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UNICITY OF MEROMORPHIC FUNCTIONS OF CLASS $\mathcal{A}$

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Abstract. Jank and Terglane gave a unicity condition of three meromorphic functions of class $\mathcal{A}$. We generalize this unicity condition for arbitrary q meromorphic functions, and prove that the condition is sharp in the cases $q = 3$ and 4. Moreover, we provide a conjecture concerning this aspect.

1. Introduction

A meromorphic function f is of class $\mathcal{A}$ if it satisfies

$$\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f).$$

It includes all moermorphic functions f satisfy either $\delta(0, f) = \delta(\infty, f) = 1$ or $\Theta(0, f) = \Theta(\infty, f) = 1$. In this paper, we study the unicity condition of q distinct meromorphic functions of class $\mathcal{A}$. Let $f_1, f_2, \ldots, f_q$ be q non-constant meromorphic functions and a a complex number. Define $\overline{N}_0(r, a, f_1, f_2, \ldots, f_q)$ to be the reduced counting function of common zeros of $f_j(z) - a$, $1 \leq j \leq q$, and we will simply use the notation $\overline{N}_0(r, a)$ if it is clear what functions we are referring to. We denote by E the set of r in $(0, \infty)$ with finite linear measure which may be variant in different place and denote by $S(r, f)$ the quantity which is $o(T(r, f))$ as $r \to \infty$, $r \notin E$.

Given meromorphic functions $f_1, f_2, \ldots, f_q$ of class $\mathcal{A}$. Define the value τ as follows:

$$\tau = \varlimsup_{\substack{r \to \infty \\ r \notin E}} \frac{\overline{N}_0(r, 1)}{\sum_{j=1}^{q} T(r, f_j)}.$$

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The main goal of this paper is to study necessary conditions for τ to ensure that $f_1, f_2, \ldots, f_q$ are distinct. Brosch [1] proved the following result.

Theorem 1. *Let $f,\ g \in \mathcal{A}$, and*

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1,f,g)}{T(r,f)+T(r,g)} > \frac{1}{3}.$$

Then either $f(z) \equiv g(z)$ or $f(z)\cdot g(z) \equiv 1$.

By the theorem, we know that if f, g are distinct meromorphic functions of class $\mathcal{A}$ and $f(z)\cdot g(z) \not\equiv 1$, then we must have

$$\tau \leq \frac{1}{3}. \tag{1.1}$$

In the case of three meromorphic functions of class $\mathcal{A}$, Jank and Terglane [3] proved the following theorem.

Theorem 2. *Let $f,\ g,\ h \in \mathcal{A}$ be three distinct meromorphic functions. Then*

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)} \leq \frac{1}{4}.$$

Also, Jank and Terglance [3] gave an example to show that the result in Theorem 2 is sharp.

To generalize the discussion above, one can ask, given q meromorphic functions, what is the necessary condition for these meromorphic functions being distinct. Observe from the above theorems, for two meromorphic functions we have $\tau \leq \frac{1}{3}$, and $\tau \leq \frac{1}{4}$ for three meromorphic functions. It is reasonable to conjecture that if $f_j \in \mathcal{A}$, $1 \leq j \leq q$, are distinct, then $\tau \leq \frac{1}{q+1}$. In fact, we will get even better conclusion as in our main theorem.

Main Theorem. Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q}$$

when q is even, and

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{2}{3q-1}$$

when q is odd.

To prove this theorem, we need some facts from Nevanlinna theory. We will review these facts in the following section. Also, we assume that the reader is familiar with the Nevanlinna theory, and we will follow the standard notations of the Nevanlinna theory of meormorphic functions [2, 4].

2. Some Fact About Meoromrphic Functions of Class $\mathcal{A}$

In this section, we list some properties of meromorphic functions of class $\mathcal{A}$. We omit the proofs of the lemmas which can be found in [5].

Lemma 3. *Let $f \in \mathcal{A}$ and $k \in \mathbb{N}$. Then*

(1) $T(r, \frac{f^{(k)}}{f}) = S(r, f)$.

(2) $T(r, f^{(k)}) = T(r, f) + S(r, f)$.

(3) $f^{(k)} \in \mathcal{A}$.

Lemma 4. *Let $f \in \mathcal{A}$ and a be a finite non-zero number. Then*

$$\overline{N}_{1)}(r, \frac{1}{f-a}) = T(r, f) + S(r, f),$$

where $\overline{N}_{1)}(r, \frac{1}{f-a})$ denotes the reduced counting function of simple zeros of $f(z) - a$.

Lemma 5. *Let $f, g \in \mathcal{A}$ be distinct and $\Delta = (\frac{f''}{f'} - \frac{2f'}{f-1}) - (\frac{g''}{g'} - \frac{2g'}{g-1})$. If $\Delta \equiv 0$, then $f(z) \cdot g(z) \equiv 1$.*

3. Main Theorem

We are ready to prove the main theorem now.

Main Theorem. Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then

$$\tau = \varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r, 1)}{\sum_{j=1}^{q} T(r, f_j)} \leq \frac{2}{3q}$$

when q is even, and

$$\tau = \overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \le \frac{2}{3q-1}$$

when q is odd.

Proof. Set

$$\Delta_{ij} = \left(\frac{f_i''}{f_i'} - \frac{2f_i'}{f_i-1}\right) - \left(\frac{f_j''}{f_j'} - \frac{2f_j'}{f_j-1}\right),$$

where $1 \le i < j \le q$. If $\Delta_{ij} \not\equiv 0$, let z_0 be a simple zero of $f_i(z)-1$ and $f_j(z)-1$, then it is easy to see that z_0 is a zero of Δ_{ij}. Denote by $\overline{N}_{(2}(r, \frac{1}{f_k-1})$ the reduced counting function of the zeros of $f_k(z)-1$ with multiplicities ≥ 2. Then, by Lemma 3 and 4, we have

$$\begin{aligned}
\overline{N}_0(r,1,f_i,f_j) &\le N(r,\frac{1}{\Delta}) + N_{(2}(r,\frac{1}{f_i-1}) + N_{(2}(r,\frac{1}{f_j-1}) \\
&\le T(r,\Delta) + O(1) + S(r,f_i) + S(r,f_j) \\
&\le N(r,\Delta) + S(r,f_i) + S(r,f_j) \\
&\le \overline{N}(r,\frac{1}{f_i-1}) - \overline{N}_0(r,1,f_i,f_j) + \overline{N}(r,\frac{1}{f_j-1}) \\
&\quad -\overline{N}_0(r,1,f_i,f_j) + S(r,f_i) + S(r,f_j) \\
&\le T(r,f_i) + T(r,f_j) - 2\overline{N}_0(r,1,f_i,f_j) + S(r,f_i) + S(r,f_j).
\end{aligned}$$

Therefore,

$$3\overline{N}_0(r,1) \le 3\overline{N}_0(r,1,f_i,f_j) \le T(r,f_i) + T(r,f_j) + S(r,f_i) + S(r,f_j).$$

Now, assume that $q = 2n$ is even. If $\Delta_{ij} \equiv 0$ and $\Delta_{ik} \equiv 0$ for $j \ne k$, then, by Lemma 5, we get $f_j \equiv f_k$ which is impossible by assumption. Therefore, there are at most n of Δ_{ij} which are identically zero and we may assume that only Δ_{12}, $\Delta_{34}, \ldots, \Delta_{(q-1)q}$ may be identically zero. Apply the above inequality to all Δ_{ij} which are nonzero and add together, we obtain

$$\left(\binom{q}{2} - \frac{q}{2}\right) 3\overline{N}_0(r,1) \le (q-2)\sum_{j=1}^{q} T(r,f_j) + \sum_{j=1}^{q} S(r,f_j)$$

Hence,

$$\tau \le \frac{2n-2}{3[n(2n-1)-n]} = \frac{1}{3n} = \frac{2}{3q}.$$

Finally, we assume that $q = 2n + 1$ is odd. By the same argument as above, we may assume that only Δ_{12}, $\Delta_{34}, \ldots$, $\Delta_{(q-2)(q-1)}$ may be identically zero and obtain the following inequality

$$\left(\binom{q}{2} - \frac{q-1}{2}\right) 3\overline{N}_0(r,1) \le (q-2)\sum_{j=1}^{q-1} T(r,f_j) + (q-1)T(r,f_q) + \sum_{j=1}^{q} S(r,f_j).$$

Since

$$\overline{N}_0(r,1) \le \overline{N}(r,\frac{1}{f_j-1}) \le T(r,f_j) + O(1),\ 1 \le j \le q-1,$$

we have

$$(q-1)\overline{N}_0(r,1) \le \sum_{j=1}^{q-1} T(r,f_j) + O(1).$$

Combine these inequalities, we have

$$\left\{3\left(\binom{q}{2} - \frac{q-1}{2}\right) + (q-1)\right\}\overline{N}_0(r,1) \le (q-1)\sum_{j=1}^{q} T(r,f_j) + \sum_{j=1}^{q} S(r,f_j).$$

Therefore,

$$\tau \le \frac{2n}{\{3[n(2n+1)-n]+2n\}} = \frac{1}{3n+1} = \frac{2}{3q-1}. \qquad \blacksquare$$

Obviously, our Main Theorem generalizes Theorem 2. An easy consequence of the Main Theorem is the following corollary.

Corollary 6. *Let $f_j \in \mathcal{A}$, $1 \le j \le q$, be distinct, where $q \ge 3$. If $\tau > \frac{2}{3q}$ when q is even or $\tau > \frac{2}{3q-1}$ when q is odd, then at least two of f_j are the same.*

The inequality in the main theorem is sharp for $q = 3, 4$. When $q = 3$, the example can be found in [3]. When $q = 4$, let f_1, f_2, f_3, f_4 be the following functions

$$f_1(z) = e^z, f_2(z) = e^{-z}, f_3(z) = e^{2z}, \text{and } f_4(z) = e^{-2z}. \tag{3.1}$$

Clearly, they are meromorphic functions of class $\mathcal{A}$ and we have

$$\overline{N}_0(r,1) = \overline{N}(r,\frac{1}{f_1-1}) = T(r,f_1) + S(r,f_1),$$

where the first equality follows from the definition of f_j, $1 \le j \le 4$, and the second one follows from Lemma 4. Moreover,

$$\begin{aligned}&T(r,f_2) = T(r,f_1) + O(1),\ T(r,f_3) = 2T(r,f_1) + O(1),\\ &\text{and } T(r,f_4) = 2T(r,f_1) + O(1).\end{aligned}$$

Therefore,

$$\tau = \varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{4} T(r,f_j)} = \lim_{r\to\infty} \frac{T(r,f_1)+S(r,f_1)}{6T(r,f_1)+O(1)} = \frac{1}{6}.$$

4. A Conjecture

Our Main Theorem says that any q distinct meromorphic functions of class $\mathcal{A}$ must satisfies

$$\begin{cases} \tau \le \dfrac{2}{3q} & \text{if } q \text{ is even,} \\ \tau \le \dfrac{2}{3q-1} & \text{if } q \text{ is odd.} \end{cases}$$

For $q = 3, 4$, this result is sharp. But for $q \ge 5$, we don't know whether it is sharp or not. As the construction of the example (3.1), we can follow exact the same pattern to construct the following examples for $q \ge 5$:

$$f_1(z) = e^z, f_2(z) = e^{-z}, \ldots, f_{2n-1}(z) = e^{nz}, f_{2n}(z) = e^{-nz} \text{ if } q = 2n,$$

and

$$f_1(z) = e^z, f_2(z) = e^{-z}, \ldots, f_{2n}(z) = e^{-nz}, f_{2n+1} = e^{(n+1)z} \text{ if } q = 2n+1.$$

Apply the same arguments as above, we obtain that

$$\tau = \begin{cases} \dfrac{4}{q(q+2)} & \text{if } q \text{ is even,} \\ \dfrac{4}{(q+1)^2} & \text{if } q \text{ is odd.} \end{cases}$$

The numbers τ match the Main Theorem in the cases $q = 3, 4$, but less than the numbers in our Main Theorem. We conjecture that the examples actually provide the sharp conditions.

Conjecture. Let $f_1, f_2, \ldots, f_q$ be q distinct meromorphic functions of class $\mathcal{A}$, where $q \geq 3$. Then

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{4}{q(q+2)}$$

when q is even, and

$$\tau = \overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}_0(r,1)}{\sum_{j=1}^{q} T(r,f_j)} \leq \frac{4}{(q+1)^2}$$

when q is odd.

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