

STOCHASTIC CALCULUS FOR SYMMETRIC MARKOV PROCESSES

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Dedicated to S. Nakao on the occasion of his 60th birthday

Using time-reversal, we introduce a stochastic integral for zero-energy additive functionals of symmetric Markov processes, extending earlier work of S. Nakao. Various properties of such stochastic integrals are discussed and an Itô formula for Dirichlet processes is obtained.

1. Introduction and framework. It is well known that stochastic integrals and Itô's formula for semimartingales play a central role in modern probability theory. However, there are many important classes of Markov processes that are not semimartingales. For example, symmetric diffusions on \mathbb{R}^d whose infinitesimal generators are elliptic operators in divergence form $\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j})$ with merely measurable coefficients need not be semimartingales. Even when X is a Brownian motion in \mathbb{R}^d and $u \in W^{1,2}(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d; dx) \mid |\nabla u| \in L^2(\mathbb{R}^d; dx)\}$, the process $u(X_t)$ is not generally a semimartingale. To study such processes, Fukushima obtained the following substitute for Itô's formula (see [7]): for $u \in W^{1,2}(\mathbb{R}^d)$,

$$(1.1) \quad u(X_t) = u(X_0) + M_t^u + N_t^u \quad \text{for } t \geq 0,$$

\mathbf{P}_x -a.s. for quasi-every $x \in \mathbb{R}^d$, where M^u is a square-integrable martingale and N^u is a continuous additive functional of zero energy. The decomposition (1.1) is called *Fukushima's decomposition* and holds for a general symmetric Markov

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process X and for $u \in \mathcal{F}$, where $(\mathcal{E}, \mathcal{F})$ is the Dirichlet space for X . In this paper, a stochastic process $\xi = \{\xi_t, t \geq 0\}$ under some σ -finite measure \mathbf{P} is called a *Dirichlet process* if ξ has locally finite quadratic variation under \mathbf{P} . The composite process $u(X)$ is a Dirichlet process under \mathbf{P}_m , where m is the Lebesgue measure on \mathbb{R}^d , as it has finite quadratic variation on compact time intervals. Nakao introduced a stochastic integral $\int_0^t f(X_s) dN_s^u$ in [14] by using a Riesz representation theorem in a suitably constructed Hilbert space. Nakao's stochastic integral played an important role in the study of lower order perturbation of diffusion processes by Lunt, Lyons and Zhang [12] and by Fitzsimmons and Kuwae [5]. However, Nakao's definition of the stochastic integral $\int_0^t f(X_s) dN^u$, requiring u to be in the domain of the Dirichlet form of X and f to be square-integrable with respect to the energy measure of u , is too restrictive to be useful in the study of lower-order perturbation for symmetric Markov processes with discontinuous sample paths, such as stable processes. Such a study requires stochastic integrals for more general integrators as well as integrands. The purpose of this paper is to present a new way of defining the stochastic integral for Dirichlet processes associated with a symmetric Markov process. Our new approach uses only the time-reversal operator for the process X and is therefore more direct and provides additional insight into stochastic integration for Dirichlet processes. This approach enables us to define $\Lambda(M)$ [see (1.5)] for any locally square-integrable martingale additive functional (MAF) M , subject to some mild conditions. Thus, it not only recovers Nakao's results in [14], but also extends them significantly. The new stochastic integral allows us to study various transforms for symmetric Markov processes, a project that is carried out in a subsequent paper [2].

A more detailed description of the current paper appears below.

Let $X = \{\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \theta_t, \zeta, \mathbf{P}_x, x \in E\}$ be an m -symmetric right Markov process with a Lusin state space E , where m is a σ -finite measure with full support on E . Its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is known to be quasi-regular (see [13]). By [1], $(\mathcal{E}, \mathcal{F})$ is quasi-homeomorphic to a regular Dirichlet space on a locally compact separable metric space. Using this quasi-homeomorphism, there is no loss of generality in assuming that X is an m -symmetric Hunt process on a locally compact metric space E such that its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(E; m)$ and that m is a positive Radon measure with full topological support on E . We assume this throughout the sequel.

Without loss of generality, we can take Ω to be the canonical path space $D([0, \infty[\rightarrow E_\Delta)$ of right-continuous, left-limited (*rcll*, for short) functions from $[0, \infty[$ to E_Δ , for which Δ is a trap [i.e., if $\omega(t) = \Delta$, then $\omega(s) = \Delta$ for all $s > t$]. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \Delta\}$ be the lifetime of X . As usual, \mathcal{F}_∞ and \mathcal{F}_t are the minimal augmented σ -algebras obtained from $\mathcal{F}_\infty^0 := \sigma\{X_s \mid 0 \leq s < \infty\}$ and $\mathcal{F}_t^0 := \sigma\{X_s \mid 0 \leq s \leq t\}$, respectively, under \mathbf{P}_x ; see the next section for more details. We sometimes use a filtration denoted by (\mathcal{M}_t) on (Ω, \mathcal{M}) in order to represent several filtrations, for example,

$(\mathcal{F}_t^0), (\mathcal{F}_{t+}^0)$ on $(\Omega, \mathcal{F}_\infty^0)$, (\mathcal{F}_t) on $(\Omega, \mathcal{F}_\infty)$ and others introduced later. We use θ_t to denote the shift operator defined by $\theta_t(\omega)(s) := \omega(t + s)$, $t, s \geq 0$. Let ω_Δ be the path starting from Δ . Then, $\omega_\Delta(s) \equiv \Delta$ for all $s \in [0, \infty[$. Note that $\theta_\zeta(\omega)(\omega) = \omega_\Delta$ if $\zeta(\omega) < \infty$, $\{\omega_\Delta\} \in \mathcal{F}_0^0 \subset \mathcal{F}_t^0$ for all $t > 0$ and $\mathbf{P}_x(\{\omega_\Delta\}) \leq \mathbf{P}_x(X_0 = \Delta) = 0$ for $x \in E$. For a Borel subset B of E , $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *exit time* of B) is an (\mathcal{F}_t) -stopping time. If B is closed, then τ_B is an (\mathcal{F}_{t+}^0) -stopping time. Also, ζ is an (\mathcal{F}_t^0) -stopping time because $\{\zeta \leq t\} = \{X_t = \Delta\} \in \mathcal{F}_t^0$, $t \geq 0$. The transition semigroup of X , $\{P_t, t \geq 0\}$, is defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta], \quad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(E; m)$; collectively, these operators form a strongly L^2 -continuous semigroup of self-adjoint contractions. The Dirichlet form associated with X is the bilinear form

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} \frac{1}{t} (u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \left\{ u \in L^2(E; m) \mid \sup_{t > 0} t^{-1} (u - P_t u, u)_m < \infty \right\}.$$

Here, we use the notation $(f, g)_m := \int_E f(x)g(x)m(dx)$.

For the reader's convenience, we recall the following definitions from [13] and [7].

DEFINITION 1.1. (i) An increasing sequence $\{F_n\}_{n \geq 1}$ of closed subsets of E is an \mathcal{E} -*nest* (or simply *nest*) if and only if $\bigcup_{n \geq 1} \mathcal{F}_{F_n}$ is \mathcal{E}_1 -dense in \mathcal{F} , where $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(E, m)}$ and

$$\mathcal{F}_{F_n} := \{u \in \mathcal{F} : u = 0 \text{ } m\text{-a.e. on } E \setminus F_n\}.$$

(ii) A subset $N \subset E$ is \mathcal{E} -*polar* if and only if there is an \mathcal{E} -nest $\{F_n\}_{n \geq 1}$ such that $N \subset \bigcap_{n \geq 1} (E \setminus F_n)$.

(iii) A function f on E is said to be *quasi-continuous* if there is an \mathcal{E} -nest $\{F_n\}_{n \geq 1}$ such that $f|_{F_n}$ is continuous on F_n for each $n \geq 1$; we denote this situation briefly by writing $f \in C(\{F_n\})$.

(iv) A statement depending on $x \in A$ is said to hold *quasi-everywhere* (q.e. in abbreviation) on A if there is an \mathcal{E} -polar set $N \subset A$ such that the statement is true for every $x \in A \setminus N$.

(v) A nearly Borel subset $N \subset E$ is called *properly exceptional* if $m(N) = 0$ and

$$\mathbf{P}_x(X_t \in E_\Delta \setminus N \text{ for } t \geq 0 \text{ and } X_{t-} \in E_\Delta \setminus N \text{ for } t > 0) = 1$$

for every $x \in E \setminus N$.

It is known (cf. [7]) that a family $\{F_n\}$ of closed sets is an \mathcal{E} -nest if and only if

$$\mathbf{P}_x \left(\lim_{n \rightarrow \infty} \tau_{F_n} = \zeta \right) = 1 \quad \text{for q.e. } x \in E.$$

It is also known that a properly exceptional set is \mathcal{E} -polar and that every \mathcal{E} -polar set is contained in a properly exceptional set. Every element u in \mathcal{F} admits a quasi-continuous m -version. We assume throughout this section that functions in \mathcal{F} are always represented by their quasi-continuous m -versions. In the sequel, the abbreviations CAF, PCAF and MAF stand for “continuous additive functional,” “positive continuous additive functional” and “martingale additive functional,” respectively; the definitions of these terms can be found in [7].

Let $\mathring{\mathcal{M}}$ and \mathcal{N}_c denote, respectively, the space of MAF’s of finite energy and the space of continuous additive functionals of zero energy. For $u \in \mathcal{F}$, Fukushima’s decomposition holds:

$$(1.2) \quad u(X_t) - u(X_0) = M_t^u + N_t^u \quad \text{for every } t \in [0, \infty[,$$

\mathbf{P}_x -a.s. for q.e. $x \in E$, where $M^u \in \mathring{\mathcal{M}}$ and $N^u \in \mathcal{N}_c$.

A positive continuous additive functional (PCAF) of X (call it A) determines a measure $\mu = \mu_A$ on the Borel subsets of E via the formula

$$(1.3) \quad \mu(f) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_m \left[\int_0^t f(X_s) dA_s \right],$$

in which $f : E \rightarrow [0, \infty]$ is Borel measurable. Here, $\uparrow \lim_{t \downarrow 0}$ indicates an increasing limit as $t \downarrow 0$. The measure μ is necessarily *smooth*, in the sense that μ charges no \mathcal{E} -polar set of X and there is an \mathcal{E} -nest $\{F_n\}$ of closed subsets of E such that $\mu(F_n) < \infty$ for each $n \in \mathbb{N}$. Conversely, given a smooth measure μ , with $A = A^\mu$. In the sequel, we refer to this bijection between smooth measures and PCAF’s as the *Revuz correspondence* and to μ as the *Revuz measure* of A^μ .

If M is a locally square-integrable martingale additive functional (MAF) of X on the random time interval $[[0, \zeta[$, then the process $\langle M \rangle$ (the dual predictable projection of $[M]$) is a PCAF (Proposition 2.8) and the associated Revuz measure [as in (1.3)] is denoted by $\mu_{\langle M \rangle}$. More generally, if M^u is the martingale part in the Fukushima decomposition of $u \in \mathcal{F}$, then $\langle M^u, M \rangle$ is a CAF locally of bounded variation and we have the associated Revuz measure $\mu_{\langle M^u, M \rangle}$, which is locally the difference of smooth (positive) measures. For $u \in \mathcal{F}$, the Revuz measure $\mu_{\langle M^u \rangle}$ of $\langle M^u \rangle$ will usually be denoted by $\mu_{\langle u \rangle}$.

Let $(N(x, dy), H_t)$ be a Lévy system for X ; that is, $N(x, dy)$ is a kernel on $(E_\Delta, \mathcal{B}(E_\Delta))$ and H_t is a PCAF with bounded 1-potential such that for any non-negative Borel function ϕ on $E_\Delta \times E_\Delta$ vanishing on the diagonal and any $x \in E_\Delta$,

$$\mathbf{E}_x \left[\sum_{s \leq t} \phi(X_{s-}, X_s) \right] = \mathbf{E}_x \left[\int_0^t \int_{E_\Delta} \phi(X_s, y) N(X_s, dy) dH_s \right].$$

To simplify notation, we will write

$$N\phi(x) := \int_{E_\Delta} \phi(x, y)N(x, dy)$$

and

$$(N\phi * H)_t := \int_0^t N\phi(X_s) dH_s.$$

Let μ_H be the Revuz measure of the PCAF H . The jumping measure J and the killing measure κ of X are then given by

$$J(dx, dy) = \frac{1}{2}N(x, dy)\mu_H(dx) \quad \text{and} \quad \kappa(dx) = N(x, \{\Delta\})\mu_H(dx).$$

These measures feature in the Beurling–Deny decomposition of \mathcal{E} : for $f, g \in \mathcal{F}$,

$$\begin{aligned} \mathcal{E}(f, g) &= \mathcal{E}^{(c)}(f, g) + \int_{E \times E} (f(x) - f(y))(g(x) - g(y))J(dx, dy) \\ &\quad + \int_E f(x)g(x)\kappa(dx), \end{aligned}$$

where $\mathcal{E}^{(c)}$ is the strongly local part of \mathcal{E} .

For $u \in \mathcal{F}$, the martingale part M_t^u in (1.2) can be decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa} \quad \text{for every } t \in [0, \infty[,$$

\mathbf{P}_x -a.s. for q.e. $x \in E$, where $M_t^{u,c}$ is the continuous part of the martingale M^u and

$$\begin{aligned} M_t^{u,j} &= \lim_{\varepsilon \downarrow 0} \left\{ \sum_{0 < s \leq t} (u(X_s) - u(X_{s-})) \mathbf{1}_{\{|u(X_s) - u(X_{s-})| > \varepsilon\}} \mathbf{1}_{\{s < \zeta\}} \right. \\ &\quad \left. - \int_0^t \left(\int_{\{y \in E: |u(y) - u(X_s)| > \varepsilon\}} (u(y) - u(X_s))N(X_s, dy) \right) dH_s \right\}, \\ M_t^{u,\kappa} &= \int_0^t u(X_s)N(X_s, \{\Delta\}) dH_s - u(X_{\zeta-}) \mathbf{1}_{\{t \geq \zeta\}} \end{aligned}$$

are the jump and killing parts of M^u , respectively. All three terms in this decomposition of M^u are elements of $\overset{\circ}{\mathcal{M}}$; see Theorem A.3.9 of [7]. The limit in the expression for $M^{u,j}$ is in the sense of convergence in the norm of the space of MAF's of finite energy and of convergence in probability under \mathbf{P}_x for q.e. $x \in E$ (see [7]).

Let $\mathcal{N}_c^* \subset \mathcal{N}_c$ denote the class of continuous additive functionals of the form $N^u + \int_0^\cdot g(X_s) ds$ for some $u \in \mathcal{F}$ and $g \in L^2(E; m)$. Nakao [14] constructed a linear map Γ from $\overset{\circ}{\mathcal{M}}$ into \mathcal{N}_c^* in the following way. It is shown in [14] that, for every $Z \in \overset{\circ}{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$(1.4) \quad \mathcal{E}_1(w, f) = \frac{1}{2} \mu_{\langle M^f + M^{f,\kappa}, Z \rangle} (E) \quad \text{for every } f \in \mathcal{F}.$$

This unique w is denoted by $\gamma(Z)$. The operator Γ is now defined by

$$(1.5) \quad \Gamma(Z)_t := N_t^{\gamma(Z)} - \int_0^t \gamma(Z)(X_s) ds \quad \text{for every } Z \in \overset{\circ}{\mathcal{M}}.$$

Nakao showed that $\Gamma(Z)$ is characterized by the following equation

$$(1.6) \quad \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{g,m}[\Gamma(Z)_t] = -\frac{1}{2} \mu_{\langle M^g + M^{g,\kappa}, Z \rangle}(E) \quad \text{for every } g \in \mathcal{F}_b.$$

Here, $\mathcal{F}_b := \mathcal{F} \cap L^\infty(E; m)$. So, in particular, we have $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Nakao [14] then used the operator Γ to define a stochastic integral

$$(1.7) \quad \int_0^t f(X_s) dN_s^u := \Gamma(f * M^u)_t - \frac{1}{2} \langle M^{f,c} + M^{f,j}, M^{u,c} + M^{u,j} \rangle_t,$$

where $u \in \mathcal{F}$, $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u \rangle})$ and $(f * M^u)_t := \int_0^t f(X_{s-}) dM_s^u$. If we define

$$\begin{aligned} \widetilde{\mathcal{N}}_c := \{N \in \mathcal{N}_c \mid N = N^u + A^\mu \text{ for some } u \in \mathcal{F} \\ \text{and some signed smooth measure } \mu\}, \end{aligned}$$

then we see, by (1.5), that $\int_0^\cdot f(X_s) dN_s^u \in \widetilde{\mathcal{N}}_c$ if $u \in \mathcal{F}$ and $f \in \mathcal{F} \cap L^2(E; \mu_{\langle u \rangle})$. However, the conditions imposed on the integrand $f(X_t)$ and on the integrator N^u in Nakao’s stochastic integral are too restrictive for certain applications, in particular the perturbation theory of general symmetric Markov processes, which requires more general integrators as well as integrands; see [2].

The purpose of this paper is to provide a new way of defining $\Gamma(M)$ and Nakao’s stochastic integral for zero-energy AF’s N^u .

For a finite rcll AF M_t , it is known (see [3], Lemma 3.2) that there is a Borel function φ on $E \times E$ with $\varphi(x, x) = 0$ for all $x \in E$ so that

$$(1.8) \quad M_t - M_{t-} = \varphi(X_{t-}, X_t) \quad \text{for every } t \in]0, \zeta[, \mathbf{P}_m\text{-a.e.}$$

Such a φ is uniquely determined up to J -negligible sets. We will call φ the *jump function* of M . When $M = M^u$, $u \in \mathcal{F}$, the jump function φ for M^u can be taken to be as $\varphi(x, y) = u(y) - u(x)$ for $(x, y) \in E \times E$, with $u(\Delta) := 0$. We have a similar result for locally square-integrable MAF’s on $[[0, \zeta[[$ [see Definition 2.5(iii)] for the definition of a locally square-integrable MAF on $[[0, \zeta[[$]. Let M be a locally square-integrable MAF on $[[0, \zeta[[$. There then exists a jump function φ on $E \times E$ for M satisfying the property (1.8) (see Corollary 2.9). Assume that

$$(1.9) \quad \int_0^t \int_E (\widehat{\varphi}^2 \mathbf{1}_{\{|\widehat{\varphi}| \leq 1\}} + |\widehat{\varphi}| \mathbf{1}_{\{|\widehat{\varphi}| > 1\}})(X_s, y) N(X_s, dy) dH_s < \infty$$

for every $t < \zeta$,

\mathbf{P}_x -a.s. for q.e. $x \in E$, where $\widehat{\varphi}(x, y) := \varphi(x, y) + \varphi(y, x)$ for $x, y \in E$. By Lemma 3.2 below, there is a unique purely discontinuous local MAF K on $\llbracket 0, \zeta \llbracket$ with

$$K_t - K_{t-} = -\widehat{\varphi}(X_{t-}, X_t) \quad \text{for } t < \zeta, \mathbf{P}_x\text{-a.s. for q.e. } x \in E.$$

Define \mathbf{P}_m -a.e. on $[0, \zeta[$,

$$\Lambda(M)_t := -\frac{1}{2}(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t) \quad \text{for } t \in [0, \zeta[$$

where r_t is the time-reversal operator at time $t > 0$. Note that since X is symmetric, the measure \mathbf{P}_m , when restricted to $\{t < \zeta\}$, is invariant under r_t . This time reversibility plays an important role in this paper. So, $\Lambda(M)$ is clearly well defined on $\llbracket 0, \zeta \llbracket$ under the σ -finite measure \mathbf{P}_m . It will be shown in Theorem 2.18 and Remark 3.4(ii) below that $\Lambda(M)$ is a continuous even AF of X on $\llbracket 0, \zeta \llbracket$ admitting m -null set. Note that when $M = M^u$ for some $u \in \mathcal{F}$, $\varphi(x, y) = u(y) - u(x)$ is antisymmetric and so $\widehat{\varphi} = 0$. Thus, \mathbf{P}_m -a.e. on $\{t < \zeta\}$,

$$\Lambda(M^u)_t := -\frac{1}{2}(M_t^u + M_t^u \circ r_t + u(X_{t-}) - u(X_t)) = N_t^u.$$

The last identity follows by applying the time-reversal operator to both sides of (1.2) and using the fact that $N_t^u \circ r_t = N_t^u$ \mathbf{P}_m -a.e. on $\llbracket 0, \zeta \llbracket$ (cf. [4], Theorem 2.1). It then follows for every $u \in \mathcal{F}$ that $\Lambda(M^u) = \Gamma(M^u)$ on $\llbracket 0, \zeta \llbracket$ \mathbf{P}_m -a.e. We will show in Theorem 3.6 below that this holds when M^u is replaced by any $M \in \overset{\circ}{\mathcal{M}}$. Therefore, under the σ -finite measure \mathbf{P}_m , Λ is a genuine extension of Nakao’s map Γ .

A function f is said to be *locally in \mathcal{F}* (denoted as $f \in \mathcal{F}_{\text{loc}}$) if there is an increasing sequence of finely open Borel sets $\{D_k, k \geq 1\}$ with $\bigcup_{k=1}^\infty D_k = E$ q.e. and for every $k \geq 1$, there is $f_k \in \mathcal{F}$ such that $f = f_k$ m -a.e. on D_k . For two subsets A, B of E , we denote $A = B$ q.e. if $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is \mathcal{E} -polar. By definition, every $f \in \mathcal{F}_{\text{loc}}$ admits a quasi-continuous m -version, so we may assume that all $f \in \mathcal{F}_{\text{loc}}$ are quasi-continuous. We then have $f = f_k$ q.e. on D_k . For $f \in \mathcal{F}_{\text{loc}}$, $M^{f,c}$ is well defined as a continuous MAF on $\llbracket 0, \zeta \llbracket$ of locally finite energy. Moreover, for $f \in \mathcal{F}_{\text{loc}}$ and a locally square-integrable MAF M on $\llbracket 0, \zeta \llbracket$,

$$t \mapsto (f * M)_t := \int_0^t f(X_{s-}) dM_s$$

is a locally square-integrable MAF on $\llbracket 0, \zeta \llbracket$. Here, for a locally square-integrable MAF M on $\llbracket 0, \zeta \llbracket$, denote by M^c its continuous part, which is also a locally square-integrable MAF on $\llbracket 0, \zeta \llbracket$ (see Theorem 8.23 in [9]).

DEFINITION 1.2 (Stochastic integral). Suppose that M is a locally square-integrable MAF on $\llbracket 0, \zeta \llbracket$ and that $f \in \mathcal{F}_{\text{loc}}$. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a jump function

for M and assume that φ satisfies condition (1.9). Define, on $\llbracket 0, \zeta \llbracket$,

$$\begin{aligned} & \int_0^t f(X_{s-}) d\Lambda(M)_s \\ & := \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c}, M^c \rangle_t \\ & \quad + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s)) \varphi(y, X_s) N(X_s, dy) dH_s, \end{aligned}$$

whenever $\Lambda(f * M)$ is well defined and the third term in the right-hand side of (3.10) is absolutely convergent.

The above stochastic integral is well defined on $\llbracket 0, \zeta \llbracket$ under the σ -finite measure \mathbf{P}_m and extends that of Nakao (1.7). [See Remark 3.9(i) and Theorem 3.10 below.] We will show in Theorem 4.7 below that it enjoys a generalized Itô formula.

2. Additive functionals. In this section, we will prove some facts about additive functionals, to be used later. We begin with some details on the completion of filtrations. Let $\mathcal{P}(E)$ be the family of all probability measures on E . For each $\nu \in \mathcal{P}(E)$, let \mathcal{F}_∞^ν (resp., \mathcal{F}_t^ν) be the \mathbf{P}_ν -completion of \mathcal{F}_∞^0 (resp., \mathbf{P}_ν -completion of \mathcal{F}_t^0 in \mathcal{F}_∞^ν) and set $\mathcal{F}_\infty := \bigcap_{\nu \in \mathcal{P}(E)} \mathcal{F}_\infty^\nu$ and $\mathcal{F}_t := \bigcap_{\nu \in \mathcal{P}(E)} \mathcal{F}_t^\nu$. Let \mathcal{F}_∞^m (resp., \mathcal{F}_t^m) be the \mathbf{P}_m -completion of \mathcal{F}_∞^0 (resp., \mathbf{P}_m -completion of \mathcal{F}_t^0 in \mathcal{F}_∞^m). Although m may not be a finite measure on E , we do have $\mathcal{F}_\infty \subset \mathcal{F}_\infty^m$, $\mathcal{F}_t \subset \mathcal{F}_t^m$ because for $g \in L^1(E; m)$ with $0 < g \leq 1$ on E satisfying $gm \in \mathcal{P}(E)$, \mathbf{P}_{gm} -negligibility is the same as \mathbf{P}_m -negligibility.

For a fixed filtration (\mathcal{M}_t) on (Ω, \mathcal{M}) , we recall the notions of (\mathcal{M}_t) -predictability, (\mathcal{M}_t) -optionality and (\mathcal{M}_t) -progressive measurability as follows (see [15] for more details). On $[0, \infty[\times \Omega$, the (\mathcal{M}_t) -predictable [resp., (\mathcal{M}_t) -optional] σ -field $\mathcal{P}(\mathcal{M}_t)$ [resp., $\mathcal{O}(\mathcal{M}_t)$] is defined as the smallest σ -field over $[0, \infty[\times \Omega$ containing all $\mathbf{P}_\nu(\mathcal{M})$ -evanescent sets for all $\nu \in \mathcal{P}(E_\Delta)$ and with respect to which all \mathcal{M}_t -adapted lcll (left-continuous, right-limited) (resp., rcll) processes are measurable. A process $\phi(s, \omega)$ on $[0, \infty[\times \Omega$ is said to be (\mathcal{M}_t) -progressively measurable provided $[0, t] \times \Omega \ni (s, \omega) \mapsto \phi(s, \omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{M}_t$ -measurable for all $t > 0$. It is well known that (\mathcal{M}_t) -predictability implies (\mathcal{M}_t) -optionality, which in turn implies (\mathcal{M}_t) -progressive measurability.

For $[0, \infty[$ -valued functions S, T on Ω with $S \leq T$, we employ the usual notation for *stochastic intervals*; for example,

$$\llbracket S, T \llbracket := \{(t, \omega) \in [0, \infty[\times \Omega \mid S(\omega) \leq t < T(\omega)\},$$

the other species of stochastic intervals being defined analogously. We write $\llbracket S \rrbracket := \llbracket S, S \rrbracket$ for the *graph* of S . Note that these are all subsets of $[0, \infty[\times \Omega$. If S and T are (\mathcal{M}_t) -stopping times, then $\llbracket S, T \rrbracket$, $\llbracket S, T \llbracket, \dots$ and $\llbracket S \rrbracket$ are (\mathcal{M}_t) -optional (see Theorem 3.16 in [9]).

DEFINITION 2.1 (AF). An (\mathcal{F}_t) -adapted [resp., (\mathcal{F}_t^m) -adapted] process $A = (A_t)_{t \geq 0}$ with values in $[-\infty, \infty]$ is said to be an *additive functional* (AF in short) (resp., *AF admitting m -null set*) if there exist a *defining set* $\Xi \in \mathcal{F}_\infty$ and an \mathcal{E} -polar (resp., m -null) set N satisfying the following conditions:

- (i) $\mathbf{P}_x(\Xi) = 1$ for all $x \in E \setminus N$;
- (ii) $\theta_t \Xi \subset \Xi$ for all $t \geq 0$; in particular, $\omega_\Delta \in \Xi$ and $\mathbf{P}_\Delta(\Xi) = 1$ because $\omega_\Delta = \theta_{\zeta(\omega)}(\omega)$ for all $\omega \in \Xi$;
- (iii) for all $\omega \in \Xi$, $A_t(\omega)$ is right-continuous with left limits on $[0, \zeta(\omega)[$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for all $t, s \geq 0$;
- (iv) for all $t \geq 0$, $A_t(\omega_\Delta) = 0$; in particular, under the additivity in (iii), $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for all $t \geq \zeta(\omega)$ and $\omega \in \Xi$.

An AF A (admitting m -null set) is called *right-continuous with left limits* (rcll AF in brief) if $A_{\zeta(\omega)-}$ exists for each $\omega \in \Xi$. An AF A (admitting m -null set) is said to be *finite* [resp., *continuous additive functional* (CAF in brief)] if $|A_t(\omega)| < \infty$, $t \in [0, \infty[$ (resp. $t \mapsto A_t(\omega)$ is continuous on $[0, \infty[$) for each $\omega \in \Xi$. A $[0, \infty[$ -valued CAF is called a *positive continuous additive functional* (PCAF in short). Two AF's A and B are called *equivalent* if there exists a common defining set $\Xi \in \mathcal{F}_\infty$ and an \mathcal{E} -polar set N such that $A_t(\omega) = B_t(\omega)$ for all $t \in [0, \infty[$ and $\omega \in \Xi$. We call $A = (A_t)_{t \geq 0}$ an AF on $[[0, \zeta[$ or a local AF (admitting m -null set) if A is (\mathcal{F}_t) -adapted and satisfies (i), (ii), (iv) and the property (iii)' in which (iii) is modified so that the additivity condition is required only for $t + s < \zeta(\omega)$. The notions of rcll AF, CAF and PCAF on $[[0, \zeta[$ are defined similarly. Two AF's on $[[0, \zeta[$, A and B , are called *equivalent* if there exists a common defining set $\Xi \in \mathcal{F}_\infty$ and an \mathcal{E} -polar set N such that $A_t(\omega) = B_t(\omega)$ for all $t \in [0, \zeta[$ and $\omega \in \Xi$.

REMARK 2.2. Any PCAF A on $[[0, \zeta[$ can be extended to a PCAF by setting

$$A_t(\omega) := \begin{cases} \lim_{u \uparrow \zeta} A_u(\omega), & \text{if } t \geq \zeta(\omega) > 0, \\ 0, & \text{if } t \geq \zeta(\omega) = 0, \end{cases}$$

for $\omega \in \Xi$ and setting $A_t(\omega) \equiv 0$ for $\omega \in \Xi^c$. The (\mathcal{F}_t) -adaptedness of this extended A holds as follows: for a fixed $T > 0$, we know $\{A_t \leq T\} \cap \{t < \zeta\} \in \mathcal{F}_t$. From this, we have the \mathcal{F}_ζ -measurability of $\{A_\zeta \leq T\}$. Indeed, $\{A_\zeta \leq T\} = \bigcap_{t \in \mathbb{Q}_+} \{A_t \leq T, t < \zeta\} \in \mathcal{F}_\zeta$ as $\{A_t \leq T, t < \zeta\} \in \mathcal{F}_\zeta$ for any $t \geq 0$. Thus, $\{A_t \leq T\} \cap \{t \geq \zeta\} = \{A_\zeta \leq T\} \cap \{t \geq \zeta\} \in \mathcal{F}_t$. Therefore, $\{A_t \leq T\} \in \mathcal{F}_t$ for any $T > 0$, which gives the (\mathcal{F}_t) -adaptedness of A . Noting that $\zeta \circ \theta_t = \zeta - t$ if $t < \zeta$ and $\zeta \circ \theta_t = 0$ if $t \geq \zeta$, we conclude that $A_\zeta = A_t + A_\zeta \circ \theta_t$ for any $t \in [0, \infty[$ on Ξ . Consequently, $A_{t+s} = A_t + A_s \circ \theta_t$ holds for any $t, s \in [0, \infty[$ on Ξ .

The following lemma is a special case of [14], Theorem 2.2.

LEMMA 2.3. *Let A, B be PCAF's such that for m -a.e. $x \in E$, $\mathbf{E}_x[A_t] = \mathbf{E}_x[B_t]$ for all $t \geq 0$ and suppose that the Revuz measure μ_A has finite total mass. A is then equivalent to B .*

REMARK 2.4. The above lemma may fail if the condition $\mu_A(E) < \infty$ is not satisfied. For example, take $E = \mathbb{R}^d$ with $d \geq 2$ and let X be Brownian motion on \mathbb{R}^d and $\mu_A(dx) = |x|^{-d-1} dx$. μ_A is then a smooth measure and corresponds to a PCAF A of X . Let $B_t = A_t + t$, which is a PCAF of X with Revuz measure $\mu_A(dx) + dx$. However,

$$\mathbf{E}_x[A_t] = \int_0^t \left(\int_{\mathbb{R}^d} p(s, x, y) |y|^{-d-1} dy \right) ds = \infty = \mathbf{E}_x[B_t]$$

for every $x \in \mathbb{R}^d \setminus \{0\}$.

Here, $p(s, x, y) = (2\pi t)^{-d/2} \exp(-|x - y|^2/(2t))$ is the transition density function of X .

As usual, if T is an (\mathcal{F}_t) -stopping time and M a process, then M^T is the stopped process defined by $M_t^T := M_{t \wedge T}$. Following [9], we give the notion of local marginals of interval type.

DEFINITION 2.5 (*Processes of interval type*). Let \mathcal{D} be a class of (\mathcal{F}_t) -adapted processes and denote by \mathcal{D}_{loc} its localization (resp., by $\mathcal{D}_{f\text{-loc}}$ its localization by a nest of finely open Borel sets); that is, $M \in \mathcal{D}_{\text{loc}}$ (resp., $M \in \mathcal{D}_{f\text{-loc}}$) if and only if there exists a sequence $M^n \in \mathcal{D}$ and an increasing sequence of stopping times T_n with $T_n \rightarrow \infty$ (resp., a nest $\{G_n\}$ of finely open Borel sets) such that $M^{T_n} = (M^n)^{T_n}$ (resp., $M_t = M_t^n$ for $t < \tau_{G_n}$) for each n . Here, a family $\{G_n\}$ of finely open Borel sets is called a *nest* if $\mathbf{P}_x(\lim_{n \rightarrow \infty} \tau_{G_n} = \zeta) = 1$ for q.e. $x \in E$. (However, see Lemma 3.1.) Clearly, $\mathcal{D} \subset \mathcal{D}_{\text{loc}}$ (resp., $\mathcal{D} \subset \mathcal{D}_{f\text{-loc}}$) and $(\mathcal{D}_{\text{loc}})_{\text{loc}} = \mathcal{D}_{\text{loc}}$ [resp., $(\mathcal{D}_{f\text{-loc}})_{f\text{-loc}} = \mathcal{D}_{f\text{-loc}}$]. If \mathcal{D} is a subclass of AF's, then so is \mathcal{D}_{loc} [for if $M \in \mathcal{D}_{\text{loc}}$, then there exist M^n and T_n as above and for each ω and $t, s \geq 0$, there exists $n \in \mathbb{N}$ with $s + t < T_n(\omega)$ and $s < T_n(\theta_t \omega)$, hence $M_{t+s}(\omega) = M_t(\omega) + M_s(\theta_t \omega)$], while $\mathcal{D}_{f\text{-loc}}$ is contained in the class of AF's on $\llbracket 0, \zeta \rrbracket$.

(i) $B \subset [0, \infty[\times \Omega$ is called a *set of interval type* if there exists a nonnegative random variable S such that for each $\omega \in \Omega$, the section $B_\omega := \{t \in [0, \infty[\mid (t, \omega) \in B\}$ is $[0, S(\omega)]$ or $[0, S(\omega)[$ and $B_\omega \neq \emptyset$.

(ii) Let B be an (\mathcal{F}_t) -optional set of interval type. A real-valued stochastic process M on B [i.e., $M\mathbf{1}_B = (M_t(\omega)\mathbf{1}_B(t, \omega))_{t \geq 0}$ is a real-valued stochastic process] is said to be *in \mathcal{D}^B* if and only if there exists $N \in \mathcal{D}$ such that $M\mathbf{1}_B = N\mathbf{1}_B$ and is said to be *locally in \mathcal{D} on B* [write $M \in (\mathcal{D}_{\text{loc}})^B$] if and only if $S := D_{B^c}$ is the debut of B^c and there exists an increasing sequence of (\mathcal{F}_t) -stopping times $\{S_n\}$ with $\lim_{n \rightarrow \infty} S_n = S$ and a sequence of $M^n \in \mathcal{D}$ such

that $B_\omega \subset \bigcup_{n=1}^\infty [0, S_n(\omega)]$ \mathbf{P}_x -a.s. $\omega \in \Omega$ and $(M\mathbf{1}_B)^{S_n} = (M^n\mathbf{1}_B)^{S_n}$ for all $n \in \mathbb{N}$ and $t \geq 0$, \mathbf{P}_x -a.s. $\omega \in \Omega$ for q.e. $x \in E$. Clearly, $\mathcal{D}^B \subset (\mathcal{D}_{\text{loc}})^B$. Moreover, $\mathcal{D}^{B_2} \subset \mathcal{D}^{B_1}$ and $(\mathcal{D}_{\text{loc}})^{B_2} \subset (\mathcal{D}_{\text{loc}})^{B_1}$ for any pair of (\mathcal{F}_t) -optional sets B_1, B_2 of interval type with $B_1 \subset B_2$.

(iii) Let B be an (\mathcal{F}_t) -optional set of interval type. We set

$$\begin{aligned} \mathcal{M}^1 := \{M \mid M \text{ is a finite rcll AF, } \mathbf{E}_x[|M_t|] < \infty, \\ \mathbf{E}_x[M_t] = 0 \text{ for } \mathcal{E}\text{-q.e. } x \in E \text{ and all } t \geq 0\} \end{aligned}$$

and speak of an element of $(\mathcal{M}^1)^B$ [resp., $(\mathcal{M}_{\text{loc}}^1)^B$] as being an *MAF on B* (resp., a *local MAF on B*). Similarly,

$$\begin{aligned} \mathcal{M} := \{M \mid M \text{ is a finite rcll AF, } \mathbf{E}_x[M_t^2] < \infty, \\ \mathbf{E}_x[M_t] = 0 \text{ for } \mathcal{E}\text{-q.e. } x \in E \text{ and all } t \geq 0\} \end{aligned}$$

and an element of \mathcal{M}^B [resp., $(\mathcal{M}_{\text{loc}})^B$] is a *square-integrable MAF on B* (resp., *locally square-integrable MAF on B*). We further set

$$\begin{aligned} \mathcal{M}^c := \{M \in \mathcal{M} \mid M \text{ is a CAF}\}, \\ \mathcal{M}^d := \{M \in \mathcal{M} \mid M \text{ is a purely discontinuous AF}\} \end{aligned}$$

and an element of $(\mathcal{M}_{\text{loc}}^c)^B$ [resp., $(\mathcal{M}_{\text{loc}}^d)^B$] is called a *locally square-integrable continuous MAF on B* (resp., *locally square-integrable purely discontinuous MAF on B*). For $M \in (\mathcal{M}_{\text{loc}})^B$, M admits a unique decomposition $M = M^c + M^d$ with $M^c \in (\mathcal{M}_{\text{loc}}^c)^B$ and $M^d \in (\mathcal{M}_{\text{loc}}^d)^B$ (see Theorem 8.23 in [9]). In these definitions, we omit the usage “on B ” when $B = [0, \infty[\times\Omega$.

For a $[0, \infty]$ -valued function R on Ω and $A \subset \Omega$, $R_A := R \cdot \mathbf{1}_A + (+\infty) \cdot \mathbf{1}_{A^c}$ is called the *restriction* of R on A . Clearly, $R \leq R_A$.

REMARK 2.6. When $B =]0, R[$ for a given (\mathcal{F}_t) -stopping time R , there is another notion of “locally in \mathcal{D} on B ,” obtained by replacing $(M\mathbf{1}_B)^{S_n} = (M^n\mathbf{1}_B)^{S_n}$ with $M^{S_n}\mathbf{1}_B = (M^n)^{S_n}\mathbf{1}_B$ in our definition; this is a weaker notion than ours because $t \mapsto \mathbf{1}_B(t, \omega)$ is decreasing and $\mathbf{1}_B(t, \omega)\mathbf{1}_B(s, \omega) = \mathbf{1}_B(t, \omega)$ for $s \leq t$ and $\omega \in \Omega$. This weaker notion is described in [15].

DEFINITION 2.7 (*MAF locally of finite energy*). Recall that $\mathring{\mathcal{M}}$ is the totality of MAF’s of finite energy, that is,

$$\mathring{\mathcal{M}} := \left\{ M \in \mathcal{M} \mid e(M) := \lim_{t \downarrow 0} \frac{1}{2t} \mathbf{E}_m[M_t^2] < \infty \right\}.$$

We say that an AF M on $]0, \zeta[$ is *locally in $\mathring{\mathcal{M}}$* (and write $M \in \mathring{\mathcal{M}}_{f\text{-loc}}$) if there exists a sequence $\{M^n\}$ in $\mathring{\mathcal{M}}$ and a nest $\{G_n\}$ of finely open Borel sets such that

$M_t = M_t^n$ for $t < \tau_{G_n}$ for each $n \in \mathbb{N}$. In case X is a diffusion process with no killing inside E , we can define the predictable quadratic variation $\langle M \rangle$ for $M \in \overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ as follows. First, note that $M_{t \wedge \tau_{G_n}}^n = M_{t \wedge \tau_{G_m}}^m$ for $n < m$ because of the continuity of M^n . Owing to the uniqueness of Doob–Meyer decomposition, we see that $\langle M^n \rangle_{t \wedge \tau_{G_n}} = \langle M^m \rangle_{t \wedge \tau_{G_n}}$. The predictable quadratic variation $\langle M \rangle$ of $M \in \overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ as a PCAF is well defined by setting $\langle M \rangle_t = \langle M^n \rangle_t$, $t < \tau_{G_n}$, $n \in \mathbb{N}$, with Remark 2.2 and by choosing an appropriate defining set and \mathcal{E} -polar set of $\langle M \rangle$, where $M^n \in \overset{\circ}{\mathcal{M}}$ and $\{G_n\}$ is a nest of finely open Borel sets such that $M_t = M_t^n$, $t < \tau_{G_n}$.

PROPOSITION 2.8. $(\mathcal{M}_{\text{loc}})^{\parallel 0, \zeta \parallel} \subset \overset{\circ}{\mathcal{M}}_{f\text{-loc}}$. More precisely, for each $M \in (\mathcal{M}_{\text{loc}})^{\parallel 0, \zeta \parallel}$, there exists a nest $\{G_k\}$ of finely open Borel sets such that $\mathbf{1}_{G_k} * M \in \overset{\circ}{\mathcal{M}}$ for each $k \in \mathbb{N}$ and the predictable quadratic variation process $\langle M \rangle$ can be constructed as a PCAF.

PROOF. Let $M \in (\mathcal{M}_{\text{loc}})^{\parallel 0, \zeta \parallel}$. There then exists an increasing sequence $\{T_n\}$ of stopping times with $\lim_{n \rightarrow \infty} T_n = \zeta$ (\mathbf{P}_x -a.s. $\omega \in \Omega$ for q.e. $x \in E$) and $M^n \in \mathcal{M}_{\text{loc}}$ such that $M_{t \wedge T_n} \mathbf{1}_{[0, \zeta[}(t \wedge T_n) = M_{t \wedge T_n}^n \mathbf{1}_{[0, \zeta[}(t \wedge T_n)$ holds for all $t \geq 0$ \mathbf{P}_x -a.s. for q.e. $x \in E$. We may assume that it holds for all $\omega \in \Omega$ by changing the sample space. Note that $[0, \zeta(\omega)[\subset \bigcup_{n=1}^\infty [0, T_n(\omega)]$ for all $\omega \in \Omega$. Hence, $M_{t \wedge T_n}^m \mathbf{1}_{[0, \zeta[}(t \wedge T_n) = M_{t \wedge T_n}^n \mathbf{1}_{[0, \zeta[}(t \wedge T_n)$ for $n < m$. As noted in Definition 2.5, we see that M is an AF on $\llbracket 0, \zeta \rrbracket$. Owing to the uniqueness of the Doob–Meyer decomposition for semimartingales on $\llbracket 0, \zeta \rrbracket$ (see [9]), we have $\langle M^m \rangle_{t \wedge T_n} \mathbf{1}_{[0, \zeta[}(t \wedge T_n) = \langle M^n \rangle_{t \wedge T_n} \mathbf{1}_{[0, \zeta[}(t \wedge T_n)$ for $n < m$. Thus, we have $\langle M^m \rangle_t = \langle M^n \rangle_t$ for $t < T_n$ and $n < m$. The predictable quadratic variation $\langle M \rangle$ of M is therefore well defined by setting $\langle M \rangle_t := \langle M^n \rangle_t$ for $t < T_n$. Setting $\langle M \rangle_t := \langle M \rangle_\zeta := \lim_{s \uparrow \zeta} \langle M \rangle_s$ for all $t \geq \zeta$, we obtain a PCAF because of Remark 2.2. Let $\mu_{\langle M \rangle}$ be the Revuz measure corresponding to $\langle M \rangle$ and $\{F_k\}$ an \mathcal{E} -nest of closed sets such that $\mu_{\langle M \rangle}(F_k) < \infty$ for each k , and let G_k be the fine interior of F_k . $\{G_k\}$ is then a nest. In view of the proofs of Theorem 5.6.1 and Lemma 5.6.2 in [7], the stochastic integral $\mathbf{1}_{G_k} * M$ is of finite energy with $\mathbf{e}(\mathbf{1}_{G_k} * M) = \frac{1}{2} \mu_{\langle M \rangle}(G_k)$ and its predictable quadratic variation $\langle \mathbf{1}_{G_k} * M \rangle$ is a PCAF. Let μ_k (resp., μ_k) be the Revuz measure corresponding to $\langle \mathbf{1}_{G_k} * M \rangle$ (resp., $\langle \mathbf{1}_{G_k} * M, M \rangle$). By Lemma 5.6.2 in [7], for $M_i \in \overset{\circ}{\mathcal{M}}$ and $f_i \in L^2(E; d\mu_{\langle M_i \rangle})$ ($i = 1, 2$), we have $f_1 f_2 d\mu_{\langle M_1, M_2 \rangle} = d\mu_{\langle f_1 * M_1, f_2 * M_2 \rangle}$, hence $\int_0^t (f_1 f_2)(X_s) d\langle M_1, M_2 \rangle_s = \langle f_1 * M_1, f_2 * M_2 \rangle_t$. From this, we see that $\langle \mu_k, f^2 \rangle = \langle \mu_k, f^2 \rangle = \langle \mathbf{1}_{G_k} \mu_{\langle M \rangle}, f^2 \rangle$ for any $f \in L^2(E; \mu_{\langle M \rangle})$; consequently, we have $\mu_k = \mu_k = \mathbf{1}_{G_k} \mu_{\langle M \rangle}$ by $\mu_{\langle M \rangle}(G_k) < \infty$. This yields $\langle \mathbf{1}_{G_k} * M \rangle_t = \langle \mathbf{1}_{G_k} * M, M \rangle_t = \int_0^t \mathbf{1}_{G_k}(X_s) d\langle M \rangle_s$ for $t < \zeta$, hence $\langle M - \mathbf{1}_{G_k} * M \rangle_t = 0$ for $t < \tau_{G_k}$. Therefore, $M_t = (\mathbf{1}_{G_k} * M)_t$ for $t < \tau_{G_k}$ and $\mathbf{1}_{G_k} * M \in \overset{\circ}{\mathcal{M}}$. \square

COROLLARY 2.9. *Let M be a locally square-integrable MAF on $[[0, \zeta[$, that is, $M \in (\mathcal{M}_{\text{loc}})^{[0, \zeta[}$. There then exists a Borel function φ on $E \times E$ with $\varphi(x, x) = 0$ for all $x \in E$ such that*

$$M_t - M_{t-} = \varphi(X_{t-}, X_t) \quad \text{for every } t \in]0, \zeta[, \mathbf{P}_m\text{-a.e.}$$

PROOF. By the proof of Proposition 2.8, there exists an \mathcal{E} -nest $\{F_k\}$ such that for each $k \in \mathbb{N}$ $M^k := \mathbf{1}_{F_k} * M \in \mathring{\mathcal{M}}$ and $M_t = M_t^k, t < \tau_{F_k}$. Let φ_k be the jump function corresponding to M^k . We then have $\varphi_k(X_{t-}, X_t) = \varphi_\ell(X_{t-}, X_t), t < \tau_{F_k}, \mathbf{P}_m\text{-a.e.}$, for $k < \ell$. From this, we see that $\varphi_k = \varphi_\ell$ J -a.e. on $F_k \times F_k$. We construct a Borel function φ on $E \times E$ in the following manner. We set $F_0 := \emptyset, \varphi(x, y) := \varphi_k(x, y)$ for $(x, y) \in F_k \times F_k \setminus (F_{k-1} \times F_{k-1}), k \in \mathbb{N}, \varphi(x, y) := 0$ if $(x, y) \in E \times E \setminus (\bigcup_{k=1}^\infty F_k \times \bigcup_{k=1}^\infty F_k)$. φ then satisfies $\varphi(x, x) = 0$ for $x \in E$. We also have $\varphi = \varphi_k$ J -a.e. on $F_k \times F_k$. Consequently, $\varphi(X_{t-}, X_t) = \varphi_k(X_{t-}, X_t), t < \tau_{F_k}, \mathbf{P}_m\text{-a.e.}$ This means that $M_t - M_{t-} = \varphi(X_{t-}, X_t), t < \tau_{F_k}, \mathbf{P}_m\text{-a.e.}$ Therefore, $M_t - M_{t-} = \varphi(X_{t-}, X_t), 0 < t < \zeta \mathbf{P}_m\text{-a.e.}$ \square

We recall the definition of the shift operator θ_s and the time-reversal operator r_t on the path space Ω . For each $s \geq 0$, the shift operator θ_s is defined by $\theta_s \omega(t) := \omega(t + s)$ for $t \in [0, \infty[$. Given a path $\omega \in \{t < \zeta\}$, the operator r_t is defined by

$$(2.1) \quad r_t(\omega)(s) := \begin{cases} \omega((t - s)-), & \text{if } 0 \leq s \leq t, \\ \omega(0), & \text{if } s \geq t. \end{cases}$$

Here, for $r > 0, \omega(r-) := \lim_{s \uparrow r} \omega(s)$ is the left limit at r and we use the convention that $\omega(0-) := \omega(0)$. For a path $\omega \in \{t \geq \zeta\}$, we set $r_t(\omega) := \omega_\Delta$. We note that

$$(2.2) \quad \begin{aligned} \lim_{s \downarrow 0} r_t(\omega)(s) &= \omega(t-) = r_t(\omega)(0) \quad \text{and} \\ \lim_{s \uparrow t} r_t(\omega)(s) &= \omega(0) = r_t(\omega)(t). \end{aligned}$$

A key consequence of the m -symmetry assumption on the Hunt process X is that the measure \mathbf{P}_m , when restricted to $\{t < \zeta\}$, is invariant under the time-reversal operator r_t .

Clearly for $t, s > 0, \theta_s : \Omega \rightarrow \Omega$ is $\mathcal{F}_{t+s}^m / \mathcal{F}_t^m$ -measurable. The following lemma deals with the measurability issue of the time-reversal operator r_t .

LEMMA 2.10. *For each $t > 0, r_t : \Omega \rightarrow \Omega$ is $\mathcal{F}_t^0 / \mathcal{F}_\infty^0$ -measurable and $\mathcal{F}_t^m / \mathcal{F}_t^m$ -measurable.*

PROOF. Let $F_i \in \mathcal{B}(E_\Delta)$ and $s_i \in [0, \infty[, i = 1, 2, \dots, n$, with $s_1 < s_2 < \dots < s_k \leq t < s_{k+1} < \dots < s_n$ for some $k \in \{1, 2, \dots, n\}$. Then, $r_t^{-1}(\bigcap_{i=1}^n X_{s_i}^{-1}(F_i)) = \bigcap_{i=1}^n (X_{s_i} \circ r_t)^{-1}(F_i)$ is equal to $\bigcap_{i=1}^k (\{X_{(t-s_i)-} \in F_i, t <$

$\zeta\} \cup \{\Delta \in F_i, t \geq \zeta\}) \cap \bigcap_{i=k+1}^n (\{X_0 \in F_i, t < \zeta\} \cup \{\Delta \in F_i, t \geq \zeta\}) \in \mathcal{F}_t^0$. Next, we show the $\mathcal{F}_t^m / \mathcal{F}_t^m$ -measurability of r_t . Take $C \in \mathcal{F}_t^m$. There then exist $D \in \mathcal{F}_t^0$ and $N \in \mathcal{F}_\infty^0$ such that $C \Delta D \subset N$ and $\mathbf{P}_m(N) = 0$. Since $\mathbf{P}_m(\{\omega_\Delta\}) = 0$, by deleting $\{\omega_\Delta\} = \{\omega \in \Omega \mid \zeta(\omega) = 0\} \in \mathcal{F}_0^0 \subset \mathcal{F}_t^0$, we may assume that $\omega_\Delta \notin C \cup D \cup N$. Then, $r_t^{-1}(C) \Delta r_t^{-1}(D) \subset r_t^{-1}(N)$, $r_t^{-1}(D), r_t^{-1}(N) \in \mathcal{F}_t^0$ and $\mathbf{P}_m(r_t^{-1}(N)) = \mathbf{P}_m(r_t^{-1}(N) \cap \{t < \zeta\}) + \mathbf{1}_N(\omega_\Delta)\mathbf{P}_m(t \geq \zeta) = \mathbf{P}_m(N \cap \{t < \zeta\}) = 0$. \square

DEFINITION 2.11. For any $t > 0$, we say that two sample paths ω and ω' are t -equivalent if $\omega(s) = \omega'(s)$ for all $s \in [0, t]$. We say that two sample paths ω and ω' are pre- t -equivalent if $\omega(s) = \omega'(s)$ for all $s \in [0, t[$.

For an rcll AF A_t adapted to $(\mathcal{F}_t^0)_{t \geq 0}$, $A_t(\omega) = A_t(\omega')$ if ω and ω' are t -equivalent $A_{t-}(\omega) = A_{t-}(\omega')$ if ω and ω' are pre- t -equivalent. These conclusions may fail to hold if the measurability conditions are not satisfied. We need the following notion.

DEFINITION 2.12 (PrAF). A process $A = (A_t)_{t \geq 0}$ with values in $\overline{\mathbb{R}} := [-\infty, \infty]$ is said to be a progressively additive functional (PrAF in short) (resp., PrAF admitting m -null set) if A is (\mathcal{F}_t) -adapted [resp., (\mathcal{F}_t^m) -adapted] and there exist defining sets $\Xi \in \mathcal{F}_\infty$, $\Xi_t \in \mathcal{F}_t$ [resp., $\Xi \in \mathcal{F}_\infty^m$, $\Xi_t \in \mathcal{F}_t^m$] for each $t > 0$ and an \mathcal{E} -polar (resp., m -null) set N satisfying the following conditions:

- (i) $\mathbf{P}_x(\Xi) = 1$ for all $x \in E \setminus N$, $\Xi \subset \Xi_t \subset \Xi_s$ for every $t > s > 0$ and $\Xi = \bigcap_{t>0} \Xi_t$;
- (ii) $\theta_t \Xi \subset \Xi$ for all $t \geq 0$ and $\theta_{t-s}(\Xi_t) \subset \Xi_s$ for all $s \in]0, t[$, and, in particular, $\omega_\Delta \in \Xi \subset \Xi_t$ and $\mathbf{P}_\Delta(\Xi) = \mathbf{P}_\Delta(\Xi_t) = 1$ under (i);
- (iii) for all $\omega \in \Xi_t$, $A(\omega)$ is defined on $[0, t[$, is right continuous on $[0, t \wedge \zeta(\omega)[$ and has left limit on $]0, t] \cap]0, \zeta(\omega)[$ such that $A_0(\omega) = 0$, $|A_s(\omega)| < \infty$ for $s \in [0, t \wedge \zeta(\omega)[$ and $A_{p+q}(\omega) = A_p(\omega) + A_q(\theta_p \omega)$ for all $p, q \geq 0$ with $p + q < t$;
- (iv) for all $t \geq 0$, $A_t(\omega_\Delta) = 0$;
- (v) for any $t > 0$ and pre- t -equivalent paths $\omega, \omega' \in \Omega$, $\omega \in \Xi_t$ implies that $\omega' \in \Xi_t$, $A_s(\omega) = A_s(\omega')$ for any $s \in [0, t[$ and $A_{s-}(\omega) = A_{s-}(\omega')$ for any $s \in]0, t[$.

Furthermore, A is called an rcll PrAF (or an rcll PrAF admitting m -null set) if, for each $t > 0$ and $\omega \in \Xi_t$, $s \mapsto A_s(\omega)$ is right-continuous on $[0, t[$ and has left-hand limits on $]0, t]$ and a PrAF (or a PrAF admitting m -null set) is said to be finite (resp., continuous) if $|A_s(\omega)| < \infty$ for all $s \in [0, t[$ (resp., continuous on $[0, t[$) for every $\omega \in \Xi_t$.

We say that an AF A on $[[0, \zeta[[$ (resp., AF A on $[[0, \zeta[[$ admitting m -null set) is a PrAF on $[[0, \zeta[[$ (resp., PrAF on $[[0, \zeta[[$ admitting m -null set) if A is (\mathcal{F}_t) -adapted [resp., (\mathcal{F}_t^m) -adapted] and there exist $\Xi \in \mathcal{F}_\infty$, $\Xi_t \in \mathcal{F}_t$ (resp., $\Xi \in \mathcal{F}_\infty^m$, $\Xi_t \in \mathcal{F}_t^m$) for each $t > 0$ and an \mathcal{E} -polar (resp., m -null) set N such that (i'), (ii), (iii'), (iv) and

(v') hold—(i'): $\mathbf{P}_x(\Xi) = 1$ for all $x \in E \setminus N$, $\Xi \subset \Xi_t$ for all $t > 0$, $\Xi = \bigcap_{t>0} \Xi_t$ and $\Xi_t \cap \{t < \zeta\} \subset \Xi_s \cap \{s < \zeta\}$ for $s < t$; (iii'): for each $\omega \in \Xi_t \cap \{t < \zeta\}$, the same conclusion as in (iii) holds; (v'): for any $t > 0$ and pre- t -equivalent paths $\omega, \omega' \in \Omega \cap \{t < \zeta\}$, the same conclusion as in (v) holds.

The notion of *rcll PrAF on $\llbracket 0, \zeta \rrbracket$* (or *rcll PrAF admitting m -null set*) is similarly defined.

REMARK 2.13. (i) Our notion of PrAF is different from what is found in Walsh [16].

(ii) Every PrAF (resp., PrAF on $\llbracket 0, \zeta \rrbracket$) is an AF (resp., AF on $\llbracket 0, \zeta \rrbracket$).

(iii) The MAF M^u and the CAF N^u of 0-energy appearing in Fukushima's decomposition (1.2) can be regarded as finite rcll PrAF's in view of the proof of Theorem 5.2.2 in [7]. In this case, the defining sets for M^u as PrAF are given by

$$\Xi := \{\omega \in \Omega \mid M_{s-}^{u_n}(\omega) \text{ converges uniformly on }]0, t] \text{ for } \forall t \geq 0$$

$$\text{for some subsequence } n_k \} \in \mathcal{F}_\infty,$$

$$\Xi_t := \{\omega \in \Omega \mid M_{s-}^{u_n}(\omega) \text{ converges uniformly on }]0, t]$$

$$\text{for some subsequence } n_k \} \in \mathcal{F}_t$$

for every $t > 0$, where $M_t^{u_n} := u_n(X_t) - u_n(X_0) - \int_0^t (u_n(X_s) - f_n(X_s)) ds$ with $f_n := n(u - nR_{n+1}u)$ and $u_n := R_1 f_n = nR_{n+1}u$. Hence, an MAF of stochastic integral type $\int_0^t g(X_{s-}) dM_s^u$ [$g, u \in \mathcal{F}$ with $g \in L^2(E; \mu_{(u)})$] can be regarded as a finite rcll PrAF. Consequently, any MAF of finite energy can also be regarded as an rcll PrAF, in view of the assertion of Lemma 5.6.3 in [7] and Lemma 2.14 below.

(iv) Every $M \in \mathring{\mathcal{M}}_{f\text{-loc}}$ can be regarded as a PrAF on $\llbracket 0, \zeta \rrbracket$, hence every $M \in \mathcal{M}_{\text{loc}}^{\llbracket 0, \zeta \rrbracket}$ is also. Since every local martingale can be written as the sum of a local martingale with bounded jumps (and hence a locally square-integrable martingale) and a local martingale of finite variation, we conclude that every local MAF is a PrAF.

LEMMA 2.14. *Let (A^n) be a sequence of finite rcll PrAF's with defining sets $\Xi^n \in \mathcal{F}_\infty$ and $\Xi_t^n \in \mathcal{F}_t$. For each $t > 0$, set*

$$\Xi_t := \left\{ \omega \in \bigcap_{n \in \mathbb{N}} \Xi_t^n \mid A^n \text{ converges uniformly on } [0, t[\right\} \in \mathcal{F}_t$$

and

$$\Xi := \left\{ \omega \in \bigcap_{n \in \mathbb{N}} \Xi^n \mid A^n \text{ converges uniformly on } [0, t[\text{ for every } t \in [0, \infty[\right\} \in \mathcal{F}_\infty.$$

Suppose that there exists an \mathcal{E} -polar set N such that $\mathbf{P}_x(\Xi) = 1$ for $x \in E \setminus N$. If we define $A_t := \varinjlim_{n \rightarrow \infty} A_t^n$ on Ω , then A is a finite rcl PrAF with its defining sets Ξ, Ξ_t .

PROOF. We only show that for any $t > 0$ and pre- t -equivalent paths $\omega, \omega', \omega \in \Xi_t$ implies that $\omega' \in \Xi_t$. Suppose that $\omega \in \Xi_t$ and ω is pre- t -equivalent to ω' . It is easy to see that $\omega' \in \bigcap_{n \in \mathbb{N}} \Xi_t^n$. We then see the uniform convergence of $A_{s-}^n(\omega') = A_{s-}^n(\omega)$ for $s \in]0, t]$. Therefore, $\omega' \in \Xi_t, A_s(\omega') = A_s(\omega)$ for $s \in [0, t[$ and $A_{s-}(\omega') = A_{s-}(\omega)$ for $s \in]0, t]$. \square

Recall that $\{\theta_t, t > 0\}$ denotes the time-shift operators on the path space for the process X .

LEMMA 2.15. For $t, s > 0$:

- (i) $\theta_t r_{t+s} \omega$ is s -equivalent to $r_s \omega$ if $t + s < \zeta(\omega)$ or $s \geq \zeta(\omega)$;
- (ii) $r_t \theta_s \omega$ is pre- t -equivalent to $r_{t+s} \omega$ and, moreover, if ω is continuous at s , then $r_t \theta_s \omega$ is t -equivalent to $r_{t+s} \omega$.

PROOF. (i) We may assume that $t + s < \zeta(\omega)$. For $v \in [0, s]$,

$$\theta_t r_{t+s} \omega(v) = \omega((s - v)-) = r_s \omega(v)$$

and so $\theta_t r_{t+s} \omega$ is s -equivalent to $r_s \omega$.

(ii) Note that $t + s < \zeta(\omega)$ is equivalent to $t < \zeta(\theta_s \omega)$. It follows from the definition, if $t + s < \zeta(\omega)$, that

$$(2.3) \quad (r_t \theta_s \omega)(v) = \begin{cases} \omega((t + s - v)-), & \text{if } 0 \leq v < t, \\ \omega(s), & \text{if } v = t, \end{cases}$$

while $r_{t+s} \omega(v) = \omega((t + s - v)-)$ for $0 \leq v \leq t$. Hence, typically, $r_t \theta_s \omega$ is only pre- t -equivalent to $r_{t+s} \omega$. \square

Fix $t > 0$. Set $\mathcal{H}_s^t := \mathcal{F}_t$ for $s \in [0, t]$ and $\mathcal{H}_s^t := \mathcal{F}_s$ for $s \in]t, \infty[$. $(\mathcal{H}_s^t)_{s \geq 0}$ is then a filtration over $(\Omega, \mathcal{F}_\infty)$ and $\mathcal{F}_s \subset \mathcal{H}_s^t$ for all $s \geq 0$.

LEMMA 2.16. The following assertions hold for any fixed $t > 0$:

- (i) if we let φ be a Borel function on $E \times E$ and set $X_{0-} := X_0$, then $[0, \infty[\times \Omega \ni (s, \omega) \mapsto \mathbf{1}_{]0, \zeta[}(s, \omega) \mathbf{1}_{\Gamma_t}(\omega) \varphi(X_{s-}(\omega), X_s(\omega))$ is (\mathcal{H}_s^t) -optional for any $\Gamma_t \in \mathcal{F}_t$;
- (ii) if we let A be an rcl PrAF with defining sets $\Xi \in \mathcal{F}_\infty, \Xi_t \in \mathcal{F}_t$ and we set $A_{0-}(\omega) := 0$ and $A_s^t(\omega) := \mathbf{1}_{\Xi_t}(\omega) (\mathbf{1}_{]0, t]}(s) A_s(\omega) + \mathbf{1}_{]t, \infty[}(s) A_t(\omega))$ for $\omega \in \Omega$, then $[0, \infty[\times \Omega \ni (s, \omega) \mapsto \mathbf{1}_{]0, \zeta[}(s, \omega) (A_s^t(\omega) - A_{s-}^t(\omega))$ is (\mathcal{H}_s^t) -optional.

PROOF. (i) Note that $\mathbf{1}_{\llbracket 0, \zeta \rrbracket}$ is (\mathcal{H}_s^t) -predictable. The assertion is clear if $\varphi = f \otimes g$ for bounded Borel functions f, g on E_Δ . The monotone class theorem for functions gives us the desired result.

(ii) Since A^t is (\mathcal{H}_s^t) -adapted and rcll on Ω and A^t_- is (\mathcal{H}_s^t) -adapted and lcll on Ω , $(s, \omega) \mapsto A_s(\omega)$ is (\mathcal{H}_s^t) -optional and $(s, \omega) \mapsto A^t_{s-}(\omega)$ is (\mathcal{H}_s^t) -predictable. Consequently, $(s, \omega) \mapsto A^t_s(\omega) - A^t_{s-}(\omega)$ is (\mathcal{H}_s^t) -optional. \square

By Lemma 3.2 of [3], for a finite rcll AF $A = (A_t)_{t \geq 0}$, there is a Borel function $\varphi: E \times E \rightarrow \mathbb{R}$ with $\varphi(x, x) = 0$ for all $x \in E$ such that

$$(2.4) \quad A_t - A_{t-} = \varphi(X_{t-}, X_t) \quad \text{for every } t \in]0, \zeta[, \mathbf{P}_m\text{-a.e.}$$

Moreover, if $\tilde{\varphi}$ is another such function, then $J(\varphi \neq \tilde{\varphi}) = 0$. As before, we refer to such a function φ as a jump function for A . Recall that if $M \in \mathcal{M}_{\text{loc}}^{\llbracket 0, \zeta \rrbracket}$, then there exists a jump function φ (unique in the above sense) so that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$ for $t \in]0, \zeta[, \mathbf{P}_m\text{-a.e.}$

LEMMA 2.17. *Let A be a finite rcll PrAF with defining sets $\{\Xi, \Xi_t, t \geq 0\}$. There then exists a real-valued Borel function φ on $E_\Delta \times E_\Delta$ with $\varphi(x, x) = 0$ for $x \in E_\Delta$ such that A with defining sets*

$$\begin{aligned} \tilde{\Xi} &:= \{\omega \in \Xi \mid A_s(\omega) - A_{s-}(\omega) = \varphi(X_{s-}(\omega), X_s(\omega)) \text{ for } s \in]0, \zeta(\omega)[\}, \\ \tilde{\Xi}_t &:= \{\omega \in \Xi_t \mid A_s(\omega) - A_{s-}(\omega) = \varphi(X_{s-}(\omega), X_s(\omega)) \text{ for } s \in]0, t[\cap]0, \zeta(\omega)[\} \end{aligned}$$

is again an rcll PrAF admitting m -null set. The analogous assertion holds for PrAF's on $\llbracket 0, \zeta \rrbracket$ and, in particular, for elements of $(\mathcal{M}_{\text{loc}})^{\llbracket 0, \zeta \rrbracket}$.

PROOF. Let $\varphi: E_\Delta \times E_\Delta \rightarrow \mathbb{R}$ be a Borel function vanishing on the diagonal and define $\tilde{\Xi}, \tilde{\Xi}_t$ in terms of φ , as above. Clearly, $\tilde{\Xi} = \bigcap_{t > 0} \tilde{\Xi}_t$, $\tilde{\Xi}_t \subset \tilde{\Xi}_s$ for $s < t$. Moreover, we see that $\theta_t \tilde{\Xi} \subset \tilde{\Xi}$ for $t \geq 0$ and $\theta_{t-s}(\tilde{\Xi}_t) \subset \tilde{\Xi}_s$ for $s < t$. For two pre- t -equivalent paths ω, ω' , we see that $\omega \in \tilde{\Xi}_t$ implies that $\omega' \in \tilde{\Xi}_t$.

By the previous lemma,

$$\Gamma := \{(s, \omega) \mid \mathbf{1}_{\llbracket 0, \zeta \rrbracket}(s, \omega) \mathbf{1}_{\Xi_t}(\omega) (A_s^t(\omega) - A^t_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega))) \neq 0\}$$

is (\mathcal{H}_s^t) -progressively measurable for any fixed $t > 0$ and the debut of Γ is

$$D_\Gamma(\omega) := \inf\{s \geq 0 \mid \mathbf{1}_{\llbracket 0, \zeta \rrbracket}(s, \omega) \mathbf{1}_{\Xi_t}(\omega) (A_s^t(\omega) - A^t_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega))) \neq 0\},$$

which is an (\mathcal{H}_s^t) -stopping time by (A5.1) in [15]. In particular,

$$\begin{aligned} \{\omega \in \Omega \mid \mathbf{1}_{\llbracket 0, \zeta \rrbracket}(s, \omega) \mathbf{1}_{\Xi_t}(\omega) (A_s(\omega) - A_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega))) = 0 \\ \text{for } s \in [0, t[\} \end{aligned}$$

$$= \{\omega \in \Omega \mid t < D_\Gamma(\omega)\} \in \mathcal{H}_t^t = \mathcal{F}_t.$$

Hence,

$$\begin{aligned} & \{\omega \in \Xi_t \mid A_s(\omega) - A_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega)) = 0 \text{ for } s \in]0, t[\cap]0, \zeta(\omega)[\} \\ &= \{\omega \in \Xi_t \mid A_s(\omega) - A_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega)) = 0 \\ & \hspace{15em} \text{for } s \in [0, t[\cap [0, \zeta(\omega)[\} \\ &= \{\omega \in \Xi_t \mid \mathbf{1}_{]0, \zeta[[}(s, \omega)(A_s(\omega) - A_{s-}(\omega) - \varphi(X_{s-}(\omega), X_s(\omega))) = 0 \\ & \hspace{15em} \text{for } s \in [0, t[\} \\ & \in \mathcal{F}_t. \end{aligned}$$

Therefore, $\tilde{\Xi}_t \in \mathcal{F}_t$ and $\tilde{\Xi} \in \mathcal{F}_\infty$. The proof for PrAF's on $]0, \zeta[[$ is similar, so we omit it. \square

The following theorem is a key to our extension of Nakao's operator Γ . Its proof is complicated by measurability issues, but the idea behind it is fairly transparent. We will use the convention $X_{0-}(\omega) := X_0(\omega)$.

THEOREM 2.18 (Dual PrAF). *Let A be a finite rcll PrAF on $]0, \zeta[[$ with defining sets Ξ, Ξ_t admitting m -null set. Suppose that there is a Borel function φ on $E \times E$ with $\varphi(x, x) = 0$ for $x \in E$ such that $\varphi(X_{s-}(\omega), X_s(\omega)) = A_s(\omega) - A_{s-}(\omega)$ for all $s \in]0, t[\cap]0, \zeta[$ and all $\omega \in \Xi_t$. Set*

$$(2.5) \quad \begin{aligned} \hat{A}_t(\omega) &:= A_t(r_t(\omega)) + \varphi(X_t(\omega), X_{t-}(\omega)) \\ & \text{for } t \in [0, \zeta(\omega)[\text{ and } \hat{A}_t(\omega) := 0 \text{ for } t \in [\zeta(\omega), \infty[. \end{aligned}$$

\hat{A} is then an rcll PrAF on $]0, \zeta[[$ admitting m -null set such that

$$\hat{A}_t = A_{t-} \circ r_t + \varphi(X_t, X_{t-}) \quad \text{and} \quad \hat{A}_t - \hat{A}_{t-} = \varphi(X_t, X_{t-})$$

for all $t \in]0, \zeta[, \mathbf{P}_m$ -a.e.

PROOF. Let $\Xi \in \mathcal{F}_\infty, \Xi_t \in \mathcal{F}_t^m, t > 0$ be the defining sets of A admitting m -null set. We easily see $r_t^{-1}(\Xi_t) \cap \{t < \zeta\} \subset r_s^{-1}(\Xi_s) \cap \{s < \zeta\}$ for $s \in]0, t[$ by use of Lemma 2.15(i) and $\theta_{t-s}\Xi_t \subset \Xi_s$.

Set $\hat{\Xi}_t := r_t^{-1}(\Xi_t)$ for $t > 0$ and $\hat{\Xi} := \bigcap_{t>0} \hat{\Xi}_t$. We then see that $\hat{\Xi} = \bigcap_{t>0, t \in \mathbb{Q}} \hat{\Xi}_t$ by use of $r_t^{-1}(\Xi_t) \cap \{t \geq \zeta\} = \{t \geq \zeta\}$ and the monotonicity of $r_t^{-1}(\Xi_t) \cap \{t < \zeta\}$. Indeed, we have $\hat{\Xi} \subset \bigcap_{t>0, t \in \mathbb{Q}} \hat{\Xi}_t \subset (\hat{\Xi}_s \cap \{s < \zeta\}) \cup \{t \geq \zeta\}$ for any $0 < s < t$ with $t \in \mathbb{Q}$. Taking the intersection over $t \in]s, \infty[\cap \mathbb{Q}$, we have $\hat{\Xi} \subset \bigcap_{t>0, t \in \mathbb{Q}} \hat{\Xi}_t \subset \hat{\Xi}_s$ for all $s > 0$, which yields the assertion.

We prove $\theta_t \hat{\Xi} \subset \hat{\Xi}$ for each $t \geq 0$, in particular, $\theta_t \hat{\Xi} \subset \theta_s \hat{\Xi}$ and, equivalently, $\theta_s^{-1} \hat{\Xi} \subset \theta_t^{-1} \hat{\Xi}$ if $s \in [0, t]$. Suppose that $\omega \in \hat{\Xi}$. Then, $r_{t+s}\omega \in \Xi_{t+s}$. If $t + s < \zeta(\omega)$, then $r_{t+s}\omega \in \Xi_s$, otherwise $r_{t+s}\omega = \omega_\Delta \in \Xi_s$. Hence, we have

$r_s\theta_t\omega \in \Xi_s$, by Lemma 2.15(ii). Therefore, $r_s\theta_t\omega \in \Xi_s$ for all $s > 0$, which implies that $\theta_t\omega \in \widehat{\Xi}$.

Next, we prove $\theta_{t-s}(\widehat{\Xi}_t) \subset \widehat{\Xi}_s$ for $s \in]0, t[$. Take $\omega \in \widehat{\Xi}_t$. Then, $r_s\theta_{t-s}\omega$ is pre- s -equivalent to $r_t\omega \in \Xi_t \subset \Xi_s$, by Lemma 2.15(ii) and hence $r_s\theta_{t-s}\omega \in \Xi_s$. Therefore, $\theta_{t-s}\omega \in \widehat{\Xi}_s$ for all $s \in]0, t[$.

From $\Xi_t \subset \mathcal{F}_t^m$, we obtain $\widehat{\Xi}_t \in \mathcal{F}_t^m$, by Lemma 2.10. Since $(\widehat{\Xi}_t)^c = r_t^{-1}((\Xi_t)^c) = r_t^{-1}((\Xi_t)^c) \cap \{t < \zeta\}$ holds by noting $\omega_\Delta \in \Xi_t$, we have $\mathbf{P}_m((\widehat{\Xi}_t)^c) = \mathbf{P}_m((\Xi_t)^c) = 0$.

By (2.2), $v \mapsto r_s(\omega)(v)$ is continuous at $v = s$. Hence, on $\widehat{\Xi}_t \cap \{t < \zeta\}$, we have $\varphi(X_{s-}, X_s) \circ r_s = \varphi(X_s, X_{s-}) \circ r_s = 0$, in particular, $A_s \circ r_s = A_{s-} \circ r_s$ for $s \in]0, t[$.

The remainder of the proof is devoted to showing that \widehat{A} is an rcll PrAF on $[0, \zeta[$ with defining sets $\widehat{\Xi}, \widehat{\Xi}_t$ such that on $\widehat{\Xi}_t \cap \{t < \zeta\}$, $\widehat{A}_s = A_{s-} \circ r_s + \varphi(X_s, X_{s-})$, $s \in]0, t[$. First, note that for $\omega \in \widehat{\Xi}$, $|\widehat{A}_t(\omega)| < \infty$ for any $t \in]0, \zeta(\omega)[$ because, by taking $T \in]t, \zeta(\omega)[$, $r_T\omega \in \Xi_T$ implies that $r_t\omega \in \Xi_t$, hence $|A_{t-}(r_t\omega)| < \infty$. Moreover, for $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\}$, we see that $r_s\omega \in \Xi_s \cap \{s < \zeta\}$ and $|A_{s-}(r_s\omega)| < \infty$ for all $0 < s < t$.

For two pre- t -equivalent paths $\omega, \omega' \in \Omega \cap \{t < \zeta\}$ with $t > 0$, we show that $\omega \in \widehat{\Xi}_t$ implies $\omega' \in \widehat{\Xi}_t$ and $\widehat{A}_s(\omega) = \widehat{A}_s(\omega')$ for $s \in [0, t[$. Recall that $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\} \subset \widehat{\Xi}_s \cap \{s < \zeta\}$ for $s \in [0, t[$ and note that ω and ω' are s -equivalent for any $s \in [0, t[$. On the other hand, $s < \zeta(\omega)$ is equivalent to $s < \zeta(\omega')$ for any $s \in [0, t[$. We then see that $r_s\omega \in \Xi_s$ is s -equivalent to $r_s\omega'$ for any $s \in [0, t[$, which implies that $r_s\omega' \in \Xi_s$ for any $]0, t[$ and $A_{s-}(r_s\omega) = A_{s-}(r_s\omega')$ for any $s \in [0, t[$.

Fix $t > 0$. On $\widehat{\Xi}_t \cap \{t < \zeta\}$ and for any $p, q > 0$ with $p + q < t$, by Lemma 2.15,

$$\begin{aligned} \widehat{A}_{p+q} &= A_{(p+q)-} \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= (A_p + A_{q-} \circ \theta_p) \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= A_p \circ r_{p+q} + A_{q-} \circ \theta_p \circ r_{p+q} + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= (A_{p-} \circ r_{p+q} + \varphi(X_{p-}, X_p) \circ r_{p+q}) + A_{q-} \circ r_q + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= (A_{p-} \circ r_p \circ \theta_q + \varphi(X_q, X_{q-})) + (\widehat{A}_q - \varphi(X_q, X_{q-})) \\ &\quad + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= (\widehat{A}_p - \varphi(X_p, X_{p-})) \circ \theta_q + \widehat{A}_q + \varphi(X_{p+q}, X_{(p+q)-}) \\ &= \widehat{A}_p \circ \theta_q + \widehat{A}_q. \end{aligned}$$

On $\widehat{\Xi}_t \cap \{t < \zeta\}$, again by Lemma 2.15 and (2.2), for any $s > 0$ and $u \in]0, s[$,

$$\begin{aligned} \widehat{A}_s - \widehat{A}_{s-u} &= \widehat{A}_u \circ \theta_{s-u} \\ &= (A_{u-} \circ r_u + \varphi(X_u, X_{u-})) \circ \theta_{s-u} \\ &= A_{u-} \circ r_u \circ \theta_{s-u} + \varphi(X_s, X_{s-}) \\ &= A_{u-} \circ r_s + \varphi(X_s, X_{s-}). \end{aligned}$$

So,

$$\lim_{u \downarrow 0} (\widehat{A}_s - \widehat{A}_{s-u}) = \varphi(X_s, X_{s-}).$$

This shows that \widehat{A} has left limit at $s \in]0, t[$ and $\widehat{A}_s - \widehat{A}_{s-} = \varphi(X_s, X_{s-})$.

To show the right continuity of \widehat{A} on $\widehat{\Xi}_t \cap \{t < \zeta\}$ at any $s \in]0, t[$, note that for any $u \in]0, t - s[$, by Lemma 2.15 and (2.2),

$$\begin{aligned} \widehat{A}_{s+u} - \widehat{A}_s &= \widehat{A}_u \circ \theta_s \\ &= (A_{u-} \circ r_u + \varphi(X_u, X_{u-})) \circ \theta_s \\ &= A_{u-} \circ r_u \circ \theta_s + \varphi(X_{s+u}, X_{(s+u)-}) \\ &= A_{u-} \circ r_{s+u} + \varphi(X_{s+u}, X_{(s+u)-}). \end{aligned}$$

Since $(A_v - A_{v-}) \circ r_{s+v} = \varphi(X_{v-}, X_v) \circ r_{s+v} = \varphi(X_s, X_{s-})$, while, by Lemma 2.15 and (2.2),

$$\begin{aligned} (A_v - A_{v-}) \circ r_{s+v} &= \lim_{u \downarrow 0} (A_v - A_{v-u}) \circ r_{s+v} \\ &= \lim_{u \downarrow 0} A_u \circ \theta_{v-u} \circ r_{s+v} \\ &= \lim_{u \downarrow 0} A_{u-} \circ r_{v+u} + \varphi(X_s, X_{s-}), \end{aligned}$$

we conclude that

$$\lim_{u \downarrow 0} A_{u-} \circ r_{s+u} = 0.$$

On the other hand, for any $s \geq 0$,

$$\begin{aligned} \lim_{u \downarrow 0} \varphi(X_{s+u}, X_{(s+u)-}) &= \lim_{u \downarrow 0} \varphi(X_{(v-u)-}, X_{v-u}) \circ r_{s+v} \\ &= \lim_{u \downarrow 0} (A_{v-u} - A_{(v-u)-}) \circ r_{s+v} \\ &= (A_{v-} - A_{v-}) \circ r_{s+v} = 0. \end{aligned}$$

Hence, we have, for $s > 0$,

$$\lim_{u \downarrow 0} (\widehat{A}_{s+u} - \widehat{A}_s) = 0.$$

In other words, \widehat{A} is right-continuous at any $s \in]0, t[$ on $\widehat{\Xi}_t \cap \{t < \zeta\}$. We also see that

$$\lim_{u < s, s \downarrow 0, u \downarrow 0} (\widehat{A}_{s+u} - \widehat{A}_s) = 0.$$

We can thus define the limit $\widehat{A}_0(\omega) := \lim_{s \downarrow 0} \widehat{A}_s(\omega)$ for $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\}$ for any $t > 0$. We also see that $\widehat{A}_0(\omega) = \lim_{s \downarrow 0} \widehat{A}_{s-}(\omega)$ for $\omega \in \widehat{\Xi}_t \cap \{t < \zeta\}$ for any $t > 0$ because $\lim_{s \downarrow 0} \varphi(X_s, X_{s-}) = 0$. Next, we prove that $\widehat{A}_0(\omega) = 0$ for $\omega \in \widehat{\Xi}_t \cap \{t <$

$\zeta\}$ for any $t > 0$. Take $\omega \in \widehat{\mathcal{E}}_t \cap \{t < \zeta\}$ for some fixed $t > 0$. It suffices to show that $\lim_{u \downarrow 0} \widehat{A}_{s-u}(\theta_u \omega) = \widehat{A}_s(\omega)$ for $s \in [0, t]$. Owing to Lemma 2.15(ii), we have

$$\begin{aligned} \widehat{A}_{s-u}(\theta_u \omega) &= A_{(s-u)-}(r_{s-u} \theta_u \omega) + \varphi(X_s(\omega), X_{s-}(\omega)) \\ &= A_{(s-u)-}(r_s \omega) + \varphi(X_s(\omega), X_{s-}(\omega)) \\ &= A_{s-u}(r_s \omega) - \varphi(X_u(\omega), X_{u-}(\omega)) + \varphi(X_s(\omega), X_{s-}(\omega)) \\ &= A_{s-u}(r_s \omega) - \widehat{A}_u(\omega) + \widehat{A}_{u-}(\omega) + \varphi(X_s(\omega), X_{s-}(\omega)) \\ &\rightarrow A_{s-}(r_s \omega) + \varphi(X_s(\omega), X_{s-}(\omega)) \quad \text{as } u \downarrow 0 \\ &= \widehat{A}_s(\omega). \end{aligned}$$

The \mathcal{F}_t^m -measurability of \widehat{A}_t is clear from (2.5). This proves the theorem. \square

3. Stochastic integral for Dirichlet processes. The following fact will be used repeatedly in this section. Since a Hunt process is quasi-left continuous, for each fixed $t > 0$, we have $X_{t-} = X_t$, \mathbf{P}_x -a.s. for every $x \in E$.

Before embarking on the definition of our stochastic integral, we prepare the following lemma for later use.

LEMMA 3.1. *The following assertions hold.*

(i) *Let $\{G_n\}$ be an increasing sequence of finely open Borel sets. The following are then equivalent:*

- (a) $\{G_n\}$ is a nest, that is, $\mathbf{P}_x(\lim_{n \rightarrow \infty} \sigma_{E \setminus G_n} \wedge \zeta = \zeta) = 1$ for q.e. $x \in E$;
- (b) $E = \bigcup_{n=1}^{\infty} G_n$ q.e.;
- (c) $\mathbf{P}_x(\lim_{n \rightarrow \infty} \sigma_{E \setminus G_n} = \infty) = 1$ for m-a.e. $x \in E$;
- (d) $\mathbf{P}_x(\lim_{n \rightarrow \infty} \sigma_{E \setminus G_n} = \infty) = 1$ for q.e. $x \in E$.

In particular, for an increasing sequence $\{F_n\}$ of closed sets, $\{F_n\}$ is an \mathcal{E} -nest if and only if $\mathbf{P}_x(\lim_{n \rightarrow \infty} \sigma_{E \setminus F_n} = \infty) = 1$ for m-a.e. $x \in E$.

(ii) *For a function f on E , $f \in \mathcal{F}_{\text{loc}}$ if and only if there exist an \mathcal{E} -nest $\{F_k\}$ of closed sets and $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}_b$ such that $f = f_k$ q.e. on F_k .*

PROOF. (i) For the implications (i)(a) \iff (i)(b), see Theorem 4.6 in [11]. The implication (i)(d) \implies (i)(a) is clear. Next, we show (i)(b) \implies (i)(c). Since each G_n is finely open, it is quasi-open by Theorem 4.6.1(i) in [7]. So, there exists a common nest $\{A_\ell\}$ of closed sets such that $(E \setminus G_n) \cap A_\ell$ is closed for all $n, \ell \in \mathbb{N}$. Set $\sigma := \lim_{n \rightarrow \infty} \sigma_{E \setminus G_n}$. We then have that for all $n \in \mathbb{N}$, $X_{\sigma_{E \setminus G_n}} \in E \setminus G_n$ \mathbf{P}_x -a.s. on $\{\sigma < \infty\}$ for q.e. $x \in E$. We have $\mathbf{P}_x(\lim_{\ell \rightarrow \infty} \sigma_{E \setminus A_\ell} = \infty) = 1$ q.e. $x \in E$. Since $\sigma(\omega) < \infty$ and $\lim_{\ell \rightarrow \infty} \sigma_{E \setminus A_\ell}(\omega) = \infty$ together imply $\sigma(\omega) < \sigma_{E \setminus A_{\ell_0}}(\omega)$ for some $\ell_0 = \ell_0(\omega) \in \mathbb{N}$, we have that there exists $\ell_0 \in \mathbb{N}$ such that $\sigma_{E \setminus G_n} <$

$\sigma_{E \setminus A_\ell}$ for all $n > \ell \geq \ell_0$, \mathbf{P}_x -a.s. on $\{\sigma < \infty\}$ for q.e. $x \in E$. This means that

$$\begin{aligned} \mathbf{P}_x(\sigma < \infty) &\leq \mathbf{P}_x\left(\varliminf_{\ell \rightarrow \infty} \{X_{\sigma_{E \setminus G_n}} \in (E \setminus G_n) \cap A_\ell \text{ for all } n > \ell, \sigma < \infty\}\right) \\ &\leq \varliminf_{\ell \rightarrow \infty} \mathbf{P}_x(X_{\sigma_{E \setminus G_n}} \in (E \setminus G_\ell) \cap A_\ell \text{ for all } n > \ell, \sigma < \infty) \\ &\leq \varliminf_{\ell \rightarrow \infty} \mathbf{P}_x(X_\sigma \in (E \setminus G_\ell) \cap A_\ell, \sigma < \infty) \\ &\leq \varliminf_{\ell \rightarrow \infty} \mathbf{P}_x(X_\sigma \in E \setminus G_\ell, \sigma < \infty) \\ &= \mathbf{P}_x\left(X_\sigma \in E \setminus \bigcup_{\ell=1}^\infty G_\ell, \sigma < \infty\right) = 0 \end{aligned}$$

for m -a.e. $x \in E$ because of the \mathcal{E} -polarity of $E \setminus \bigcup_{\ell=1}^\infty G_\ell$, where we use the quasi-left continuity of X up to ∞ and the closedness of $(E \setminus G_\ell) \cap A_\ell$. The implication (i)(c) \iff (i)(d) follows from the fact that $x \mapsto \mathbf{P}_x(\sigma < \infty)$ is the limit of a decreasing sequence of excessive functions and Lemma 4.1.7 in [7].

(i) The ‘‘if’’ part is clear by (i) because $\tau_{F_k} = \tau_{G_k}$, where G_k is the fine interior of F_k . We only prove the ‘‘only if’’ part. Take $f \in \mathcal{F}_{\text{loc}}$. There then exist $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}$ and an increasing sequence $\{G_k\}$ of finely open sets with $E = \bigcup_{k=1}^\infty G_k$ q.e. such that $f = f_k$ m -a.e. on G_k . We may take $f_k \in \mathcal{F}_b$ for each $k \in \mathbb{N}$ by replacing f_k with $(-k) \vee f_k \wedge k$ and G_k with $G_k \cap \{|f| < k\}$. Note that f and f_k are quasi-continuous, so $f = f_k$ q.e. on G_k . Taking an \mathcal{E} -quasi-closure $\overline{G_k}^\mathcal{E}$ of G_k , we have $f = f_k$ q.e. on $\overline{G_k}^\mathcal{E}$ (see [10] for the definition of \mathcal{E} -quasi-closure). Let $\{A_n\}$ be a common \mathcal{E} -nest of closed sets such that for each $k, n \in \mathbb{N}$, $\overline{G_k}^\mathcal{E} \cap A_n$ is closed. Set $F_k := \overline{G_k}^\mathcal{E} \cap A_k$. By (i), $\{G_k\}$ is a nest, hence $\overline{G_k}^\mathcal{E}$ is a nest of q.e. finely closed sets because of $\tau_{G_k} \leq \tau_{\overline{G_k}^\mathcal{E}}$. Here, we recognize $\overline{G_k}^\mathcal{E}$ as a finely closed Borel sets by deleting an \mathcal{E} -polar set. Since $\{A_n\}$ is a nest of closed sets, $\{F_k\}$ is also, that is, $\mathbf{P}_m(\lim_{k \rightarrow \infty} \tau_{F_k} \neq \zeta) = 0$. Therefore, $\{F_k\}$ is an \mathcal{E} -nest of closed sets. We easily see that for each $k \in \mathbb{N}$, $f = f_k$ q.e. on F_k . \square

Recall that any locally square-integrable MAF M on $[[0, \zeta[$ admits a jump function φ on $E \times E$ with $\varphi(x, x) = 0$ for $x \in E$ such that $\Delta M_t = \varphi(X_{t-}, X_t)$ for $t \in]0, \zeta[$, \mathbf{P}_m -a.e. When $M \in \mathring{\mathcal{M}}$, we can strengthen this statement by replacing $]0, \zeta[$ with $]0, \infty[$ in view of Fukushima’s decomposition and the combination of Theorem 5.2.1 and Lemma 5.6.3 in [7].

LEMMA 3.2. *Let ϕ be a Borel function on $E_\Delta \times E_\Delta$ satisfying $\phi(x, x) = 0$ for all $x \in E_\Delta$.*

(i) *Suppose that*

$$N(\mathbf{1}_{E \times E}(|\phi|^2 \wedge |\phi|))\mu_H \in S.$$

There then exists a unique, purely discontinuous local MAF K on $\llbracket 0, \zeta \llbracket$ [i.e., $K \in (\mathcal{M}_{\text{loc}}^1)^{\llbracket 0, \zeta \llbracket}$] such that $K_t - K_{t-} = \phi(X_{t-}, X_t)$ for all $t \in [0, \zeta[$, \mathbf{P}_x -a.s. for q.e. $x \in E$.

(ii) If

$$N(\mathbf{1}_{E \times E_{\Delta}}(|\phi|^2 \wedge |\phi|))\mu_H \in S,$$

then K can be taken to be a local MAF (i.e., $K \in \mathcal{M}_{\text{loc}}^1$) and $K_t - K_{t-} = \phi(X_{t-}, X_t)$ for all $t \in [0, \infty[$, \mathbf{P}_x -a.s. for q.e. $x \in E$.

PROOF. The proof of (ii) is similar to that of (i), so we only prove (i). By martingale theory (see, e.g. [9]), the hypothesis implies that the compensated process

$$\begin{aligned} K_t^{(2)} := & \sum_{0 < s \leq t} \phi(X_{s-}, X_s) \mathbf{1}_{\{|\phi(X_{s-}, X_s)| > 1\}} \mathbf{1}_{\{s < \zeta\}} \\ & - \int_0^t \int_E N(X_s, dy) \phi(X_s, y) \mathbf{1}_{\{|\phi(X_s, y)| > 1\}} dH_s \end{aligned}$$

is a local MAF of X of finite variation on $\llbracket 0, \zeta \llbracket$ and

$$\begin{aligned} K_t^{(1)} := & \lim_{\varepsilon \rightarrow 0} \left(\sum_{0 < s \leq t} \phi(X_{s-}, X_s) \mathbf{1}_{\{\varepsilon < |\phi(X_{s-}, X_s)| \leq 1\}} \mathbf{1}_{\{s < \zeta\}} \right. \\ & \left. - \int_0^t \int_E N(X_s, dy) \phi(X_s, y) \mathbf{1}_{\{\varepsilon < |\phi(X_s, y)| \leq 1\}} dH_s \right) \end{aligned}$$

is a purely discontinuous locally square-integrable MAF of X on $\llbracket 0, \zeta \llbracket$. Thus, $K := K^{(1)} + K^{(2)}$ is a purely discontinuous MAF on $\llbracket 0, \zeta \llbracket$ with jump function ϕ . The uniqueness is clear from martingale theory. \square

DEFINITION 3.3. Let M be a local MAF on $\llbracket 0, \zeta \llbracket$ [i.e., $M \in (\mathcal{M}_{\text{loc}}^1)^{\llbracket 0, \zeta \llbracket}$] with jump function φ . Assume that for q.e. $x \in E$, \mathbf{P}_x -a.s.

$$(3.1) \quad \int_0^t \int_E (\widehat{\varphi}^2 \mathbf{1}_{\{|\widehat{\varphi}| \leq 1\}} + |\widehat{\varphi}| \mathbf{1}_{\{|\widehat{\varphi}| > 1\}})(X_s, y) N(X_s, dy) dH_s < \infty$$

for every $t < \zeta$,

where $\widehat{\varphi}(x, y) := \varphi(x, y) + \varphi(y, x)$. Define, \mathbf{P}_m -a.e. on $\llbracket 0, \zeta \llbracket$,

$$(3.2) \quad \Lambda(M)_t := -\frac{1}{2}(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t) \quad \text{for } t \in [0, \zeta[$$

where K_t is the purely discontinuous local MAF on $\llbracket 0, \zeta \llbracket$ with

$$(3.3) \quad K_t - K_{t-} = -\widehat{\varphi}(X_{t-}, X_t) \quad \text{for every } t < \zeta, \mathbf{P}_x\text{-a.s.}$$

for q.e. $x \in E$.

REMARK 3.4. (i) The condition (3.1) is nothing but $N(\mathbf{1}_{E \times E}(|\widehat{\varphi}|^2 \wedge |\widehat{\varphi}|))\mu_H \in S$. In particular, condition (3.1) is satisfied by the jump function of any element of $(\mathcal{M}_{\text{loc}})^{\llbracket 0, \zeta \rrbracket}$.

(ii) It follows from Remark 2.13(iv) and Theorem 2.18 that $\Lambda(M)$ is a continuous PrAF admitting m -null set on $\llbracket 0, \zeta \rrbracket$. (This is because Remark 2.13(iv) and Theorem 2.18 imply that the process defined on $[0, \zeta[$ by $B_t := M_t \circ r_t + \varphi(X_t, X_{t-})$ is an rcll PrAF, with left-limit process $B_{t-} = B_t - \varphi(X_t, X_{t-})$. It follows that $\Lambda(M)$ is rcll on $[0, \zeta[$ and that $\Lambda(M)_{t-} = \Lambda(M)_t$ for all $t \in]0, \zeta[$, \mathbf{P}_m -a.e.) Note that $-K_t := \sum_{s \leq t} \widehat{\varphi}(X_{s-}, X_s) \mathbf{1}_{\{s < \zeta\}} - \int_0^t \int_E \widehat{\varphi}(X_s, y) N(X_s, dy) dH_s$, $t < \zeta$, satisfies $K_t = K_t \circ r_t$ \mathbf{P}_m -a.e. on $\{t < \zeta\}$ for fixed $t > 0$. In view of Theorem 2.18, it is then clear from the definition that Λ is a linear operator that maps local MAF's on $\llbracket 0, \zeta \rrbracket$ satisfying condition (3.1) into even CAF's on $\llbracket 0, \zeta \rrbracket$ admitting m -null set.

(iii) If $\{M^n, n \geq 1\}$ is a sequence of MAF's having finite energy and converging in probability to M , then it is easy to see that $M_t^n \circ r_t, \varphi^n(X_{t-}, X_t) = M_t^n - M_{t-}^n$ and $\varphi^n(X_t, X_{t-})$ converge to $M_t \circ r_t, \varphi(X_{t-}, X_t) = M_t - M_{t-}$ and $\varphi(X_t, X_{t-})$ in probability, respectively, under \mathbf{P}_m . Hence, we have that $\Lambda(M^n)_t$ converges to $\Lambda(M)_t$ in measure for each $t > 0$.

(iv) For $u \in \mathcal{F}$,

$$\Lambda(M^u)_t = -\frac{1}{2}(M_t^u + M_t^u \circ r_t + u(X_{t-}) - u(X_t)) = N_t^u,$$

\mathbf{P}_m -a.e. on $\{t < \zeta\}$, for each fixed $t \geq 0$. The first equality above is just the definition of $\Lambda(M^u)$, while the second follows by applying r_t to both sides of (1.2) because $X_t = X_{t-}$ and $N^u \circ r_t = N_t^u$, \mathbf{P}_m -a.e. on $\{t < \zeta\}$. (The last property is proved in [4], Theorem 2.1, when X is a diffusion, but the same proof works for general symmetric Markov process X .) Since both $\Lambda(M^u)_t$ and N_t^u are continuous in t , we even have, \mathbf{P}_m -a.e.,

$$\Lambda(M^u)_t = N_t^u \quad \text{for all } t < \zeta.$$

We are going to show that $\Lambda(M)$ defined above coincides on $[0, \zeta[$ with $\Gamma(M)$ defined in (1.5) by Nakao when M is an MAF of finite energy. An AF Z is called *even* (resp., *odd*) if and only if $Z_t \circ r_t = Z_t$ (resp., $Z_t \circ r_t = -Z_t$), \mathbf{P}_m -a.e. on $\{t < \zeta\}$ for each $t > 0$. For an rcll process Z with $Z_0 = 0$ and $T > 0$, we define

$$R_T Z_t := (R_T Z)_t := Z_{T-} - Z_{(T-t)-} \quad \text{for } 0 \leq t \leq T,$$

with the convention $Z_{0-} = Z_0 = 0$. Note that $R_T Z_t$ so defined is an rcll process in $t \in [0, T]$.

LEMMA 3.5. *Suppose that Z is an rcll PrAF. Then, \mathbf{P}_m -a.e. on $\{T < \zeta\}$,*

$$(3.4) \quad R_T Z_t = \begin{cases} Z_t \circ r_T, & \text{if } Z \text{ is even,} \\ -Z_t \circ r_T, & \text{if } Z \text{ is odd,} \end{cases} \quad \text{for every } t \in [0, T].$$

PROOF. Let Z be an rcll PrAF and let $T > 0$. By Lemma 2.15,

$$(3.5) \quad Z_t \circ r_T = (Z_T - Z_{T-t} \circ \theta_t) \circ r_T = Z_T \circ r_T - Z_{T-t} \circ r_{T-t}$$

for all $t < T$.

When Z is even,

$$Z_t \circ r_T = Z_T - Z_{T-t} = Z_{T-} - Z_{(T-t)-} = R_T Z_t,$$

\mathbf{P}_m -a.e. on $\{T < \zeta\}$ for each fixed $0 \leq t < T$. Since both sides are right-continuous in $t \in [0, T[$, we have, \mathbf{P}_m -a.e., $R_T Z_t = Z_t \circ r_T$ for every $t \in [0, T]$. When Z is an odd AF of Z , (3.4) can be proven similarly. \square

THEOREM 3.6. For an MAF M of finite energy, $\Lambda(M)$ defined above coincides on $\llbracket 0, \zeta \llbracket$ with $\Gamma(M)$ defined in (1.5), \mathbf{P}_m -a.e.

PROOF. For $u \in \mathcal{F}$ and $0 < t < T$, since N^u is an even CAF, by Lemma 3.5,

$$\begin{aligned} (M_t^u + 2N_t^u) \circ r_T &= (u(X_t) - u(X_0) + N_t^u) \circ r_T \\ &= u(X_{(T-t)-}) - u(X_{T-}) + N_{T-}^u - N_{(T-t)-}^u \\ &= M_{(T-t)-}^u - M_{T-}^u \\ &= -R_T M_t^u. \end{aligned}$$

Since both $(M_t^u + 2N_t^u) \circ r_T$ and $R_T M_t^u$ are right-continuous in t , we have, \mathbf{P}_m -a.e. on $\{T < \zeta\}$,

$$(3.6) \quad R_T M_t^u = -(M_t^u + 2N_t^u) \circ r_T \quad \text{for every } t \in [0, T].$$

For $u \in \mathcal{D}(\mathcal{L}) \subset \mathcal{F}$ and $v \in \mathcal{F}_b$, define $M_t = \int_0^t v(X_{s-}) dM_s^u$, which is an MAF of finite energy. Note that, since $u \in \mathcal{D}(\mathcal{L})$, $N_t^u = \int_0^t \mathcal{L}u(X_s) ds$ is a continuous process of finite variation. For each fixed $0 < t < T$ and $n \geq 1$, define $t_i = it/n$ and $s_i = T - t + t_i$. Using the standard Riemann sum approximation of the Itô integral and of the covariance process $[M^v, M^u]$, we have, \mathbf{P}_m -a.e. on $\{T < \zeta\}$,

$$\begin{aligned} &M_T - M_{T-t} + [M^v, M^u]_T - [M^v, M^u]_{T-t} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=0}^{n-1} v(X_{s_i})(M_{s_{i+1}}^u - M_{s_i}^u) + (M_{s_{i+1}}^v - M_{s_i}^v)(M_{s_{i+1}}^u - M_{s_i}^u) \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=0}^{n-1} v(X_{s_{i+1}})(M_{s_{i+1}}^u - M_{s_i}^u) - (N_{s_{i+1}}^v - N_{s_i}^v)(M_{s_{i+1}}^u - M_{s_i}^u) \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} v(X_{s_{i+1}})(M_{s_{i+1}}^u - M_{s_i}^u) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} v(X_{T-t+i})(R_T M_{t-t_i}^u - R_T M_{t-t_{i+1}}^u) \\
 &= \lim_{n \rightarrow \infty} \left(\sum_{i=0}^{n-1} v(X_{t-t_{i+1}})(M_{t-t_{i+1}}^u - M_{t-t_i}^u + 2N_{t-t_{i+1}}^u - 2N_{t-t_i}^u) \right) \circ r_T \\
 &= - \left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T,
 \end{aligned}$$

where in the third equality, we used the fact that N^u has zero energy, while in the second to the last equality, we used (3.6). Note that the stochastic integral involving N^u in the last equality is just the Lebesgue–Stieltjes integral since N^u is of finite variation. Also, note that $X_t = X_{t-}$, \mathbf{P}_m -a.e., for each fixed $t > 0$. So, we have, for each fixed $t < T$, \mathbf{P}_m -a.e. on $\{T < \zeta\}$,

$$R_T M_t + R_T[M^v, M^u]_t = - \left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T.$$

Since both sides are right-continuous in $t \in [0, T]$, we have, \mathbf{P}_m -a.e. on $\{T < \zeta\}$,

$$\begin{aligned}
 (3.7) \quad R_T M_t + R_T[M^v, M^u]_t &= - \left(\int_0^t v(X_{s-}) d(M_s^u + 2N_s^u) \right) \circ r_T \\
 &\text{for every } t \in [0, T].
 \end{aligned}$$

By [14], Theorem 3.1 and (1.7),

$$\int_0^t v(X_{s-}) dN_s^u = \int_0^t v(X_s) dN_s^u = \Gamma(M)_t - \frac{1}{2} \langle M^{v,c} + M^{v,j}, M^{u,c} + M^{u,j} \rangle_t.$$

It follows that \mathbf{P}_m -a.e. on $\{T < \zeta\}$,

$$\begin{aligned}
 R_T M_t + R_T[M^v, M^u]_t &= -(M_t + 2\Gamma(M)_t - \langle M^{v,c} + M^{v,j}, M^{u,c} + M^{u,j} \rangle_t) \circ r_T \\
 &= -(M_t + 2\Gamma(M)_t - \langle M^{v,c}, M^{u,c} \rangle_t - \langle M^{v,j}, M^{u,j} \rangle_t) \circ r_T \\
 &\text{for all } t \leq T.
 \end{aligned}$$

Recall that

$$\begin{aligned}
 [M^v, M^u]_t &= \langle M^{v,c}, M^{u,c} \rangle_t + \sum_{s \leq t} (M_s^v - M_{s-}^v)(M_s^u - M_{s-}^u) \\
 &= \langle M^{v,c}, M^{u,c} \rangle_t + \sum_{s \leq t} (v(X_s) - v(X_{s-}))(u(X_s) - u(X_{s-})).
 \end{aligned}$$

Taking $t = T$ and noting that both $\Gamma(M)$ and $\langle M^{v,c}, M^{u,c} \rangle$ are continuous even AF's, we have, from above, that, \mathbf{P}_m -a.e. on $\{t < \zeta\}$,

$$(3.8) \quad \Gamma(M)_t = -\frac{1}{2}(M_t + M_t \circ r_t + v(X_t)(u(X_{t-}) - u(X_t)) + K_t),$$

where

$$K_t = \sum_{s \leq t} (v(X_s) - v(X_{s-})) (u(X_s) - u(X_{s-})) - \langle M^{v,j}, M^{u,j} \rangle_t$$

is the purely discontinuous MAF with $K_t - K_{t-} = (v(X_t) - v(X_{t-})) (u(X_t) - u(X_{t-}))$. Note that the right-hand side of (3.8) is right-continuous on $[0, \zeta[$, \mathbf{P}_m -a.e. [cf. Remark 3.4(ii)]. Also, observe that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$, where $\varphi(x, y) = v(x)(u(y) - u(x))$, and that

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}).$$

This shows that $\Gamma(M)_t = \Lambda(M)_t$, \mathbf{P}_m -a.e. on $\{t < \zeta\}$ for each fixed $t \geq 0$. Since both processes are continuous in $t \in [0, \zeta[$, we have, \mathbf{P}_m -a.e.,

$$\Gamma(M) = \Lambda(M) \quad \text{on } [0, \zeta[$$

for an MAF M of the form $M_t = \int_0^t v(X_{s-}) dM_s^u$ with $u \in \mathcal{D}(\mathcal{L})$ and $v \in \mathcal{F}_b$. By Lemma 5.4.5 in [6], such MAF's form a dense subset in the space of MAF's having finite energy. Thus, by Lemma 3.1 in Nakao [14] and Remark 3.4(iii), we have, for a general MAF M of finite energy, $\Gamma(M)_t = \Lambda(M)_t$ \mathbf{P}_m -a.e. on $\{t < \zeta\}$ for every fixed $t \geq 0$. Since both processes are continuous in $t \in [0, \zeta[$, it follows that $\Gamma(M) = \Lambda(M)$ on $\llbracket 0, \zeta \rrbracket$, \mathbf{P}_m -a.e. \square

THEOREM 3.7. *Let M be a locally square-integrable MAF on $\llbracket 0, \zeta \rrbracket$ with jump function φ . Suppose that φ satisfies condition (3.1). Then, for every $t > 0$,*

$$(3.9) \quad \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} (\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n})^2 = 0,$$

where the convergence is in \mathbf{P}_{gm} -measure on $\{t < \zeta\}$ for any $g \in L^1(E; m)$ with $0 < g \leq 1$ m -a.e.

PROOF. By (1.5) and Theorem 3.6, (3.9) clearly holds when M is an MAF of finite energy. For a locally square-integrable MAF M on $\llbracket 0, \zeta \rrbracket$, there is an \mathcal{E} -nest $\{F_k\}$ of closed sets such that $\mathbf{1}_{F_k} * M \in \overset{\circ}{\mathcal{M}}$ for each $k \geq 1$ in view of the proof of Proposition 2.8 and so (3.9) holds with $\mathbf{1}_{F_k} * M$ in place of M . For each fixed $k \geq 1$,

$$\Lambda(M)_t = \Lambda(\mathbf{1}_{F_k} * M)_t - \frac{1}{2} K_t^k, \quad \mathbf{P}_m\text{-a.e. on } [0, \tau_{F_k}[,$$

where K_t^k is a purely discontinuous local MAF on $\llbracket 0, \zeta \rrbracket$ with

$$K_t^k - K_{t-}^k = \mathbf{1}_{F_k^c}(X_{t-})\varphi(X_{t-}, X_t) + \mathbf{1}_{F_k^c}(X_t)\varphi(X_t, X_{t-}) \quad \text{for } t < \zeta.$$

Since $\mathbf{1}_{F_k} * M \in \overset{\circ}{\mathcal{M}}$, we have

$$\int_E N(\mathbf{1}_{F_k \times E} \varphi^2) d\mu_H = \int_E N(\mathbf{1}_{E \times F_k} \varphi^2) d\mu_H < \infty.$$

Consequently, by Lemma 3.2, we have the existence of a purely discontinuous local MAF on $\llbracket 0, \zeta \rrbracket$ with jumps given by $\mathbf{1}_{F_k}(X_{t-})\varphi(X_{t-}, X_t) + \mathbf{1}_{F_k}(X_t)\varphi(X_t, X_{t-})$, $t < \zeta$. So, we obtain the existence of such K_t^k . Since the square bracket of K^k is given by $\sum_{s \leq t} \mathbf{1}_{F_k^c}(X_{s-})\varphi^2(X_{s-}, X_s) + \mathbf{1}_{F_k^c}(X_s)\varphi^2(X_s, X_{s-})$ and it vanishes at $t < \tau_{F_k}$, we have, for each fixed $t > 0$,

$$\lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} (\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n})^2 = 0 \quad \text{in } \mathbf{P}_{gm}\text{-measure on } \{t < \tau_{F_k}\}.$$

Passing to the limit as $k \uparrow \infty$ establishes (3.9). \square

We are now in a position to define stochastic integrals against $\Lambda(M)$ as integrator. Note that, for $f \in \mathcal{F}_{loc}$, $M^{f,c}$ is well defined as a continuous MAF on $\llbracket 0, \zeta \rrbracket$ of locally finite energy (see Theorem 8.2 in [9]). Moreover, for $f \in \mathcal{F}_{loc}$ and a locally square-integrable MAF M on $\llbracket 0, \zeta \rrbracket$,

$$t \mapsto (f * M)_t := \int_0^t f(X_{s-}) dM_s$$

is a locally square-integrable MAF on $\llbracket 0, \zeta \rrbracket$.

DEFINITION 3.8 (Stochastic integral). Suppose that M is a locally square-integrable MAF on $\llbracket 0, \zeta \rrbracket$ and $f \in \mathcal{F}_{loc}$. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a jump function for M and assume that φ satisfies condition (3.1). Define, \mathbf{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket$,

$$\begin{aligned} & \int_0^t f(X_{s-}) d\Lambda(M)_s \\ (3.10) \quad & := \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c}, M^c \rangle_t \\ & + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s)) \varphi(y, X_s) N(X_s, dy) dH_s, \end{aligned}$$

whenever $\Lambda(f * M)$ is well defined and the third term in the right-hand side of (3.10) is absolutely convergent.

REMARK 3.9. (i) Under the above condition, the stochastic integral is clearly well defined on $\llbracket 0, \zeta \rrbracket$ under \mathbf{P}_m and is a PrAF of X admitting m -null set.

(ii) Here are some sufficient conditions for every term on the right-hand side of (3.10) to be well defined. In addition to the conditions in Definition 3.8, we assume that, \mathbf{P}_m -a.e.,

$$(3.11) \quad \int_0^t \int_E (f(X_s) - f(y))^2 N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta$$

and that

$$(3.12) \quad \int_0^t \int_E \varphi(y, X_s)^2 N(X_s, dy) dH_s < \infty \quad \text{for every } t < \zeta.$$

The first and third terms on the right-hand side of (3.10) are then well defined. This is because $N(\mathbf{1}_{E \times E}|\widehat{\varphi})\mu_H \in S$ implies that $N(\mathbf{1}_{E \times E}|f\widehat{\varphi})\mu_H \in S$, and

$$f(x)\varphi(x, y) + f(y)\varphi(y, x) = f(x)\widehat{\varphi}(x, y) + (f(y) - f(x))\varphi(y, x),$$

so $\Lambda(f * M)$ is well defined on $[0, \zeta[$ in view of the condition (3.1) for $f * M$, (3.11) and (3.12). Condition (3.11) is satisfied when f is a bounded function in \mathcal{F}_{loc} or $f \in \mathcal{F}$. This is because, when $f \in \mathcal{F}$, the left-hand side of (3.11) is just $\langle M^{f,j} \rangle_t$. When f is a bounded function in \mathcal{F}_{loc} , by Lemma 3.1(i), there exist a nest $\{F_n \mid n \in \mathbb{N}\}$ of closed sets and a sequence of functions $\{f_n \mid n \in \mathbb{N}\} \subset \mathcal{F}_b$ such that $f = f_n$ q.e. on F_n for every $n \geq 1$. Note that for each $n \geq 1$, $M^{f_n,d}$ is a square-integrable, purely discontinuous martingale and

$$M_t^{f_n,d} - M_{t-}^{f_n,d} = f_n(X_t) - f_n(X_{t-}).$$

So, $t \mapsto \sum_{s \leq t} (f_n(X_s) - f_n(X_{s-}))^2$ is \mathbf{P}_x -integrable for q.e. $x \in E$. Since f is bounded, we have, for each $n \geq 1$, that

$$\begin{aligned} t \mapsto & \sum_{s \leq t \wedge \tau_{F_n}} (f(X_s) - f(X_{s-}))^2 \\ &= \sum_{s < t \wedge \tau_{F_n}} (f(X_s) - f(X_{s-}))^2 + (f(X_{t \wedge \tau_{F_n}}) - f(X_{t \wedge \tau_{F_n}-}))^2 \\ &= \sum_{s < t \wedge \tau_{F_n}} (f_n(X_s) - f_n(X_{s-}))^2 + (f(X_{t \wedge \tau_{F_n}}) - f(X_{t \wedge \tau_{F_n}-}))^2 \end{aligned}$$

is an increasing process and is \mathbf{P}_x -integrable for each fixed $t \geq 0$ for q.e. $x \in E$. Similarly, $A_t := \sum_{s \leq t} (f(X_s) - f(X_{s-}))^2$ is *locally integrable* in the sense of Definition 5.18 in [9]. Indeed, for a stopping time $T_n := \inf\{t > 0 \mid A_t > n\}$, $A_{T_n} = A_{T_n-} + (f(X_{T_n}) - f(X_{T_n-}))^2$ is bounded, hence \mathbf{P}_x -integrable for q.e. $x \in E$. Note that the dual predictable projection of A_t is nothing but $\int_0^t \int_{E_\Delta} (f(X_s) - f(y))^2 N(X_s, dy) dH_s$. The dual predictable projection of $\sum_{s \leq t \wedge \tau_{F_n}} (f(X_s) - f(X_{s-}))^2$ is then given by $\int_0^{t \wedge \tau_{F_n}} \int_{E_\Delta} (f(X_s) - f(y))^2 N(X_s, dy) dH_s$ from Corollary 5.24 in [9], which is \mathbf{P}_x -integrable for q.e. $x \in E$. This implies that (3.11) holds for every $t < \tau_{F_n}$. Therefore, (3.11) holds for every $t < \zeta$.

Condition (3.12) is satisfied when M^d is \mathbf{P}_m -square-integrable. Indeed,

$$\begin{aligned} \mathbf{E}_m \left[\sum_{s \leq t} \varphi^2(X_s, X_{s-}) : t < \zeta \right] &= \mathbf{E}_m [[M^d]_t \circ r_t : t < \zeta] \\ &= \mathbf{E}_m [[M^d]_t : t < \zeta] < \infty. \end{aligned}$$

Corollary 4.5 in [8] then tells us that

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbf{E}_m \left[\sum_{s \leq t} \varphi^2(X_s, X_{s-}) : t < \zeta \right] = \lim_{t \rightarrow 0} \frac{1}{t} \mathbf{E}_m \left[\sum_{s \leq t} \varphi^2(X_s, X_{s-}) \right],$$

which implies that

$$\mathbf{E}_m \left[\int_0^t \int_E \varphi(y, X_s)^2 N(X_s, dy) dH_s \right] < \infty$$

for all $t > 0$ by way of its subadditivity. Hence, we obtain (3.12).

(iii) Suppose that $f \in \mathcal{F}$. Let K_t be a purely discontinuous local MAF on $\llbracket 0, \zeta \llbracket$ with $K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-})$ on $]0, \zeta[$. Then,

$$\langle M^{f,j}, M^j + K \rangle_t = - \int_0^t \int_E (f(y) - f(X_s)) \varphi(y, X_s) N(X_s, dy) dH_s.$$

In this case, (3.10) can be rewritten as

$$(3.13) \quad \int_0^t f(X_{s-}) d\Lambda(M)_s = \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c} + M^{f,j}, M^c + M^j + K \rangle_t$$

on $]0, \zeta[$. So, when $M = M^u$ for some $u \in \mathcal{F}$ and $f \in \mathcal{F} \cap L^2(E; \mu_{(u)})$, $\int_0^t f(X_{s-}) d\Lambda(M)_s$ on $]0, \zeta[$ is just the $\int_0^t f(X_s) \circ d\Gamma(M)_s$ defined by (1.7). This shows that the stochastic integral given in Definition 3.8 extends Nakao's definition (1.7) of stochastic integral first introduced in [14].

THEOREM 3.10. *The stochastic integral in (3.10) is well defined. That is, if M and \tilde{M} are two locally square-integrable MAF's on $\llbracket 0, \zeta \llbracket$ such that all conditions in Definition 3.3 for M and \tilde{M} are satisfied and $\Lambda(M) \equiv \Lambda(\tilde{M})$ on $\llbracket 0, \zeta \llbracket$, then for every $f \in \mathcal{F}_{\text{loc}}$ for which $\int_0^t f(X_{s-}) d\Lambda(M)_s$ and $\int_0^t f(X_{s-}) d\Lambda(\tilde{M})_s$ are well defined, we have, \mathbf{P}_m -a.e.,*

$$\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f(X_{s-}) d\Lambda(\tilde{M})_s \quad \text{on }]0, \zeta[.$$

PROOF. It is equivalent to show that

$$\int_0^t f(X_{s-}) d\Lambda(M - \tilde{M})_s = 0 \quad \text{on }]0, \zeta[.$$

By taking M to be $M - \tilde{M}$, we may and will assume that $\tilde{M} = 0$. Moreover, a localization argument allows us to assume that f is bounded. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a jump function for M . Let K_t be the purely discontinuous local MAF on $\llbracket 0, \zeta \llbracket$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}) \quad \text{for } t < \zeta.$$

Since $\Lambda(M) \equiv 0$, we have

$$(3.14) \quad M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t = 0 \quad \text{on }]0, \zeta[.$$

Thus, by (3.5) and (3.14), on $\{T < \zeta\}$,

$$(3.15) \quad \begin{aligned} M_t \circ r_T &= M_T \circ r_T - M_{T-t} \circ r_{T-t} \\ &= -M_T - K_T + M_{T-t} + K_{T-t} \\ &\quad - \varphi(X_T, X_{T-}) + \varphi(X_{T-t}, X_{(T-t)-}) \end{aligned}$$

for every $t \in [0, T]$. Using the standard Riemann sum approximation and (3.15), we have, for $f \in \mathcal{F}$,

$$\begin{aligned} & (f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ &= -(f * M)_t - (f * K)_t - [M^f, M + K]_t \\ &= -(f * M)_t - (f * K)_t - \langle M^{f,c}, M^c \rangle_t \\ &\quad + \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) \end{aligned}$$

\mathbf{P}_m -a.e. on $\{t < \zeta\}$ for each fixed $t \geq 0$. Consequently, we have, for $f \in \mathcal{F}_{\text{loc}}$, \mathbf{P}_m -a.e. for all $t \in [0, \zeta[$,

$$\begin{aligned} & (f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ (3.16) \quad &= -(f * M)_t - (f * K)_t - \langle M^{f,c}, M^c \rangle_t \\ &\quad + \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}), \end{aligned}$$

since both sides are right-continuous in $t \in [0, \zeta[$. Let \tilde{K} be the purely discontinuous local MAF on $\llbracket 0, \zeta \rrbracket$ with

$$\tilde{K}_t - \tilde{K}_{t-} = -f(X_{t-})\varphi(X_{t-}, X_t) - f(X_t)\varphi(X_t, X_{t-}) \quad \text{for all } t \in [0, \zeta[.$$

Then, for $f \in \mathcal{F}_{\text{loc}}$, we have, by (3.16),

$$\begin{aligned} \Lambda(f * M)_t &= -\frac{1}{2}((f * M)_t + (f * M) \circ r_t + f(X_t)\varphi(X_t, X_{t-}) + \tilde{K}_t) \\ &= \frac{1}{2} \left(\int_0^t f(X_{s-}) dK_s + \langle M^{f,c}, M^c \rangle_t \right. \\ &\quad \left. - \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \tilde{K}_t \right). \end{aligned}$$

Thus,

$$\begin{aligned} & \int_0^t f(X_{s-}) d\Lambda(M)_s \\ &= \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c}, M^c \rangle_t \\ &\quad + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s \\ &= \frac{1}{2} \int_0^t f(X_{s-}) dK_s - \frac{1}{2} \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \frac{1}{2} \tilde{K}_t \\ &\quad + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s. \end{aligned}$$

Note that

$$(3.17) \quad \begin{aligned} \tilde{K}_t &= -\sum_{s \leq t} (f(X_{s-})\varphi(X_{s-}, X_s) + f(X_s)\varphi(X_s, X_{s-})) \\ &\quad + \int_0^t \int_E (f(X_s)\varphi(X_s, y) + f(y)\varphi(y, X_s))N(X_s, dy) dH_s \end{aligned}$$

and that

$$(3.18) \quad K_t = \lim_{\varepsilon \rightarrow 0} \left(-\sum_{s \leq t} (\widehat{\varphi} \mathbf{1}_{\{|\widehat{\varphi}| > \varepsilon\}})(X_{s-}, X_s) + (N(\widehat{\varphi} \mathbf{1}_{\{|\widehat{\varphi}| > \varepsilon\}}) * H)_t \right),$$

where $\widehat{\varphi}(x, y) := \varphi(x, y) + \varphi(y, x)$. It follows that

$$\int_0^t f(X_{s-}) d\Lambda(M)_s = 0 \quad \text{for all } t < \zeta,$$

\mathbf{P}_m -a.e. \square

REMARK 3.11. The above proof actually shows that if $\Lambda(M) = \Lambda(\widetilde{M})$ on $[0, T] \cap [0, \zeta[$, then, \mathbf{P}_m -a.e.,

$$\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f(X_{s-}) d\Lambda(\widetilde{M})_s \quad \text{on } [0, T] \cap [0, \zeta[.$$

4. Further study of the stochastic integral.

THEOREM 4.1. Suppose that $f \in \mathcal{F}_{\text{loc}}$ and that M is a locally square-integrable MAF on $]\!]\!][0, \zeta[$ satisfying (3.1) such that $\Lambda(M)$ is a continuous process A of finite variation on $]\!]\!][0, \zeta[$. Assume that the stochastic integral $t \mapsto \int_0^t f(X_{s-}) d\Lambda(M)_s$ is well defined. Then, \mathbf{P}_m -a.e.,

$$\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f(X_s) dA_s \quad \text{on } [0, \zeta[,$$

where the integral on the right-hand side is the Lebesgue–Stieltjes integral.

PROOF. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a Borel function with $\varphi(x, x) = 0$ for $x \in E$ such that $\varphi(X_{t-}, X_t) = M_t - M_{t-}$ for $t \in [0, \zeta[$, \mathbf{P}_m -a.e. Let K_t be the purely discontinuous local MAF on $]\!]\!][0, \zeta[$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}) \quad \text{for } t \in]0, \zeta[.$$

Since $\Lambda(M) = A$ on $[0, \zeta[$,

$$M_t \circ r_t + \varphi(X_t, X_{t-}) = -M_t - K_t - 2A_t \quad \text{for all } t \in [0, \zeta[.$$

Thus, by (3.5), for every $T > t > 0$, on $\{T < \zeta\}$,

$$(4.1) \quad \begin{aligned} M_t \circ r_T &= -M_T - K_T - 2A_T + M_{T-t} + K_{T-t} \\ &\quad + 2A_{T-t} - \varphi(X_T, X_{T-}) + \varphi(X_{T-t}, X_{(T-t)-}). \end{aligned}$$

Now, fix $f \in \mathcal{F}_{loc}$; as before, we may assume, without loss of generality, that f is bounded. Using the standard Riemann sum approximation, we obtain, on $\{t < \zeta\}$,

$$\begin{aligned} & (f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ &= -(f * M)_t - (f * K)_t - 2(f * A)_t - [M^f, M + K + 2A]_t \\ &= -(f * M)_t - (f * K)_t - 2(f * A)_t - \langle M^{f,c}, M^c \rangle_t \\ & \quad + \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}). \end{aligned}$$

Consequently, we have, \mathbf{P}_m -a.e. for all $t \in [0, \zeta[$,

$$(4.2) \quad \begin{aligned} & (f * M)_t \circ r_t + f(X_t)\varphi(X_t, X_{t-}) \\ &= -(f * M)_t - (f * K)_t - 2(f * A)_t - \langle M^{f,c}, M^c \rangle_t \\ & \quad + \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) \end{aligned}$$

since both sides are right-continuous in $t \in [0, \zeta[$. Let \tilde{K} be the purely discontinuous local MAF on $[[0, \zeta[[$ with

$$\tilde{K}_t - \tilde{K}_{t-} = -f(X_{t-})\varphi(X_{t-}, X_t) - f(X_t)\varphi(X_t, X_{t-}) \quad \text{for all } t \in [0, \zeta[.$$

Then, by (4.2),

$$\begin{aligned} \Lambda(f * M)_t &= -\frac{1}{2}((f * M)_t + (f * M) \circ r_t + f(X_t)\varphi(X_t, X_{t-}) + \tilde{K}_t) \\ &= \frac{1}{2} \left(\int_0^t f(X_{s-}) dK_s + 2 \int_0^t f(X_{s-}) dA_s + \langle M^{f,c}, M^c \rangle_t \right. \\ & \quad \left. - \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \tilde{K}_t \right). \end{aligned}$$

Thus,

$$\begin{aligned} & \int_0^t f(X_{s-}) d\Lambda(M)_s \\ &= \Lambda(f * M)_t - \frac{1}{2} \langle M^{f,c}, M^c \rangle_t \\ & \quad + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s \\ &= \frac{1}{2} \int_0^t f(X_{s-}) dK_s + \int_0^t f(X_{s-}) dA_s \\ & \quad - \frac{1}{2} \sum_{s \leq t} (f(X_s) - f(X_{s-}))\varphi(X_s, X_{s-}) - \frac{1}{2} \tilde{K}_t \\ & \quad + \frac{1}{2} \int_0^t \int_E (f(y) - f(X_s))\varphi(y, X_s) N(X_s, dy) dH_s. \end{aligned}$$

It now follows from (3.17)–(3.18) that, \mathbf{P}_m -a.e.,

$$\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f(X_{s-}) dA_s \quad \text{for all } t \in]0, \zeta[.$$

This proves the theorem. \square

Note that if $f, g \in \mathcal{F}_{\text{loc}}$, then $fg \in \mathcal{F}_{\text{loc}}$.

THEOREM 4.2. *Let $f, g \in \mathcal{F}_{\text{loc}}$ and let M be a locally square-integrable MAF on $]0, \zeta[$ satisfying (3.1). Then \mathbf{P}_m -a.e.,*

$$(4.3) \quad \begin{aligned} & \int_0^t g(X_{s-}) d\left(\int_0^s f(X_{r-}) d\Lambda(M)_r\right) \\ &= \int_0^t f(X_{s-})g(X_{s-}) d\Lambda(M)_s \quad \text{for every } t < \zeta, \end{aligned}$$

whenever all of the integrals involved are well defined.

PROOF. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a Borel function with $\varphi(x, x) = 0$ for $x \in E$ such that, \mathbf{P}_m -a.e.,

$$\varphi(X_{t-}, X_t) = M_t - M_{t-} \quad \text{for all } t \in]0, \zeta[.$$

Let K_t and \tilde{K}_t be the purely discontinuous local MAF's on $]0, \zeta[$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}) \quad \text{for } t \in]0, \zeta[$$

and

$$\tilde{K}_t - \tilde{K}_{t-} = -f(X_{t-})\varphi(X_{t-}, X_t) - f(X_t)\varphi(X_t, X_{t-}) \quad \text{for } t \in]0, \zeta[,$$

respectively. The left-hand side of (4.3) is then equal to

$$\begin{aligned} & \int_0^t g(X_{s-}) d\Lambda(f * M)_s - \frac{1}{2} \int_0^t g(X_{s-}) d\langle M^{f,c}, M^c \rangle_s \\ &+ \frac{1}{2} \int_0^t \int_E g(X_s)(f(y) - f(X_s))\varphi(y, X_s)N(X_s, dy) dH_s \\ &= \Lambda(fg * M)_t - \frac{1}{2} \langle M^{g,c}, (f * M)^c \rangle_t \\ &+ \frac{1}{2} \int_0^t \int_E (g(y) - g(X_s))f(y)\varphi(y, X_s)N(X_s, dy) dH_s \\ &- \frac{1}{2} \int_0^t g(X_{s-}) d\langle M^{f,c}, M^c \rangle_s \\ &+ \frac{1}{2} \int_0^t \int_E g(X_s)(f(y) - f(X_s))\varphi(y, X_s)N(X_s, dy) dH_s \\ &= \Lambda(fg * M)_t - \frac{1}{2} \langle M^{fg,c}, M^c \rangle_t \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{2} \int_0^t \int_E (f(y)g(y) - f(X_s)g(X_s))\varphi(y, X_s)N(X_s, dy) dH_s \\ &= \int_0^t f(X_{s-})g(X_{s-}) d\Lambda(M)_s. \end{aligned}$$

This proves the theorem. \square

Let \mathcal{J} denote the class of stochastic processes that can be written as the sum of an (\mathcal{F}_t) -semimartingale Y and $\Lambda(M)$ for a locally square-integrable MAF M on $[[0, \zeta[$ satisfying the condition of Definition 3.3. The last two theorems imply that the following stochastic integral is well defined for integrators $Z \in \mathcal{J}$.

DEFINITION 4.3. For $f \in \mathcal{F}_{loc}$ and $Z = Y + \Lambda(M) \in \mathcal{J}$, define on $[0, \zeta[$,

$$\int_0^t f(X_{s-}) dZ_s := \int_0^t f(X_{s-}) dY_s + \int_0^t f(X_{s-}) d\Lambda(M)_s,$$

whenever the latter stochastic integral is well defined.

To establish Itô’s formula, we need the following result.

THEOREM 4.4. Let $f \in \mathcal{F}_{loc}$ and let M be a locally square-integrable MAF on $[[0, \zeta[$ such that $\int_0^\cdot f(X_{s-}) d\Lambda(M)$ is well defined on $[0, \zeta[$. Then, for every $t > 0$, \mathbf{P}_m -a.e. on $\{t < \zeta\}$,

$$(4.4) \quad \int_0^t f(X_{s-}) d\Lambda(M)_s = \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} f(X_{\ell t/n})(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}).$$

Here, the convergence is in measure with respect to \mathbf{P}_{gm} on $\{t < \zeta\}$ for every $g \in L^1(E; m)$ with $0 < g \leq 1$ m -a.e.

PROOF. By (3.5), $M_s \circ r_t = M_t \circ r_t - M_{t-s} \circ r_{t-s}$ for all $s < t$. Let $\varphi : E \times E \rightarrow \mathbb{R}$ be a Borel function with $\varphi(x, x) = 0$ for $x \in E$ such that $\varphi(X_{t-}, X_t) = M_t - M_{t-}$ for all $t \in [0, \zeta[$. Let K be the purely discontinuous local MAF on $[[0, \zeta[$ with

$$K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-}) \quad \text{for } t \in]0, \zeta[.$$

Then, for each fixed $t > 0$, \mathbf{P}_m -a.e. on $\{t < \zeta\}$,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} f(X_{\ell t/n})(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}) \\ &= -\frac{1}{2}(f * M)_t - \frac{1}{2}(f * K)_t \\ &\quad + \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} f(X_{\ell t/n})(M_{(\ell+1)t/n} \circ r_{(\ell+1)t/n} - M_{\ell t/n} \circ r_{\ell t/n}) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}(f * M)_t - \frac{1}{2}(f * K)_t \\
 &\quad - \frac{1}{2} \lim_{n \rightarrow \infty} \left[\sum_{\ell=0}^{n-1} f(X_{(\ell+1)t/n})(M_{(\ell+1)t/n} - M_{\ell t/n}) \right] \circ r_t \\
 &= -\frac{1}{2}(f * M)_t - \frac{1}{2}(f * K)_t - \frac{1}{2}(f * M)_t \circ r_t - \frac{1}{2}[M^f, M]_t \circ r_t \\
 &= -\frac{1}{2}(f * M)_t - \frac{1}{2}(f * K)_t - \frac{1}{2}(f * M)_t \circ r_t - \frac{1}{2}\langle M^{f,c}, M^c \rangle_t \\
 &\quad - \frac{1}{2} \sum_{s \leq t} (f(X_{s-}) - f(X_s))\varphi(X_s, X_{s-}) \\
 &= \Lambda(f * M)_t + \frac{1}{2}\tilde{K}_t - \frac{1}{2}(f * K)_t - \frac{1}{2}\langle M^{f,c}, M^c \rangle_t \\
 &\quad - \frac{1}{2} \sum_{s \leq t} (f(X_{s-}) - f(X_s))\varphi(X_s, X_{s-}) \\
 &= \int_0^t f(X_{s-}) d\Lambda(M)_s,
 \end{aligned}$$

where \tilde{K} in the penultimate equality is the purely discontinuous local MAF on $[[0, \zeta[$ with $\tilde{K}_s - \tilde{K}_{s-} = -f(X_{s-})\varphi(X_{s-}, X_s) - f(X_s)\varphi(X_s, X_{s-})$ for $s \in]0, \zeta[$. □

REMARK 4.5. (i) Theorem 4.4 immediately implies Theorems 3.10 and 4.1.

(ii) By (3.9),

$$(4.5) \quad \int_0^t f(X_{s-}) d\Lambda(M)_s = \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} f(X_{(\ell+1)t/n})(\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n})$$

holds in P_{g_m} -measure on $\{t < \zeta\}$ for any $g \in L^1(E; m)$ with $0 < g \leq 1$ m -a.e. Hence, we could denote this stochastic integral by either $\int_0^t f(X_s) d\Lambda(M)_s$ or $\int_0^t f(X_s) \circ d\Lambda(M)_s$. Here, $\int_0^t f(X_s) \circ d\Lambda(M)_s$ is the Fisk–Stratonovich type integral: for $t < \zeta$

$$\begin{aligned}
 (4.6) \quad &\int_0^t f(X_s) \circ d\Lambda(M)_s \\
 &:= \lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} \frac{f(X_{(\ell+1)t/n}) + f(X_{\ell t/n})}{2} (\Lambda(M)_{(\ell+1)t/n} - \Lambda(M)_{\ell t/n}).
 \end{aligned}$$

(iii) For any $f \in \mathcal{F}_{loc}$ and \mathbf{P}_m -square-integrable MAF M , by way of the Riemann sum approximation (4.4), we can extend the stochastic integral $\int_0^t f(X_{s-}) d\Lambda(M)_s$ without imposing further conditions. Indeed, let $\{G_\ell\}$ be a nest of finely open Borel sets and $f_\ell \in \mathcal{F}_b$ with $f = f_\ell$ m -a.e. on G_ℓ [see the explanation for the condition (3.11) in Remark 3.9]. By (4.4), we see

$\int_0^t f_n(X_{s-}) d\Lambda(M)_s = \int_0^t f_\ell(X_{s-}) d\Lambda(M)_s$ for $t < \tau_{G_n}$ and $n < \ell$. We can then define $\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f_n(X_{s-}) d\Lambda(M)_s$ for $t < \tau_{G_n}$ for each $n \in \mathbb{N}$ and, consequently, for all $t < \zeta$ \mathbf{P}_m -a.e. More strongly, for $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{F}_{\text{loc}}$, we can define $\int_0^t f(X_{s-}) d\Lambda(M)_s$ for all $t \in [0, \infty[$ \mathbf{P}_m -a.e. Indeed, by Remark 3.9(ii), our stochastic integral $f_n * \Lambda(M)$ for $M \in \overset{\circ}{\mathcal{M}}$ agrees with that defined by Nakao [14] on $[0, \zeta[$ \mathbf{P}_m -a.e., while the latter is defined as a CAF of X for all $t \geq 0$. This implies that $\lim_{s \uparrow \zeta} (f_n * \Lambda(M))_s$ exists and is finite \mathbf{P}_m -a.e. After we extend our definition of stochastic integral $f_n * \Lambda(M)$ beyond $[0, \zeta[$ by

$$(f_n * \Lambda(M))_t = (f_n * \Lambda(M))_\zeta = \lim_{s \uparrow \zeta} (f_n * \Lambda(M))_s \quad \text{for } t \geq \zeta,$$

$f_n * \Lambda(M)$ becomes a CAF of X on $[0, \infty[$ \mathbf{P}_m -a.e. With this extension for each $n < \ell$, we have $\int_0^t f(X_{s-}) d\Lambda(M)_s = \int_0^t f_\ell(X_{s-}) d\Lambda(M)_s$ for $t < \sigma_{E \setminus G_n}$, \mathbf{P}_m -a.e. Owing to Lemma 3.1(i) and the existence of the limit $\lim_{t \uparrow \zeta} \int_0^t f(X_{s-}) d\Lambda(M)_s$ \mathbf{P}_m -a.e., we obtain the stochastic integral $\int_0^t f(X_{s-}) d\Lambda(M)_s$, on $[0, \infty[$, \mathbf{P}_m -a.e. for any $f \in \mathcal{F}_{\text{loc}}$ and $M \in \overset{\circ}{\mathcal{M}}$, extending the stochastic integral of Nakao [14].

Remark 4.5(iii) says that the stochastic integral $f * \Lambda(M)_t := \int_0^t f(X_{s-}) d\Lambda(M)_s$ can be defined for $t \in [0, \infty[$, \mathbf{P}_m -a.e., for every $f \in \mathcal{F}_{\text{loc}}$ and $M \in \overset{\circ}{\mathcal{M}}$. We shall refine this statement from m -almost every starting point $x \in E$ to quasi-every $x \in E$.

LEMMA 4.6. *For $f \in \mathcal{F}_{\text{loc}}$ and $M \in \overset{\circ}{\mathcal{M}}$, the stochastic integral $f * \Lambda(M)_t := \int_0^t f(X_{s-}) d\Lambda(M)_s$ can be defined for all $t \in [0, \infty[$, \mathbf{P}_x -a.s. for q.e. $x \in E$, in particular, $f * \Lambda(M)$ is a CAF of X on $[0, \infty[$.*

PROOF. Since $f \in \mathcal{F}_{\text{loc}}$, we have $\{f_k \mid k \in \mathbb{N}\} \subset \mathcal{F}_b$ and a nest $\{G_k \mid k \in \mathbb{N}\}$ of finely open Borel sets such that $f = f_k$ q.e. on G_k . We know that the stochastic integral $f_k * \Lambda(M)$ is defined \mathbf{P}_x -a.s. for q.e. $x \in E$. Let Ξ_k be the defining set admitting an \mathcal{E} -polar set for the CAF $f_k * \Lambda(M)$ of zero energy and set

$$\begin{aligned} \Xi := & \left\{ \omega \in \bigcap_{k=1}^{\infty} \Xi_k \mid \text{for any } k, \ell \in \mathbb{N} \text{ with } k < \ell, \right. \\ & \int_0^t f_k(X_{s-}(\omega)) d\Lambda(M)_s(\omega) \\ & \left. = \int_0^t f_\ell(X_{s-}(\omega)) d\Lambda(M)_s(\omega) \text{ for } t < \sigma_{E \setminus G_k}(\omega) \right\}. \end{aligned}$$

Then, $\mathbf{P}_x(\Xi^c) = 0$, m -a.e. $x \in E$. Hence, for each $s > 0$, $\mathbf{P}_x(\theta_s^{-1}(\Xi^c)) = \mathbf{P}_s(\mathbf{P}(\Xi^c))(x) = 0$ for q.e. $x \in E$. Setting $\widehat{\Xi} := \bigcap_{k=1}^{\infty} \Xi_k \cap \bigcap_{s \in \mathbb{Q}_{++}} \theta_s^{-1}(\Xi)$, we

have $\mathbf{P}_x(\widehat{\Xi}) = 1$ for q.e. $x \in E$. For $\omega \in \widehat{\Xi}$ with $t < \sigma_{E \setminus G_k}(\omega)$, we can find small $s_0(=s_0(\omega)) > 0$ such that $t + s_0 < \sigma_{E \setminus G_k}(\omega)$. We then see that $t < \sigma_{E \setminus G_k}(\theta_s \omega)$ for any rational $s \in]0, s_0[$. Hence, for such ω , we have for $k < \ell$ and any rational $s \in]0, s_0[$

$$\int_s^{t+s} f_k(X_{v-}(\omega)) d\Lambda(M)_v(\omega) = \int_s^{t+s} f_\ell(X_{v-}(\omega)) d\Lambda(M)_v(\omega).$$

Letting $s \rightarrow 0$ and noting that $\omega \in \Xi_k, k \in \mathbb{N}$, we have that for $k < \ell, f_k * \Lambda(M)_t = f_\ell * \Lambda(M)_t$ for $t < \sigma_{E \setminus G_k}, \mathbf{P}_x$ -a.s. for q.e. $x \in E$. By Lemma 3.1(i), we know that $\mathbf{P}_x(\lim_{k \rightarrow \infty} \sigma_{E \setminus G_k} = \infty) = 1$ for q.e. $x \in E$. Therefore, we obtain that the stochastic integral $f * \Lambda(M)$ defined as in Remark 4.5(4.5) can be established \mathbf{P}_x -a.s. for q.e. $x \in E$. This completes the proof. \square

THEOREM 4.7 (Generalized Itô formula). *Suppose that $\Phi \in C^2(\mathbb{R}^d)$ and $u = (u_1, \dots, u_d) \in \mathcal{F}^d$. Then, for q.e. $x \in E, \mathbf{P}_x$ -a.s. for all $t \in [0, \infty[$,*

$$\begin{aligned} & \Phi(u(X_t)) - \Phi(u(X_0)) \\ &= \sum_{k=1}^d \int_0^t \frac{\partial \Phi}{\partial x_k}(u(X_{s-})) du_k(X_s) \\ (4.7) \quad &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_{s-})) d\langle M^{u_i,c}, M^{u_j,c} \rangle_s \\ &+ \sum_{s \leq t} \left(\Phi(u(X_s)) - \Phi(u(X_{s-})) \right. \\ &\quad \left. - \sum_{k=1}^d \frac{\partial \Phi}{\partial x_k}(u(X_{s-}))(u_k(X_s) - u_k(X_{s-})) \right). \end{aligned}$$

PROOF. Note that both sides appearing in (4.7) are \mathbf{P}_x -a.s. defined for q.e. $x \in E$ in view of Lemma 4.6. First, we show this Itô formula (4.7) under \mathbf{P}_m for a fixed $t \geq 0$. Note that $\Phi \circ u \in \mathcal{F}_{\text{loc}}$ and that

$$u_k(X_t) = u_k(X_0) + M_t^{u_k} + N_t^{u_k} = u_k(X_0) + M_t^{u_k} + \Lambda(M^{u_k})_t.$$

This version of Itô’s formula follows from Theorems 3.7 and 4.4 by a line of reasoning similar to that used to prove Itô’s formula for semimartingales (cf. [9]). Since both sides in (4.7) are right-continuous, (4.7) holds under \mathbf{P}_m .

Second, we refine the starting point. Recall that Ω consists of rcll paths. Let $I_t(\omega)$ be the difference of the left-hand side and the right-hand side of (4.7). Let Ξ be the intersection of all of the defining sets of AF’s appearing in the formula and $\{\omega \in \Omega \mid I_t(\omega) = 0, \forall t \in [0, \infty[\}$. Then, $\mathbf{P}_x(\Xi^c) = 0, m$ -a.e. $x \in E$. Let $\widehat{\Xi}$ be the intersection of the defining sets of AF’s appearing in the formula

and $\bigcap_{s \in \mathbb{Q}_{++}} \theta_s^{-1}(\Xi)$. We then have $\mathbf{P}_x(\widehat{\Xi}) = 1$ for q.e. $x \in E$, as in the proof of Lemma 4.6. Take $\omega \in \widehat{\Xi}$. For any positive rational $s > 0$, we then have $I_t(\theta_s \omega) = 0$, that is,

$$\begin{aligned} & \Phi(u(X_{t+s}(\omega))) - \Phi(u(X_s(\omega))) \\ &= \sum_{k=1}^d \int_s^{t+s} \frac{\partial \Phi}{\partial x_k}(u(X_{v-}(\omega))) du_k(X_v(\omega)) \\ & \quad + \frac{1}{2} \sum_{i,j=1}^d \int_s^{t+s} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(u(X_{v-}(\omega))) d\langle M^{u_i \cdot c}, M^{u_j \cdot c} \rangle_v(\omega) \\ & \quad + \sum_{s < v \leq t+s} \left(\Phi(u(X_v(\omega))) - \Phi(u(X_{v-}(\omega))) \right. \\ & \quad \quad \left. - \sum_{k=1}^d \frac{\partial \Phi}{\partial x_k}(u(X_{v-}(\omega)))(u_k(X_v(\omega)) - u_k(X_{v-}(\omega))) \right). \end{aligned}$$

Letting $s \rightarrow 0$ and using the right-continuity of $s \mapsto u(X_s)$ and stochastic integrals, we have $I_t(\omega) = 0$. This completes the proof. \square

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