 1$ then we reason as in (5.10). In the case $0 < y \leq x \leq 1$ we have

$$x^{-\beta} \leq y^{-\beta} \ln\left(\frac{3}{x}\right). \tag{5.11}$$

Again we proceed as in (5.10) finishing the proof. $\square$

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