CONTINUOUS FIRST-PASSAGE PERCOLATION AND CONTINUOUS GREEDY PATHS MODEL: LINEAR GROWTH

BY JEAN-BAPTISTE GOUÉRÉ AND RÉGINE MARCHAND

Université d'Orléans and Université Henri Poincaré Nancy 1

We study a random growth model on $\mathbb{R}^d$ introduced by Deijfen. This is a continuous first-passage percolation model. The growth occurs by means of spherical outbursts with random radii in the infected region. We aim to find conditions on the distribution of the random radii to determine whether the growth of the process is linear or not. To do so, we compare this model with a continuous analogue of the greedy lattice paths model and transpose results for greedy paths from the lattice setting to the continuous setting.

1. Introduction and statement of the main results. We study a random growth model on $\mathbb{R}^d$, $d \geq 1$, introduced by Deijfen in [3]. The model can be thought of as describing the spread of an infection in a continuous medium. We fix an initially infected region S_0 in $\mathbb{R}^d$ (with positive Lebesgue measure) and a distribution μ on $(0, +\infty)$. Let us denote by S_t the random subset of $\mathbb{R}^d$ that corresponds to the infected region at time t and by $|S_t|$ its Lebesgue measure. The random growth process $(S_t)_{t \geq 0}$ is a Markov process whose dynamics is as follows. Given S_t, we wait an exponentially distributed random time with mean $|S_t|^{-1}$. We then add to S_t a random ball, whose center is chosen uniformly on S_t and whose radius is chosen accordingly to the law μ.

In [3] Deijfen proved an asymptotic shape result, namely, the almost sure convergence of $t^{-1} S_t$ toward a deterministic Euclidean ball. This convergence holds as soon as the growth of S_t is not superlinear. She provided a sufficient condition for this behavior of the growth: the boundedness of the support of μ. This condition was weakened by Deijfen, Häggström and Bagley in [2] to the existence of an exponential moment for μ.

In this work we aim to find a necessary and sufficient condition on the distribution μ of the radii of added balls to ensure the asymptotic shape result for this growth process. In Theorem 1.1 we prove the following:

- If

$$(1) \qquad \int_0^{+\infty} \left(\int_x^{+\infty} r\mu(dr) \right)^{1/d} dx < +\infty,$$

then the growth is not superlinear and, therefore, the asymptotic shape result holds.

- On the other hand, if

$$(2) \qquad \int_0^{+\infty} r^{d+1} \mu(dr) = +\infty,$$

then the growth is superlinear and, therefore, the asymptotic shape result does not hold.

In dimension $d = 1$, conditions (1) and (2) are exclusive and thus give a necessary and sufficient condition for the linear growth. Unfortunately, in dimension $d \geq 2$, there is a gap between these two conditions that we did not manage to fill. Note, however, that if there exists $\varepsilon > 0$ such that

$$\int_1^{+\infty} r^{d+1} (\ln r)^{d+\varepsilon} \mu(dr) < +\infty,$$

then (1) holds. The gap is therefore reasonably sharp.

To establish the sufficient condition (1), we introduce and study a continuous analogue to the greedy lattice paths model introduced by Cox, Gandolfi, Griffin and Kesten in [1]. In Theorem 1.2 we give a necessary condition and a sufficient condition for the integrability of the supremum of mean weights of paths in the continuous setting. Those results mimic similar ones in the discrete setting.

A comparison between Deijfen's model and those continuous greedy paths then enables us to conclude. Note that the gap between (1) and (2) comes directly from a similar gap for the continuous lattice paths model, a gap which is itself similar to the one existing for the greedy lattice paths model.

In the following, the dimension $d \geq 1$ is fixed. On $\mathbb{R}^d$, we denote by $\|\cdot\|$ the Euclidean norm, and by B_r the closed Euclidean ball centered at the origin with radius r.

1.1. *Deijfen's model.* Let us first recall the growth model introduced by Deijfen in [3]. Instead of using the original construction of the process, we use the construction given later in [4] by Deijfen and Häggström. This makes the analogy with first-passage percolation clearer. We follow the presentation of Gouéré in [7].

We fix a probability measure μ on $(0, +\infty)$. We also fix χ, a Poisson point process on $\mathbb{R}^d \times [0, +\infty) \times (0, +\infty)$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d \times [0, +\infty)$ by the distribution μ on $(0, +\infty)$.

Let us consider the complete directed graph with vertex set $\mathbb{R}^d$. We associate a passage time τ with each edge as follows:

(1) For all $x \in \mathbb{R}^d$, we let $\tau(x, x) = 0$.

(2) For each point $(c, t, r) \in \chi$—where c, t and r respectively belong to $\mathbb{R}^d$, $[0, +\infty)$ and $(0, +\infty)$—and for each vertex $y \in (c + B_r) \setminus \{c\}$, we let $\tau(c, y) = t$.

(3) For all edges (x, y) to which we have not yet assigned any passage time, we let $\tau(x, y) = +\infty$.

To say it in words, for a point (c, t, r) in the Poisson process χ, t represents the time needed to travel from the center c to each point of the ball centered in c with radius r, while outside the balls, it takes an infinite time to travel.

If a and b are two points of $\mathbb{R}^d$, we call path from a to b any finite sequence $\pi = (a = x_0, \ldots, x_k = b)$ of distinct points of $\mathbb{R}^d$. We denote by $\mathcal{C}(a, b)$ the set of such paths. With each path $\pi = (x_0, \ldots, x_k)$ we associate a passage time defined by

$$T(\pi) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

If A is a subset of $\mathbb{R}^d$ and x is a point of $\mathbb{R}^d$, we can then define the time $T(A, x)$ needed to cover x starting from A by

$$T(A, x) = \inf\{T(\pi) : a \in A, \ \pi \in \mathcal{C}(a, x)\}.$$

Finally, if we start the process with the unit Euclidean ball B centered at the origin, we can define the set S_t of covered points at time t by

$$S_t = \{x \in \mathbb{R}^d : T(B, x) \le t\}.$$

Relying on Kingman's subadditive ergodic theorem and using the isotropy of the model, one can establish the existence of a real $\lambda \ge 0$ such that the following convergence holds almost surely:

$$\lim_{\|x\| \to +\infty} \frac{T(B, x)}{\|x\|} = \lambda.$$

This result is contained in the paper by Deijfen [3]. The growth is linear if λ, which is the inverse of the speed, is positive. In such a case, one can easily deduce the following asymptotic shape result: almost surely, for every $\varepsilon > 0$, for all large enough t, one has

$$B_{(1-\varepsilon)/\lambda} \subset \frac{S_t}{t} \subset B_{(1+\varepsilon)/\lambda}.$$

To prove the positivity of λ when μ admits an exponential moment, Bagley, Deijfen and Häggström introduce a new process that grows faster than Deijfen's one and whose linear growth is easier to prove. This new process can be roughly described as follows. Assume that we have started with a set A_0 and that at time t we have added balls $A_1, \ldots, A_n$. With each set A_i we associate an exponential clock with mean $|A_i|^{-1}$. These clocks are independent. We wait for the first clock to ring. If it is clock i, then we choose a point uniformly in A_i and add a ball centered at this point with random radius. The projection on the first-coordinate axis of this new process is a one dimensional spatial branching process whose linear growth, when μ admits an exponential moment, is well known.

Our conditions for the linear growth of Deijfen's model are the following ones.

THEOREM 1.1.   1. *If*

$$(3) \qquad \int_0^{+\infty} \left(\int_x^{+\infty} r\mu(dr) \right)^{1/d} dx < +\infty,$$

then there exists $\lambda \in (0, +\infty)$ *such that, almost surely, for all* $\varepsilon > 0$, *for all large enough* t, *one has*

$$B_{(1-\varepsilon)/\lambda} \subset \frac{S_t}{t} \subset B_{(1+\varepsilon)/\lambda}.$$

2. *If*

$$(4) \qquad \int_0^{+\infty} r^{d+1} \mu(dr) = +\infty,$$

then, almost surely, for all $M > 0$, *for all large enough* t, *one has*

$$B_M \subset \frac{S_t}{t}.$$

REMARKS.   1. Note that in dimension $d = 1$, if $\int_{(0,+\infty)} r^2 \mu(dr) < +\infty$, one can compute explicitly the speed of the growth. With the notation of the theorem, one finds $\lambda^{-1} = \frac{1}{2} \int_0^{+\infty} r^2 \mu(dr)$. We sketch a proof of this result in Appendix.

2. To see how quantities scale, let us prove that if one multiplies the radii by 2, then one multiplies the speed by 2^{d+1}. Let us consider the following Poisson point processes:

$$\chi_1 = \{(c, t, 2r), (c, t, r) \in \chi\},$$
$$\chi_2 = \{(2^{-1}c, 2^{-1}t, r), (c, t, r) \in \chi\},$$
$$\chi_3 = \{(c, 2^{-(d+1)}t, r), (c, t, r) \in \chi\}.$$

With those points processes one can, as we have done with χ, associate passage times to paths. We denote them by T_1, T_2 and T_3. Let $\pi = (x_0, \ldots, x_n)$ be a path that originates in 0. Then

$$T_1(2\pi) \|2x_n\|^{-1} = T_2(\pi) \|x_n\|^{-1},$$
$$T_2(\pi) \overset{\text{law}}{=} T_3(\pi),$$
$$T_3(\pi) = 2^{-(d+1)} T(\pi).$$

(The second property results of the equality in law of the Poisson point processes χ_2 and χ_3, which itself results of the equality of their intensities.) This gives the announced scaling.

Some of the ideas of the proof. In the end of this subsection we would like to explain how trying to rule out the possibility of an infinite speed in Deijfen's model naturally leads to introduce continuous greedy paths. We fix a small $\alpha > 0$ and consider the "fast" balls:

$$(5) \qquad \xi_\alpha = \{(c, r) \colon \exists t \leq \alpha r \colon (c, t, r) \in \chi\}.$$

Roughly speaking, the speed in these balls is at least $1/\alpha$, while outside these "fast" balls, the speed is less than $1/\alpha$. More precisely, consider a path $\pi = (x_0, \dots, x_n)$ such that $T(\pi)$ is finite. Then, by the definition of T, every point x_i in π—but the last—is the first coordinate of a point (x_i, t_i, r_i) in χ. Moreover, we have $\|x_{i+1} - x_i\| \leq r_i$ and

$$T(\pi) = \sum_{i=0}^{n-1} t_i.$$

If the path uses only "slow" balls, then

$$T(\pi) = \sum_{i=0}^{n} t_i \geq \sum_{i=0}^{n} \alpha r_i \geq \sum_{i=0}^{n} \alpha \|x_{i+1} - x_i\| = \alpha |\pi|,$$

where $|\pi|$ is the length of the path π, that is, the sum of the Euclidean length of its segments.

If it also uses a "fast" ball (x_i, t_i, r_i), the portion between x_i and x_{i+1} is traveled through at high speed. By considering that this portion is traveled through at infinite speed, we obtain (the sums are over visited "fast" balls)

$$T(\pi) \geq \alpha \Big(|\pi| - \sum \|x_{i+1} - x_i\|\Big)$$

$$\geq \alpha \Big(|\pi| - \sum r_i\Big) \quad \text{and then} \quad \frac{T(\pi)}{|\pi|} \geq \alpha \Big(1 - \frac{1}{|\pi|} \sum r_i\Big).$$

We are therefore led to bound from above the following kind of quantity:

$$\frac{1}{|\pi|} \sum_{x_i \in \pi, (x_i, r_i) \in \xi_\alpha} r_i.$$

In the greedy lattice paths model one assigns random weights to points in $\mathbb{Z}^d$ and tries to bound average weights of paths (see Section 1.2). The previous expression naturally leads to define and study the continuous analogue of greedy paths discussed in Section 1.3. This crude link between Deijfen's model and continuous greedy paths will be detailed in Section 2.3. The proof of Theorem 1.1 is given in Section 2.4.

1.2. *Greedy lattice paths model.* In this subsection $d \geq 2$.

One first gives to points c of $\mathbb{Z}^d$ i.i.d. positive random weights $r(c)$ with common law ν. A path is here a finite sequence of distinct points of $\mathbb{Z}^d$ such that the Euclidean distance between any two consecutive points is 1 and the length of a path is naturally the sum of the Euclidean lengths of its segments. With each path one associates a weight which is the sum of the weights of its points. If n is a positive integer, we denote by A_n the supremum of the weights of all paths with length n that originates in 0. In [1] the authors show that if there exists a real $\varepsilon > 0$ such that

$$\int_1^{+\infty} r^d (\ln r)^{d+\varepsilon} \nu(dr) < +\infty,$$

then there exists a real $M < +\infty$ such that

$$\limsup_{n \to +\infty} \frac{A_n}{n} \leq M \qquad \text{a.s.}$$

This result was improved in [5] by Gandolfi and Kesten: under the same condition, A_n/n converges almost surely and in L^1 toward a finite constant. Martin, in [8], obtains the same results under a weaker assumption and with a much simpler proof:

$$\text{if } \int_0^{+\infty} \nu([r, +\infty))^{1/d} < +\infty,$$

$$\text{then } A_n/n \text{ converges a.s. and in } L^1 \text{ to a finite constant.}$$

As an intermediate step, he shows that

$$(6) \qquad \text{if } \int_0^\infty \nu([r, +\infty))^{1/d} < +\infty, \qquad \text{then } \sup \mathbb{E}\left(\frac{A_n}{n}\right) < +\infty.$$

Deriving this property in a continuous setting will turn out to be sufficient for our purpose.

On the other hand, from results in [1] and [8], one knows that

$$\text{if } \int_0^{+\infty} r^d \nu(dr) = +\infty, \qquad \text{then } A_n/n \text{ almost surely goes to} +\infty.$$

1.3. *Continuous greedy paths.* In our continuous analogue the points of the lattice $\mathbb{Z}^d$ are replaced by the points of a homogeneous Poisson point process on $\mathbb{R}^d$. Fix a finite measure ν on $(0, +\infty)$, and consider a Poisson point process ξ on $\mathbb{R}^d \times (0, +\infty)$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by the measure ν. We denote by Ξ the projection of ξ on $\mathbb{R}^d$: the point process Ξ is thus a Poisson point process on $\mathbb{R}^d$ with intensity $\nu((0, +\infty))$ times the Lebesgue measure on $\mathbb{R}^d$. If x is a point of Ξ, we denote by $r(x)$ the only positive real number such that $(x, r(x))$ belongs to ξ. Thus, given Ξ, the weights $(r(x))_{x \in \Xi}$ are i.i.d.

with common law $\nu((0, +\infty))^{-1}\nu$ [if $\nu((0, +\infty))$ is positive]. For points $x \in \mathbb{R}^d$ that are not in Ξ, we set $r(x) = 0$.

A path is a finite sequence of distinct points of $\mathbb{R}^d$. (In the lattice model, consecutive points of a path are required to be nearest neighbors. We do not require such a condition in our continuous model, which is therefore not an exact analogue of the lattice model.) We denote by $|\pi|$ its length, that is, the sum of the Euclidean length of its segments. We define the weight $A(\pi)$ of a path $\pi = (x_0, \ldots, x_n)$ by

$$A(\pi) = \sum_{i=0}^{n} r(x_i).$$

We are interested in the finiteness of the supremum S of the mean weights of paths, defined as

$$(7) \qquad\qquad S = \sup\left\{\frac{A(\pi)}{|\pi|}\right\},$$

where the supremum is taken over all paths whose length is positive and that originate in 0. In order to emphasize the dependence of S on ξ, we shall sometimes use the notation $S(\xi)$. We also introduce, for $l > 0$,

$$(8) \qquad\qquad S_l = \sup\left\{\frac{A(\pi)}{|\pi|}\right\},$$

where the supremum is now taken over all paths whose length is larger than l and that originate in 0.

As we will mainly consider paths starting from 0, to avoid extra discussion on the status of the origin, we will always work on the full event $\{0 \notin \Xi\}$. Let us also notice that we do not change S if we take the supremum over all paths $(x_0, \ldots, x_n)$ such that, in addition to the previous requirements, x_i belongs to Ξ for all $i \geq 1$. This can be seen by the triangular inequality. We shall use this remark when convenient without further reference.

We state in the following theorem sufficient conditions for S to be either integrable or a.s. infinite. These conditions are similar to the ones obtained for the discrete setting.

THEOREM 1.2. *Assume $d \geq 2$.*
1. *If $\int_0^{+\infty} \nu([r, +\infty))^{1/d}\, dr < +\infty$, then $\mathbb{E}S < +\infty$.*
2. *If $\int_0^{+\infty} r^d \nu(dr) = +\infty$, then S is a.s. infinite.*

REMARK. If $d = 1$, then $\mathbb{E}(S)$ is infinite as soon as $\nu((0, +\infty))$ is positive: the contribution of the first positive point already has an infinite mean. Indeed, denote by X the smallest positive point of Ξ. This is an exponential random variable with parameter $\nu((0, +\infty))$. Moreover, $r(X)$ is distributed according to $\nu((0, +\infty))^{-1}\nu$ and is independent of X. As a consequence, $\mathbb{E}(r(X)X^{-1})$ is infinite and, therefore, $\mathbb{E}(S)$ is infinite. We shall therefore be led to study $\mathbb{E}(\lim_{l \to +\infty} S_l)$ when $d = 1$ in Section 2.2.

We conclude this subsection by giving some ideas of the proof. The second item of the theorem is rather straightforward [actually, the supremum of $r(x)\|x\|^{-1}$, $x \in \Xi$, is already a.s. infinite]. The proof of the first item follows the proof of the corresponding result in the lattice setting by Martin [8]. The first step consists in studying the case where ν is the Dirac mass at point 1, in which case S is integrable. For a general measure ν, one then distinguishes between the contribution of the points $x \in \Xi$ according to the value of the radius $r(x)$. Using the fact that the supremum of a sum is less than the sum of the supremum, this allows to obtain the following upper bound [it is derived in details in (9)]:

$$\mathbb{E}S(\xi) \leq \int_0^{+\infty} \mathbb{E}S(\xi^\rho)\,d\rho,$$

where $\xi^\rho = \{(c, 1) : c \in \Xi \text{ and } r(c) \geq \rho\}$. The point is then that, by a scaling argument for Poisson point processes, one can express $\mathbb{E}S(\xi^\rho)$ as the product of $\nu([\rho, +\infty))^{1/d}$ by the expectancy of S in the case where ν is the Dirac mass at point 1. This scaling argument, which is straightforward in the continuous setting, explains the role played by the dimension d. The proof is given in Section 2.1.

Note also that the link between the measure ν in the continuous greedy paths model and the distribution μ in Deijfen's model is presented in Section 2.3 (and thus the link between conditions in Theorem 1.2 and in Theorem 1.1).

1.4. *Links with the Boolean model of continuum percolation and further intuition.*   In this part we point out some links with the Boolean model of continuum percolation and give some further intuition. Fix $\alpha > 0$ and denote by Σ the union of the balls $c + B_r$, $(c, r) \in \xi_\alpha$ [ξ_α is defined by (5)]. This is the Boolean model of continuum percolation driven by ξ_α. We can define a first-passage percolation process on the complete nonoriented graph of $\mathbb{R}^d$ as follows: the time needed to travel along an edge xy is α times the one-dimensional Lebesgue measure of $[x, y] \setminus \Sigma$. In other words, one travels at speed α^{-1} outside Σ and at speed $+\infty$ inside Σ. Let us call it the Boolean first-passage percolation process. By coupling, we can see that the speed in this Boolean first-passage percolation process is larger than in Deijfen's one.

The intensity of ξ_α is the product of the Lebesgue measure on $\mathbb{R}^d$ by the measure $\nu_\alpha(dr) = \alpha r \mu(dr)$, which is also α times the product of the Lebesgue measure on $\mathbb{R}^d$ by the measure $\nu_1(dr) = r \mu(dr)$. By a result of Gouéré [6], one knows that the connected component of Σ that contains the origin is almost surely bounded for small enough α if and only if $\int r^d \nu_1(dr)$ is finite, that is, if and only if $\int r^{d+1} \mu(dr)$ is finite.

This suggests that, when $\int r^{d+1} \mu(dr)$ is finite, a constant and positive proportion of any path should lie outside Σ. If this result were true, then the speed in the Boolean first-passage percolation process—and therefore the speed in Deifen's model—would be finite as soon as $\int r^{d+1} \mu(dr)$ is finite. Unfortunately, we do

not know whether the finiteness of $\int r^{d+1}\mu(dr)$ is sufficient or not to bound away from 0 the proportion of length that any path spends outside Σ.

On the other hand, if $\int r^{d+1}\mu(dr)$ is infinite, "fast" balls with speed larger than $1/\alpha$ percolate for all $\alpha > 0$. This very roughly suggests that the speed in Deifjen's model is at least $1/\alpha$ for every $\alpha > 0$, which means that the growth is superlinear.

To conclude, let us notice that one can also compare the Boolean first-passage percolation process with the continuous greedy paths. The Boolean first-passage percolation process is actually an intermediate model between Deijfen's one and the continuous greedy paths model. With the results of this paper about continuous greedy paths, one can therefore show that the growth in the Boolean first-passage percolation process is not superlinear for small enough α as soon as $\int_0^{+\infty} \nu_1([r, +\infty))^{1/d}\,dr$ is finite. This implies that, under the same conditions, the Boolean model driven by ξ_α does not percolate. This is almost the main result of [6]: nonpercolation of the Boolean model for small enough α when $\int_0^{+\infty} r^d \nu_1(dr)$ is finite.

2. Proofs.

2.1. *Continuous greedy paths*: *proof of Theorem* 1.2. We keep the notation and objects introduced in Section 1.3. We begin with the case of a deterministic radius equal to 1 and we denote by δ_1 the Dirac mass at point 1.

LEMMA 2.1. *If $d \geq 2$ and $\nu = \delta_1$, then $\mathbb{E}S < +\infty$.*

PROOF. Let $\alpha_0 > 0$ be such that

$$\forall \alpha \geq \alpha_0 \qquad \int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|)\,dx < 1.$$

Let $\alpha \geq \alpha_0$ be fixed and fix also an integer $k \geq 1$. Let us denote by $B(k, \alpha)$ the set of all finite sequences $(x_1, \ldots, x_k)$ of distinct points of Ξ such that

$$k \geq \alpha\left(\|x_1\| + \sum_{i=2}^k \|x_i - x_{i-1}\|\right).$$

For a finite set A, we denote by card(A) its cardinality. We have

$$\mathbb{P}\big(B(k, \alpha) \neq \varnothing\big) \leq \mathbb{E}(\text{card}(B(k, \alpha)))$$

$$= \int_{(\mathbb{R}^d)^k} \mathbf{1}_{k \geq \alpha(\|x_1\| + \sum_{i=2}^k \|x_i - x_{i-1}\|)}\,dx_1 \cdots dx_k$$

$$\leq \int_{(\mathbb{R}^d)^k} \exp\left(k - \alpha\left(\|x_1\| + \sum_{i=2}^k \|x_i - x_{i-1}\|\right)\right)dx_1 \cdots dx_k$$

$$= \left(\int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|)\,dx\right)^k.$$

Let us denote by $F(\alpha)$ the following event: there exists a path π originating in 0, with positive length and whose points, except 0, belong to Ξ, such that the inequality $A(\pi) \geq \alpha|\pi|$ holds. Decomposing on the number of points in the path, we get

$$\{S > \alpha\} \subset F(\alpha) = \bigcup_{k=1}^{+\infty} \{B(k, \alpha) \neq \varnothing\},$$

and thus,

$$\mathbb{P}(S > \alpha) \leq \sum_{k=1}^{+\infty} \left(\int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|) \, dx \right)^k$$

$$\leq \int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|) \, dx \left(1 - \int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|) \, dx \right)^{-1}$$

$$\leq \int_{\mathbb{R}^d} \exp(1 - \alpha\|x\|) \, dx \left(1 - \int_{\mathbb{R}^d} \exp(1 - \alpha_0\|x\|) \, dx \right)^{-1}$$

$$\leq \alpha^{-d} \int_{\mathbb{R}^d} \exp(1 - \|x\|) \, dx \left(1 - \int_{\mathbb{R}^d} \exp(1 - \alpha_0\|x\|) \, dx \right)^{-1},$$

which is an integrable function of α since $d \geq 2$. $\square$

We next give the scaling argument announced in the introduction. Denote by ξ_ν a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ with intensity the product of Lebesgue's measure on $\mathbb{R}^d$ by the positive finite measure ν on $(0, +\infty)$.

LEMMA 2.2.   *For any $m > 0$, $S(\xi_{m\nu})$ has the same law as $m^{1/d} S(\xi_\nu)$.*

PROOF.   We just need to notice that the random set $\{(m^{1/d} c, r) : (c, r) \in \xi_{m\nu}\}$ is a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ with intensity the product of Lebesgue's measure on $\mathbb{R}^d$ by the positive finite measure ν on $(0, +\infty)$. $\square$

PROOF OF THEOREM 1.2.   1. Assume that

$$\int_0^{+\infty} \nu([\rho, +\infty))^{1/d} \, d\rho < +\infty,$$

and let us prove that $\mathbb{E}S < +\infty$.

First, we need to make the process with the Dirac mass appear, by, in a certain manner, decomposing on the different values of the support of ν. In fact, the useful

way to do so is to use the classical trick $r = \int_0^{+\infty} \mathbf{1}_{r \geq \rho} \, d\rho$: we have

$$\mathbb{E}S = \mathbb{E}\left(\sup_{\pi=(0,x_1,\ldots,x_n)} \int_0^{+\infty} \frac{\sum_i \mathbf{1}_{r(x_i) \geq \rho}}{|\pi|} \, d\rho \right)$$

(9)
$$\leq \int_0^{+\infty} \mathbb{E}\left(\sup_{\pi=(0,x_1,\ldots,x_n)} \frac{\sum_i \mathbf{1}_{r(x_i) \geq \rho}}{|\pi|} \right) d\rho$$

$$\leq \int_0^{+\infty} \mathbb{E}S(\xi^\rho) \, d\rho,$$

where ξ^ρ is the point process on $\mathbb{R}^d \times (0, +\infty)$ defined by

$$\xi^\rho = \{(c, 1) : c \in \Xi \text{ and } r(c) \geq \rho\}.$$

Notice that ξ^ρ is a Poisson point process whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by $\nu([\rho, +\infty))\delta_1$.

Then, we use the scaling property: if $\widetilde{\xi}$ is a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by δ_1, then

(10)
$$\mathbb{E}S(\xi^\rho) = \nu([\rho, +\infty))^{1/d}\mathbb{E}S(\widetilde{\xi}).$$

Indeed, if $\nu([\rho, +\infty)) = 0$, then the equality is straightforward, while if $\nu([\rho, +\infty)) \neq 0$, it is a simple application of Lemma 2.2.

Finally, from (9) and (10) we get

(11)
$$\mathbb{E}S \leq \int_0^{+\infty} \nu([\rho, +\infty))^{1/d}\mathbb{E}S(\widetilde{\xi}) \, d\rho < +\infty,$$

by the integrability assumption on ν and Lemma 2.1.

2. Assume that

$$\int_0^{+\infty} r^d \nu(dr) = +\infty,$$

and let us prove that $S = +\infty$ a.s.

Let $M > 0$ and consider the following point process:

(12)
$$\{(c, r) : (c, r) \in \xi, r \geq M\|c\| \text{ and } c \neq 0\}.$$

The cardinality of this point process is distributed according to a Poisson law with parameter

$$\int_{\mathbb{R}^d} dc \int_0^{+\infty} \mathbf{1}_{r \geq M\|c\|} \nu(dr) = \int_0^{+\infty} |B_{rM^{-1}}|\nu(dr) = |B_{M^{-1}}| \int_0^{+\infty} r^d \nu(dr).$$

By our assumption on ν, this equals infinity and, therefore, this point process is almost surely nonempty. But if (c, r) is a point of this point process, then, by considering the path $(0, c)$, we get $S \geq M$ a.s., which concludes the proof. $\quad\square$

2.2. *Continuous greedy paths in dimension $d = 1$.* To tackle the one dimensional case, where $\mathbb{E}(S) = +\infty$, we will use the asymptotic behavior of S_l stated in the following easy result.

LEMMA 2.3. *If $d = 1$ and $\int_0^{+\infty} r\nu(dr)$ is finite, then*

$$\lim_{l \to +\infty} S_l \leq 2 \int_0^{+\infty} r\nu(dr) \qquad a.s.$$

PROOF. Let $l > 0$ and $\pi = (x_0, \ldots, x_n) \in \Pi_l$, where Π_l is the set of paths whose length is larger than l and that originate in 0. As $x_0 = 0$, all the x_i belong to $[-|\pi|, |\pi|]$. We then have

$$A(\pi) \leq \sum_{x \in \Xi \cap [-|\pi|, |\pi|]} r(x).$$

Therefore,

(13) $$\lim S_l \leq \limsup \frac{1}{l} \sum_{x \in \Xi \cap [-l,l]} r(x).$$

Recall the following:

- Ξ is a Poisson point process with intensity $\nu((0, +\infty))$ times the Lebesgue measure. This implies that the number of points in $\Xi \cap [-l, l]$ is Poisson distributed with mean $2l\nu((0, +\infty))$.
- Given Ξ, the sequence $(r(x))_{x \in \Xi}$ is an i.i.d. sequence of random variables distributed according to $\nu((0, +\infty))^{-1}\nu$.

Therefore, the right-hand side of (13) is a.s.

$$2\nu((0, +\infty)) \int_{(0,+\infty)} r\nu((0, +\infty))^{-1}\nu(dr).$$

This concludes the proof. □

2.3. *A link between Deijfen's model and continuous greedy paths.* Let us recall that the Poisson point process χ, driving Deijfen's model, has been introduced in Section 1.1. In this subsection we fix a real $\alpha > 0$. We consider the following point process on $\mathbb{R}^d \times (0, +\infty)$:

$$\xi_\alpha = \{(c, r) : \exists t \leq \alpha r : (c, t, r) \in \chi\}.$$

In other words, ξ_α is the projection on $\mathbb{R}^d \times (0, +\infty)$ of the intersection of χ with the Borel set

$$\{(c, t, r) \in \mathbb{R}^d \times [0, +\infty) \times (0, +\infty) : t \leq \alpha r\}.$$

Let us notice that ξ_α is a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by the finite measure ν_α on $(0, +\infty)$ defined by

$$\nu_\alpha(dr) = \alpha r \mu(dr).$$

We consider the continuous greedy paths model driven by ξ_α. Except for the name of this point process, we keep the notation and objects defined in Sections 1.3 and 1.1.

The set ξ_α corresponds to the "fast" balls, that is, the balls where the infection progresses with a speed larger than $1/\alpha$. The next lemma gives the link between the travel time in Deijfen's model and the functional S in the continuous greedy paths model driven by ξ_α: roughly speaking, outside the balls in ξ_α, the travel time between two points is at least α times the Euclidean distance between the two points, and the existence of "fast" balls gives a correction term controlled by $S(\xi_\alpha)$.

LEMMA 2.4. *For all $x \in \mathbb{R}^d \setminus B$, for all $\alpha > 0$, one has*

$$\frac{T(B, x)}{\|x\|} \geq \alpha \left(1 - S_{\|x\|}(\xi_\alpha) - \frac{1}{\|x\|} \right).$$

PROOF.   Let $x \in \mathbb{R}^d \setminus B$. Let $\pi = (x_0, \ldots, x_n)$ be a path from B to x. (In other words, x_0 belongs to B and x_n equals x.) Therefore, in order to prove the lemma, it is sufficient to check the following inequality:

$$\frac{T(\pi)}{\|x\|} \geq \alpha \left(1 - S_{\|x\|}(\xi_\alpha) - \frac{1}{\|x\|} \right).$$

We assume that for every $i \in \{1, \ldots, n\}$, $x_i \neq 0$ [otherwise, if $x_i = 0$, one uses the inequality $T(\pi) \geq T(x_i, \ldots, x_n)$ and works with the path $(x_i, \ldots, x_n)$].

We extend π in a path $\tilde{\pi}$ starting from 0 by adding, if necessary, a first point $x_{-1} = 0$ to π. As $x_n = x$, the length of the path $\tilde{\pi}$ is at least $\|x\|$. To establish the lemma, it is therefore sufficient to prove that

$$\frac{T(\pi)}{\|x\|} \geq \alpha \left(1 - \frac{A(\tilde{\pi})}{|\tilde{\pi}|} - \frac{1}{\|x\|} \right).$$

Let $i \in \{0, \ldots, n-1\}$. Let us show the following inequality:

(14) $\tau(x_i, x_{i+1}) \geq \alpha \|x_i - x_{i+1}\| - \alpha r(x_i).$

Remember that if there exists r_i (a.s. unique) such that $(x_i, r_i) \in \xi_\alpha$, then $r(x_i) = r_i$, and $r(x_i) = 0$ in any other case. Two cases arise:

(1) If $\tau(x_i, x_{i+1})$ is infinite, (14) is obvious.
(2) Otherwise, there exist r_i, t_i such that (x_i, t_i, r_i) belongs to χ and $\|x_i - x_{i+1}\| \leq r_i$, which implies $\tau(x_i, x_{i+1}) = t_i$.

- If $t_i > \alpha r_i$, then $(x_i, r_i) \notin \xi_\alpha$, and $r(x_i) = 0$: thus, (14) holds.
- If $t_i \leq \alpha r_i$, then $(x_i, r_i) \in \xi_\alpha$ and, thus, $r(x_i) = r_i$, which gives (14).

As $\alpha r(x_n)$ is nonnegative, summing (14) for $i \in \{0, \ldots, n-1\}$ implies that $T(\pi) \geq \alpha(|\pi| - A(\pi))$. Remember that a.s., $0 \notin \Xi$, and, thus, that $r(0) = 0$ a.s. As

$$\|x_0 - x_{-1}\| - r(x_{-1}) = \|x_0\| - r(0) \leq 1,$$

we obtain [whether $\tilde{\pi} = (x_{-1}, \ldots, x_n)$ or $\tilde{\pi} = (x_0, \ldots, x_n)$] that $T(\pi) \geq \alpha(|\tilde{\pi}| - A(\tilde{\pi}) - 1)$. From $|\tilde{\pi}| \geq \|x_n - 0\| = \|x\|$, we then deduce

$$(15) \qquad \frac{T(\pi)}{\|x\|} \geq \frac{T(\pi)}{|\tilde{\pi}|} \geq \alpha\left(1 - \frac{A(\tilde{\pi})}{|\tilde{\pi}|} - \frac{1}{|\tilde{\pi}|}\right) \geq \alpha\left(1 - \frac{A(\tilde{\pi})}{|\tilde{\pi}|} - \frac{1}{\|x\|}\right).$$

The lemma follows. $\square$

2.4. *Deijfen's model*: *Proof of Theorem* 1.1.   Let us recall the following result from Deijfen [3].

THEOREM 2.5.   *There exists a constant $\lambda \geq 0$ such that the following convergence holds almost surely and in L^1:*

$$\lim_{\|x\| \to +\infty} \frac{T(B, x)}{\|x\|} = \lambda.$$

If $\lambda > 0$, then, almost surely, for all $\varepsilon > 0$ and for all large enough positive real t, one has

$$B_{\lambda^{-1}(1-\varepsilon)t} \subset S_t \subset B_{\lambda^{-1}(1+\varepsilon)t}.$$

If $\lambda = 0$, then, almost surely, for all $M > 0$ and for all large enough positive real t, one has

$$B_{Mt} \subset S_t.$$

Actually, Theorem 2.5 is not explicitly stated in [3]. Nevertheless, it can be easily proven using ideas and results of Deijfen's paper. A complete proof can be found in the first version of our paper on the Mathematics ArXiv.

The following lemma will enable us to prove the first part of Theorem 1.1. We have stated it in such a way that its proof does not rely on Theorem 2.5.

LEMMA 2.6.

$$If \int_0^{+\infty} \left(\int_x^{+\infty} r\mu(dr)\right)^{1/d} dx < +\infty \qquad then \liminf_{\|x\| \to +\infty} \frac{\mathbb{E}T(B, x)}{\|x\|} > 0.$$

PROOF.   Recall that measures ν_α, $\alpha > 0$, are defined in Section 2.3 and that

$$(16) \qquad \nu_\alpha(dr) = \alpha\nu_1(dr) = \alpha r\mu(dr).$$

By assumption, the following condition holds:

$$\int_0^{+\infty} \nu_1([x, +\infty))^{1/d}\, dx < +\infty.$$

Let us first consider the case $d \geq 2$. By Theorem 1.2, we then get that $\mathbb{E}S(\xi_1)$ is finite. Using (16), Lemma 2.2 ensures that $\mathbb{E}S(\xi_\alpha) = \alpha^{1/d}\mathbb{E}S(\xi_1)$. As $S_{\|x\|}(\xi_\alpha) \leq S(\xi_\alpha)$, Lemma 2.4 implies then that, for all real $\alpha > 0$,

$$(17) \qquad \liminf_{\|x\| \to +\infty} \frac{\mathbb{E}T(B, x)}{\|x\|} \geq \alpha\big(1 - \mathbb{E}S(\xi_\alpha)\big) = \alpha\big(1 - \alpha^{1/d}\mathbb{E}S(\xi_1)\big).$$

But this quantity is positive as soon as α is small enough.

Let us assume now that $d = 1$. The assumption of the lemma guarantees that $\int_0^{+\infty} r^2\mu(dr) < +\infty$. By Lemma 2.4, we get

$$\liminf_{\|x\| \to +\infty} \frac{T(B, x)}{\|x\|} \geq \alpha\left(1 - \lim_{l \to +\infty} S_l(\xi_\alpha)\right).$$

By Lemma 2.3, we then get, almost surely,

$$\liminf_{\|x\| \to +\infty} \frac{T(B, x)}{\|x\|} \geq \alpha\left(1 - 2\int_0^{+\infty} r\nu_\alpha(dr)\right) = \alpha\left(1 - 2\alpha\int_0^{+\infty} r^2\mu(dr)\right).$$

By the Fatou lemma, we get

$$\liminf_{\|x\| \to +\infty} \frac{\mathbb{E}T(B, x)}{\|x\|} \geq \alpha\left(1 - 2\alpha\int_0^{+\infty} r^2\mu(dr)\right).$$

But this quantity is positive as soon as α is small enough. This ends the proof of the lemma. $\square$

REMARK.   By optimizing in α in equation (17), we get, for $d \geq 2$, the following lower bound for the inverse of the speed:

$$\lambda \geq \frac{d^d}{(d+1)^{d+1}}\left(\frac{1}{\mathbb{E}S(\xi_1)}\right)^d.$$

Using the bound in (11), we obtain that, if $\widetilde{\xi}$ is a Poisson point process on $\mathbb{R}^d \times (0, +\infty)$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by δ_1, then

$$\mathbb{E}S(\xi_1) \leq \mathbb{E}S(\widetilde{\xi})\int_0^{+\infty}\left(\int_\rho^{+\infty} r\mu(dr)\right)^{1/d} d\rho.$$

Finally, there exists a positive constant C_d that only depends on the dimension d such that

$$\lambda \geq C_d \left(\int_0^{+\infty} \left(\int_\rho^{+\infty} r\mu(dr) \right)^{1/d} d\rho \right)^{-d}.$$

The result is still true for $d = 1$.

The following lemma will enable us to prove the second part of Theorem 1.1.

LEMMA 2.7.

$$If \int_0^{+\infty} r^{d+1} \mu(dr) = +\infty, \qquad then \lim_{x \to +\infty} \frac{T(B,x)}{\|x\|} = 0 \qquad a.s. \ and \ in \ L^1.$$

PROOF.   By Theorem 2.5, one can fix a real $\beta > 0$ such that, almost surely, for all x such that $\|x\|$ is large enough, the following inequality holds:

(18) $$T(B,x) \leq \beta \|x\|.$$

Let $M > 0$ and write $A = \{(c,t,r) \in \mathbb{R}^d \times [0,+\infty) \times (0,+\infty) : t \leq \|c\| \leq rM^{-1}\}$. The cardinality of $\chi \cap A$ is distributed according to a Poisson law with parameter

$$\int_{\mathbb{R}^d} dc \int_{[0,+\infty)} dt \int_{(0,+\infty)} \mu(dr)\mathbf{1}_{(c,t,r) \in A}$$

$$= \int_{(0,+\infty)} \mu(dr) \int_{B_{rM^{-1}}} dc\, \|c\|$$

$$= \int_{(0,+\infty)} \mu(dr) r^{d+1} \int_{B_{M^{-1}}} dc\, \|c\| = +\infty,$$

because of the assumption made about μ. Therefore, the cardinality of $\chi \cap A$ is almost surely infinite. Let s be a positive real and write $A_s = \{(c,t,r) \in A : \|c\| \leq s\}$. The cardinality of $\chi \cap A_s$ is distributed according to a Poisson law with parameter

$$\int_{\mathbb{R}^d} dc \int_{[0,+\infty)} dt \int_{(0,+\infty)} \nu(dr)\mathbf{1}_{(c,t,r) \in A_s} \leq |B_s|s < +\infty.$$

The cardinality of $\chi \cap A_s$ is therefore almost surely finite. The two preceding observations imply that, almost surely, there exists a sequence $(c_n, t_n, r_n)_n$ with values in $\chi \cap A$ such that $\|c_n\|$ goes to infinity.

As Mc_n belongs to $c_n + B_{r_n}$, we have $\tau(c_n, Mc_n) = t_n$, which leads, using (18) and the definition of A, to

$$T(B, Mc_n) \leq T(B, c_n) + t_n \leq \beta\|c_n\| + \|c_n\| = (1+\beta)\|c_n\|,$$

(for large enough n) and thus,

$$\frac{T(B, Mc_n)}{M\|c_n\|} \leq \frac{1+\beta}{M}, \qquad \text{which implies } \liminf_{\|x\| \to \infty} \frac{T(B,x)}{\|x\|} \leq \frac{1+\beta}{M}.$$

Theorem 2.5 enables us then to conclude the proof.   □

PROOF OF THEOREM 1.1.   Just apply Theorem 2.5, Lemma 2.6 and Lemma 2.7.   □

APPENDIX: COMPUTATION OF THE SPEED IN DIMENSION $d = 1$

We assume that $d = 1$ and that $\int_0^{+\infty} r^2 \mu(dr)$ is finite. Our aim in this section is to sketch a proof of the following result: almost surely, for all $\varepsilon > 0$ and for all t large enough, we have

$$[-v(1 - \varepsilon)t, v(1 - \varepsilon)t] \subset S_t \subset [-v(1 + \varepsilon)t, v(1 + \varepsilon)t],$$

where

$$v = \frac{1}{2} \int_0^{+\infty} r^2 \mu(dr) = \frac{\mathbb{E}(R^2)}{2}.$$

We come back to the initial description of the process by Deijfen in [3]. We keep the same point process χ, but a point $(c, t, r) \in \chi$ has now the following interpretation: the ball centered at c with radius r becomes infected at time t if its center has been infected before time t. The equivalence of the two descriptions follows from the properties of the exponential law. In order to simplify the notation, instead of setting $S_0 = [-1, 1]$, we set $S_0 = [-2, 0]$. We denote by $(S_t)_t$ the continuous first-passage percolation process starting from S_0 and driven by χ. We denote by ρ_t the rightmost point in S_t. By symmetry arguments, it is sufficient to prove that $\rho_t t^{-1}$ converges almost surely to v.

• In this step, we show

$$\limsup_{t \to +\infty} \rho_t t^{-1} \leq v \qquad \text{a.s.}$$

In order to prove this result, we introduce a process $(\overline{S}_t)_t$ that stochastically dominates $(S_t)_t$. It has the same evolution by outbursts as $(S_t)_t$—in particular, it uses the same Poisson point process χ—but it starts from $\overline{S}_0 = (-\infty, 0]$. Denote by $\overline{\rho}_t$ the rightmost point in $\overline{S}_t$. It suffices to show that $\overline{\rho}_t t^{-1}$ converges almost surely to v.

Denote by $\chi^+ = \{(c, t, r) \in \chi : c \leq 0, \ r + c > 0\}$ the set of efficient balls. It is a Poisson point process with intensity measure

$$\mathbf{1}_{c \in (-r, 0]} \, dc \, dt \, \mu(dr).$$

Its projection on the t coordinate is a homogeneous Poisson point process on $\mathbb{R}^+$ with finite intensity

$$\int_0^{+\infty} \mu(dr) \int_{-r}^0 dc = \int_0^{+\infty} r \mu(dr) = \mathbb{E}R.$$

Consider the first efficient outburst time T_1 (i.e., the leftmost point of the previous point process). The random variable T_1 follows the exponential law with mean $(\mathbb{E}R)^{-1}$. Let (C_1, R_1) be such that $(C_1, T_1, R_1) \in \chi^+$. It is a random variable with law

$$(\mathbb{E}R)^{-1}\mathbf{1}_{c\in(-r,0]}\,dc\,\mu(dr).$$

We have $\overline{\rho}_{T_1} = C_1 + R_1$. We can then compute the mean increase at time T_1:

$$\mathbb{E}(\overline{\rho}_{T_1}) = (\mathbb{E}R)^{-1}\int_0^{+\infty}\mu(dr)\int_{-r}^0 dc\,(c+r) = \frac{\mathbb{E}(R^2)}{2\mathbb{E}R}.$$

Set $T_0 = 0$ and denote by $(T_i)_{i\geq1}$ the sequence of efficient outburst times. For example, T_2 is the t coordinate of the point of $\{(c,t,r)\in\chi:c\leq\overline{\rho}_{T_1},\ r+c>\overline{\rho}_{T_1},\ t\geq T_1\}$ whose t coordinate is minimal. The increments $(\overline{\rho}_{T_i}-\overline{\rho}_{T_{i-1}})_{i\geq1}$ and $(T_i-T_{i-1})_{i\geq1}$ are independent sequences of i.i.d. random variables. Thus,

$$\frac{\overline{\rho}_{T_k}}{k}\to\mathbb{E}(\overline{\rho}_{T_1}) = \frac{\mathbb{E}(R^2)}{2\mathbb{E}R}\quad\text{and}\quad\frac{T_k}{k}\to\mathbb{E}(T_1) = \frac{1}{\mathbb{E}R}$$

almost surely when k goes to infinity. As $(T_i)_{i\geq1}$ goes almost surely to infinity, we can find for any $t>0$ a random k such that $T_k\leq t<T_{k+1}$. For such a k, we have

$$\frac{\overline{\rho}_{T_k}}{T_{k+1}}\leq\frac{\overline{\rho}_t}{t}\leq\frac{\overline{\rho}_{T_{k+1}}}{T_k}.$$

This ends this step.
- In this step we show

$$\liminf_{t\to+\infty}\rho_t t^{-1}\geq v\qquad\text{a.s.}$$

We introduce a process $(\underline{S}_t)_t$ that is stochastically dominated by $(S_t)_t$. It also starts from $[-2,0]$, uses the same outburst process χ, but increases only by the right-hand side (the contribution of added balls on the left-hand side are erased). Denote by $\underline{\rho}_t$ its rightmost point at time t. It suffices to prove

$$\liminf_{t\to+\infty}\underline{\rho}_t t^{-1}\geq v\qquad\text{a.s.}$$

Set $T_0 = 0$. We define (C_1, T_1, R_1) as in the previous step. The first efficient outburst for $(\underline{S}_t)_t$ occurs at time T_1 or later. We have

$$\underline{\rho}_{T_1} = (C_1 + R_1)\mathbf{1}_{C_1\geq-2}.$$

Let T_2 be the t-coordinate of the point of $\{(c,t,r)\in\chi:c\leq\underline{\rho}_{T_1},\ r+c>\underline{\rho}_{T_1},\ t\geq T_1\}$ whose t coordinate is minimal. Let C_2 and R_2 be such that (C_2,T_2,R_2) belongs to χ. We define in the same way (C_i,T_i,R_i) for all $i\geq3$.

Notice that $(C_i - \underline{\rho}_{T_{i-1}}, T_i - T_{i-1}, R_i)_{i \geq 1}$ is a sequence of i.i.d. random variables with the same law as (C_1, T_1, R_1). Moreover, for all $i \geq 1$, we have

$$\underline{\rho}_{T_i} - \underline{\rho}_{T_{i-1}} = (C_i - \underline{\rho}_{T_{i-1}} + R_i)\mathbf{1}_{C_i \geq -2}.$$

We write this as follows:

$$\underline{\rho}_{T_i} - \underline{\rho}_{T_{i-1}} = (C_i - \underline{\rho}_{T_{i-1}} + R_i)\mathbf{1}_{C_i - \underline{\rho}_{T_{i-1}} \geq -2 - \underline{\rho}_{T_{i-1}}}.$$

In particular, we have

$$\underline{\rho}_{T_i} - \underline{\rho}_{T_{i-1}} \geq (C_i - \underline{\rho}_{T_{i-1}} + R_i)\mathbf{1}_{C_i - \underline{\rho}_{T_{i-1}} \geq -2},$$

from which we can conclude that $\underline{\rho}_{T_i}$ converges to $+\infty$. Therefore, for all $s > 0$, almost surely and for large enough i, we have

$$\underline{\rho}_{T_i} - \underline{\rho}_{T_{i-1}} \geq (C_i - \underline{\rho}_{T_{i-1}} + R_i)\mathbf{1}_{C_i - \underline{\rho}_{T_{i-1}} \geq -s}.$$

Consequently, for all $s > 0$ and almost surely, we have

$$\liminf_{i \to +\infty} \frac{\underline{\rho}_{T_i}}{i} \geq \liminf \frac{\sum_{j=1}^{i}(C_j - \underline{\rho}_{T_{j-1}} + R_j)\mathbf{1}_{C_j - \underline{\rho}_{T_{j-1}} \geq -s}}{i}$$
$$= \mathbb{E}\big((C_1 + R_1)\mathbf{1}_{C_1 \geq -s}\big).$$

We let s go to infinity and end the proof as in the first step.

• The result follows from the previous two steps. Let us notice that the proof of the previous step also allows us to prove again that the speed is infinite when $\mathbb{E}(R^2)$ is infinite.

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LABORATOIRE DE MATHÉMATIQUES
  APPLICATIONS ET PHYSIQUE
  MATHÉMATIQUE D'ORLÉANS
  UMR 6628
UNIVERSITÉ D'ORLÉANS
BP 6759
45067 ORLÉANS CEDEX 2
FRANCE
E-MAIL: Jean-Baptiste.Gouere@univ-orleans.fr

INSTITUT ELIE CARTAN NANCY (MATHÉMATIQUES)
UNIVERSITÉ HENRI POINCARÉ NANCY 1
CAMPUS SCIENTIFIQUE
BP 239
54506 VANDOEUVRE-LÈS-NANCY CEDEX
FRANCE
E-MAIL: Regine.Marchand@iecn.u-nancy.fr