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MARTINGALE STRUCTURE OF SKOROHOD INTEGRAL PROCESSES

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Let the process $\{Y_t, t \in [0,1]\}$ have the form $Y_t = \delta(u\mathbf{1}_{[0,t]})$, where δ stands for a Skorohod integral with respect to Brownian motion and u is a measurable process that verifies some suitable regularity conditions. We use a recent result by Tudor to prove that Y_t can be represented as the limit of linear combinations of processes that are products of forward and backward Brownian martingales. Such a result is a further step toward the connection between the theory of continuous-time (semi)martingales and that of anticipating stochastic integration. We establish an explicit link between our results and the classic characterization (owing to Duc and Nualart) of the chaotic decomposition of Skorohod integral processes. We also explore the case of Skorohod integral processes that are time-reversed Brownian martingales and provide an "anticipating" counterpart to the classic optional sampling theorem for Itô stochastic integrals.

1. Introduction. Let $(C_{[0,1]}, \mathcal{C}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$ be the canonical space, where $\mathbb{P}$ is the law of a standard Brownian motion started from zero, and write $X = \{X_t : t \in [0,1]\}$ for the coordinate process. In this paper we investigate some properties of *Skorohod integral processes* defined with respect to X, that is, measurable stochastic processes with the form

$$Y_t = \int_0^1 u_s \mathbf{1}_{[0,t]}(s)\, dX_s = \int_0^t u_s\, dX_s, \qquad t \in [0,1], \tag{1}$$

where $\{u_s : s \in [0,1]\}$ is a suitably regular (and not necessarily adapted) process that verifies

$$\mathbb{E}\left[\int_0^1 u_s^2\, ds\right] < +\infty \tag{2}$$

and the stochastic differential dX has to be interpreted in the Skorohod sense (as defined in [15]; see the discussion below, as well as [11] or [9], Chapters 1 and 3, for basic results concerning Skorohod integration). It is well known that if u_s is adapted to the natural filtration of X (denoted $\{\mathcal{F}_s : s \in [0,1]\}$) and satisfies (2), then Y_t is a stochastic integral process in the Itô sense (as defined, e.g., in [14]),

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and therefore Y_t is a square-integrable $\mathcal{F}_t$-martingale. In general, the martingale property of Y_t fails when u_s is not $\mathcal{F}_s$-adapted, and Y_t may have a path behavior that is very different from that of classical Itô stochastic integrals (see [1], for examples of anticipating integral processes with very irregular trajectories). However, Tudor [16] proved that the class of Skorohod integral processes (when the integrand u is sufficiently regular) coincides with the set of *Skorohod–Itô integrals*, that is, processes that admit the representation

$$Y_t = \int_0^t \mathbb{E}[v_s | \mathcal{F}_{[s,t]^c}]\, dX_s, \qquad t \in [0,1], \tag{3}$$

where v is measurable and satisfies (2), $\mathcal{F}_{[s,t]^c} := \mathcal{F}_s \vee \sigma\{X_1 - X_r : r \geq t\}$ and for each fixed t, the stochastic integral is in the usual Itô sense (indeed, for fixed t, X_s is a standard Brownian motion on $[0,t]$, with respect to the enlarged filtration $s \mapsto \mathcal{F}_{[s,t]^c}$).

The principal aim of this paper is to use representation (3) to provide an exhaustive characterization of Skorohod integral processes in terms of products of *forward* and *backward* Brownian martingales. In particular, we shall prove that a process Y_t has representation (1) [or, equivalently, (3)] if and only if Y_t is the limit, in an appropriate norm, of linear combinations of stochastic processes of the type

$$Z_t = M_t \times N_t, \qquad t \in [0,1],$$

where M_t is a centered (forward) $\mathcal{F}_t$-martingale and N_t is an $\mathcal{F}_{[0,t]^c}$-backward martingale (i.e., for any $0 \leq s < t \leq 1$, $N_t \in \mathcal{F}_{[0,t]^c}$ and $\mathbb{E}[N_s | \mathcal{F}_{[0,t]^c}] = N_t$). Such a representation accounts in particular for the well-known property of Skorohod integral processes (see, e.g., [9], Lemma 3.2.1),

$$\mathbb{E}[Y_t - Y_s | \mathcal{F}_{[s,t]^c}] = 0 \qquad \text{for every } s < t, \tag{4}$$

which, in the anticipating calculus, plays a somewhat analogous role as the martingale property in the Itô calculus. We will see in the subsequent discussion that our characterization of processes such as Y_t complements some classic results contained in [5], where the authors study the multiple Wiener integral expansion of Skorohod integral processes.

The paper is organized as follows. In Section 2 we introduce some notation and discuss preliminary issues concerning the Malliavin calculus. In Section 3 the main results of the paper are stated and proved. In Section 4 we establish an explicit link between our results and those contained in [5]. In Section 5 we concentrate on a special class of Skorohod integral processes, whose elements can be represented as *time-reversed Brownian martingales*, and we state sufficient conditions so that such processes are semimartingales in their own filtration. Finally, Section 6 discusses some relationships between processes such as (1) and stopping times.

2. Notation and preliminaries. Let $L^2([0,1], dx) = L^2([0,1])$ be the Hilbert space of square-integrable functions on $[0,1]$. In what follows, the notation

$$X = \{X(f) : f \in L^2([0,1])\}$$

indicates an *isonormal Gaussian process* on $L^2([0,1])$, that is, X is a centered Gaussian family indexed by the elements of $L^2([0,1])$, defined on some (complete) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that $\mathbb{E}[X(f)X(g)] = \int_0^1 f(x)g(x)\,dx$ for every $f, g \in L^2([0,1])$. We also introduce the standard Brownian motion $X_t = X(\mathbf{1}_{[0,t]})$, $t \in [0,1]$, and denote $L^2(\mathbb{P})$ the space of square-integrable functionals of X. The usual notation of Malliavin calculus is adopted throughout the sequel (see [9]): for instance, D and δ denote the (Malliavin) derivative operator and the Skorohod integral with respect to the Wiener process X. For $k \geq 1$ and $p \geq 2$, $\mathbb{D}^{k,p}$ denotes the space of k times differentiable functionals of X, endowed with the norm $\|\cdot\|_{k,p}$, whereas $\mathbb{L}^{k,p} = L^p([0,1]; \mathbb{D}^{k,p})$. Note that $\mathbb{L}^{k,p} \subset \mathrm{Dom}(\delta)$, the domain of δ. Now take a Borel subset A of $[0,1]$ and denote by $\mathcal{F}_A$ the σ-field generated by random variables with the form $X(f)$, where $f \in L^2([0,1])$ is such that its support is contained in A. We recall that if $F \in \mathcal{F}_A$ and $F \in \mathbb{D}^{1,2}$, then

$$D_t F(\omega) = 0 \qquad \text{on } A^c \times \Omega. \tag{5}$$

We will also need the integration by parts formula

$$\delta(Fu) = F\delta(u) - \int_{[0,1]} D_s F u_s \, ds \tag{6}$$

p.s.-$\mathbb{P}$, whenever $u \in \mathrm{Dom}(\delta)$ and $F \in \mathbb{D}^{1,2}$ are such that $\mathbb{E}(F^2 \int_{[0,1]} u_s^2\, ds) < \infty$.

Eventually, let us introduce, for further reference, the families of σ-fields

$$\begin{aligned} \mathcal{F}_t &= \sigma\{X_h : h \leq t\}, && t \in [0,1], \\ \mathcal{F}_{[s,t]^c} &= \sigma\{X_h : h \leq s\} \vee \sigma\{X_1 - X_h : h \geq t\}, && 0 \leq s < t \leq 1, \end{aligned}$$

and observe that, to simplify the notation, we will write $\mathcal{F}_{[0,t]^c} = \mathcal{F}_{t^c}$, so that $\mathcal{F}_{[s,t]^c} = \mathcal{F}_{t^c} \vee \mathcal{F}_s$.

3. Skorohod integral processes and martingales. Let $L_0^2(\mathbb{P})$ denote the space of zero mean square-integrable functionals of X. We write $Y \in \mathbf{BF}$ to indicate that the measurable stochastic process $Y = \{Y_t : t \in [0,1]\}$ can be represented as a finite linear combination of processes with the form

$$Z_t = \mathbb{E}[H_1 | \mathcal{F}_t] \times \mathbb{E}[H_2 | \mathcal{F}_{t^c}] = M_t \times N_t, \qquad t \in [0,1], \tag{7}$$

where $H_1 \in L_0^2(\mathbb{P})$ and $H_2 \in L^2(\mathbb{P})$. Note that M in (7) is a forward (centered) Brownian martingale, whereas N is a backward Brownian martingale. For every measurable process $G = \{G_t : t \in [0,1]\}$, we also introduce the notation

$$V(G) = \sup_{\pi} \mathbb{E}\left[\sum_{j=0}^{m-1} (G_{t_j} - G_{t_{j+1}})^2\right], \tag{8}$$

where π runs over all partitions of [0, 1] with the form $0 = t_0 < t_1 < \cdots < t_m = 1$. The following result shows that **BF** is in some sense dense in the class of Skorohod integral processes.

THEOREM 1. *Let $u \in \mathbb{L}^{k,p}$ with $k \geq 3$ and $p > 2$. Then there exists a sequence of processes*

$$\{Z_t^{(r)} : t \in [0, 1]\}, \qquad r \geq 1,$$

with the following properties:

(i) *For every r, $Z^{(r)} \in \mathbf{BF}$.*
(ii) *For every r, $Z_t^{(r)} = \int_0^t \mathbb{E}[v_\alpha^{(r)} | \mathcal{F}_{[\alpha,t]^c}]\, dX_\alpha$, $t \in [0, 1]$, where $v^{(r)} \in \mathbb{L}^{k-2,p}$.*
(iii) *For every r, $V(Z^{(r)}) < +\infty$ and $\lim_{r\to\infty} V(\delta(u\mathbf{1}_{[0,\cdot]}) - Z^{(r)}) = 0$.*

Note that points (i) and (iii) of Theorem 1 imply that $Z^{(r)}$ converges to $\delta(u\mathbf{1}_{[0,\cdot]})$ uniformly in $L^2(\mathbb{P})$. This implies that the convergence takes also place in the sense of finite-dimensional distributions. Before proving Theorem 1, we need to state two simple results.

LEMMA 2. *Fix $k \geq 1$ and $p \geq 2$. Let A_1 and A_2 be two disjoint subsets of [0, 1], and let $\mathcal{F}_{A_i}$, $i = 1, 2$, be the σ-field generated by random variables of the form $X(h\mathbf{1}_{A_i})$, $h \in L^2([0, 1])$. Suppose that $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$ and also $F \in \mathbb{D}^{k,p}$. Then F is the limit in $\mathbb{D}^{k,p}$ of linear combinations of smooth random variables of the type*

$$G = G_1 \times G_2, \tag{9}$$

where, for $i = 1, 2$, G_i is smooth and $\mathcal{F}_{A_i}$-measurable.

PROOF. By definition, every $F \in \mathbb{D}^{k,p}$ can be approximated in the space $\mathbb{D}^{k,p}$ by a sequence of smooth polynomial functionals of the type

$$P_m = p_{n_{(m)}}\big(X\big(h_1^{(m)}\big), \ldots, X\big(h_{n_{(m)}}^{(m)}\big)\big), \qquad m \geq 1,$$

where, for every m, $n_{(m)} \geq 1$, $p_{n_{(m)}}$ is a polynomial in $n_{(m)}$ variables and, for $j = 1, \ldots, n_{(m)}$, $h_j^{(m)} \in L^2([0, 1])$. It is also easily checked that $\mathbb{E}[P_m | \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}] \in \mathbb{D}^{k,p}$ for every m and, since $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$,

$$\mathbb{E}\big[P_m | \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\big] \to F$$

in $\mathbb{D}^{k,p}$. To conclude, it is sufficient to prove that every random variable of the kind

$$Z = \mathbb{E}\big[(X(h_1))^{k_1} \cdots (X(h_n))^{k_n} | \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}\big],$$

where $h_j \in L^2([0,1])$ and $k_j \geq 1$, can be represented as a linear combination of random variables such as (9). To see this, write $A_3 = [0,1]\backslash(A_1 \cup A_2)$ and use twice the binomial formula to obtain

$$
\begin{aligned}
(X(h_j))^{k_j} &= \sum_{l=0}^{k_j} \binom{k_j}{l} (X(h_j \mathbf{1}_{A_1}))^{k_j - l} (X(h_j \mathbf{1}_{A_2 \cup A_3}))^l \\
&= \sum_{l=0}^{k_j} \sum_{a=0}^{l} \binom{k_j}{l} \binom{l}{a} (X(h_j \mathbf{1}_{A_1}))^{k_j - l} (X(h_j \mathbf{1}_{A_2}))^{l-a} (X(h_j \mathbf{1}_{A_3}))^a,
\end{aligned}
$$

thus implying that the functional $(X(h_1))^{k_1} \cdots (X(h_n))^{k_n}$ is a linear combination of random variables of the type

$$
H = \prod_{j=1}^{n} (X(h_j \mathbf{1}_{A_1}))^{\gamma_{1,j}} (X(h_j \mathbf{1}_{A_2}))^{\gamma_{2,j}} (X(h_j \mathbf{1}_{A_3}))^{\gamma_{3,j}},
$$

where $\gamma_{i,j} \geq 0$, $j = 1, \dots, n$, $i = 1, 2, 3$. To conclude, use independence to obtain

$$
\begin{aligned}
&\mathbb{E}[H | \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}] \\
&\quad = \mathbb{E}\Biggl[\prod_{j=1}^{n} (X(h_j \mathbf{1}_{A_3}))^{\gamma_{3,j}}\Biggr] \times \prod_{j=1}^{n} (X(h_j \mathbf{1}_{A_1}))^{\gamma_{1,j}} \prod_{j=1}^{n} (X(h_j \mathbf{1}_{A_2}))^{\gamma_{2,j}}
\end{aligned}
$$

and, therefore, the desired conclusion. $\square$

REMARK. Suppose that $F = I_n^X(h)$, $n \geq 1$, where I_n^X stands for a multiple Wiener integral of order n. Then $F \in \mathbb{D}^{k,p}$ for every $k \geq 1$ and $p \geq 2$. Moreover, the isometric properties of multiple integrals imply that F can be approximated in $\mathbb{D}^{k,2}$, and therefore in $\mathbb{D}^{k,p}$ for every $p \geq 2$, by linear combinations of random variables with the form $H_n(X(h))$, where H_n is a Hermite polynomial of the nth order and h is an element of $L^2([0,1])$. In particular, if $F \in \mathcal{F}_{A_1} \vee \mathcal{F}_{A_2}$ as in the statement of Lemma 2, the arguments contained in the above proof entail that F is the limit in $\mathbb{D}^{k,p}$ of linear combinations of random variables of the type $G = G_1 \times G_2$, where, for $i = 1, 2$, G_i is an $\mathcal{F}_{A_i}$-measurable polynomial functional of order $\gamma_i \geq 0$ such that $\gamma_1 + \gamma_2 \leq n$.

The proof of the following result is trivial and is therefore omitted.

LEMMA 3. *Fix $k \geq 1$ and $p \geq 2$, as well as a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$. Then, for every finite collection $\{F_j : j = 1, \dots, n\}$ of elements of $\mathbb{D}^{k,p}$, the process*

$$
u_t = \sum_{j=0}^{n-1} F_j \mathbf{1}_{(t_j, t_{j+1})}(t)
$$

is an element $\mathbb{L}^{k,p}$. *Moreover, if* $F_j^m \to_{m\to+\infty} F_j$ *in* $\mathbb{D}^{k,p}$, *then, as* $m \to +\infty$, *the sequence of processes*

$$u_t^m = \sum_{j=0}^{n-1} F_j^m \mathbf{1}_{(t_j,t_{j+1})}(t)$$

converges to u *in* $\mathbb{L}^{k,p}$.

PROOF OF THEOREM 1. It is well known (see, e.g., [5]) that the process $t \mapsto Y_t = \delta(u\mathbf{1}_{[0,t]})$ is such that $V(Y) < +\infty$. Moreover, according to Proposition 1 in [16], Y admits the (unique) representation

$$Y_t = \int_0^t \mathbb{E}[v_\alpha | \mathcal{F}_{[\alpha,t]^c}]\, dX_\alpha, \qquad t \in [0,1], \tag{10}$$

where $v \in \mathbb{L}^{k-2,p}$. Now, for every partition π of the type $0 = t_0 < \cdots < t_n = 1$, we introduce the step process

$$v_t^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}[v_s | \mathcal{F}_{[t_i,t_{i+1}]^c}]\, ds \right) \mathbf{1}_{(t_i,t_{i+1})}(t), \qquad t \in [0,1], \tag{11}$$

and we recall that $v^\pi \in \mathbb{L}^{k-2,p}$ and that v^π converges to v in $\mathbb{L}^{k-2,p}$ whenever the mesh of π, denoted $|\pi|$, converges to zero. Now define $Y_t^\pi = \int_0^t \mathbb{E}[v_\alpha^\pi | \mathcal{F}_{[\alpha,t]^c}]\, dX_\alpha$. From the calculations contained in [16], Proposition 2, we deduce that

$$V(Y - Y^\pi) \le \|v - v^\pi\|_{1,2}^2 \tag{12}$$

and, therefore, that $V(Y^\pi) < +\infty$ and $V(Y - Y^\pi)$ converges to zero as $|\pi| \to 0$. Now fix a partition π and, for $i = 0, \ldots, n-1$, write

$$F_i^\pi := \frac{1}{t_{i+1}-t_i} \left(\int_{t_i}^{t_{i+1}} \mathbb{E}[v_s | \mathcal{F}_{[t_i,t_{i+1}]^c}]\, ds \right) \in \mathcal{F}_{[t_i,t_{i+1}]^c}. \tag{13}$$

Since for every i and every s such that $t_i \le s \le t_{i+1}$ and $s < t$,

$$\mathbb{E}[F_i^\pi | \mathcal{F}_{[s,t]^c}] = \mathbb{E}[F_i^\pi | \mathcal{F}_{[s,t]^c \cap [t_i,t_{i+1}]^c}] = \mathbb{E}[F_i^\pi | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}],$$

we obtain, using properties (6) and (5),

$$\begin{aligned} Y_t^\pi &= \sum_{i=0}^{n-1} \int_0^t \mathbf{1}_{[t_i,t_{i+1}]}(s) \mathbb{E}[F_i^\pi | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}]\, dX_s \\ &= \sum_{i=0}^{n-1} \mathbb{E}[F_i^\pi | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}] (X_{t\wedge t_{i+1}} - X_{t_i}) \mathbf{1}_{(t \ge t_i)} \\ &= \sum_{i=0}^{n-1} Z_t^{(\pi,i)}, \end{aligned}$$

where $Z_t^{(\pi,i)} = \mathbb{E}[F_i^\pi | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}](X_{t\wedge t_{i+1}} - X_{t_i})\mathbf{1}_{(t\geq t_i)}$. Now fix $i = 0,\dots,n-1$. Since F_i^π is $\mathcal{F}_{[t_i,t_{i+1}]^c}$-measurable and $F_i \in \mathbb{D}^{k-2,p}$, thanks to Lemma 2 in the special case $A_1 = (0,t_i)$ and $A_2 = (t_{i+1},1)$, the random variable F_i^π is the limit in the space $\mathbb{D}^{k-2,p}$ of a sequence of random variables of the type

$$G_m^{(i,\pi)} = \sum_{k=1}^{M_m} G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)}, \qquad m \geq 1, \tag{14}$$

where, for every m, $M_m \geq 1$ and, for every k, $G_{m,k}^{(i,\pi,1)}$ and $G_{m,k}^{(i,\pi,2)}$ are smooth and such that $G_{m,k}^{(i,\pi,1)} \in \mathcal{F}_{t_i}$ and $G_{m,k}^{(i,\pi,2)} \in \mathcal{F}_{t_{i+1}^c}$. This implies, thanks to Lemma 3, that the process

$$v_t^{m,\pi} = \sum_{i=0}^{n-1} G_m^{(i,\pi)} \mathbf{1}_{(t_i,t_{i+1})}(t), \qquad t \in [0,1],$$

converges to v^π in $\mathbb{L}^{k-2,p}$ and, therefore, due to an inequality similar to (12), for every π the sequence of processes

$$\begin{aligned}
Y_t^{m,\pi} &= \sum_{i=0}^{n-1} \int_0^t \mathbf{1}_{[t_i,t_{i+1}]}(s) \mathbb{E}\big[G_m^{(i,\pi)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big]\, dX_s \\
&= \sum_{i=0}^{n-1} \mathbb{E}\big[G_m^{(i,\pi)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big]\big(X_{t\wedge t_{i+1}} - X_{t_i}\big)\mathbf{1}_{(t\geq t_i)} \\
&= \sum_{i=0}^{n-1} \sum_{k=1}^{M_m} \mathbb{E}\big[G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big]\big(X_{t\wedge t_{i+1}} - X_{t_i}\big)\mathbf{1}_{(t\geq t_i)} \\
&= \sum_{i=0}^{n-1} \sum_{k=1}^{M_m} U_t^{(m,k,\pi,i)}, \qquad m \geq 1,
\end{aligned}$$

is such that $V(Y^{m,\pi}) < +\infty$ and $\lim_{m\to+\infty} V(Y^\pi - Y^{m,\pi}) = 0$. We now show that $U^{(m,k,\pi,i)} \in \mathbf{BF}$. As a matter of fact,

$$\begin{aligned}
U_t^{(m,k,\pi,i)} &= \mathbb{E}\big[G_{m,k}^{(i,\pi,1)} G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big]\big(X_{t\wedge t_{i+1}} - X_{t_i}\big)\mathbf{1}_{(t\geq t_i)} \\
&= \big[G_{m,k}^{(i,\pi,1)}\big(X_{t\wedge t_{i+1}} - X_{t_i}\big)\mathbf{1}_{(t\geq t_i)}\big] \times \mathbb{E}\big[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big] \\
&= M_t \times N_t.
\end{aligned} \tag{15}$$

Eventually, observe that $M_t = \int_0^t H_s\, dX_s$, where $H_s = G_{m,k}^{(i,\pi,1)}\mathbf{1}_{(t_i,t_{i+1})}(s)$, and therefore, since H_s is $\mathcal{F}_s$-predictable, M_t is a Brownian martingale such that $M_0 = 0$. On the other hand,

$$\begin{aligned}
N_t &= \mathbb{E}\big[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big] = \mathbb{E}\big[\mathbb{E}\big[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{t_{i+1}^c}\big] | \mathcal{F}_{[t_i,t_{i+1}\vee t]^c}\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{t_{i+1}^c}\big] | \mathcal{F}_{(t_{i+1}\vee t)^c}\big] = \mathbb{E}\big[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{(t_{i+1}\vee t)^c}\big]
\end{aligned} \tag{16}$$

and also

$$
\begin{aligned}
N_t &= \mathbb{E}[\mathbb{E}[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{t_{i+1}^c}] | \mathcal{F}_{(t_{i+1}\vee t)^c}] \\
&= \mathbb{E}[\mathbb{E}[G_{m,k}^{(i,\pi,2)} | \mathcal{F}_{t_{i+1}^c}] | \mathcal{F}_{t^c}] \\
&= \mathbb{E}[N_0 | \mathcal{F}_{t^c}],
\end{aligned} \tag{17}
$$

so that N_t is a backward martingale such that $N_1 = \mathbb{E}[G_{m,k}^{(i,\pi,2)}]$. As a consequence, we obtain that $U^{(m,k,\pi,i)}$, and therefore $Y^{m,\pi}$, is an element of **BF**. We have therefore shown that for every $r \geq 1$ there exists a partition $\pi(r)$ and a number $m(r, \pi(r))$ such that $V(Y - Y^{\pi(r)}) \leq 1/(4r)$ and also $V(Y^{\pi(r),m(r,\pi(r))} - Y^{\pi(r)}) \leq 1/(4r)$. To conclude, set $Z^{(r)} := Y^{\pi(r),m(r,\pi(r))}$ and observe that

$$
V(Y - Z^{(r)}) \leq 2[V(Y - Y^{\pi(r)}) + V(Y^{\pi(r),m(r,\pi(r))} - Y^{\pi(r)})] \leq \frac{1}{r}. \qquad \square
$$

The next result contains a converse to Theorem 1.

THEOREM 4. *Let the sequence $Z^{(n)} \in$ **BF**, $n \geq 1$, be such that $V(Z^{(n)}) < +\infty$ and*

$$
\lim_{n,m\to+\infty} V(Z^{(n)} - Z^{(m)}) = 0.
$$

Then there exists a process $\{Y_t : t \in [0, 1]\}$ such that:

(i) *Y_t admits a Skorohod integral representation*;
(ii) *$V(Y) < +\infty$ and $\lim_{n\to+\infty} V(Z^{(n)} - Y) = 0$.*

PROOF. We first prove point (ii). Consider the trivial partition $t_0 = 0$, $t_1 = 1$. Then the assumptions in the statement (remember that $Z_0^{(n)} = 0$) imply that $Z_1^{(n)}$ is a Cauchy sequence in $L^2(\mathbb{P})$. Moreover, since for every $t \in (0, 1)$,

$$
\lim_{n,m\to+\infty} \mathbb{E}[(Z_t^{(n)} - Z_t^{(m)})^2 + (Z_t^{(n)} - Z_t^{(m)} - (Z_1^{(n)} - Z_1^{(m)}))^2] = 0,
$$

we readily obtain that for every $t \in [0, 1]$ there exists $Y_t \in L^2(\mathbb{P})$ such that $Y_0 = 0$ and also $Z_t^{(n)} \to Y_t$ in $L^2(\mathbb{P})$. Now fix $\varepsilon > 0$. It follows from the assumptions that there exists $N \geq 1$ such that for every $n, m > N$ and for every partition $0 = t_0 < \cdots < t_M = 1$,

$$
\mathbb{E}\left[\sum_{j=0}^{M-1} ((Z_{t_{j+1}}^{(n)} - Z_{t_{j+1}}^{(m)}) - (Z_{t_j}^{(n)} - Z_{t_j}^{(m)}))^2\right] \leq \varepsilon
$$

and, therefore, letting m go to infinity, we obtain that for $n > N$,

$$
\sup_{\pi} \mathbb{E}\left[\sum_{j=0}^{M-1} ((Z_{t_{j+1}}^{(n)} - Y_{t_{j+1}}) - (Z_{t_j}^{(n)} - Y_{t_j}))^2\right] = V(Z^{(n)} - Y) \leq \varepsilon,
$$

which entails $\lim_{n\to+\infty} V(Z^{(n)} - Y) = 0$. To conclude the proof of (ii), observe that, for $n > N$ as before,

$$V(Y) \le 2\big(V\big(Y - Z^{(n)}\big) + V\big(Z^{(n)}\big)\big) \le 2\big(\varepsilon + V\big(Z^{(n)}\big)\big) < +\infty.$$

Thanks to Proposition 2.3 in [5], to show point (i) it is now sufficient to prove that for any $s < t$,

$$\mathbb{E}\big[Y_t - Y_s | \mathcal{F}_{[s,t]^c}\big] = 0,$$

which is easily proved by using L^2 convergence as well as the fact that for every process Z_t as in (7) we have

$$\mathbb{E}\big[Z_t - Z_s | \mathcal{F}_{[s,t]^c}\big] = N_t \mathbb{E}\big[M_t | \mathcal{F}_{[s,t]^c}\big] - M_s \mathbb{E}\big[N_s | \mathcal{F}_{[s,t]^c}\big] = 0. \qquad \square$$

4. Representation of finite chaos Skorohod integral processes. We say that the process $Y = \{Y_t : t \in [0,1]\}$ is a *finite chaos Skorohod integral process of order* $N \ge 0$ (written: $Y \in \mathbf{FS}_N$) if $Y_t = \delta(u\mathbf{1}_{[0,t]})$ for some Skorohod integrable process $u_\alpha(\omega) \in L^2([0,1] \times \Omega)$ such that, for each $\alpha \in [0,1]$, the random variable u_α belongs to $\bigoplus_{j=0,\dots,N} C_j$, where C_j represents the jth Wiener chaos associated to X. Note that if $Y \in \mathbf{FS}_N$, then, for each t, $Y_t \in \bigoplus_{j=0,\dots,N+1} C_j$. We also define $\mathbf{FS} = \bigcup_{N\ge 0} \mathbf{FS}_N$. The aim of this paragraph is to discuss the relationships between the results of the previous section and the representation of the elements of the class **FS** introduced in [5]. To this end, we need some further notation (note that our formalism is essentially analogous to that contained in the first part of [5]).

For every $M \ge 2$ and every $1 \le m \le M$, we write $\mathbf{j}_{(m)} \subset \{1,\dots,M\}$ to indicate that the vector $\mathbf{j}_{(m)} = (j_1,\dots,j_m)$ has integer-valued components such that $1 \le j_1 < j_2 < \cdots < j_m \le M$. Note that $\mathbf{j}_{(M)} = (1,\dots,M)$. We set $\mathbf{j}_{(0)} = \varnothing$ by definition and also, given $\mathbf{x}_M = (x_1,\dots,x_M) \in [0,1]^M$ and $\mathbf{j}_{(m)} = (j_1,\dots,j_m) \subset \{1,\dots,M\}$,

$$\mathbf{x}_{\mathbf{j}_{(m)}} := \big(x_{j_1},\dots,x_{j_m}\big), \qquad \mathbf{x}_{\mathbf{j}_{(0)}} := 0.$$

We use the following notation: (a) For every permutation $\sigma^M = \{\sigma(1),\dots,\sigma(M)\}$ of $\{1,\dots,M\}$, we set

$$\Delta_M^{\sigma^M} := \big\{(x_1,\dots,x_M) \in [0,1]^M : 0 < x_{\sigma(M)} < \cdots < x_{\sigma(1)} < 1\big\}$$

and also write

$$\Delta_M^{\sigma_0^M} := \Delta_M = \{(x_1,\dots,x_M) \in [0,1]^M : 0 < x_M < \cdots < x_1 < 1\}$$

for the simplex contained in $[0,1]^M$. (b) For every $m = 0,\dots,M$ and $\mathbf{j}_{(m)} \subset \{1,\dots,M\}$,

$$\Delta_M^{\mathbf{j}_{(m)}} := \Big\{(x_1,\dots,x_M) \in (0,1)^M : \max_{i\in\mathbf{j}_{(m)}} (x_i) < \min_{l\in\{1,\dots,M\}\setminus\mathbf{j}_{(m)}} (x_l)\Big\},$$

where $\max_{i\in\varnothing}(x_i) := 0$ and $\min_{l\in\varnothing}(x_l) := 1$. (c) For every $t \in [0,1]$ and every $\mathbf{j}_{(m)} \subset \{1,\dots,M\}$,

$$\Delta_M^{\mathbf{j}_{(m)}}(t) := \Big\{(x_1,\dots,x_M) \in (0,1)^M : \max_{i\in\mathbf{j}_{(m)}}(x_i) < t < \min_{l\in\{1,\dots,M\}\setminus\mathbf{j}_{(m)}}(x_l)\Big\}.$$

(d) For every $t \in [0,1]$,

$$A_{M,m}(t) = \bigcup_{\mathbf{j}_{(m)}\subset\{1,\dots,M\}} \Delta_M^{\mathbf{j}_{(m)}}(t).$$

REMARK. Note that $\Delta_M^{\mathbf{j}_{(0)}} = \Delta_M^{\mathbf{j}_{(M)}} = (0,1)^M$ and, in general, for every $m = 0,\dots,M$ and every $\mathbf{j}_{(m)} \subset \{1,\dots,M\}$,

$$\Delta_M^{\mathbf{j}_{(m)}} = \bigcup_{t\in\mathbb{Q}\cap(0,1)} \Delta_M^{\mathbf{j}_{(m)}}(t).$$

We have also the relationships

$$A_{M,M}(t) = \Delta_M^{\mathbf{j}_{(M)}}(t) = (0,t)^M, \qquad A_{M,0}(t) = \Delta_M^{\mathbf{j}_{(0)}}(t) = (t,1)^M.$$

Moreover, if $t \in \{0,1\}$ and $0 < m < M$, then $A_{M,m}(t) = \varnothing$.

The following result corresponds to properties (B1)–(B3) in [5].

PROPOSITION 5. *Fix $M \geq 2$ and $0 \leq m \leq M$, and let the previous notation prevail. Then* (i)

$$\bigcup_{\mathbf{j}_{(m)}\subset\{1,\dots,M\}} \Delta_M^{\mathbf{j}_{(m)}} = [0,1]^M, \qquad \textit{a.e.-Leb},$$

where Leb stands for Lebesgue measure; (ii) *if* $\mathbf{i}_{(m)}, \mathbf{j}_{(m)} \subset \{1,\dots,M\}$, *then* $\Delta_M^{\mathbf{j}_{(m)}} \cap \Delta_M^{\mathbf{i}_{(m)}} \neq \varnothing$ *if and only if* $\mathbf{i}_{(m)} = \mathbf{j}_{(m)}$; (iii) *for any $t \in [0,1]$, if $m \neq m'$ and $0 \leq m, m' \leq M$, then $A_{M,m}(t) \cap A_{M,m'}(t) = \varnothing$ and also*

$$\bigcup_{m=0,\dots,M} A_{M,m}(t) = [0,1]^M, \qquad \textit{a.e.-Leb}.$$

The next fact is a combination of Theorems 1.3 and 2.1 in [5], and gives a univocal characterization of the chaos expansion of the elements of **FS**. Note that in the following we will write $L_s^2([0,1]^k)$, $k \geq 2$, to indicate the set of symmetric functions on $[0,1]^k$ that are square integrable with respect to Lebesgue measure. Moreover, for any $k \geq 2$ and $f \in L_s^2([0,1]^k)$, the symbol $I_k^X(f)$ will denote the standard multiple Wiener–Itô integral (of order k) of f with respect to X (see, e.g., [9, 10] for definitions). We will also use the notation $L_s^2([0,1]) = L^2([0,1])$ and, for $f \in L^2([0,1])$, $I_1^X(f) = X(f)$.

THEOREM 6 (Duc and Nualart). *Let the above notation prevail and fix* $N \geq 0$. *Then the process* $Y = \{Y_t : t \in [0,1]\}$ *is an element of* $\mathbf{FS}_N$ *if and only if there exists a* (*unique*) *collection of kernels* $\{f_{l,q} : 1 \leq q \leq l \leq N+1\}$ *such that* $f_{l,q} \in L_s^2([0,1]^l)$ *for every* $1 \leq q \leq l \leq N+1$ *and*

$$Y_t = \sum_{l=1}^{N+1} \sum_{q=1}^{l} I_l^X\big(f_{l,q}\mathbf{1}_{A_{l,q}(t)}\big), \qquad t \in [0,1]. \tag{18}$$

Moreover, if condition (18) *is satisfied*,

$$\sum_{l=1}^{N+1} l! \sum_{q=0}^{l-1} \|f_{l,q} - f_{l,q+1}\|^2 \leq V(Y) < +\infty, \tag{19}$$

where $V(Y)$ *is defined according to* (8) *and* $f_{l,0} := 0$.

The link between the objects introduced in this paragraph and those of the previous section is given by the following lemma.

LEMMA 7. *Fix* $m, n \geq 0$ *and, for every* $r \geq 1$, *take a natural number* $M_r \geq 1$, *as well as two collections of kernels*

$$\{h_j^{(u,r)} : 1 \leq u \leq M_r;\, j = 1, \dots, m\}, \qquad \{g_i^{(u,r)} : 1 \leq u \leq M_r;\, i = 1, \dots, n\},$$

where $h_j^{(u,r)} \in L_s^2([0,1]^j)$ *and* $g_i^{(u,r)} \in L_s^2([0,1]^i)$ *for every* i, j, *and a set of real numbers*

$$\{b^{(u,r)} : 1 \leq u \leq M_r\}.$$

For every $t \in [0,1]$ *and* $r \geq 1$, *we define*

$$\begin{aligned} Z_t^{(r)} &:= \sum_{u=1}^{M_r} Z_t^{(u,r)} \\ &= \sum_{u=1}^{M_r} \left(\sum_{j=1}^{m} I_j^X\big(h_j^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes j}\big) \right) \times \left(b^{(u,r)} + \sum_{i=1}^{n} I_i^X\big(g_i^{(u,r)} \mathbf{1}_{(t,1)}^{\otimes i}\big) \right). \end{aligned} \tag{20}$$

Then (i) *for every* $r \geq 1$, $V(Z^{(r)}) < +\infty$; (ii) *if*

$$\lim_{r,r' \uparrow +\infty} V\big(Z^{(r)} - Z^{(r')}\big) = 0,$$

there exists a process $Y = \{Y_t : t \in [0,1]\}$ *such that*

$$Y_0 = 0, \qquad V(Y) < +\infty \quad \textit{and} \quad \lim_{r \uparrow +\infty} V\big(Z^{(r)} - Y\big) = 0 \tag{21}$$

and, moreover, there exists a unique collection of kernels $f_{l,q} \in L_s^2([0,1]^l)$ *such that, for every* $t \in [0,1]$, Y_t *admits the representation*

$$Y_t = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1 \le q \le l \wedge m} I_l^X(\mathbf{1}_{A_{l,q}(t)} f_{l,q}), \qquad t \in [0,1], \tag{22}$$

where, for every $k \ge 1$, *we adopt the notation* $\sum_{k \le q \le 0} := 0$. *In particular,* $Y \in \mathbf{FS}_{n+m-1}$.

PROOF. If m or n is equal to zero, the statement can be proved by standard arguments. Now suppose $n, m \ge 1$, and fix $r \ge 1$ and $u = 1, \dots, M_r$. The multiplication formula for multiple Wiener integrals yields

$$Z^{(u,r)} = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1 \le q \le l \wedge m} I_l^X\big(\widetilde{(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}) \otimes_0 (g_{l-q}^{(u,r)}} \mathbf{1}_{(t,1)}^{\otimes l-q})\big),$$

where $g_0^{(u,r)} := b^{(u,r)}$ and the tilde ($\tilde{}$) stands for symmetrization. Note that if $q = l$, then $l \le m$ and

$$I_l^X\big(\widetilde{(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}) \otimes_0 (g_{l-q}^{(u,r)}} \mathbf{1}_{(t,1)}^{\otimes l-q})\big) = b^{(u,r)} I_l^X\big((h_l^{(u,r)} \mathbf{1}_{A_{l,l}(t)})\big).$$

On the other hand, when $1 \le q < l$, for every $\mathbf{x}_l \in [0,1]^l$,

$$\begin{aligned}
&\widetilde{(h_q^{(u,r)} \mathbf{1}_{(0,t)}^{\otimes q}) \otimes_0 (g_{l-q}^{(u,r)}} \mathbf{1}_{(t,1)}^{\otimes l-q}) \\
&\quad = \binom{l}{q}^{-1} \sum_{\mathbf{j}_{(q)} \subset \{1,\dots,l\}} h_q^{(u,r)}(\mathbf{x}_{\mathbf{j}_{(q)}}) g_{l-q}^{(u,r)}(\mathbf{x}_{\{1,\dots,l\}\setminus \mathbf{j}_{(q)}}) \\
&\qquad\qquad \times \mathbf{1}_{[0,t)^q}(\mathbf{x}_{\mathbf{j}_{(q)}}) \mathbf{1}_{(t,1]^{l-q}}(\mathbf{x}_{\{1,\dots,l\}\setminus \mathbf{j}_{(q)}}) \\
&\quad = \binom{l}{q}^{-1} \mathbf{1}_{A_{l,q}(t)}(\mathbf{x}_l) \\
&\qquad\quad \times \sum_{\mathbf{j}_{(q)} \subset \{1,\dots,l\}} h_q^{(u,r)}(\mathbf{x}_{\mathbf{j}_{(q)}}) g_{l-q}^{(u,r)}(\mathbf{x}_{\{1,\dots,l\}\setminus \mathbf{j}_{(q)}}) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}(\mathbf{x}_l).
\end{aligned}$$

Since the function

$$\mathbf{x}_l \mapsto \sum_{\mathbf{j}_{(q)} \subset \{1,\dots,l\}} h_q^{(u,r)}(\mathbf{x}_{\mathbf{j}_{(q)}}) g_{l-q}^{(u,r)}(\mathbf{x}_{\{1,\dots,l\}\setminus \mathbf{j}_{(q)}}) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}(\mathbf{x}_l)$$

is symmetric, we immediately deduce that, for every $r \ge 1$, the family of random variables

$$\{Z_t^{(r)} : t \in (0,1)\},$$

as defined in (20), admits a representation of the form (22), namely

$$Z_t^{(r)} = \sum_{l=1}^{m+n} \sum_{(l-n)\vee 1\le q\le l\wedge m} I_l^X\big(\mathbf{1}_{A_{l,q}(t)} f_{l,q}^{(r)}\big), \tag{23}$$

where

$$f_{l,q}^{(r)}(\mathbf{x}_l) := \binom{l}{q}^{-1} \sum_{u=1}^{M_r} \sum_{\mathbf{j}_{(q)}\subset\{1,\dots,l\}} h_q^{(u,r)}(\mathbf{x}_{\mathbf{j}_{(q)}}) g_{l-q}^{(u,r)}(\mathbf{x}_{\{1,\dots,l\}\setminus\mathbf{j}_{(q)}}) \mathbf{1}_{\Delta_l^{\mathbf{j}_{(q)}}}(\mathbf{x}_l).$$

Point (i) in the statement now follows from Theorem 6 and (23). Now suppose that

$$\lim_{r,r'\to+\infty} V\big(Z^{(r)} - Z^{(r')}\big) = 0.$$

Then the existence of a process Y that satisfies (21) follows from the same arguments contained in the proof of Theorem 4. Moreover, relation (19) implies immediately that for every l and q, the family $\{f_{l,q}^{(r)} : r \ge 1\}$ is a Cauchy sequence in $L_s^2([0,1]^l)$. Since $Y_t = L^2\text{-}\lim_{r\to+\infty} Z_t^{(r)}$ for every t, the conclusion is obtained by standard arguments. $\square$

Now, for every $p \ge 0$, call $\mathbf{BF}_p$ the subset of the class $\mathbf{BF}$, as defined through (7), composed of processes of the form (20) and such that $n+m \le p$. We have, therefore, the following proposition:

PROPOSITION 8. *Fix $N \ge 0$ and consider a measurable process $Y = \{Y_t : t \in [0,1]\}$. Then the following conditions are equivalent*:

1. *There exists $Y \in \mathbf{FS}_N$.*
2. *There exists a sequence $Z^{(r)} \in \mathbf{BF}_{N+1}$, $r \ge 1$, such that* $\lim_{r\to+\infty} V(Z^{(r)} - Y) = 0$.

PROOF. The implication $2 \Longrightarrow 1$ is an immediate consequence of Lemma 7 and Theorem 6. To deal with the opposite direction, suppose that $Y_t = \delta(u\mathbf{1}_{[0,t]})$, $t \in [0,1]$, where $u_\alpha(\omega) \in L^2([0,1]\times\Omega)$ is such that, for every $\alpha \in [0,1]$, $u_\alpha \in \bigoplus_{j=0,\dots,N} C_j$. Note that $u \in \mathbb{L}^{k,p}$ for every $k \ge 1$ and $p > 2$, and we can, therefore, take up the same line of reasoning and notation as in the proof of Theorem 1. In particular, according to Proposition 1 in [16], we know that Y admits the representation $Y_t = \int_0^t \mathbb{E}[v_\alpha | \mathcal{F}_{[\alpha,t]^c}]\, dX_\alpha$, where the process $v_\alpha = u_\alpha + \int_0^\alpha D_\alpha u_s\, dX_s$, $\alpha \in [0,1]$, is also such that $v_\alpha \in \bigoplus_{j=0,\dots,N} C_j$ for every α. By linearity, this implies that for every partition $\pi = \{0 = t_0 < \cdots < t_n = 1\}$ the random variables F_i^π, $i = 0,\dots,n-1$, as defined in (13), are such that $F_i^\pi \in \bigoplus_{j=0,\dots,N} C_j$. According to the remark following Lemma 2, every F_i^π is the limit, say in $\mathbb{D}^{3,3}$, of a sequence of random variables with the form

$$G_m^{(i,m)} = \sum_{k=1}^{M_m} G_{m,k}^{(i,\pi,1)} \times G_{m,k}^{(i,\pi,2)}, \qquad m \ge 1,$$

where $M_m \geq 1$ for every m and also

$$G_{m,k}^{(i,\pi,1)} = a + \sum_{l=1}^{\gamma_1} I_l^X\big(h_l \mathbf{1}_{(0,t_j)^l}\big),$$

$$G_{m,k}^{(i,\pi,2)} = b + \sum_{r=1}^{\gamma_2} I_r^X\big(g_r \mathbf{1}_{(t_{j+1},1)^r}\big),$$

where all dependencies on i, π, m and k have been dropped in the second members, and $\gamma_1 + \gamma_2 \leq N + 1$. By using relations (16) and (17), we see immediately that the process $U_t^{(m,k,\pi,i)}$, $t \in [0, 1]$, is an element of $\mathbf{BF}_{N+1}$ and the conclusion is obtained as in the proof of Theorem 1. $\square$

5. Skorohod integrals as time-reversed Brownian martingales. Now fix $k \geq 3$ and $p > 2$, take $u \in \mathbb{L}^{k,p}$ and note $Y_t = \delta(u\mathbf{1}_{[0,t]})$. Suppose, moreover, that the process $v_\alpha \in \mathbb{L}^{k-2,p}$ that appears in formula (10) is such that $v_\alpha = D_\alpha F$ for some $F \in \mathbb{D}^{1,2}$ (we refer to [9], page 40) for a characterization of such processes in term of their Wiener–Itô expansion). Then, according to the *generalized Clark–Ocone formula* stated in [11],

$$Y_t = \int_0^t \mathbb{E}\big[D_\alpha F | \mathcal{F}_{[\alpha,t]^c}\big]\, dX_\alpha = F - \mathbb{E}[F | \mathcal{F}_{t^c}], \qquad t \in [0, 1]. \tag{24}$$

As made clear by the following discussion, a process of the type $Y_t = F - \mathbb{E}[F | \mathcal{F}_{t^c}]$ can be easily represented as a *time-reversed Brownian martingale*. The principal aim of this section is to establish sufficient conditions to have that Y_t is a semimartingale in its own filtration (the reader is referred to [16], for further applications of (24) to Skorohod integration).

To this end, for every $f \in L^2([0, 1])$ we define $\widehat{f}(x) = f(1 - x)$, so that the transformation $f \mapsto \widehat{f}$ is an isomorphism of $L^2([0, 1])$ into itself. Such an operator can be extended to the space $L_s^2([0, 1]^n)$—that is, the space of square-integrable and symmetric functions on $[0, 1]^n$—by setting

$$\widehat{f}_n(x_1, \dots, x_n) = f(1 - x_1, \dots, 1 - x_n)$$

for every $f_n \in L_s^2([0, 1]^n)$, thus obtaining an isomorphism of $L_s^2([0, 1]^n)$ into itself. We also set, for $f \in L^2([0, 1])$, $\widehat{X}(f) = X(\widehat{f})$ and eventually

$$\widehat{X} = \{\widehat{X}(f) : f \in L^2([0, 1])\}.$$

Of course, $\widehat{X}$ is an isonormal Gaussian process on $L^2([0, 1])$ and the random function

$$\widehat{X}_t = \widehat{X}\big(\mathbf{1}_{[0,t]}\big) = X_1 - X_{1-t}, \qquad t \in [0, 1],$$

is again a standard Brownian motion. As usual, given $n \geq 1$ and $h_n \in L_s^2([0, 1]^n)$, $I_n^X(h_n)$ and $I_n^{\widehat{X}}(h_n)$ stand for the multiple Wiener–Itô integrals of h_n, respectively, with respect to X and $\widehat{X}$ (see [9]). The following lemma will be useful throughout the sequel.

LEMMA 9. *Let $F \in L^2(\mathbb{P})$ have the Wiener–Itô expansion $F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^X(f_n)$. Then*

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^{\widehat{X}}(\widehat{f}_n).$$

PROOF. By density, one can consider functionals of the form $F = I_n^X(f^{\otimes n})$, $n \geq 1$, where $f \in L^2([0,1])$ and $f^{\otimes n}(x_1, \dots, x_n) = f(x_1) \cdots f(x_n)$. In this case, it is well known that $F = n! H_n(X(f))$, where H_n is the nth Hermite polynomial as defined in [9], Chapter 1, and therefore

$$F = n! H_n(\widehat{X}(\widehat{f})) = I_n^{\widehat{X}}(\widehat{f}^{\otimes n}) = I_n^{\widehat{X}}(\widehat{f^{\otimes n}}),$$

thus proving the claim. $\square$

We now introduce the filtration

$$\widehat{\mathcal{F}}_t = \sigma\{\widehat{X}_h : h \leq t\}, \qquad t \in [0,1].$$

Note that

$$\begin{aligned} \mathcal{F}_{[s,t]^c} &= \mathcal{F}_s \vee \widehat{\mathcal{F}}_{1-t}, \\ \mathcal{F}_{t^c} &= \widehat{\mathcal{F}}_{1-t}. \end{aligned} \tag{25}$$

PROPOSITION 10. *Let $\{Y_t : t \in [0,1]\}$ be a measurable process.*

1. *The following conditions are equivalent*:

(i) *There exists $F \in L^2(\mathbb{P})$ such that $Y_t = F - \mathbb{E}(F | \mathcal{F}_{t^c})$.*
(ii) *There exists a square-integrable $\widehat{\mathcal{F}}_t$-martingale $\{\widehat{M}_t : t \in [0,1]\}$ such that $Y_t = \widehat{M}_1 - \widehat{M}_{1-t}$.*
(iii) *There exists an $\widehat{\mathcal{F}}_\alpha$-predictable process $\{\widehat{\phi}_\alpha : \alpha \in [0,1]\}$ such that $\mathbb{E}(\int_0^1 \widehat{\phi}_\alpha^2 \, d\alpha) < +\infty$ and $Y_t = \int_{1-t}^1 \widehat{\phi}_\alpha \, d\widehat{X}_\alpha$.*
(iv) *There exist kernels $f_n \in L_s^2([0,1]^n)$, $n \geq 1$, such that*

$$Y_t = \sum_{n=1}^{\infty} I_n^X\big(f_n(1 - \mathbf{1}_{[t,1]}^{\otimes n})\big) = \sum_{n=1}^{\infty} I_n^{\widehat{X}}\big(\widehat{f}_n(1 - \mathbf{1}_{[0,1-t]}^{\otimes n})\big),$$

where the convergence of the series takes place in $L^2(\mathbb{P})$.

2. *Let any one of conditions* (i)–(iv) *be verified*, *let F be given by* (i) *and let the f_n's be given by* (iv). *Then*

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^X(f_n) = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^{\widehat{X}}(\widehat{f}_n).$$

3. *Under the assumptions of point* 2, *suppose moreover that* F *is an element of* $\mathbb{D}^{1,2}$ *and let* $\widehat{\phi}$ *be given by* (iii). *Then*

$$\widehat{\phi}_\alpha = \mathbb{E}[D_{1-\alpha}F(X)|\widehat{\mathcal{F}}_\alpha], \qquad \alpha \in [0,1], \tag{26}$$

where $DF(X)$ *is the usual Malliavin derivative of* F, *which is regarded as a functional of* X.

REMARK. Note that (26) appears also in [17], formula (4.4), where it is obtained by completely different arguments.

PROOF OF PROPOSITION 10. If (i) is verified, then (ii) holds thanks to (25), by defining $\widehat{M}_t = \mathbb{E}(F|\widehat{\mathcal{F}}_t)$. On the other hand, (ii) implies (iii) due to the predictable representation property of $\widehat{X}$. Of course, if (iii) is verified, then

$$\begin{aligned} Y_t &= \int_{1-t}^1 \widehat{\phi}_\alpha \, d\widehat{X}_\alpha \\ &= \int_0^1 \widehat{\phi}_\alpha \, d\widehat{X}_\alpha - \int_0^{1-t} \widehat{\phi}_\alpha \, d\widehat{X}_\alpha \\ &= F - \mathbb{E}[F|\widehat{\mathcal{F}}_{1-t}], \end{aligned}$$

where $F = \int_0^1 \widehat{\phi}_\alpha \, d\widehat{X}_\alpha$, thus proving the implication (iii) $\Longrightarrow$ (i). Now, let (i) be verified and let F have the representation

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n^X(f_n).$$

We can apply Lemma 1.2.4 in [9] to obtain that

$$\begin{aligned} Y_t &= \sum_{n=1}^{\infty} I_n^X(f_n) - \mathbb{E}\left[\sum_{n=1}^{\infty} I_n^X(f_n)|\mathcal{F}_{t^c}\right] \\ &= \sum_{n=1}^{\infty} I_n^X(f_n) - \sum_{n=1}^{\infty} I_n^X(f_n \mathbf{1}_{[t,1]}^{\otimes n}), \end{aligned} \tag{27}$$

thus giving immediately (i) $\Longrightarrow$ (iv) [the second equality in (iv) is a consequence of Lemma 9]. The opposite implication may be obtained by reading formula (27) backward. The proof of point 2 is now immediate. To deal with point 3, observe that if F is derivable in the Malliavin sense as a functional of X, then F is also derivable as a functional of $\widehat{X}$, and the two derivative processes must verify

$$D_\alpha F(\widehat{X}) = D_{1-\alpha}F(X), \qquad \text{a.e.-}d\alpha \otimes d\mathbb{P},$$

where $DF(\widehat{X})$ stands for the Malliavin derivative of F, which is regarded as a functional of $\widehat{X}$. As a matter of fact, let F_k be a sequence of polynomial functionals with the form $F_k = p(X(h_1), \dots, X(h_m))$, where p is a polynomial in m variables (note that p, m and the h_j's may, in general, depend on k), that converges to F in $L^2(\mathbb{P})$ and satisfies

$$\mathbb{E}\left[\int_0^1 \left(\sum_{j=1}^m \frac{\partial}{\partial x_j} p\big(X(h_1), \dots, X(h_m)\big) h_j(x) - D_x F(X)\right)^2 dx\right] \to 0.$$

Then $p(X(h_1), \dots, X(h_m)) = p(\widehat{X}(\widehat{h}_1), \dots, \widehat{X}(\widehat{h}_m))$ and also

$$\mathbb{E}\left[\int_0^1 \left(\sum_{j=1}^m \frac{\partial}{\partial x_j} p\big(\widehat{X}(\widehat{h}_1), \dots, \widehat{X}(\widehat{h}_m)\big) \widehat{h}_j(x) - D_{1-x} F(X)\right)^2 dx\right] \to 0,$$

thus immediately giving the desired conclusion. The proof of point 3 is achieved by using the Clark–Ocone formula (see [4] and [13]). $\square$

EXAMPLE. Let $F = H_n(X(h))$, where H_n is the nth Hermite polynomial and h is such that $\|h\| = 1$. Then, thanks to Proposition 10, point 2, the process $Y_t = F - \mathbb{E}[F | \mathcal{F}_{t^c}]$ has the representation

$$\begin{aligned} Y_t &= \frac{1}{n!}\big[I_n^X(h^{\otimes n}) - I_n^X\big(h^{\otimes n}\mathbf{1}_{[t,1]}^{\otimes n}\big)\big] \\ &= H_n(X(h)) - \|h\mathbf{1}_{[0,t]}\|^n I_n^X\left(\left(\frac{h\mathbf{1}_{[t,1]}}{\|h\mathbf{1}_{[t,1]}\|}\right)^{\otimes n}\right) \\ &= H_n(X(h)) - \|h\mathbf{1}_{[t,1]}\|^n H_n\left(\frac{X(h\mathbf{1}_{[t,1]})}{\|h\mathbf{1}_{[t,1]}\|}\right) \end{aligned} \tag{28}$$

as well as

$$Y_t = \int_{1-t}^1 \mathbb{E}[H_{n-1}(X(h)) | \widehat{\mathcal{F}}_\alpha] h(1-\alpha)\, d\widehat{X}_\alpha.$$

Formula (28) generalizes the obvious relationships (corresponding to the case $n = 1$ and $h = \mathbf{1}_{[0,1]}$)

$$X_1 - \mathbb{E}[X_1 | \mathcal{F}_{t^c}] = X_t = \widehat{X}_1 - \widehat{X}_{1-t}.$$

Given a filtration $\{\mathcal{G}_t : t \in [0,1]\}$ and two adapted, cadlag processes U_t, and V_t, we write $[U, V] = \{[U, V]_t : t \in [0,1]\}$ to indicate the *quadratic covariation process* of U and V (if it exists). This means that $[U, V]$ is the cadlag $\mathcal{G}_t$-adapted process of bounded variation such that, for every $t \in [0,1]$ and for every sequence of (possibly random) partitions of $[0,t]$—say $\tau_n = \{0 < t_{1,n} < \cdots < t_{M_n,n} = t\}$—with mesh tending to zero, the sequence

$$\lim_n \left[U_0 V_0 + \sum_{i=0}^{M_n - 1} \big(U_{t_{i+1,n}} - U_{t_{i,n}}\big)\big(V_{t_{i+1,n}} - V_{t_{i,n}}\big)\right] = [U, V]_t,$$

where the convergence is in probability and is uniform on compacts. The next result uses quadratic covariations to characterize processes of the form $t \mapsto (F - \mathbb{E}[F|\mathcal{F}_{t^c}])$ in terms of semimartingales.

PROPOSITION 11. *Let F and $\{Y_t : t \in [0,1]\}$ satisfy any one of conditions* (i)–(iv) *in Proposition* 10, *fix $k \geq 1$ and let $\widehat{\phi}_\alpha$, as in Proposition* 10 *point* 1(iii), *be cadlag and of the form*

$$\widehat{\phi}_\alpha = \Phi\big(\alpha; \widehat{X}\big(g_1 \mathbf{1}_{[0,\alpha]}\big), \dots, \widehat{X}\big(g_k \mathbf{1}_{[0,\alpha]}\big)\big),$$

where Φ is a measurable function on $[0,1] \times \Re^k$ and $g_j \in L^2([0,1])$, $j = 1, \dots, k$. If there exists the quadratic covariation process $[\widehat{\phi}, \widehat{X}]$, then Y_t is a semimartingale on $[0,1]$ in its own filtration and, moreover,

$$Y_t = \int_0^t \widehat{\phi}_{1-\alpha}\, dX_\alpha - [\widehat{\phi}, \widehat{X}]_1 + [\widehat{\phi}, \widehat{X}]_{1-t}. \tag{29}$$

PROOF. The proof is directly inspired by Theorem 3.3 in [6]. Let $t \in (0,1]$ and $\tau = \{1 - t = s_0 < \cdots < s_n = 1\}$ be a deterministic partition of $[1-t, 1]$. Then, when the mesh of τ converges to zero, Y_t is (uniformly) the limit in probability of

$$\sum_{i=0}^{n-1} \widehat{\phi}_{s_i} \big(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\big).$$

Now note that since $\widehat{X}(g_j \mathbf{1}_{[0,1-\alpha]}) = X(\widehat{g}_j) - X(\widehat{g}_j \mathbf{1}_{[0,\alpha]})$, $j = 1, \dots, k$, the process $\alpha \mapsto \widehat{\phi}_{1-\alpha}$ is left-continuous and adapted to the filtration

$$\mathcal{H}_\alpha = \sigma(X_h, h \leq \alpha) \vee \sigma\big(X_1, X(\widehat{g}_1), \dots, X(\widehat{g}_k)\big), \qquad \alpha \in [0,1].$$

Therefore, since X_t is classically an $\mathcal{H}_t$-semimartingale (see [2]), the stochastic integral in (29) is well defined as the limit in probability of the sequence

$$\sum_{i=0}^{n-1} \widehat{\phi}_{1-t_{i+1}} \big(X_{t_i} - X_{t_{i+1}}\big) = \sum_{i=0}^{n-1} \widehat{\phi}_{s_{i+1}} \big(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\big),$$

where $t_i = 1 - s_i$. Eventually, we observe that the finite variation process $t \mapsto [\widehat{\phi}, \widehat{X}]_1 - [\widehat{\phi}, \widehat{X}]_{1-t}$ is by definition the limit in probability (as the mesh of τ converges to zero) of

$$\sum_{i=0}^{n-1} \big(\widehat{\phi}_{s_{i+1}} - \widehat{\phi}_{s_i}\big)\big(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}\big)$$

and, therefore, it is an $\mathcal{H}_t$-semimartingale, being an adapted process of finite variation. To prove the adaptation, just observe that if $1 - t \leq s \leq 1$, then

$$\begin{aligned}\widehat{\phi}_s &= \Phi\big(\alpha; X(\widehat{g}_1) - X\big(\widehat{g}_1 \mathbf{1}_{[0,1-s]}\big), \dots, X(\widehat{g}_k) - X\big(\widehat{g}_k \mathbf{1}_{[0,1-s]}\big)\big) \\ &\in \sigma(X_h, h \leq t) \vee \sigma\big(X_1, X(\widehat{g}_1), \dots, X(\widehat{g}_k)\big).\end{aligned}$$

As a consequence of the above discussion, the quantity

$$Y_t - \int_0^t \widehat{\phi}_{1-\alpha}\, dX_\alpha + [\phi, \widehat{X}]_1 - [\phi, \widehat{X}]_{1-t}$$

is the limit in probability of

$$\sum_{i=0}^{n-1} \widehat{\phi}_{s_i}(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}) - \sum_{i=0}^{n-1} \widehat{\phi}_{s_{i+1}}(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}) + \sum_{i=0}^{n-1} (\widehat{\phi}_{s_{i+1}} - \widehat{\phi}_{s_i})(\widehat{X}_{s_{i+1}} - \widehat{X}_{s_i}),$$

which equals zero for every τ. To conclude, observe that Y_t is the sum of two $\mathcal{H}_t$-semimartingales: therefore, it is itself an $\mathcal{H}_t$-semimartingale and, consequently, by Stricker's theorem, it is a semimartingale in its own filtration. $\square$

Now we state a (classic) sufficient condition for the existence of the quadratic covariation process $[\widehat{\phi}, \widehat{X}]$.

PROPOSITION 12. *Under the assumptions and notation of Proposition 11, suppose that the function Φ is of class C^1 in $[0, 1] \times \Re^k$. Then the quadratic covariation process $[\widehat{\phi}, \widehat{X}]$ exists.*

PROOF. This is an application of Theorem 5 in [8], page 359. The vector

$$\gamma_\alpha := (\alpha, \widehat{X}_\alpha, \widehat{X}(g_1 \mathbf{1}_{[0,\alpha]}), \dots, \widehat{X}(g_k \mathbf{1}_{[0,\alpha]}))$$

is indeed a $(k+2)$-dimensional $\widehat{\mathcal{F}}_\alpha$-semimartingale. Now define

$$\Phi^*(\alpha, x_1, \dots, x_{k+1}) = \Phi(\alpha, x_2, \dots, x_{k+1}), \qquad (\alpha, x_1, \dots, x_{k+1}) \in [0, 1] \times \Re^{k+1}.$$

Since the assumptions imply that Φ^* is of class C^1 in $[0, 1] \times \Re^{k+1}$ and $\widehat{\phi}_\alpha = \Phi^*(\gamma_\alpha)$, the quadratic variation process $\alpha \mapsto [\widehat{\phi}, \widehat{\phi}]_\alpha$ exists, as do the processes $[\widehat{X}, \widehat{X}]$ and $[\widehat{\phi} + \widehat{X}, \widehat{\phi} + \widehat{X}]$. It follows that $[\widehat{\phi}, \widehat{X}]$ exists, thanks to the polarization identity

$$[\widehat{\phi}, \widehat{X}]_\alpha = \tfrac{1}{2}\{[\widehat{\phi} + \widehat{X}, \widehat{\phi} + \widehat{X}]_\alpha - [\widehat{X}, \widehat{X}]_\alpha - [\widehat{\phi}, \widehat{\phi}]_\alpha\}, \qquad \alpha \in [0, 1]. \qquad \square$$

6. Anticipating integrals and stopping times. For the sake of completeness, in this section we explore some links between Skorohod integral processes and the family of stopping times. Classically, the stopping times are strongly related to the martingale theory. For instance, fix a filtration $\mathcal{U}_t$ as well as a $\mathcal{U}_t$-stopping time T: it is well known from the *optional sampling theorem* (see, e.g., [3]) that, for any $\mathcal{U}_t$-martingale M_t, the stopped process $t \mapsto M_{T \wedge t}$ is again a martingale for the filtration $t \mapsto \mathcal{U}_{T \wedge t}$ of events determined prior to T. It is also well known that a stopped Itô integral at the stopping time T coincides with the Itô integral on the random interval $[0, T]$. In this section we prove a variant of the optional sampling theorem for Skorohod integral processes and we discuss what happens if one samples such a process at a random time. For a discussion in this direction, see also the paper by Nualart and Thieullen [12]. We keep the notation of the previous sections and consider anticipating integral processes given by

$$Y_t = \delta\big(u\mathbf{1}_{[0,t]}(\cdot)\big),$$

where $u\mathbf{1}_{[0,t]}$ belongs to $\mathrm{Dom}(\delta)$ for every $t \in [0,1]$. Given two stopping times S, T for the filtration $\mathcal{F}_t$, we denote by $\mathcal{F}_T$, respectively, $\mathcal{F}_S$, the σ-field of the events determined prior to T, respectively S.

We have the following *optional sampling theorem.*

PROPOSITION 13. *If S, T are $\mathcal{F}_t$-stopping times such that $S \le T$ a.s., it holds that*

$$E[Y_T - Y_S | \mathcal{F}_S] = 0. \tag{30}$$

PROOF. Let us first consider as in Karatzas and Shreve [7] two sequences of stopping times $(S_n)_n$, $(T_n)_n$ taking on a countable number of values in the dyadic partition of [0, 1] and such that $S_n \to S$, $T_n \to T$ and

$$S \le S_n, \qquad T \le T_n \quad \text{and} \quad S_n \le T_n.$$

As in [3], page 325, using the fact that the process $(\mathbb{E}(Y_t|\mathcal{F}_t))_t$ is a martingale, we can prove that $\int_A Y_{S_n}\,d\mathbb{P} = \int_A Y_{T_n}\,d\mathbb{P}$ for every $A \in \mathcal{F}_{S_n}$. We follow next the lines of the proof of Theorem 1.3.22 in [7], observing that the sequence $(Y_{S_n})_n$ is uniformly integrable. This is a consequence of the bound

$$\sup_t \mathbb{E}Y_t^2 \le \sup_t \big(\mathbb{E}(Y_1 - Y_t)^2 + \mathbb{E}Y_t^2\big) \le V(Y). \qquad \square$$

The next result is a version of Theorem 2.5 of [12].

PROPOSITION 14. *Let $u \in \mathbb{L}^{1,p}$, $p > 4$, and let T be a stopping time for the filtration $\mathcal{F}_t$. Then $u\mathbf{1}_{[0,T]}$ belongs to* $\mathrm{Dom}(\delta)$ *and it holds that*

$$\delta\big(u\mathbf{1}_{[0,t]}\big)|_{t=T} = \delta\big(u\mathbf{1}_{[0,T]}\big). \tag{31}$$

PROOF. Since, for u as in the statement, the process $t \mapsto \int_0^t \mathbb{E}(u_s)\,dX_s$ is a continuous, square-integrable Gaussian $\mathcal{F}_t$-martingale, we can assume, without loss of generality, that $\mathbb{E}(u_t) = 0$ for every $t \in [0,1]$. We first prove property (31) for the approximation u^π given by (11):

$$u_t^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} E\big(u_s | \mathcal{F}_{[t_i,t_{i+1}]^c}\big)\,ds \right) \mathbf{1}_{[t_i,t_{i+1}]}(t).$$

Let us consider the sum

$$S = \sum_{i=0}^{n-1} F_i \big(X_{T\wedge t_{i+1}} - X_{T\wedge t_i}\big)$$
$$= \sum_{i=0}^{n-1} F_i \delta\big(\mathbf{1}_{[0,T]}\mathbf{1}_{[t_i,t_{i+1}]}\big),$$

where

$$F_i = \frac{1}{t_{i+1}-t_i}\bigg(\int_{t_i}^{t_{i+1}} E\big(u_s|\mathcal{F}_{[t_i,t_{i+1}]^c}\big)\,ds\bigg).$$

Using relation (6) [note that all hypotheses are satisfied, i.e., $F_i \in \mathbb{D}^{1,2}$, $\mathbf{1}_{[0,T]}\mathbf{1}_{[t_i,t_{i+1}]} \in \mathrm{Dom}(\delta)$, being adapted, and $\mathbb{E}(F^2 \int_0^1 \mathbf{1}_{[0,T]}(s)\mathbf{1}_{[t_i,t_{i+1}]}(s)\,ds) \le \mathbb{E}(F^2) < \infty$] and (5), we obtain that $u^\pi \mathbf{1}_{[0,T]} \in \mathrm{Dom}(\delta)$ and

$$\delta\big(u^\pi \mathbf{1}_{[0,T]}\big) = S = \sum_{i=0}^{n-1} F_i \big(X_{t\wedge t_{i+1}} - X_{t\wedge t_i}\big)|_{t=T}$$
$$= \delta\big(u^\pi \mathbf{1}_{[0,t]}\big)|_{t=T}.$$

Now recall that, for every partition π, the process u^π is an element of $\mathbb{L}^{1,p}$ and also, when $|\pi| \to 0$,

$$\begin{aligned} &u^\pi \to u && \text{in } \mathbb{L}^{1,p}, \\ &u^\pi \mathbf{1}_{[0,T]} \to u\mathbf{1}_{[0,T]} && \text{in } L^2([0,1]\times\Omega), \\ &\delta\big(u^\pi \mathbf{1}_{[0,t]}\big) \to \delta\big(u\mathbf{1}_{[0,t]}\big) && \text{in } L^2(\mathbb{P}) \text{ for every } t \in [0,1]. \end{aligned} \tag{32}$$

Fix a sequence of partitions π such that $|\pi| \to 0$. From (32), we deduce immediately that there exists a finite constant $K > 0$, not depending on π, such that

$$\int_0^1 \mathbb{E}\bigg[\bigg|\int_0^1 (D_s u_t^\pi)^2\,ds\bigg|^{p/2}\bigg]\,dt < K \qquad \text{for every } \pi.$$

Moreover, since $\mathbb{E}(u_t^\pi) = 0$ for every t, we can use the same line of reasoning as in the proof of Nualart [10], Proposition 5.1.1 and deduce the existence of a finite constant $K' > 0$ such that, for every $s, t \in [0,1]$ and every π,

$$\mathbb{E}\big[\big|\delta\big(u^\pi \mathbf{1}_{[0,t]}\big) - \delta\big(u^\pi \mathbf{1}_{[0,s]}\big)\big|^p\big] \le K' \times |t-s|^{p/2-1}.$$

As a consequence, by applying, for instance, [10], Lemma 5.3.1 and since T takes values in $[0,1]$ by construction, we deduce that, as $|\pi| \to 0$,

$$\delta\big(u^\pi \mathbf{1}_{[0,T]}\big) = \delta\big(u^\pi \mathbf{1}_{[0,t]}\big)|_{t=T} \to \delta\big(u\mathbf{1}_{[0,t]}\big)|_{t=T} \qquad \text{in } L^p(\mathbb{P}).$$

We conclude by the basic lemma for the convergence of Skorohod integrals that $u\mathbf{1}_{[0,T]} \in \mathrm{Dom}(\delta)$ and (31) holds. $\square$

REMARK. Note that in [12], Theorem 2.5, Nualart and Thieullen proved the relationship, for every $\mathcal{F}_t$-stopping time T and for every $u \in \mathrm{Dom}(\delta)$,

$$\delta(u\mathbf{1}_{[0,T]}) = \delta(u\mathbf{1}_{[0,t]})|_{t=T^+},$$

where $\delta(u\mathbf{1}_{[0,t]})|_{t=T^+}$ is defined as

$$\delta(u\mathbf{1}_{[0,t]})|_{t=T^+} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_T^{T+\varepsilon} \delta(u\mathbf{1}_{[0,s]})\, ds$$

when the above limit exists in $L^2(\mathbb{P})$. The obtention of result (31) is due to the use of the approximating processes (11) for which the limit can be explicitly computed. Note that, with our method, we do not need to introduce any special assumption on T. On the other hand, we are forced to assume a stronger hypothesis on the integrand u, that is, $u \in \mathbb{L}^{1,p}$, $p > 4$, instead of $u \in \mathrm{Dom}(\delta)$.

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