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0–1 LAWS FOR REGULAR CONDITIONAL DISTRIBUTIONS

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Let $(\Omega, \mathcal{B}, P)$ be a probability space, $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field, and μ a regular conditional distribution for P given $\mathcal{A}$. Necessary and sufficient conditions for $\mu(\omega)(A)$ to be 0–1, for all $A \in \mathcal{A}$ and $\omega \in A_0$, where $A_0 \in \mathcal{A}$ and $P(A_0) = 1$, are given. Such conditions apply, in particular, when $\mathcal{A}$ is a tail sub-σ-field. Let $H(\omega)$ denote the $\mathcal{A}$-atom including the point $\omega \in \Omega$. Necessary and sufficient conditions for $\mu(\omega)(H(\omega))$ to be 0–1, for all $\omega \in A_0$, are also given. If $(\Omega, \mathcal{B})$ is a standard space, the latter 0–1 law is true for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes.

1. Introduction and motivations. Let $(\Omega, \mathcal{B}, P)$ be a probability space and $\mathcal{A} \subset \mathcal{B}$ a sub-σ-field. A *regular conditional distribution* (r.c.d.) for P given $\mathcal{A}$ is a mapping $\mu : \Omega \to \mathbb{P}$, where $\mathbb{P}$ denotes the set of probability measures on $\mathcal{B}$, such that $\mu(\cdot)(B)$ is a version of $E(I_B \mid \mathcal{A})$ for all $B \in \mathcal{B}$. A σ-field is *countably generated* (c.g.) in case it is generated by one of its countable subclasses. In the sequel, it is assumed that P admits a r.c.d. given $\mathcal{A}$ and

$$\mu \text{ denotes a } \textit{fixed} \text{ r.c.d., for } P \text{ given } \mathcal{A}, \text{ and } \mathcal{B} \text{ is c.g.} \tag{1}$$

Moreover,

$$H(\omega) = \bigcap_{\omega \in A \in \mathcal{A}} A$$

is the atom of $\mathcal{A}$ including the point $\omega \in \Omega$.

Heuristically, conditioning to $\mathcal{A}$ should mean conditioning to the atom of $\mathcal{A}$ which actually occurs, say $H(\omega)$, and the probability of $H(\omega)$ given $H(\omega)$ should be 1. If this interpretation is agreed, μ should be everywhere *proper*, that is, $\mu(\omega)(A) = I_A(\omega)$ for all $A \in \mathcal{A}$ and $\omega \in \Omega$. Though $\mu(\cdot)(A) = I_A(\cdot)$ a.s. for *fixed* $A \in \mathcal{A}$, however, μ can behave quite inconsistently with properness.

Say that $\mathcal{A}$ is c.g. under P in case the trace σ-field $\mathcal{A} \cap C = \{A \cap C : A \in \mathcal{A}\}$ is c.g. for some $C \in \mathcal{A}$ with $P(C) = 1$. If $\mathcal{A}$ is c.g. under P then

$$\mu(\omega)(A) = I_A(\omega) \qquad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0, \tag{2}$$

where, here and in what follows, A_0 designates some set of $\mathcal{A}$ with $P(A_0) = 1$. In general, the exceptional set A_0^c cannot be removed. Further, (2) implies that

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$\mathcal{A} \cap A_0$ is c.g. See [2, 3] and Theorem 1 of [4]. In other terms, not only everywhere properness is to be weakened into condition (2), but the latter holds if and only if $\mathcal{A}$ is c.g. under P.

If $\mathcal{A}$ fails to be c.g. under P, various weaker notions of r.c.d. have been investigated. Roughly speaking, in such notions, μ is asked to be everywhere proper but σ-additivity and/or measurability are relaxed. See [1] and references therein. However, not very much is known on r.c.d.'s, regarded in the usual sense, when $\mathcal{A}$ is not c.g. under P (one exception is [11]). In particular, when (2) fails, one question is whether some of its consequences are still available.

In this paper, among these consequences, we focus on:

$$\mu(\omega)(H(\omega)) \in \{0, 1\} \qquad \text{for all } \omega \in A_0, \tag{3}$$

$$\mu(\omega)(A) \in \{0, 1\} \qquad \text{for all } A \in \mathcal{A} \text{ and } \omega \in A_0. \tag{4}$$

Note that (4) implies (3) in case $H(\omega) \in \mathcal{A}$ for all $\omega \in A_0$, and that, for (3) to make sense, one needs to assume $H(\omega) \in \mathcal{B}$ for all $\omega \in A_0$.

Both conditions (3) and (4) worth some attention.

Investigating (3) can be seen as a development of the seminal work of [4]. The conjecture that (3) holds (under mild conditions) is supported by those examples in the literature where $\mathcal{B}$ is c.g. In these examples, in fact, either $\mu(\omega)(H(\omega)) = 1$ a.s. or $\mu(\omega)(H(\omega)) = 0$ a.s. See, for instance, [11].

Condition (4) seems to have been neglected so far, though it is implicit in some ideas of [7] and [5]. In any case, (4) holds in a number of real situations and can be attached a clear heuristic meaning. As to the latter, fix $\omega_0 \in \Omega$. Since $\mu(\omega_0)$ comes out by conditioning on $\mathcal{A}$, one could expect that μ is a r.c.d. for $\mu(\omega_0)$ given $\mathcal{A}$, too. Condition (4) grants that this is true, provided $\omega_0 \in A_0$. More precisely, letting $M = \{Q \in \mathbb{P} : \mu \text{ is a r.c.d. for } Q \text{ given } \mathcal{A}\}$, condition (4) is equivalent to

$$\mu(\omega) \in M \qquad \text{for all } \omega \in A_0;$$

see Theorem 12.

Since $\mathcal{B}$ is c.g., $P(\mu \neq \nu) = 0$ for any other r.c.d. ν, and this is basic for (3) and (4). For some time, we guessed that $\mathcal{B}$ c.g. is enough for (3) and (4). Instead, as we now prove, some extra conditions are needed. Let

$$\mathcal{N} = \{B \in \mathcal{B} : P(B) = 0\}.$$

EXAMPLE 1 [*A failure of condition* (3)]. Let $\Omega = \mathbb{R}$, $\mathcal{B}$ the Borel σ-field, Q a probability measure on $\mathcal{B}$ vanishing on singletons, and $P = \frac{1}{2}(Q + \delta_0)$. If $\mathcal{A} = \sigma(\mathcal{N})$, then $\mu = P$ a.s. and $H(\omega) = \{\omega\}$ for all ω, so that $H(0) = \{0\} \notin \mathcal{A}$ and $\mu(0)\{0\} = P\{0\} = \frac{1}{2}$.

Incidentally, Example 1 exhibits also a couple of (perhaps unexpected) facts. Unless $H(\omega) \in \mathcal{A}$ for all $\omega \in A_0$, (4) does not imply (3). Further, it may be that $\mu(\omega)(H(\omega)) < 1$, for a single point ω ($\omega = 0$ in Example 1), even though $H(\omega) \in \mathcal{B}$ and $\mu(\omega)(A) = I_A(\omega)$ for all $A \in \mathcal{A}$.

EXAMPLE 2 [*A failure of condition* (4)]. Let $\Omega = \mathbb{R}^2$, $\mathcal{B}$ the Borel σ-field, and $P = Q \times Q$ where Q is the $N(0, 1)$ law on the real Borel sets. Denoting $\mathcal{G}$ the σ-field on Ω generated by $(x, y) \mapsto x$, a (natural) r.c.d. for P given $\mathcal{G}$ is $\mu((x, y)) = \delta_x \times Q$. With such a μ, condition (4) fails if $\mathcal{A}$ is taken to be $\mathcal{A} = \sigma(\mathcal{G} \cup \mathcal{N})$. In fact, μ is also a r.c.d. for P given $\mathcal{A}$, and for all (x, y) one has $\{x\} \times [0, \infty) \in \mathcal{A}$ while

$$\mu\big((x, y)\big)\big(\{x\} \times [0, \infty)\big) = \tfrac{1}{2}.$$

Note also that (3) holds in this example, since $H(\omega) = \{\omega\}$ and $\mu(\omega)\{\omega\} = 0$ for all $\omega \in \Omega$.

This paper provides necessary and sufficient conditions for (3) and (4). Special attention is devoted to the particular case where $\mathcal{A}$ is a *tail sub-σ-field*, that is, $\mathcal{A}$ is the intersection of a nonincreasing sequence of *countably generated* sub-σ-fields. The main results are Theorems 3, 4, 8 and 15. Theorem 15 states that (4) is always true whenever $\mathcal{A}$ is a tail sub-σ-field. Theorems 3, 4 and 8 deal with condition (3). One consequence of Theorem 4 is that, when $(\Omega, \mathcal{B})$ is a standard space, (3) holds for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes.

2. When regular conditional distributions are 0–1 on the (appropriate) atoms of the conditioning σ-field.

This section deals with condition (3). It is split into three subsections.

2.1. *Basic results.* For any map $\nu : \Omega \to \mathbb{P}$, we write $\sigma(\nu)$ for the σ-field generated by $\nu(B)$ for all $B \in \mathcal{B}$, where $\nu(B)$ stands for the real function $\omega \mapsto \nu(\omega)(B)$. Since $\mathcal{B}$ is c.g., $\sigma(\nu)$ is c.g., too. In particular, $\sigma(\mu)$ is c.g. Let

$$\mathcal{A}_P = \{A \subset \Omega : \exists A_1, A_2 \in \mathcal{A} \text{ with } A_1 \subset A \subset A_2 \text{ and } P(A_2 - A_1) = 0\}$$

be the completion of $\mathcal{A}$ with respect to $P|\mathcal{A}$. The only probability measure on $\mathcal{A}_P$ agreeing with P on $\mathcal{A}$ is still denoted by P. Further, in case $H(\omega) \in \mathcal{B}$ for all ω, we let

$$f_B(\omega) = \mu(\omega)\big(B \cap H(\omega)\big) \qquad \text{for } \omega \in \Omega \text{ and } B \in \mathcal{B},$$
$$f = f_\Omega, \qquad S = \{f > 0\}.$$

We are in a position to state our first characterization of (3).

THEOREM 3. *Suppose* (1) *holds and $H(\omega) \in \mathcal{B}$ for all ω. For each $U \in \mathcal{A}$ such that the trace σ-field $\mathcal{A} \cap U$ is c.g., there is $U_0 \in \mathcal{A}$ with $U_0 \subset U$, $P(U - U_0) = 0$ and $f(\omega) = 1$ for all $\omega \in U_0$. Moreover, if $S \in \mathcal{A}_P$, then condition* (3) *is equivalent to each of the following conditions* (a)–(b):

(a) f_B *is* $\mathcal{A}_P$*-measurable for all* $B \in \mathcal{B}$;
(b) $\mathcal{A} \cap U$ *is c.g. for some* $U \in \mathcal{A}$ *with* $U \subset S$ *and* $P(S - U) = 0$.

PROOF. Suppose $\mathcal{A} \cap U$ is c.g. for some $U \in \mathcal{A}$ and define

$$\mathcal{A}_0 = \{(A \cap U) \cup F : A \in \mathcal{A}, F = \varnothing \text{ or } F = U^c\}.$$

Let $H_0(\omega)$ be the $\mathcal{A}_0$-atom including ω. Then, $\mathcal{A}_0$ is c.g. and $H_0(\omega) = H(\omega)$ for $\omega \in U$. A r.c.d. for P given $\mathcal{A}_0$ is $\mu_0(\omega) = I_U(\omega)\mu(\omega) + I_{U^c}(\omega)\alpha$, where $\alpha(\cdot) = P(\cdot \mid U^c)$ if $P(U) < 1$ and α is any fixed element of $\mathbb{P}$ if $P(U) = 1$. Since $\mathcal{A}_0$ is c.g., there is $K \in \mathcal{A}_0$ with $P(K) = 1$ and $\mu_0(\omega)(H_0(\omega)) = 1$ for all $\omega \in K$. Since $f(\omega) = \mu(\omega)(H(\omega)) = \mu_0(\omega)(H_0(\omega)) = 1$ for each $\omega \in K \cap U$, it suffices to let $U_0 = K \cap U$.

Next, suppose $S \in \mathcal{A}_P$ and take $C, D \in \mathcal{A}$ such that $C \subset S$, $D \subset S^c$ and $P(C \cup D) = 1$.

"(3) $\Rightarrow$ (a)." Let $A = A_0 \cap C$, where $A_0 \in \mathcal{A}$ is such that $P(A_0) = 1$ and $f(\omega) \in \{0, 1\}$ for all $\omega \in A_0$. Fix $B \in \mathcal{B}$. Since $f_B \le f = 0$ on D and $f_B = \mu(B)$ on A, one obtains $I_{A \cup D} f_B = I_A f_B = I_A \mu(B)$. Thus, f_B is $\mathcal{A}_P$-measurable.

"(a) $\Rightarrow$ (b)." Given any $\alpha \in \mathbb{P}$, define the map $\nu : \Omega \to \mathbb{P}$ by

$$\nu(\omega)(B) = I_S(\omega)\frac{f_B(\omega)}{f(\omega)} + I_{S^c}(\omega)\alpha(B), \qquad \omega \in \Omega, B \in \mathcal{B}.$$

Then, (a) implies that $\sigma(\nu) \subset \mathcal{A}_P$. Fix a countable field $\mathcal{B}_0$ generating $\mathcal{B}$, and, for each $B \in \mathcal{B}_0$, take a set $A_B \in \mathcal{A}$ such that $P(A_B) = 1$ and $I_{A_B}\nu(B)$ is $\mathcal{A}$-measurable. Define $U = (\bigcap_{B \in \mathcal{B}_0} A_B) \cap C$ and note that $U \in \mathcal{A}$, $U \subset S$ and $P(S - U) = 0$. Since $I_U \nu(B)$ is $\mathcal{A}$-measurable for each $B \in \mathcal{B}_0$, it follows that $\sigma(\nu) \cap U \subset \mathcal{A} \cap U$. Since $A \cap U = \{\nu(A) = 1\} \cap U$ for all $A \in \mathcal{A}$, then $\mathcal{A} \cap U \subset \sigma(\nu) \cap U$. Hence, $\mathcal{A} \cap U = \sigma(\nu) \cap U$ is c.g.

"(b) $\Rightarrow$ (3)." By the first assertion of the theorem, since $U \in \mathcal{A}$ and $\mathcal{A} \cap U$ is c.g., there is $U_0 \in \mathcal{A}$ with $U_0 \subset U$, $P(U - U_0) = 0$ and $f = 1$ on U_0. Define $A_0 = U_0 \cup D$ and note that $A_0 \in \mathcal{A}$ and $f \in \{0, 1\}$ on A_0. Since $U \subset S$ and $P(S - U) = 0$, one also obtains $P(A_0) = P(U_0) + P(D) = P(S) + P(S^c) = 1$. $\square$

A basic condition for existence of disintegrations is that

$$G = \{(x, y) \in \Omega \times \Omega : H(x) = H(y)\}$$

belongs to $\mathcal{B} \otimes \mathcal{B}$; see [1]. Such a condition also plays a role in our main characterization of (3). Let

$$\mathcal{A}^* = \{B \in \mathcal{B} : B \text{ is a union of } \mathcal{A}\text{-atoms}\}.$$

THEOREM 4. *If* (1) *holds and* $G \in \mathcal{B} \otimes \mathcal{B}$, *then* $H(\omega) \in \mathcal{B}$ *for all* $\omega \in \Omega$ *and* f_B *is* $\mathcal{A}^*$*-measurable for all* $B \in \mathcal{B}$. *If in addition* $S \in \mathcal{A}_P$, *then each of conditions* (3), (a) *and* (b) *is equivalent to*

(c) $\mathcal{A} \cap U = \mathcal{A}^* \cap U$ *for some* $U \in \mathcal{A}$ *with* $U \subset S$ *and* $P(S - U) = 0$.

PROOF. For $C \subset \Omega \times \Omega$, let $C_\omega = \{u \in \Omega : (\omega, u) \in C\}$ be the ω-section of C. Suppose (1) holds and $G \in \mathcal{B} \otimes \mathcal{B}$. Then, $H(\omega) = G_\omega \in \mathcal{B}$ for all ω. By a monotone class argument, the map $\omega \mapsto \mu(\omega)(C_\omega)$ is $\mathcal{B}$-measurable whenever $C \in \mathcal{B} \otimes \mathcal{B}$. Letting $C = G \cap (\Omega \times B)$, where $B \in \mathcal{B}$, implies that f_B is $\mathcal{B}$-measurable. Since f_B is constant on each $\mathcal{A}$-atom, it is in fact $\mathcal{A}^*$-measurable. (Note that $\mathcal{A}^*$-measurability of f_B does not require $\mathcal{B}$ c.g.) Next, suppose also that $S \in \mathcal{A}_P$. By Theorem 3, conditions (3), (a) and (b) are equivalent. Suppose (c) holds, and fix $B \in \mathcal{B}$ and a Borel set $I \subset \mathbb{R}$. Since $\{f_B \in I\} \in \mathcal{A}^*$, condition (c) yields $\{f_B \in I\} \cap U \in \mathcal{A}$, and $\{f_B \in I\} \cap (S - U) \in \mathcal{A}_P$ due to $(S - U) \in \mathcal{A}_P$ and $P(S - U) = 0$. Thus (assuming $0 \in I$ to fix ideas),

$$\{f_B \in I\} = S^c \cup \big(\{f_B \in I\} \cap (S - U)\big) \cup (\{f_B \in I\} \cap U) \in \mathcal{A}_P,$$

so that (a) holds. Conversely, suppose (a) holds. For each $B \in \mathcal{B}$, there is $A_B \in \mathcal{A}$ such that $P(A_B) = 1$ and $I_{A_B} f_B$ is $\mathcal{A}$-measurable. Letting $A = \bigcap_{B \in \mathcal{B}_0} A_B$, where $\mathcal{B}_0$ is a countable field generating $\mathcal{B}$, it follows that $A \in \mathcal{A}$, $P(A) = 1$ and $I_A f_B$ is $\mathcal{A}$-measurable for all $B \in \mathcal{B}$. Since $S \in \mathcal{A}_P$ and $P(A) = 1$, there is $U \in \mathcal{A}$ with $U \subset A \cap S$ and $P(S - U) = 0$. Given $B \in \mathcal{A}^*$, on noting that $f_B = I_B f$, one obtains

$$B \cap U = \{I_B f > 0\} \cap U = \{f_B > 0\} \cap U = (\{f_B > 0\} \cap A) \cap U \in \mathcal{A}.$$

Hence $\mathcal{A} \cap U = \mathcal{A}^* \cap U$, that is, condition (c) holds. $\square$

We now state a couple of corollaries to Theorem 4. The first covers in particular the case where the $\mathcal{A}$-atoms are the singletons, while the second (and more important) applies to various real situations.

COROLLARY 5. *Suppose* (1) *holds*, $S \in \mathcal{A}_P$ *and* $\mathcal{A}$, $\mathcal{B}$ *have the same atoms. Then*, (3) *holds if and only if* $\mathcal{A} \cap U = \mathcal{B} \cap U$ *for some* $U \in \mathcal{A}$ *with* $U \subset S$ *and* $P(S - U) = 0$.

PROOF. Since $\mathcal{A}$, $\mathcal{B}$ have the same atoms and $\mathcal{B}$ is c.g.,

$$G = \{(x, y) : x \text{ and } y \text{ are in the same } \mathcal{B}\text{-atom}\} \in \mathcal{B} \otimes \mathcal{B}.$$

Therefore, it suffices applying Theorem 4 and noting that $\mathcal{A}^* = \mathcal{B}$. $\square$

COROLLARY 6. *If* (1) *holds*, $G \in \mathcal{B} \otimes \mathcal{B}$ *and* $\mathcal{A} \cap C = \mathcal{A}^* \cap C$, *for some* $C \in \mathcal{A}$ *with* $P(C) = 1$, *then condition* (3) *holds*.

PROOF. Since $C \in \mathcal{A}$ and $S \cap C \in \mathcal{A}^* \cap C = \mathcal{A} \cap C$, then $S \cap C \in \mathcal{A}$. Since $P(C) = 1$, it follows that $S \in \mathcal{A}_P$ and (c) holds with $U = S \cap C$. Thus, (3) follows from Theorem 4. $\square$

As shown in [4], if $(\Omega, \mathcal{B})$ is a standard space (Ω Borel subset of a Polish space and $\mathcal{B}$ the Borel σ-field on Ω), then $G \in \mathcal{B} \otimes \mathcal{B}$ and $\mathcal{A}^* = \mathcal{A}$ for various classically interesting sub-σ-fields $\mathcal{A}$, including tail, symmetric, invariant, as well as some sub-σ-fields connected with continuous time processes. In view of Corollary 6, *condition* (3) *holds in case* $(\Omega, \mathcal{B})$ *is a standard space and* $\mathcal{A}$ *is any one of the above-mentioned sub-*σ*-fields.*

2.2. *Tail sub-*σ*-fields.* When condition (3) holds, the next step is determining those ω's satisfying $f(\omega) = 1$. Suppose the assumptions of Corollary 6 are in force [so that (3) holds and f is $\mathcal{A}_P$-measurable] and define

$$\mathcal{U} = \{U \in \mathcal{A} : \mathcal{A} \cap U \text{ is c.g.}\} \cup \{\varnothing\}.$$

Since $\mathcal{U}$ is closed under countable unions, some $A \in \mathcal{U}$ meets $P(A) = \sup\{P(U) : U \in \mathcal{U}\}$. By the first assertion in Theorem 3, $P(A - \{f = 1\}) = 0$. Taking U as in condition (b) and noting that $U \in \mathcal{U}$, one also obtains $P(\{f = 1\} - A) = P(U - A) = 0$. Therefore, A is the set we are looking for, in the sense that

$$P(\{f = 1\} \Delta A) = 0.$$

Incidentally, the above remarks provide also a criterion for deciding whether μ is *maximally improper* according to [11]. Under the assumptions of Corollary 6, in fact, μ is maximally improper precisely when $P(S) = 0$. Hence,

$$\mu \text{ is maximally improper} \quad \Leftrightarrow \quad P(U) = 0 \text{ for all } U \in \mathcal{U}.$$

Some handy description of the members of $\mathcal{U}$, thus, would be useful. Unfortunately, such a description is generally hard to be found. We now discuss a particular case.

Let $\mathcal{A}$ be a *tail sub-*σ*-field*, that is, $\mathcal{A} = \bigcap_{n \ge 1} \mathcal{A}_n$ where $\mathcal{A}_n$ is a *countably generated* σ-field and $\mathcal{B} \supset \mathcal{A}_n \supset \mathcal{A}_{n+1}$ for all $n \ge 1$. As already noted, the assumptions of Corollary 6 hold for such an $\mathcal{A}$ if $(\Omega, \mathcal{B})$ is a standard space. More generally, it is enough that:

LEMMA 7. *If* $\mathcal{A}$ *is a tail sub-*σ*-field*, (1) *holds and*

$$\textit{for each } n, \textit{ there is a r.c.d. } \mu_n \textit{ for } P \textit{ given } \mathcal{A}_n,$$

then $G \in \mathcal{B} \otimes \mathcal{B}$ *and* $\mathcal{A} \cap C = \mathcal{A}^* \cap C$ *for some* $C \in \mathcal{A}$ *with* $P(C) = 1$.

PROOF. Since $G_n := \{(x, y) \in \Omega \times \Omega : x \text{ and } y \text{ are in the same } \mathcal{A}_n\text{-atom}\} \in \mathcal{A}_n \otimes \mathcal{A}_n$, Proposition 1 of [4] implies $G = \bigcup_n G_n \in \mathcal{B} \otimes \mathcal{B}$. For each n, since $\mathcal{A}_n$ is c.g., there is $C_n \in \mathcal{A}_n$ such that $P(C_n) = 1$ and $\mu_n(\omega)(A) = I_A(\omega)$ whenever $A \in \mathcal{A}_n$ and $\omega \in C_n$. Define $C = \bigcup_{n \ge 1} \bigcap_{j \ge n} C_j$ and note that $C \in \mathcal{A}$ and $P(C) = 1$. Fix $B \in \mathcal{A}^*$. Since B is a union of $\mathcal{A}_n$-atoms whatever n is,

$$\lim_n \mu_n(\omega)(B) = I_B(\omega) \qquad \text{for all } \omega \in C.$$

Thus, $B \cap C = \{\lim_n \mu_n(B) = 1\} \cap C \in \mathcal{A} \cap C$. $\square$

Each $\mathcal{A}_n$, being c.g., can be written as $\mathcal{A}_n = \sigma(X_n)$ for some $X_n : \Omega \to \mathbb{R}$. Since $\mathcal{A}_n \supset \mathcal{A}_j$ for $j \geq n$, it follows that $\mathcal{A}_n = \sigma(X_n, X_{n+1}, \ldots)$. Thus, $\mathcal{A}$ admits the usual representation

$$\mathcal{A} = \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$$

for some sequence (X_n) of real random variables. In particular,

$$H(\omega) = \{\exists n \geq 1 \text{ such that } X_j = X_j(\omega) \text{ for all } j \geq n\} \in \mathcal{A}$$

so that $\mathcal{A}$ includes its atoms. Note also that a c.g. sub-σ-field is tail while the converse need not be true. In fact, for a σ-field $\mathcal{F}$ to be not c.g., it is enough that $\mathcal{F}$ supports a 0–1 valued probability measure Q such that $Q(F) = 0$ whenever $F \in \mathcal{F}$ and F is an $\mathcal{F}$-atom; see Theorem 1 of [4]. Thus, for instance, $\mathcal{A} = \bigcap_n \sigma(X_n, X_{n+1}, \ldots)$ is not c.g. in case (X_n) is i.i.d. and X_1 has a nondegenerate distribution.

To find usable characterizations of $\mathcal{U}$ is not an easy task. Countable unions of $\mathcal{A}$-atoms belong to $\mathcal{U}$, but generally they are not all the elements of $\mathcal{U}$. For instance, if $\Omega = \mathbb{R}^\infty$ and X_n is the nth coordinate projection, then

$$U = \{\exists n \geq 1 \text{ such that } X_j = X_n \text{ for all } j \geq n\}$$

is an uncountable union of $\mathcal{A}$-atoms. However, $\mathcal{A} \cap U$ is c.g. since $\mathcal{A} \cap U = \sigma(L) \cap U$ where $L = \limsup_n X_n$.

Another possibility could be selecting a subclass $\mathbb{Q} \subset \mathbb{P}$ and showing that $U \in \mathcal{U}$ if and only if $U \in \mathcal{A}$ and $Q(U) = 0$ for each $Q \in \mathbb{Q}$. We do not know whether some (nontrivial) characterization of this type is available. Here, we just note that

$$\mathbb{Q}_0 = \{Q \in \mathbb{P} : (X_n) \text{ is i.i.d. and } X_1 \text{ has a nondegenerate distribution, under } Q\}$$

does not work (though the "only if" implication is true, in view of Theorem 1 of [4]). As an example, $U := \{X_n \to 0\} \notin \mathcal{U}$ even though $U \in \mathcal{A}$ and $Q(U) = 0$ for all $Q \in \mathbb{Q}_0$. To see that $U \notin \mathcal{U}$, let X_n be the nth coordinate projection on $\Omega = \mathbb{R}^\infty$, and let P_U be a probability measure on the Borel sets of Ω which makes (X_n) independent and each X_n uniformly distributed on $(0, \frac{1}{n})$. Then $P_U(U) = 1$ and, when restricted to $\mathcal{A} \cap U$, P_U is a 0–1 probability measure such that $P_U(H(\omega)) = 0$ for each $\omega \in U$. Hence, Theorem 1 of [4] implies that $\mathcal{A} \cap U$ is not c.g.

A last note is that $P(S)$ can assume any value between 0 and 1. For instance, take $U \in \mathcal{U}$ and $P_1, P_2 \in \mathbb{P}$ such that: (i) $P_1(U) = P_2(U^c) = 1$; (ii) P_2 is 0–1 on $\mathcal{A}$ with $P_2(H(\omega)) = 0$ for all ω. Define $P = uP_1 + (1-u)P_2$ where $u \in (0, 1)$. A r.c.d. for P given $\mathcal{A}$ is $\mu(\omega) = I_U(\omega)\mu_1(\omega) + I_{U^c}(\omega)P_2$, where μ_1 denotes a r.c.d. for P_1 given $\mathcal{A}$. Since $U \in \mathcal{U}$, Theorem 3 implies $\mu_1(\omega)(H(\omega)) = 1$ for P_1-almost all $\omega \in U$. Thus, $P(S) = P(U) = u$.

2.3. *Miscellaneous results.* A weaker version of (3) lies in asking $\mu(\cdot)(H(\cdot))$ to be 0–1 over a set of $\mathcal{A}^*$, but not necessarily of $\mathcal{A}$, that is

(3*) There is $B_0 \in \mathcal{A}^*$ with $P(B_0) = 1$ and $\mu(\omega)(H(\omega)) \in \{0, 1\}$ for all $\omega \in B_0$. Suitably adapted, the proofs of Theorems 3 and 4 yield a characterization of (3*) as well. Recall $\mathcal{N} = \{B \in \mathcal{B} : P(B) = 0\}$ and note that

$$\sigma(\mathcal{A} \cup \mathcal{N}) = \{B \in \mathcal{B} : \mu(B) = I_B \text{ a.s.}\}.$$

THEOREM 8. *Suppose* (1) *holds and* $G \in \mathcal{B} \otimes \mathcal{B}$. *Then, condition* (3*) *implies* $S \in \sigma(\mathcal{A} \cup \mathcal{N})$. *Moreover, if* $S \in \sigma(\mathcal{A} \cup \mathcal{N})$, *then*

$$(3*) \quad \Leftrightarrow \quad (b*) \quad \Leftrightarrow \quad (c*)$$

where:

(b*) $\mathcal{A} \cap V$ *is c.g. for some* $V \in \mathcal{A}^*$ *with* $V \subset S$ *and* $P(S - V) = 0$;
(c*) $\mathcal{A} \cap V = \mathcal{A}^* \cap V$ *for some* $V \in \mathcal{A}^*$ *with* $V \subset S$ *and* $P(S - V) = 0$.

PROOF. If (3*) holds, then $\mu(S) = 1$ on $B_0 \cap S$, and since $P(B_0) = 1$ one obtains

$$E(\mu(S)I_{S^c}) = P(S) - E\big(\mu(S)I_{B_0}I_S\big) = P(S) - E\big(I_{B_0}I_S\big) = 0.$$

Thus, $\mu(S) = I_S$ a.s., that is, $S \in \sigma(\mathcal{A} \cup \mathcal{N})$. Next, suppose that $S \in \sigma(\mathcal{A} \cup \mathcal{N})$.

"(3*) $\Rightarrow$ (c*)." Define $V = B_0 \cap S$ and note that $B \cap V = \{\mu(B) = 1\} \cap V \in \mathcal{A} \cap V$ for all $B \in \mathcal{A}^*$.

"(c*) $\Rightarrow$ (b*)." Fix $\alpha \in \mathbb{P}$ and define $\nu(\omega)(B) = I_V(\omega)\frac{f_B(\omega)}{f(\omega)} + I_{V^c}(\omega)\alpha(B)$ for all $\omega \in \Omega$ and $B \in \mathcal{B}$. Then, $\sigma(\nu) \subset \mathcal{A}^*$. Further, $\nu(\omega)(H(\omega)) = 1$ for all $\omega \in V$, so that $B \cap V = \{\nu(B) = 1\} \cap V$ for all $B \in \mathcal{A}^*$. Hence, (c*) implies that $\mathcal{A} \cap V = \mathcal{A}^* \cap V = \sigma(\nu) \cap V$ is c.g.

"(b*) $\Rightarrow$ (3*)." Let $\mathcal{A}_0 = \{(A \cap V) \cup F : A \in \mathcal{A}, F = \emptyset \text{ or } F = V^c\}$. Since $\mu(V) = \mu(S) = I_S = I_V$ a.s., for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one obtains

$$E(I_A I_V \mu(B)) = E\big(I_A \mu(B \cap V)\big) = P\big((A \cap V) \cap B\big).$$

So, $\mu_0(\omega) = I_V(\omega)\mu(\omega) + I_{V^c}(\omega)\alpha$ is a r.c.d. for P given $\mathcal{A}_0$, where $\alpha(\cdot) = P(\cdot \mid V^c)$ if $P(V) < 1$ and α is any fixed element of $\mathbb{P}$ if $P(V) = 1$. Since $\mathcal{A}_0$ is c.g., there is $K \in \mathcal{A}_0$ with $P(K) = 1$ and $\mu_0(\omega)(H_0(\omega)) = 1$ for all $\omega \in K$, where $H_0(\omega)$ denotes the $\mathcal{A}_0$-atom including ω. Hence, it suffices to let $B_0 = (K \cap V) \cup S^c$ and noting that $H_0(\omega) = H(\omega)$ and $\mu_0(\omega) = \mu(\omega)$ for all $\omega \in V$. $\square$

One consequence of Theorem 8 is that, if (1) holds and $G \in \mathcal{B} \otimes \mathcal{B}$, then condition (3*) is equivalent to $\mu(S) = I_S$ a.s. and $P(0 < f \leq \frac{1}{2}) = 0$. In fact,

$$A \cap \{f > \tfrac{1}{2}\} = \{\mu(A) > \tfrac{1}{2}\} \cap \{f > \tfrac{1}{2}\} \qquad \text{for all } A \in \mathcal{A},$$

so that $\mathcal{A} \cap \{f > \frac{1}{2}\} = \sigma(\mu) \cap \{f > \frac{1}{2}\}$ is c.g. Hence, if $P(0 < f \leq \frac{1}{2}) = 0$, condition (b*) holds with $V = \{f > \frac{1}{2}\}$.

Finally, we give one more condition for (3). Though seemingly simple, it is hard to be tested in real problems.

PROPOSITION 9. *If* (1) *holds and* $H(\omega) \in \mathcal{B}$ *for all* ω, *a sufficient condition for* (3) *is*

$$\mu(x)(H(y)) = 0 \qquad \text{whenever } H(x) \neq H(y). \tag{5}$$

PROOF. As stated in the forthcoming Lemma 10, since $\sigma(\mu)$ is c.g. and μ is also a r.c.d. for P given $\sigma(\mu)$, there is a set $T \in \sigma(\mu)$ such that $P(T) = 1$ and $\mu(\omega)(\mu = \mu(\omega)) = 1$ for each $\omega \in T$. Let $A_0 = T$ and fix $\omega \in S$. Then, $\mu(\omega) = \mu(x)$ if $x \in H(\omega)$ [since $\sigma(\mu) \subset \mathcal{A}$] and $\mu(\omega) \neq \mu(x)$ if $x \notin H(\omega)$ since in the latter case (5) yields

$$\mu(x)(H(\omega)) = 0 < f(\omega) = \mu(\omega)(H(\omega)).$$

Thus, $H(\omega) = \{\mu = \mu(\omega)\}$. If $\omega \in T \cap S = A_0 \cap S$, this implies

$$\mu(\omega)(H(\omega)) = \mu(\omega)\big(\mu = \mu(\omega)\big) = 1. \qquad \square$$

3. When regular conditional distributions are 0–1 on the conditioning σ-field.

In this section, condition (4) is shown to be true whenever $\mathcal{A}$ is a tail sub-σ-field. Moreover, two characterizations of (4) and a result in the negative [i.e., a condition for (4) to be false] are given.

We begin by recalling a few simple facts about $\sigma(\mu)$.

LEMMA 10. *If* (1) *holds, then* $\sigma(\mu)$ *is c.g.,* μ *is a r.c.d. for* P *given* $\sigma(\mu)$, *and there is a set* $T \in \sigma(\mu)$ *with* $P(T) = 1$ *and*

$$\mu(\omega)\big(\mu = \mu(\omega)\big) = 1 \qquad \text{for all } \omega \in T.$$

Moreover,

$$\mathcal{A} = \sigma\big(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N})\big).$$

PROOF. Since $\sigma(\mu) \subset \mathcal{A}$, μ is a r.c.d. given $\sigma(\mu)$. Since $\mathcal{B}$ is c.g., $\sigma(\mu)$ is c.g. with atoms of the form $\{\mu = \mu(\omega)\}$. Hence, there is $T \in \sigma(\mu)$ with $P(T) = 1$ and $\mu(\omega)(\mu = \mu(\omega)) = 1$ for all $\omega \in T$. Finally, since

$$A = \big(\{\mu(A) = 1\} \cap \{\mu(A) = I_A\}\big) \cup \big(A \cap \{\mu(A) \neq I_A\}\big)$$

for all $A \in \mathcal{A}$, it follows that $\mathcal{A} \subset \sigma(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N})) \subset \mathcal{A}$. $\square$

By Lemma 10, $\mu(\omega)$ is 0–1 on $\sigma(\mu)$ for each $\omega \in T$. Since $\mathcal{A} = \sigma(\sigma(\mu) \cup (\mathcal{A} \cap \mathcal{N}))$, condition (4) can be written as

$$\mu(\omega)(A) \in \{0, 1\} \qquad \text{for all } \omega \in A_0 \text{ and } A \in \mathcal{A} \text{ with } P(A) = 0.$$

In particular, (4) *holds whenever P is atomic on $\mathcal{A}$*, in the sense that there is a countable partition $\{A_1, A_2, \dots\}$ of Ω satisfying $A_j \in \mathcal{A}$ and $P(A \cap A_j) \in \{0, P(A_j)\}$ for all $j \geq 1$ and $A \in \mathcal{A}$. In this case, in fact, $\mu(\omega) \ll P$ for each ω in some set $C \in \mathcal{A}$ with $P(C) = 1$.

Slightly developing the idea underlying Example 2, we next give a sufficient condition for (4) to be false.

PROPOSITION 11. *Suppose* (1) *holds and $P(\mu = \mu(\omega)) = 0$ for all ω. Then,*

$$F = \{\omega : \mu(\omega) \text{ is not } 0\text{–}1 \text{ on } \mathcal{B}\} \quad \text{and} \quad F_0 = \{\omega : \mu(\omega) \text{ is nonatomic on } \mathcal{B}\}$$

belong to $\sigma(\mu)$. Moreover, if $\mathcal{N} \subset \mathcal{A}$, then

$$\mu(\omega) \text{ is not } 0\text{–}1 \text{ on } \mathcal{A} \text{ for each } \omega \in F \cap T, \text{ and}$$

$$\mu(\omega) \text{ is nonatomic on } \mathcal{A} \text{ for each } \omega \in F_0 \cap T$$

with T as in Lemma 10. *In particular, condition* (4) *fails if $P(F) > 0$.*

PROOF. Since $\mathcal{B}$ is c.g., it is clear that $F \in \sigma(\mu)$, while $F_0 \in \sigma(\mu)$ is from [6] (see Corollary 2.13, page 1214). Suppose now that $\mathcal{N} \subset \mathcal{A}$. Let $\omega \in F \cap T$. Since $\omega \in F$, there is $B_\omega \in \mathcal{B}$ with $\mu(\omega)(B_\omega) \in (0, 1)$. Define $A_\omega = B_\omega \cap \{\mu = \mu(\omega)\}$. Since $\mathcal{N} \subset \mathcal{A}$ and $P(A_\omega) \leq P(\mu = \mu(\omega)) = 0$, then $A_\omega \in \mathcal{A}$. Since $\omega \in T$,

$$\mu(\omega)(A_\omega) = \mu(\omega)(B_\omega) \in (0, 1)$$

so that $\mu(\omega)$ is not 0–1 on $\mathcal{A}$. Finally, fix $\omega \in F_0 \cap T$ and $\varepsilon > 0$. Since $\omega \in F_0$, there is a finite partition $\{B_{1,\omega}, \dots, B_{n,\omega}\}$ of Ω such that $B_{i,\omega} \in \mathcal{B}$ and $\mu(\omega)(B_{i,\omega}) < \varepsilon$ for all i. As above, letting $A_{i,\omega} = B_{i,\omega} \cap \{\mu = \mu(\omega)\}$, one obtains $A_{i,\omega} \in \mathcal{A}$ and $\mu(\omega)(A_{i,\omega}) = \mu(\omega)(B_{i,\omega}) < \varepsilon$. Hence, $\mu(\omega)$ is nonatomic on $\mathcal{A}$ since $\mu(\omega)(\mu \neq \mu(\omega)) = 0$. $\square$

Even if $\mathcal{N}$ is not contained in $\mathcal{A}$, Proposition 11 applies at least to $\mathcal{A}' = \sigma(\mathcal{A} \cup \mathcal{N})$. Under mild conditions, μ is even nonatomic on $\mathcal{A}'$ with probability $P(F_0)$. Thus, a lot of r.c.d.'s give rise to a failure of (4) on some sub-σ-field $\mathcal{A}'$. Since we are conditioning to $\mathcal{A}$ (and not to $\mathcal{A}'$), this fact is not essential. On the other hand, it suggests that (4) is a rather delicate condition.

If P is invariant under a countable collection of measurable transformations and $\mathcal{A}$ is the corresponding invariant sub-σ-field, then (4) holds; see [9]. This well-known fact is generalized by our first characterization of (4).

THEOREM 12. *Suppose* (1) *holds and let $M = \{Q \in \mathbb{P} : \mu$ is a r.c.d. for Q given $\mathcal{A}\}$. Then, Q is an extreme point of M if and only if $Q \in M$ and Q is* 0–1 *on $\mathcal{A}$, and in that case $Q = \mu(\omega)$ for some $\omega \in \Omega$. Moreover, for each $\omega \in T$* (*with T as in Lemma* 10), *the following statements are equivalent*:

(i) $\mu(\omega)(A) \in \{0, 1\}$ *for all* $A \in \mathcal{A}$;

(ii) $\mu(\omega)$ *is an extreme point of* M;
(iii) $\mu(\omega) \in M$.

In particular, condition (4) *holds if and only if, for some* $A_0 \in \mathcal{A}$ *with* $P(A_0) = 1$,

$$\mu(\omega) \in M \quad \textit{for all } \omega \in A_0.$$

PROOF. Fix $Q \in M$. If $Q(A) \in (0, 1)$ for some $A \in \mathcal{A}$, then

$$Q(\cdot) = Q(A)Q(\cdot \mid A) + \big(1 - Q(A)\big)Q(\cdot \mid A^c),$$

and Q is not extreme since $Q(\cdot \mid A)$ and $Q(\cdot \mid A^c)$ are distinct elements of M. Suppose now that $Q = uQ_1 + (1 - u)Q_2$, where $u \in (0, 1)$ and $Q_1 \neq Q_2$ are in M. Since two elements of M coincide if and only if they coincide on $\mathcal{A}$, there is $A \in \mathcal{A}$ with $Q_1(A) \neq Q_2(A)$, and this implies $Q(A) \in (0, 1)$. Hence, $Q \in M$ is extreme if and only if it is 0–1 on $\mathcal{A}$. In particular, if Q is extreme then it is 0–1 on the c.g. σ-field $\sigma(\mu)$, so that $Q(\mu = \mu(\omega)) = 1$ for some $\omega \in \Omega$; see Theorem 1 of [4]. Thus,

$$Q(B) = \int \mu(x)(B)\, Q(dx) = \mu(\omega)(B) \qquad \text{for all } B \in \mathcal{B}.$$

This concludes the proof of the first part. As to the second one, fix $\omega \in T$, and let A and B denote arbitrary elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. Since $\omega \in T$,

$$\int_A \mu(x)(B)\mu(\omega)(dx) = \int_{A \cap \{\mu = \mu(\omega)\}} \mu(x)(B)\mu(\omega)(dx) = \mu(\omega)(A)\mu(\omega)(B).$$

"(i) $\Rightarrow$ (ii)." By what already proved, it is enough showing that $\mu(\omega) \in M$, and this depends on $\mu(\omega)(A \cap B) = \mu(\omega)(A)\mu(\omega)(B) = \int_A \mu(x)(B)\, \mu(\omega)(dx)$.

"(ii) $\Rightarrow$ (iii)." Obvious.

"(iii) $\Rightarrow$ (i)." Under (iii), $\mu(\omega)(A \cap B) = \int_A \mu(x)(B)\, \mu(\omega)(dx) = \mu(\omega) \times (A)\mu(\omega)(B)$, and letting $B = A$ yields $\mu(\omega)(A) = \mu(\omega)(A)^2$. $\square$

Next characterization of (4) stems from a result of [8], Lemma 2A, page 391.

THEOREM 13 (Fremlin). *Let* X *be an Hausdorff topological space*, $\mathcal{F}$ *a* σ*-field on* X *including the open sets*, Q *a complete Radon probability measure on* $\mathcal{F}$, *and* $\mathcal{C}_0$ *a class of pairwise disjoint* Q*-null elements of* $\mathcal{F}$. *Then*,

$$\bigcup_{C \in \mathcal{C}} C \in \mathcal{F} \textit{ for all } \mathcal{C} \subset \mathcal{C}_0 \quad \Longleftrightarrow \quad Q\Bigg(\bigcup_{C \in \mathcal{C}_0} C\Bigg) = 0.$$

Say that P is *perfect* in case each $\mathcal{B}$-measurable function $h : \Omega \to \mathbb{R}$ meets $P(h \in I) = 1$ for some real Borel set $I \subset h(\Omega)$. For P to be perfect, it is enough that Ω is an universally measurable subset of a Polish space and $\mathcal{B}$ the Borel σ-field on Ω. In the present framework, since $\mathcal{B}$ is c.g., Theorem 13 applies precisely when P is perfect. We are now able to state our second characterization of (4). It is of possible theoretical interest even if of little practical use.

THEOREM 14. *Suppose* (1) *holds and* P *is perfect, define*

$$\mathcal{A}(\omega) = \{A \in \mathcal{A} : \mu(\omega)(A) \in \{0, 1\}\} \qquad \omega \in \Omega,$$

and let Γ_0 *denote the class of those* σ*-fields* $\mathcal{G} \subset \mathcal{A}$ *with* $\mathcal{G} \neq \mathcal{A}$. *Then, condition* (4) *holds if and only if*

$$\bigcup_{\mathcal{G} \in \Gamma} \{\omega : \mathcal{A}(\omega) = \mathcal{G}\} \in \mathcal{A}_P \qquad \text{for all } \Gamma \subset \Gamma_0. \tag{6}$$

PROOF. If $\mu(\omega)$ is 0–1 on $\mathcal{A}$ for all $\omega \in A_0$, where $A_0 \in \mathcal{A}$ and $P(A_0) = 1$, then (6) follows from

$$\{\omega : \mathcal{A}(\omega) = \mathcal{G}\} \subset A_0^c \qquad \text{for all } \mathcal{G} \in \Gamma_0.$$

Conversely, suppose (6) holds. Let X be the partition of Ω in the atoms of $\mathcal{B}$. The elements of $\mathcal{B}$ are unions of elements of X, so that $\mathcal{B}$ can be regarded as a σ-field on X. Let $(X, \mathcal{F}, Q)$ be the completion of $(X, \mathcal{B}, P)$. Since $\mathcal{B}$ is c.g., under a suitable distance, X is separable metric and $\mathcal{B}$ the corresponding Borel σ-field; see [2]. Since P is perfect, P is Radon by a result of Sazonov (Theorem 12 of [10]), so that Q is Radon, too. Next, define $C_{\mathcal{G}} = \{\omega : \mathcal{A}(\omega) = \mathcal{G}\}$ for $\mathcal{G} \in \Gamma_0$, $U_A = \{\omega : \mu(\omega)(A) \in (0, 1)\}$ for $A \in \mathcal{A}$, and $U = \{\omega : \mathcal{A}(\omega) \neq \mathcal{A}\}$ (all regarded as subsets of X). For each $\mathcal{G} \in \Gamma_0$ there is $A \in \mathcal{A}$ with $C_{\mathcal{G}} \subset U_A$. Since $U_A \in \mathcal{A}$ and $P(U_A) = 0$, then $C_{\mathcal{G}} \in \mathcal{F}$ and $Q(C_{\mathcal{G}}) = 0$. Hence, $\mathcal{C}_0 = \{C_{\mathcal{G}} : \mathcal{G} \in \Gamma_0\}$ is a collection of pairwise disjoint Q-null elements of $\mathcal{F}$ satisfying $U = \bigcup_{\mathcal{G} \in \Gamma_0} C_{\mathcal{G}}$. By (6), Theorem 13 yields $Q(U) = 0$. Finally, since $U \in \mathcal{A}_P$, $Q(U) = 0$ implies $U \subset A$ for some $A \in \mathcal{A}$ with $P(A) = 0$. Thus, to get (4), it suffices to let $A_0 = A^c$. $\square$

Finally, by a martingale argument, we prove that (4) holds when $\mathcal{A}$ is a tail sub-σ-field. This is true, in addition, even though $\mathcal{B}$ fails to be c.g.

THEOREM 15. *Let* $\mathcal{A} = \bigcap_{n \geq 1} \mathcal{A}_n$, *where* $\mathcal{B} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$ *and* $\mathcal{A}_n$ *is a c.g.* σ*-field for each* n. *Given a r.c.d.* μ, *for* P *given* $\mathcal{A}$, *there is a set* $A_0 \in \mathcal{A}$ *such that* $P(A_0) = 1$ *and* $\mu(\omega)(A) \in \{0, 1\}$ *for all* $A \in \mathcal{A}$ *and* $\omega \in A_0$.

PROOF. First recall that a probability measure $Q \in \mathbb{P}$ is 0–1 on $\mathcal{A}$ if (and only if) $\sup_{A \in \mathcal{A}_n} |Q(A \cap B) - Q(A)Q(B)| \to 0$, as $n \to \infty$, for all $B \in \mathcal{A}_1$. Also, given any field $\mathcal{F}_n$ such that $\mathcal{A}_n = \sigma(\mathcal{F}_n)$, the "sup" can be taken over $\mathcal{F}_n$, that is,

$$\sup_{A \in \mathcal{A}_n} |Q(A \cap B) - Q(A)Q(B)| = \sup_{A \in \mathcal{F}_n} |Q(A \cap B) - Q(A)Q(B)|.$$

Now, since the $\mathcal{A}_n$ are c.g., there are countable fields $\mathcal{F}_n$ satisfying $\mathcal{A}_n = \sigma(\mathcal{F}_n)$ for all n. Let

$$V_n^B(\omega) = \sup_{A \in \mathcal{F}_n} |\mu(\omega)(A \cap B) - \mu(\omega)(A)\mu(\omega)(B)|, \qquad n \geq 1, B \in \mathcal{A}_1, \omega \in \Omega.$$

Since $\mathcal{F}_n$ is countable, V_n^B is an $\mathcal{A}$-measurable random variable for all n and B. It is enough proving that

$$V_n^B \to 0 \qquad \text{a.s., as } n \to \infty, \text{ for all } B \in \mathcal{A}_1. \tag{7}$$

Suppose in fact (7) holds and define

$$A_0 = \left\{\omega : \lim_n V_n^B(\omega) = 0 \text{ for each } B \in \mathcal{F}_1\right\}.$$

Since $\mathcal{F}_1$ is countable, $A_0 \in \mathcal{A}$ and (7) implies $P(A_0) = 1$. Fix $\omega \in A_0$. Since $\mathcal{A}_1 = \sigma(\mathcal{F}_1)$, given $B \in \mathcal{A}_1$ and $\varepsilon > 0$, there is $C \in \mathcal{F}_1$ such that $\mu(\omega)(B\Delta C) < \varepsilon$. Hence,

$$\begin{aligned} V_n^B(\omega) \le{}& \sup_{A\in\mathcal{F}_n} |\mu(\omega)(A\cap B) - \mu(\omega)(A\cap C)| \\ &+ \sup_{A\in\mathcal{F}_n} |\mu(\omega)(A\cap C) - \mu(\omega)(A)\mu(\omega)(C)| \\ &+ \sup_{A\in\mathcal{F}_n} |\mu(\omega)(A)\mu(\omega)(C) - \mu(\omega)(A)\mu(\omega)(B)| \\ \le{}& V_n^C(\omega) + 2\mu(\omega)(B\Delta C) < V_n^C(\omega) + 2\varepsilon \qquad \text{for all } n. \end{aligned}$$

Since $\omega \in A_0$ and $C \in \mathcal{F}_1$, it follows that

$$\limsup_n V_n^B(\omega) \le 2\varepsilon + \limsup_n V_n^C(\omega) = 2\varepsilon \qquad \text{for all } B \in \mathcal{A}_1 \text{ and } \varepsilon > 0.$$

Therefore, $\mu(\omega)$ is 0–1 on $\mathcal{A}$. It remains to check condition (7). Fix $B \in \mathcal{A}_1$, take any version of $E(I_B \mid \mathcal{A}_n)$ and define $Z_n = E(I_B \mid \mathcal{A}_n) - \mu(B)$. Then, $|Z_n| \le 2$ a.s. for all n, and the martingale convergence theorem yields $Z_n \to 0$ a.s. Further, for fixed $n \ge 1$ and $A \in \mathcal{F}_n$, one obtains

$$\begin{aligned} &\bigl|E(I_A I_B \mid \mathcal{A}) - E(I_A \mid \mathcal{A})E(I_B \mid \mathcal{A})\bigr| \\ &\quad = \bigl|E(I_A E(I_B \mid \mathcal{A}_n) \mid \mathcal{A}) - E(I_A E(I_B \mid \mathcal{A}) \mid \mathcal{A})\bigr| \\ &\quad = \bigl|E(I_A Z_n \mid \mathcal{A})\bigr| \le E(|Z_n| \mid \mathcal{A}) \qquad \text{a.s.} \end{aligned}$$

Since $\mathcal{F}_n$ is countable, it follows that

$$V_n^B = \sup_{A\in\mathcal{F}_n} \bigl|E(I_A I_B \mid \mathcal{A}) - E(I_A \mid \mathcal{A})E(I_B \mid \mathcal{A})\bigr| \le E(|Z_n| \mid \mathcal{A}) \to 0 \qquad \text{a.s.}$$ □

As noted in Section 2.2, a tail sub-σ-field includes its atoms so that (4) implies (3). Thus, by Theorem 15, condition (3) holds provided $\mathcal{A}$ is a tail sub-σ-field and P admits a r.c.d. μ given $\mathcal{A}$, even if the other assumptions of Lemma 7 fail. [In fact, such assumptions grant something more than (3)].

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