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# SOME FIXED POINT THEOREMS FOR HYBRID CONTRACTIONS IN UNIFORM SPACE

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**Abstract.** In this paper we prove new fixed point theorems for multi valued mappings with an implicit relations on complete uniform space.

## 1. Introduction

Uniform spaces form a natural extension of metric spaces. An exact analogue of the well-known Banach contraction principle in uniform spaces was obtained independently by Acharya [1], Gheorghiu [8] and Tarafdar [25]. Since then a number of fixed point theorems for single-valued and multi-valued mappings using various contractive conditions in this setting have been obtained ([2, 7, 12-17, 19, 25, 26, 28-33]). In this paper we first prove a fixed point theorem for a multi-valued mapping from an orbitally complete uniform space to its hyperspace. Subsequently, an application to locally convex spaces is also presented.

Let $(X, u)$ be a uniform space. A family $P = \{d_i : i \in I_0\}$ of pseudometrics on $X$ with indexing set $I_0$, is called an associated family for the uniformity $u$ if the family

$$\beta = \{V\,(i, r) : i \in I_0, r > 0\}$$

where

$$V(i, r) = \{(x, y) : x, y \in X, d_i(x, y) < r\}\,,$$

is a subbase for the uniformity $u$. We may assume $\beta$ itself to be base by adjoining finite intersections of members of $\beta$, if necessary. The corresponding family of pseudometrics is called an augmented associated family for $u$. An augmented

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associated family for $u$ will be denoted by $P^*$. For details the reader is referred to Tarafdar [25] and Thron [27]. Now onward, unless otherwise stated, $X$ will denote a uniform space $(X, u)$ defined by $P^*$.

Let $A$ be a nonempty subset of a uniform space $X$. Define

$$\Delta^*(A) = \sup\{d_i(x, y) : x, y \in A, i \in I_0\},$$

where $\{d_i : i \in I_0\} = P^*$. Then $\Delta^*(A)$ is called an augmented diameter of $A$. Further, $A$ is said to be $P^*$-bounded if $\Delta^*(A) < \infty$ (see [12]). Let

$$2^X = \{A : A \text{ is a nonempty } P^* - \text{bounded subset of } X\}.$$

For any nonempty subsets $A$ and $B$ of $X$, define

$$d_i(x, A) = \inf\{d_i(x, a) : a \in A, i \in I_0\}$$

$$\delta_i(A, B) = \sup\{d_i(a, b) : a \in A, b \in B, i \in I_0\}.$$

The function $\delta_i$ satisfies the following conditions

(i) $\delta_i(A, B) = \delta_i(B, A) \geq 0$, $\delta_i(A, B) = 0$ implies that $A = B$ and this set consists only one point.

(ii) $\delta_i(A, B) \leq \delta_i(A, C) + \delta_i(C, B)$ for $A, B, C \in 2^X$.

Also, if $A = \{a\}$ we write $\delta_i(A, B) = \delta_i(a, B)$ and furthermore $B = \{b\}$ we write $\delta_i(A, B) = \delta_i(a, b) = d_i(a, b)$.

A sequence $\{A_n\}$ of sets in $2^X$ is said to converge to the subset $A$ of $X$ if the following two conditions are satisfied:

(i) For each point in $a$ in $A$, there is a sequence $\{a_n\}$ such that $a_n \in A_n$ for all $n$ and $a_n \to a$.

(ii) For every $\varepsilon > 0$, there is an integer $N$ such that $A_n \subseteq A_\varepsilon$ for $n \geq N$, where

$$A_\varepsilon = \bigcup_{x \in A} U(x) = \{y \in X : d_i(x, y) < \varepsilon \text{ for some } x \text{ in } A,\ i \in I_0\}.$$

In such a case, $A$ is said to be limit of the sequence $\{A_n\}$ and we write $\lim_{n\to\infty} A_n = A$ or $A_n \to A$ as $n \to \infty$.

The mapping $F : X \to 2^X$ is said to be continuous at $x_0 \in X$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to $x$, the sequence $\{Fx_n\}$ in $2^X$ converges to $Fx$ in $2^X$. We say that $F$ is a continuous mapping of $X$ into $2^X$ if $F$ is continuous at each point $x$ in $X$.

The usual definition of a fixed point $x$ of a set valued mapping $F$ is that $x \in Fx$. A good reference, for theorems in this setting is the paper by [3, 5, 6, 16, 17, 24].

For $A, B \in 2^X$ we define

$$H_i(A, B) = \max\{\sup_{x \in A} d_i(x, B), \sup_{x \in B} d_i(x, A)\}.$$

Let $(X, u)$ be a uniform space and let $U \in u$ be an arbitrary entourage. For each subset $A$ of $X$, define

$$U[A] = \{y \in X : (x, y) \in U \text{ for some } x \in A\}.$$

The uniformity $2^u$ on $2^X$ is defined by the base

$$2^\beta = \{\tilde{U} : U \in u\}$$

where

$$\tilde{U} = \{(A, B) \in 2^X \times 2^X : A \times B \subseteq U\} \cup \Delta$$

(Here $\Delta$ denotes the diagonal of $X \times X$).

The augmented associated family $P^*$ also induces a uniformity $u^*$ on $2^X$ defined by the base

$$\beta^* = \{V^*(i, r) : i \in I_0, r > 0\},$$

where

$$V^*(i, r) = \{(A, B) \in 2^X \times 2^X : \delta_i(A, B) < \varepsilon\} \cup \Delta.$$

The uniformities $2^u$ and $u^*$on $2^X$ are uniformly isomorphic. The space $(2^X, u^*)$ is thus a uniform space called the hyperspace of $(X, u)$.

Let $S$ and $T$ be two self mapping of $(X, u)$. $S$ and $T$ to be weakly commuting if $d_i(STx, TSx) \le d_i(Tx, Sx)$ for all $x$ in $X$. $S$ and $T$ to be compatible if $\lim_n d_i(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_n Sx_n = \lim_n Tx_n = x$ for some $x \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implications are reversible.

**Definition 1.** Let $S : X \to 2^X$ be a set valued function and let $I : X \to X$ be a single valued function. We say that $S$ and $I$ commute weakly if

$$H_i(SIx, ISx) \le \delta_i(Ix, Sx)$$

for $x$ in $X$.

**Definition 2.** Let $S : X \to 2^X$ a set valued function and let $I : X \to X$ be a single valued function. We say that $S$ and $I$ are compatible if $\lim_n H_i(SIx_n, ISx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_n \delta_i(Ix_n, Sx_n) = 0$. In particular, $H_i(SIx, ISx) = 0$ if $\delta_i(Ix, Sx) = 0$ by taking $x_n = x$ for all $n$.

**Definition 3.** A set valued $S : X \to 2^X$ is said to be continuous if for any sequence $\{x_n\}$ in $X$ with $\{x_n\} \to x$ as $n \to \infty$, we have $\lim_n H_i(Sx_n, Sx) = 0$.

For fixed point theory of multi valued mappings, we refer to Hicks [9], Hicks and Rhoades [10] and references therein (also, see [18] for some related results).

## 2. Implicit Relations

Let $\Im$ be the set of real continuous functions $F(t_1, ..., t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ satisfying the following conditions:

(I1) F is nonincreasing in variables $t_2, ..., t_6$.

(I2) There exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with
[(Ia)] $F(u, v, v, u, u + v, 0) \leq 0$
or
[(Ib)] $F(u, v, u, v, 0, u + v) \leq 0$
we have $u \leq hv$.

(I3) $F(u, ..., u) > 0, \forall u > 0$.

**Example 1.** $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$ where $k \in (0, 1)$.

(I1) Obviously.

(I2) Let $u > 0$ be and $F(u, v, v, u, u + v, 0) = u - k \max\{v, v, u, \frac{1}{2}(u + v) \leq 0$. If $u \geq v$ then $u \leq ku < u$, a contradiction. Thus $u < v$ and $u \leq kv = hv$ where $h = k \in (0, 1)$. Let $u > 0$ and $F(u, v, u, v, 0, u + v) \leq 0$ then $u \leq hv$. If $u = 0$ then $u \leq hv$.

(I3) $F(u, ..., u) = u(1 - k) > 0, \forall u > 0$.

**Example 2.** $F(t_1, ..., t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, t_4 t_6\} - c_3 t_5 t_6$ where $c_1 > 0, c_2, c_3 \geq 0; c_1 + 2c_2 < 1$ and $c_1 + c_2 + c_3 < 1$.

(I1) Obviously.

(I2) Let $u > 0$ be and $F(u, v, v, u, u+v, 0) = u^2 - c_1 \max\{u^2, v^2\} - c_2 v(u+v) \leq 0$. If $u \geq v$ then $u^2(1 - c_1 - c_2) \leq 0$ which implies $c_1 + 2c_2 \geq 1$, a contradiction. Thus $u < v$ and $u \leq (c_1+2c_2)v = hv$, where $h = c_1+2c_2 < 1$. Let $u > 0$ and $F(u, v, u, v, 0, u + v) \leq 0$ then $u \leq hv$. If $u = 0$ then $u \leq hv$.

(I3) $F(u, ..., u) = u^2(1 - c_1 - c_2 - c_3) > 0, \forall u > 0$.

**Example 3.** $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - d(t_5 t_6)$, where $a > 0; b, c, d \geq 0$ and $a + b + c + d < 1$.

(I1) Obviously.

(I2) Let $u > 0$ be and $F(u, v, v, u, u + v, 0) = u^2 - u(av + bv + cu) \leq 0$ then $u \leq \frac{a+b}{1-c}v = h_1 v$ where $h_1 = \frac{a+b}{1-c} < 1$. Let $u > 0$ be and $F(u, v, u, v, 0, u + v) = uh_2 \leq -u(av + bu + cv) \leq 0$ which implies $u \leq \frac{a+c}{1-b}v = h_2 v$ where $h_2 = \frac{a+c}{1-b} < 1$. Therefore, $u \leq hv$ where $h = \max\{h_1, h_2\} < 1$. If $u = 0$ then $u \leq hv$.

(I3) $F(u, ..., u) = u^2(1 - a - b - c) > 0, \forall u > 0$.

**Example 4.** $F(t_1, ..., t_6) = t_1^3 - at_1^2 t_2 - bt_1 t_3 t_4 - ct_5^2 t_6 - dt_5 t_6^2$ where $a > 0; b, c, d \geq 0$ and $a + b + c + d < 1$.

(I1) Obviously.

(I2) Let $u > 0$ be and $F(u, v, v, u, u + v, 0) = u^3 - au^2 v - bu^2 v \leq 0$. Then $u \leq (a + b)v = hv$, where $h = a + b < 1$. If $u > 0$ and $F(u, v, u, v, 0, u + v) \leq 0$ then $u \leq hv$. If $u = 0$ then $u \leq hv$.

(I3) $F(u, ..., u) = u^3(1 - a - b - c - d) > 0, \forall u > 0$.

The purpose of this paper is to prove some fixed point theorems for hybrid contractions satisfying an implicit relations on uniform spaces.

## 3. Fixed Point Theorems

**Theorem 1.** *Let $(X, u)$ be a Hausdorff uniform space and let $I, J$ be two single valued mappings from $X$ into itself, $S, T : X \to 2^X$ be two set valued mappings satisfying the inequality*

(1) $F(\delta_i(Sx,Ty), d_i(Ix,Jy), \delta_i(Ix,Sx), \delta_i(Jy,Ty), d_i(Ix,Ty), d_i(Jy,Sx)) \leq 0$

*for all $x, y$ in $X$ and $i \in I_0$, where $F$ satisfies property I3. Then $S, T, I, J$ have at most one common fixed point.*

*Proof.* Let $y \in X$ be a common fixed point of $I, J, S$ and $T$. By (1) we have

$$F(\delta_i(Sy, Ty), d_i(Iy, Jy), \delta_i(Iy, Sy), \delta_i(Jy, Ty), d_i(Iy, Ty), d_i(Jy, Sy))$$

$$= F(\delta_i(Sy, Ty), 0, \delta_i(y, Sy), \delta_i(y, Ty), 0, 0) \leq 0$$

and thus

$$F(\delta_i(Sy, Ty), ..., (\delta_i(Sy, Ty)) \leq 0$$

a contradiction of I3 if $\delta_i(Sy, Ty) \neq 0$. Thus $\delta_i(Sy, Ty) = 0$. Since $y \in Sy$ and $y \in Ty$ then $Sy = Ty = \{y\}$.

Suppose that $I, J, S, T$ have to common fixed points $z$ and $y$. Then by (1) we have successively:

$$F(\delta_i(Sy,Tz), d_i(Iy,Jz), \delta_i(Iy,Sy), \delta_i(Jz,Tz), d_i(Iy,Tz), d_i(Jz,Sy)) \leq 0$$

$$F(d_i(y,z), d_i(y,z), 0, 0, d_i(y,z), d_i(y,z)) \leq 0$$

and

$$F(d_i(y,z), ..., d_i(y,z)) \leq 0$$

a contradiction of I3 if $y \neq z$. Thus $y = z$. ■

**Theorem 2.** *Let $(X, u)$ be a complete Hausdorff uniform space and let $I, J$ be two single valued mappings from $X$ into itself and $S, T : X \to 2^{X}$ be two set valued mappings satisfying the conditions:*

- $(a)$ *$S(X) \subset J(X)$ and $T(X) \subset I(X)$,*
- $(b)$ *$I$ or $J$ is continuous,*
- $(c)$ *$S$ and $I$ as well $T$ and $J$ are compatible,*
- $(d)$ *the inequality (1) holds for all $x, y$ in $X$ and $i \in I_0$, where $F \in \Im$, then $S, T, I$ and $J$ have a unique common fixed point $z$ in $X$. Moreover, $Sz = Tz = \{z\} = Iz = Jz$.*

*Proof.* Suppose $x_0$ an arbitrary point in $X$. Then since (a) holds we can define a sequence $\{x_n\}$ recursively as follows $Jx_{2n+1} \in Sx_{2n} = z_{2n}$; $Ix_{2n+2} \in Tx_{2n+1} = z_{2n+1}$.

Let $U \in u$ be an arbitrary entourage. Since $\beta$ is a base for $u$, there exists $V(i,r) \in \beta$ such that $V(i,r) \subseteq U$. By (1) we have successively

$$F(\delta_i(Sx_{2n}, Tx_{2n+1}), d_i(Ix_{2n}, Jx_{2n+1}), \delta_i(Ix_{2n}, Sx_{2n}), \delta_i(Jx_{2n+1}, Tx_{2n+1}),$$
$$d_i(Ix_{2n}, Tx_{2n+1}), d_i(Jx_{2n+1}, Sx_{2n})) \leq 0$$

$$F(\delta_i(z_{2n}, z_{2n+1}), \delta_i(z_{n-1}, z_{2n}), \delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n}, z_{2n+1}), \delta_i(z_{2n-1}, z_{2n+1}), 0) \leq 0$$

$$F(\delta_i(z_{2n}, z_{2n+1}), \delta_i(z_{n-1}, z_{2n}), \delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n}, z_{2n+1}),$$
$$\delta_i(z_{2n-1}, z_{2n}) + \delta_i(z_{2n}, z_{2n+1}), 0)) \leq 0.$$

By Ia we have

$$\delta_i(z_{2n}, z_{2n+1}) \leq h\delta_i(z_{2n-1}, z_{2n}).$$

Similarly, we have successively

$$F(\delta_i(Sx_{2n}, Tx_{2n-1}), d_i(Ix_{2n}, Jx_{2n-1}), \delta_i(Ix_{2n}, Sx_{2n}), \delta_i(Jx_{2n-1}, Tx_{2n-1}),$$
$$d_i(Ix_{2n}, Tx_{2n-1}), d_i(Jx_{2n-1}, Sx_{2n})) \leq 0$$

$$F(\delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n-1}, z_{2n-2}), \delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n-2}, z_{2n-1}), 0, \delta_i(z_{2n-2}, z_{2n})) \leq 0$$

$$F(\delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n-1}, z_{2n-2}), \delta_i(z_{2n-1}, z_{2n}), \delta_i(z_{2n-2}, z_{2n-1}), 0,$$
$$\delta_i(z_{2n-2}, z_{2n-1}) + \delta_i(z_{2n-1}, z_{2n})) \leq 0.$$

By Ib we have

$$\delta_i(z_{2n-1}, z_{2n}) \leq h\delta_i(z_{2n-2}, z_{2n-1})$$

and so $\delta_i(z_{2n}, z_{2n+1}) \leq h^{2n}\delta_i(x_0, Sx_0)$.

By a routine calculation follows that for $r > 0$, there is $n_0(r) \in \mathcal{N}$ such that for $m, n \geq n_0(r)$ we have $\delta_i(z_m, z_n) < r$.

Let $u_n \in z_n, u_m \in z_m$, since $d_i(u_n, u_m) \leq \delta_i(z_m, z_n) < r$ and hence $(u_n, u_m) \in U$ for all $n, m \geq n_0(r)$. Therefore the sequence $\{u_n\}$ is Cauchy sequence in the $d_i-$ uniformity on $X$.

Let $S_p = \{U_n : n \geq n_0(r)\}$ for all positive integer $n_0(r)$ and let $\beta$ be the filter basis $\{S_p : p = 1, 2, ...\}$. Then since $\{u_n\}$ is a $d_i-$ Cauchy sequence for each $i \in I_0$, it is easy to see that the filter basis $\beta$ is Cauchy filter in the uniform space $(X, u)$. To see this we first note that family $\{V(i, r) : i \in I_0, r > 0\}$ is a base $u$ as $P^* = \{d_i : i \in I_0\}$. Now, since $\{u_n\}$ is a $d_i-$ Cauchy sequence in $X$, there exists a positive integer $n_0(r)$ such that $d_i(u_n, u_m) < r$ for $m \geq n_0(r), n \geq n_0(r)$. This implies that $S_p \times S_p \subset V(i, r)$. Thus given any $U \in u$, we can find a $S_p \in \beta$ such that $S_p \times S_p \subset U$. Hence $\beta$ is a Cauchy filter in $(X, u)$. Since $(X, u)$ complete uniform space, the Cauchy filter $\beta = \{S_p\}$ converges to some point say $z$ in $X$. The point $z$ is independent of the choice of $u_n$. So $Ix_{2n} \to z, Jx_{2n+1} \to z, \delta_i(Sx_n, z) \to 0$ and $\delta_i(Tx_{2n+1}, z) \to 0$ as $n \to \infty$. Assume that $I$ is continuous. Then we have $I^2x_{2n} \to Iz$ and $\delta_i(ISx_n, Iz) \to 0$ as $n \to \infty$. Since $S$ and $I$ are compatible and $\delta_i(Ix_n, Sx_{2n}) \to 0$ as $n \to \infty$, we have

$$\delta_i(SIx_n, Iz) = H_i(SIx_{2n}, Iz) \leq H_i(SIx_{2n}, ISx_{2n}) + H_i(ISx_{2n}, Iz)$$
$$\leq H_i(SIx_{2n}, ISx_{2n}) + \delta_i(ISx_{2n}, Iz)$$

and so $\delta_i(SIx_{2n}, Iz) \to 0$ as $n \to \infty$.

For any $n \in \mathcal{N}$, we have from (1)

$$F(\delta_i(SIx_{2n}, Tx_{2n+1}), d_i(I^2x_{2n}, Jx_{2n+1}), \delta_i(I^2x_{2n+1}, SIx_{2n}), \delta_i(Jx_{2n+1}, Tx_{2n+1}),$$
$$d_i(I^2x_{2n}, Tx_{2n+1}), d_i(Jx_{2n+1}, SIx_{2n})) \leq 0.$$

As $n \to \infty$ we get

$$F(d_i(Iz, z), d_i(Iz, z), 0, 0, d_i(Iz, z), d_i(Iz, z)) \leq 0$$

which implies

$$F(d_i(Iz, z), d_i(Iz, z), d_i(Iz, z), d_i(Iz, z), d_i(Iz, z), d_i(Iz, z)) \leq 0$$

a contradiction of I3 if $d_i(Iz, z) \neq 0$. Thus $z = Iz$.

Similarly, for any $n \in \mathcal{N}$, we have

$$F(\delta_i(Sz, Tx_{2n+1}), d_i(Iz, Jx_{2n+1}), \delta_i(Iz, Sz), \delta_i(Jx_{2n+1}, Tx_{2n+1}),$$
$$d_i(Iz, Tx_{2n+1}), d_i(Jx_{2n+1}, Sz)) \leq 0.$$

As $n \to \infty$, we get

$$F(\delta_i(Sz, z), 0, \delta_i(z, Sz), 0, 0, \delta_i(z, Sz)) \leq 0.$$

By Ib follows that $\delta_i(Sz, z) = 0$ and thus $Sz = \{z\}$.

By $S(X) \subset J(X)$ there exists $w \in X$ such that $Jw = z \in Sz$. Then $TJw = Tz$. Now by (1) we have successively

$$F(\delta_i(Sz, Tw), d_i(Iz, Jw), \delta_i(Iz, Sz), \delta_i(Jw, Tw), d_i(Iz, Tw), d_i(Jw, Sz)) \leq 0$$
$$F(\delta_i(z, Tw), 0, 0, \delta_i(z, Tw), \delta_i(z, Tw), 0) \leq 0.$$

By Ia follows that $\delta_i(z, Tw) = 0$. Since $T$ and $J$ are compatible and $\delta_i(Tw, Jw) = 0$, we get $\delta_i(Tw, Jw) = H_i(TJw, JTw) = 0$. It implies $Tz = Jz$. By (1) we have successively

$$F(\delta_i(Sz, Tz), d_i(Iz, Jz), \delta_i(Iz, Sz), \delta_i(Jz, Tz), d_i(Iz, Tz), d_i(Jz, Sz)) \leq 0$$
$$F(\delta_i(z, Tz), \delta_i(z, Tz), 0, 0, \delta_i(z, Tz), \delta_i(z, Tz)) \leq 0$$
$$F(\delta_i(z, Tz), \delta_i(z, Tz), \delta_i(z, Tz), \delta_i(z, Tz), \delta_i(z, Tz), \delta_i(z, Tz)) \leq 0$$

a contradiction of I3 if $\delta_i(z, Tz) \neq 0$. Thus $\delta_i(z, Tz) = 0$ which implies $Tz = \{z\}$.

Hence the point $z$ is a common fixed point of $S, T, I$ and $J$ with $Sz = Tz = \{z\}$. By Theorem 1, $z$ is the unique common fixed point of $I, J, S$ and $T$. The proof for $J$ continuous is similar. ■

**Theorem 3.** *Let $(X, u)$ be a complete Hausdorff uniform space and let $I, J$ be mappings from $X$ into itself and for any $a \in A$, $S_a, T_a : X \to 2^X$ be set valued mappings with $\bigcup_{a\in A} S_a(X) \subset J(X)$ and $\bigcup_{a\in A} T_a(X) \subset I(X)$ such that*

$$F(\delta_i(S_a x, T_b y), d_i(Ix, Jy), \delta_i(Ix, S_a x), \delta_i(Jy, T_b y), d_i(Ix, T_b y), d_i(Jy, S_a x)) \leq 0 \tag{2}$$

*for all $x, y$ in $X$ and $i \in I_0$, $a, b \in A$ where $F \in \Im$. If for all $a$ in $A$, $S_a$ and $I$, $T_a$ and $J$ are compatible and if either $I$ or $J$ is continuous then $\{S_a\}_{a\in A}$, $\{T_a\}_{a\in A}$, $I$ and $J$ have a unique common fixed point $z$ in $X$. Moreover $S_a z = T_a z = \{z\}$ for all $a$ in $A$.*

*Proof.* Using the result of Theorem 2, we have that for any $a \in A$ there is a unique point $z_a$ in $X$ such that $Iz_a = Jz_a = \{z_a\}$ and $S_a z_a = T_a z_a = \{z_a\}$. Now for all $a, b$ in $A$, we have by (2)

$$\begin{aligned} & F(\delta_i(S_a z_a, T_b z_b), d_i(Iz_a, Jz_b), \delta_i(Iz_a, S_a z_a), \\ & \delta_i(Jz_b, T_b z_b), d_i(Iz_a, T_b z_b), d_i(Jz_b, S_a z_a)) \leq 0 \\ = & F(d_i(z_a, z_b), d_i(z_a, z_b), 0, 0, d_i(z_a, z_b), d_i(z_b, z_a)) \leq 0 \end{aligned}$$

and

$$F(d_i(z_a, z_b), d_i(z_a, z_b), d_i(z_a, z_b), d_i(z_a, z_b), d_i(z_a, z_b), d_i(z_b, z_a)) \leq 0$$

which implies by I3 that $z_a = z_b$. ■

**Theorem 4.** *Let $(X, u)$ be a complete Hausdorff uniform space and let $I, J$ be mappings from $X$ into itself $S, T : X \to 2^X$ be set valued mappings satisfying the conditions:*

- $(a)$ *$S(X) \subset J(X)$ and $T(X) \subset I(X)$,*
- $(b)$ *$S$ is continuous,*
- $(c)$ *$S$ and $I$ as well $T$ and $J$ are compatible,*
- $(d)$ *the inequality (1) holds for all $x, y$ in $X$,*
- $(e)$ *$\delta_i(Sx, Sx) \leq \delta_i(x, Sx)$ holds for all $x, y$ in $X$ and $i \in I_0$, then $S, T, I$ and $J$ have a unique common fixed point $z$ in $X$. Further $Sz = Tz = Iz = Jz = \{z\}$.*

*Proof.* Define the sequence $\{x_n\}$ as in Theorem 2 so that $Ix_{2n} \to z$, $Jx_{2n+1} \to z$, $\delta_i(Sx_{2n}, z) \to 0$ and $\delta_i(Tx_{2n+1}, z) \to 0$ as $n \to \infty$ and so $\delta_i(Sx_{2n}, Ix_{2n}) \to 0$ as $n \to \infty$. Since $S$ is continuous we have $H_i(SIx_{2n}, Sz) \to 0$ and $H_i(SJx_{2n+1}, Sz) \to 0$ as $n \to \infty$. now by the inequality

$$H_i(ISx_{2n}, Sz) \leq H_i(ISx_{2n}, SIx_{2n}) + H_i(SIx_{2n}, Sz)$$

and the fact that $S$ and $I$ are compatible we get $H_i(ISx_{2n}, Sz) \to 0$ as $n \to \infty$. Since $Jx_{2n+1} \in Sx_{2n}$, by (1) we have successively

$$\begin{gathered} F(\delta_i(SJx_{2n+1}, Tx_{2n+1}), d_i(IJx_{2n+1}, Jx_{2n+1}), \delta_i(IJx_{2n+1}, SJx_{2n+1}), \\ \delta_i(Jx_{2n+1}, Tx_{2n+1}), d_i(IJx_{2n+1}, Tx_{2n+1}), d_i(Jx_{2n+1}, SJx_{2n+1})) \leq 0 \\ F(\delta_i(SJx_{2n+1}, Tx_{2n+1}), \delta_i(ISx_{2n}, Tx_{2n+1}), \delta_i(ISx_{2n}, SJx_{2n+1}), \\ \delta_i(Jx_{2n+1}, Tx_{2n+1}), \delta_i(ISx_{2n}, Tx_{2n+1}), d_i(Jx_{2n+1}, SJx_{2n+1})) \leq 0. \end{gathered}$$

Passing the limit as $n \to \infty$ we get

$$F(\delta_i(Sz, z), \delta_i(Sz, z), \delta_i(Sz, z), 0, \delta_i(Sz, z), \delta_i(Sz, z)) \le 0$$

and by (e) and I1

$$F(\delta_i(Sz, z), \delta_i(Sz, z), \delta_i(Sz, z), \delta_i(Sz, z), \delta_i(Sz, z), \delta_i(Sz, z)) \le 0$$

a contradiction if $\delta_i(Sz, z) \ne 0$. It follows that $Sz = \{z\}$. Let $z'$ be a point in $X$ with $Jz' = z = Sz$ we have successively

$$F(\delta_i(SJx_{2n+1}, Tz'), d_i(IJx_{2n+1}, Jz'), \delta_i(IJx_{2n+1}, SJx_{2n+1}), \delta_i(Jz', Tz'),$$
$$d_i(IJx_{2n+1}, Tz'), d_i(Jx_{2n+1}, SJx_{2n+1})) \le 0$$
$$F(\delta_i(SJx_{2n+1}, Tz'), \delta_i(ISx_{2n}, Jz'), \delta_i(ISx_{2n}, Jz'), \delta_i(Jz', Tz'),$$
$$\delta_i(ISx_{2n}, Tz'), d_i(Jx_{2n+1}, SJx_{2n+1})) \le 0.$$

Then as $n \to \infty$ we get

$$F(\delta_i(z, Tz'), 0, 0, \delta_i(z, Tz'), \delta_i(z, Tz'), 0) \le 0$$

which implies by Ia that $\delta_i(z, Tz') = 0$. By the fact that $T$ and $J$ are compatible and $\delta_i(Tz', Jz') = \delta_i(z, z) = 0$ we have $H_i(JTz', TJz') = 0$ hence $JTz' = Jz = TJz' = Tz$. By (1) we have

$$F(\delta_i(Sx_{2n}, Tz), d_i(Ix_{2n}, Jz), \delta_i(Ix_{2n}, Sx_{2n}),$$
$$\delta_i(Jz, Tz), d_i(Ix_{2n}, Tz), d_i(Jz, Sx_{2n})) \le 0.$$

Then as $n \to \infty$ we get

$$F(\delta_i(z, Tz), \delta_i(z, Tz), 0, 0, \delta_i(z, Tz), \delta_i(z, Tz)) \le 0$$

which implies

$$F(\delta_i(z, Tz), ..., \delta_i(z, Tz)) \le 0$$

a contradiction of I3 if $\delta_i(z, Tz) \ne 0$. thus $Tz = \{z\} = Jz$.

Now select a point $z''$ in $X$ with $Iz'' = z = Tz$. Thus by (1) we have successively

$$F(\delta_i(Sz'', Tz), d_i(Iz'', Jz), \delta_i(Iz'', Sz''), \delta_i(Jz, Tz), d_i(Iz'', Tz), d_i(Jz, Sz'')) \le 0$$

$$F(\delta_i(Sz'', z), \delta_i(z, Sz), \delta_i(z, Sz''), \delta_i(z, z), \delta_i(z, z), \delta_i(z, Sz'')) \le 0$$

which implies

$$F(\delta_i(Sz'', z), ..., \delta_i(z, Sz'')) \le 0$$

a contradiction of I3 if $\delta_i(Sz'', z) \neq 0$. Thus $z = Sz''$, $ISz'' = Iz$ and $\delta_i(Sz'', Iz'') = 0$. Since $S$ and $I$ are compatible, so $H_i(ISz'', SIz'') = 0$ and $z = Sz = SIz'' = ISz'' = Iz$ that is $Iz = z$. This proves that the point $z$ is a common fixed point of $S, T, I$ and $J$ with $Sz = Tz = \{z\}$. The uniqueness of the common fixed point of $I, J, S$ and $T$ follows from Theorem 1. ■

**Remark 1.** If we replace the uniform space $(X, u)$ in Theorem 1-Theorem 4 by a metric space (i.e. a metricable uniform space), then Theorem 1-Theorem 4 of V. Popa and D. Türkoglu [18] will follow as special cases of our results.

## 4. Application to Locally Convex Spaces

Let $(X, \tau)$ be a locally convex linear topological space whose topology is $\tau$ generated by a family of seminorms $\{p_i : i \in I_0\}$ so that the collection

$$\{V(i, r) : \ i \in I_0, \ r > 0\},$$

where $V(i, r) = \{x \in X : p_i(x) < r\}$ is a neighborhood base for $\tau$. Then the family $P^* = \{\ p_i : i \in I_0\}$ is called an augmented associated family for $\tau$.

Now, for each $i \in I_0$, the function $d_i : X \times X \to \mathbb{R}$ defined by $d_i(x, y) = p_i(x - y)$ for all $x, y \in X$ is a pseudometric on $X$. Thus the family $P^* = \{p_i : i \in I_0\}$ determines a unique uniformity $u$ on $X$ and the uniform topology of $X$ coincides with the locally convex topology $\tau$ of the space (see Shaefer [23]).

For any nonempty subsets $A$ and $B$ of $X$, we have

$$\begin{aligned} d_i(x, A) &= \inf\{p_i(x - a) : a \in A, i \in I_0\}, \\ \delta_i(A, B) &= \sup\{p_i(a - b) : a, \ b \in A, i \in I_0\}. \end{aligned} \tag{3}$$

Then using an idea of Tarafdar [26] we have the following result as an application of Theorem 2-Theorem 4.

**Theorem 5.** *Let $I$, $J$ be single valued functions of a complete Hausdorff locally convex linear topological space $X$ into $X$ and $S$, $T$ be set valued functions of a locally convex linear topological $(X, \tau)$ into $2^X$ satisfying the conditions of Theorem [ 2 -4] with $d_i$ and $\delta_i$ as indicated above* (3). *Then $S, T, I, J$ have at most one common fixed point.*

**Remark 2.** When the results of Remark 1 and Examples 1,2,3,4 are considered, consequently, including some fixed point theorems can be obtained.

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