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CORRECTION

IMPROPER REGULAR CONDITIONAL DISTRIBUTIONS

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A strict inequality appears in Definition 6 where a weak inequality is needed. We reproduce Definition 6 here.

DEFINITION 6. Fix ω and consider those A such that $\omega \in A \in \mathcal{A}$. If for some $\omega \in A \in \mathcal{A}$, $P(A|\mathcal{A})(\omega) = 0$, say that $P(\cdot|\mathcal{A})$ is *maximally improper at* ω. Otherwise, if for each $\omega \in A \in \mathcal{A}$, $1 \geq P(A|\mathcal{A})(\omega) > 0$, say that the rcd is *modestly proper at* ω.

At the bottom of page 1614, we are not precise in the definition of a Borel space. The condition should have read that there is a one-to-one measurable function with measurable inverse between $(\Omega, \mathcal{B})$ and $(E, \mathcal{E})$, where E is a Borel subset of the reals and $\mathcal{E}$ is the Borel σ-field of subsets of E. After the remaining corrections below, our use of the term "Borel space" conforms with this definition.

Some conditions were left out of Theorem 4 and Lemma 3. The proof of Lemma 3 also had some errors that made it almost impossible to follow. Finally, the proof of Theorem 4 was said to be straightforward from Theorem 3. We include here the restatements of both results with the missing conditions, the revised proof of Lemma 3, and a proof of Lemma 4. The only application of Lemma 4 given in the original paper is to the proof of Corollary 2. The additional conditions given here are satisfied in that case.

THEOREM 4. *Assume that $\mathcal{A}$ is an atomic sub-σ-field of $\mathcal{B}$. Let $(\Theta, \mathcal{D})$ be a Borel space, with a probability measure μ. For each $\theta \in \Theta$, let P_θ be a probability on $\mathcal{B}$ such that for every $B \in \mathcal{B}$, $P_\theta(B)$ is a $\mathcal{D}$-measurable function of θ. Let $P(\cdot)$ be defined on $\mathcal{B}$ by $P(\cdot) = \int_\Theta P_\theta(\cdot)\, d\mu(\theta)$. Assume that, for μ-almost all θ, $P_\theta(\cdot|\mathcal{A})$ is a maximally improper rcd for P_θ and that it is $\mathcal{A} \otimes \mathcal{D}$-measurable as a function of (ω, θ). Also, assume that the set*

$$B^* = \{(\omega, \theta) : P_\theta(\cdot|\mathcal{A}) \text{ is maximally improper at } \omega\},$$

is in $\mathcal{A} \otimes \mathcal{D}$. Then there is a maximally improper version of $P(\cdot|\mathcal{A})$.

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LEMMA 3. *Let* $(\Theta, \mathcal{D})$ *be a Borel space, with a probability measure* μ. *For each* $\theta \in \Theta$, *let* P_θ *be a probability on* $\mathcal{B}$ *such that for every* $B \in \mathcal{B}$, $P_\theta(B)$ *is a* $\mathcal{D}$-*measurable function of* θ. *Define the probability* P *on* $\mathcal{B}$ *by* $P(B) = \int_\Theta P_\theta(B)\,d\mu(\theta)$. *Let* $\mathcal{A}$ *be a sub-*σ-*field of* $\mathcal{B}$. *Also, let* $P_\theta(\cdot|\mathcal{A})$ *denote an rcd for each* P_θ *that is* $\mathcal{A} \otimes \mathcal{D}$-*measurable as a function of* (ω, θ). *Then, for each* ω *there exists a probability* ν_ω *on* $\mathcal{D}$ *such that for all* $B \in \mathcal{B}$

$$\int_\Theta P_\theta(B|\mathcal{A})(\omega)\,d\nu_\omega(\theta) \tag{1}$$

is a version of $P(B|\mathcal{A})$. *Also, these versions form an rcd.*

PROOF. Let $\mathcal{E}$ be the product σ-field $\mathcal{B} \otimes \mathcal{D}$. For each $E \in \mathcal{E}$, define

$$E_\theta = \{\omega : (\omega, \theta) \in E\},$$
$$E^\omega = \{\theta : (\omega, \theta) \in E\},$$

the θ- and ω-sections of E. Standard arguments like those of Billingsley ([1], Section 18) allow us to conclude that $E_\theta \in \mathcal{B}$ for all θ, and $P_\theta(E_\theta)$ is a $\mathcal{D}$-measurable function of θ. Define

$$Q(E) = \int_\Theta P_\theta(E_\theta)\,d\mu(\theta),$$

which is easily seen to be a probability on $\mathcal{E}$. Let $\pi_1(\omega, \theta) = \omega$ and $\pi_2(\omega, \theta) = \theta$ be the coordinate projections, which are $\mathcal{E}$-measurable. Let $\mathcal{A}' = \pi_1^{-1}(\mathcal{A})$ and $\mathcal{D}' = \pi_2^{-1}(\mathcal{D})$, which are sub-$\sigma$-fields of $\mathcal{E}$. Every $\mathcal{A}'$-measurable function must be an $\mathcal{A}$-measurable function of π_1. Because $(\Theta, \mathcal{D})$ is a Borel space, there exists an rcd for π_2 given $\mathcal{A}'$ relative to Q, $Q(\cdot|\mathcal{A}')$. We will denote $Q(\pi_2^{-1}(D)|\mathcal{A}')(\omega, \theta)$ by $\nu_\omega(D)$. In similar fashion to the arguments earlier in the proof, $\nu_\omega(E^\omega)$ is $\mathcal{A}$-measurable as a function of ω for all $E \in \mathcal{E}$. Define

$$Q_0(E) = \int \nu_\omega(E^\omega)\,dP(\omega).$$

For each $A \in \mathcal{A}$ and $D \in \mathcal{D}$, we have

$$Q_0(A \times D) = \int I_A \nu_\omega(D)\,dP(\omega) = Q(A \times D).$$

It follows that $Q_0 = Q$ on all of $\mathcal{A} \otimes \mathcal{D}$.

For each ω, (1) is a probability. We need to show that it is $\mathcal{A}$-measurable as a function of ω. We have assumed that $P_\theta(\cdot|\mathcal{A})(\omega)$ is $\mathcal{A} \otimes \mathcal{D}$ measurable, so we can approximate it from below by a sequence $\{\phi_n\}_{n=1}^\infty$ of nonnegative simple functions. In similar fashion to the argument at the beginning of this proof, $\nu_\omega(E^\omega)$ is $\mathcal{A}$-measurable for all $E \in \mathcal{A} \otimes \mathcal{D}$. It follows that $\int \phi_n(\omega, \theta)\,d\nu_\omega(\theta)$ is $\mathcal{A}$-measurable for each n, and (1) is a limit of $\mathcal{A}$-measurable functions.

To complete the proof, we show that, for each $A \in \mathcal{A}$ and $\mathcal{B} \in \mathcal{B}$, the integral of (1) over A equals $P(A \cap B)$:

$$\begin{aligned}\int_A \int_\Theta P_\theta(B|\mathcal{A})(\omega)\, d\nu_\omega(\theta)\, dP(\omega) &= \int I_A(\omega) P_\theta(B|\mathcal{A})(\omega)\, dQ_0(\omega,\theta) \\ &= \int I_A(\omega) P_\theta(B|\mathcal{A})(\omega)\, dQ(\omega,\theta) \\ &= \int\int I_A(\omega) P_\theta(B|\mathcal{A})(\omega)\, dP_\theta(\omega)\, d\mu(\theta) \\ &= \int P_\theta(A\cap B)\, d\mu(\theta) = P(A\cap B),\end{aligned}$$

where the first equality is from the definition of Q_0, the second follows from the fact that $Q_0 = Q$ on $\mathcal{A} \otimes \mathcal{D}$, the third is from the definition of Q, the fourth is from the definition of $P_\theta(\cdot|\mathcal{A})$ and the last is the meaning of P_θ. $\square$

PROOF OF THEOREM 4. Because $\mathcal{A}$ is atomic, $P_\theta(\cdot|\mathcal{A})$ is maximally improper at ω if and only if $P_\theta(a(\omega)|\mathcal{A})(\omega) = 0$, where $a(\omega)$ is the $\mathcal{A}$-atom containing ω. Hence, we can rewrite the set B^* as

$$B^* = \{(\omega,\theta) : P_\theta(a(\omega)|\mathcal{A})(\omega) = 0\},$$

whose θ-sections satisfy

$$B^*_\theta = \{\omega : P_\theta(a(\omega)|\mathcal{A})(\omega) = 0\} \in \mathcal{B}.$$

For each θ such that $P_\theta(\cdot|\mathcal{A})$ is maximally improper, B^*_θ has inner P_θ measure 1. Hence $P_\theta(B^*_\theta) = 1$, a.e. $[\mu]$. By standard arguments, $P_\theta(B^*_\theta)$ is $\mathcal{D}$-measurable, and it follows that

$$Q(B^*) = \int_\Theta P_\theta(B^*_\theta)\, d\mu(\theta) = 1,$$

where Q was constructed in the proof of Lemma 3.

Similarly, the ω-sections of B^* satisfy

$$B^{*\omega} = \{\theta : P_\theta(a(\omega)|\mathcal{A})(\omega) = 0\} \in \mathcal{D}.$$

For each ω, let ν_ω be the measure from Lemma 3. Then $\nu_\omega(B^{*\omega})$ is $\mathcal{D}$-measurable. Since $B^* \in \mathcal{A} \otimes \mathcal{D}$, we have

$$1 = Q(B^*) = Q_0(B^*) = \int_\Omega \nu_\omega(B^{*\omega})\, dP(\omega),$$

where Q_0 was constructed in the proof of Lemma 3. So, there is a set $C \in \mathcal{B}$ with $P(C) = 1$ and for all $\omega \in C$, $\nu_\omega(B^{*\omega}) = 1$. It follows that, for each $\omega \in C$, there is a set $E(\omega) \in \mathcal{D}$ with $\nu_\omega(E(\omega)) = 1$ such that $P_\theta(a(\omega)|\mathcal{A})(\omega) = 0$ for all $\theta \in E(\omega)$. Let $P(\cdot|\mathcal{A})$ be the version guaranteed by Lemma 3. Then, for each $\omega \in C$,

$$P(a(\omega)|\mathcal{A})(\omega) = \int_\Theta P_\theta(a(\omega)|\mathcal{A})(\omega)\, d\nu_\omega(\theta) = 0.$$

This means that $P(\cdot|\mathcal{A})$ is maximally improper. $\square$

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