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APPROXIMATION OF ABSTRACT QUASILINEAR EVOLUTION EQUATIONS IN THE SENSE OF HADAMARD

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Abstract. An approximation theorem is given for abstract quasilinear evolution equations in the sense of Hadamard. A stability condition is proposed under which a sequence of approximate solutions converges to the solution. The result obtained in this paper is a generalization of an approximation theorem of regularized semigroups and is applied to an approximation problem for a degenerate Kirchhoff equation.

1. Introduction

This paper is devoted to an approximation theorem for the Cauchy problem of the quasilinear evolution equation

$$(\mathrm{QE};\, u_0) \qquad \begin{cases} u'(t) = A(u(t))u(t) & \text{for } t \in [0,T] \\ u(0) = u_0 \in D_0 \end{cases}$$

in a real Banach space X equipped with norm $\|\cdot\|_X$. Here $\{A(w); w \in D\}$ is a family of closed linear operators in X such that

$$(1.1) \qquad D(A(w)) \supset Y \quad \text{for } w \in D,$$

$$(1.2) \qquad A \text{ is strongly continuous on } D \text{ in } B(Y,E),$$

and D is a closed subset of Y which is continuously embedded in E. The spaces E and X_0 are real Banach spaces continuously embedded in X and D_0 is a subset of X_0 satisfying the following relation.

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$$
\begin{array}{ccccccc}
D & \subset & Y & \subset & E & \subset & X \\
\cup & & & & & & \cup \\
D_0 & & & \subset & & & X_0
\end{array}
\tag{1.3}
$$

According to the device due to Kato [13], let Z be another Banach space and S an operator in $B(Y,Z)$ such that there exists $c_S > 0$ satisfying the inequality

$$\|u\|_X + \|Su\|_Z \le c_S \|u\|_Y \qquad \text{for } u \in Y. \tag{1.4}$$

The Cauchy problem (QE; u_0) is said to be *well-posed in the sense of Hadamard* if for each $u_0 \in D_0$ there exists a unique solution u in the class $C([0,T];D) \cap C^1([0,T];E)$ satisfying the following continuous dependence of solutions on their initial data:

$$\|u(t) - v(t)\|_X \le M\|u(0) - v(0)\|_{X_0} \qquad \text{for } t \in [0,T].$$

For the autonomous case, there exists a vast literature on Hadamard well-posed problems, which are closely related with the theory of distribution semigroups. For instance, see Krein and Khazan [15] and Fattorini [8]. Recently, Hadamard well-posed problems were studied using the theory of integrated semigroups ([1, 2, 14, 22]) or regularized semigroups ([4-6, 18, 25]). The theory of regularized semigroups was also used to deal with generation theorems for various classes of semigroups and distribution semigroups in a unified way ([20, 25]). To extend the theory of regularized semigroups so that it may be applied to quasilinear equations, the well-posedness of the Cauchy problem (QE; u_0) in the sense of Hadamard was studied in [26], and the Kato theorem [12, 13] in the special case where $X_0 = E = X$ was also generalized.

It is natural to try to compute solutions numerically and to discuss the question of convergence which arises in that case. We are interested in studying such a problem in an operator-theoretical fashion. In the autonomous case, such a problem has been studied by interpreting as the problem of strong convergence of the semigroups generated by a given sequence of infinitesimal generators or the problem of approximation of a semigroup of operators by a sequence of discrete semigroups. The former is applied to the method of lines for concrete problems and the latter is closely related with finite difference approximations.

Both problems were discussed by Trotter [27], Chernoff [3], Kurtz [16] and Kato [11] for semigroups of class (C_0). These results were extended to the cases of several classes of semigroups ([7, 9, 24]). Although the case of integrated semigroups or regularized semigroups was discussed and the problem of strong convergence of the integrated semigroups or regularized semigroups generated by a given sequence of generators was studied intensively ([17, 21, 29]), a few attempt has been made to study the problem of approximation of an integrated semigroups or regularized semigroup by a sequence of discrete semigroups. To our knowledge, the case of local regularized semigroups was studied by Piskarev et. al. [23] to investigate an ill-posed problem. (See also Guidetti et. al. [10] and Melnikova et. al. [19].)

The purpose of this paper is to extend the above-mentioned results, by discussing an approximation theorem for the Cauchy problem of the quasilinear evolution equation (QE; u_0) in the sense of Hadamard. In fact, the final part of Section 2 contains an application of the main theorem (Theorem 1) to an approximation of local regularized semigroups.

To attain our objective, we consider an approximation of the solution of (QE; u_0) by the sequence $\{u_n\}$ of solutions of the problems

$$(u_n(t+h_n) - u_n(t))/h_n = A_n(u_n(t))u_n(t),$$

where $A_n(w)$ is an appropriate approximation to $A(w)$ and $\{h_n\}$ is a null sequence of positive numbers as $n \to \infty$. If a family $\{C_n(w); w \in D_n\}$ is defined by $C_n(w) = I + h_n A_n(w)$ for $w \in D_n$, then the solution u_n is given by $u_n(t) = u_{i,n}$ for $t \in [ih_n, (i+1)h_n) \cap [0, T]$ and $i = 0, 1, \ldots, K_n$, where $\{u_{i,n}\}_{i=0}^{K_n}$ is a sequence in D_n such that

$$u_{i,n} = C_n(u_{i-1,n})u_{i-1,n}$$

for $i = 1, 2, \ldots, K_n$ and K_n is the greatest integer such that $h_n K_n \leq T$. The feature of this paper is to propose the stability condition (H4) for the family $\{C_n(w); w \in D_n\}$ under which the sequence $\{u_n\}$ converges to the solution of (QE;u_0) as $n \to \infty$.

In Section 3, we give a key estimate (Lemma 1) on the difference between the solution of the Euler forward difference equation governed by a "quasilinear generator" B with time scale h and the solution of the quasilinear evolution equation governed by B. Section 4 presents an application of the main theorem to an approximation problem of a degenerate Kirchhoff equation.

2. Basic Hypotheses and the Main Theorem

In this section we make basic hypotheses with some comments and state the main theorem. The purpose of this paper is to discuss an approximation problem which arises when the solution of concrete problem is computed numerically. To do this, without discussing the solvability of the problem (QE;u_0) we concentrate on studying an approximation problem under the following hypothesis.

(H1) *For each* $u_0 \in D_0$, *the* (QE;u_0) *has a unique solution* $u \in C([0,T]; D) \cap C^1([0,T]; E)$.

The well-posedness of the Cauchy problem (QE;u_0) in the sense of Hadamard was studied in [26]. The special case where $X_0 = X = E$ corresponds to the Kato theory [12, 13]. An approximation theorem for Kato's quasilinear evolution equations may be derived from the main theorem (Theorem 1) by considering the special case where $X_{0,n} = X_n = E_n$ for $n \geq 1$ in the following setting.

The following hypothesis is an abstract version of the fact that a finite difference approximation to a differential operator in a space of functions defined on a domain in $\mathbb{R}^N$ acts on a different space like a space of discrete functions defined only at certain grid points. This idea is due to Trotter [27] and Kurtz [16].

(H2) *For each $n \geq 1$, there exist three Banach spaces E_n, X_n and $X_{0,n}$ and two subsets D_n and $D_{0,n}$ of X_n satisfying the relation*

$$\begin{array}{ccccc} D_n & \subset & E_n & \subset & X_n \\ \cup & & & & \cup \\ D_{0,n} & & \subset & & X_{0,n}, \end{array}$$

and another Banach space Z_n such that the following four conditions are satisfied:

(H2-i) *There exists a sequence $\{P_{X_n}\}$ of operators such that $P_{X_n} \in B(X, X_n)$ for $n \geq 1$ and $\lim_{n\to\infty} \|P_{X_n}u\|_{X_n} = \|u\|_X$ for $u \in X$.*

(H2-ii) *There exists a sequence $\{P_{E_n}\}$ of operators such that $P_{E_n} \in B(E, E_n)$ for $n \geq 1$ and $\lim_{n\to\infty} \|P_{E_n}u\|_{E_n} = \|u\|_E$ for $u \in E$.*

(H2-iii) *There exists a sequence $\{P_{X_{0,n}}\}$ of operators such that $P_{X_{0,n}} \in B(X_0, X_{0,n})$ for $n \geq 1$ and $\lim_{n\to\infty} \|P_{X_{0,n}}u\|_{X_{0,n}} = \|u\|_{X_0}$ for $u \in X_0$.*

(H2-iv) *There exists a sequence $\{P_{Z_n}\}$ of operators such that $P_{Z_n} \in B(Z, Z_n)$ for $n \geq 1$ and $\lim_{n\to\infty} \|P_{Z_n}u\|_{Z_n} = \|u\|_Z$ for $u \in Z$.*

In applications, the sets D_0 and D are taken as the set of initial data and the union of all positive orbits of solutions corresponding to the initial data, respectively. These sets need to be approximated in the following sense.

(H3) *There exists a sequence $\{S_n\}$ of operators such that $S_n \in B(E_n, Z_n)$ for $n \geq 1$ and the following conditions are satisfied:*

(H3-i) *For each $u \in D$, there exists a sequence $\{u_n\}$ such that $u_n \in D_n$ for $n \geq 1$, $\lim_{n\to\infty} \|P_{X_n}u - u_n\|_{X_n} = 0$ and $\lim_{n\to\infty} \|P_{Z_n}Su - S_n u_n\|_{Z_n} = 0$.*

(H3-ii) *For each $u \in D_0$, there exists a sequence $\{u_n\}$ such that $u_n \in D_{0,n}$ for $n \geq 1$, $\lim_{n\to\infty} \|P_{X_{0,n}}u - u_n\|_{X_{0,n}} = 0$, $\lim_{n\to\infty} \|P_{X_n}u - u_n\|_{X_n} = 0$ and $\lim_{n\to\infty} \|P_{Z_n}Su - S_n u_n\|_{Z_n} = 0$.*

The following is a stability condition proposed in this paper.

(H4) *Let* $\{h_n\}$ *be a null sequence of positive numbers as* $n \to \infty$. *For each* $n \geq 1$, *let* $\{C_n(w); w \in D_n\}$ *be a family in* $B(X_n)$ *satisfying the following conditions:*

(H4-i) *If* $x_0 \in D_{0,n}$, *then there exists a sequence* $\{x_i\}_{i=1}^{K_n}$ *in* D_n *such that* $x_i = C_n(x_{i-1})x_{i-1}$ *for* $1 \leq i \leq K_n$, *where* K_n *is the greatest integer such that* $h_n K_n \leq T$.

(H4-ii) *There exist* $M \geq 1$ *and* $p \geq 1$, *independent of* n, *such that if* $x_0 \in D_{0,n}$, $\{x_i\}_{i=1}^{K_n}$ *is a sequence in* D_n *satisfying* $x_i = C_n(x_{i-1})x_{i-1}$ *for* $1 \leq i \leq K_n$, $w_0 \in X_{0,n}$, $\{w_i\}_{i=1}^{K_n}$ *is a sequence in* X_n *satisfying* $w_i = C_n(x_{i-1})w_{i-1} + h_n f_i$ *for* $1 \leq i \leq K_n$ *and* $\{f_i\}_{i=1}^{K_n}$ *is a sequence in* E_n, *then the inequality*

$$\|w_i\|_{X_n}^p \leq M \left(\|w_0\|_{X_{0,n}}^p + h_n \sum_{l=1}^{i} \|f_l\|_{E_n}^p \right)$$

holds for $1 \leq i \leq K_n$.

(H4-iii) $C_n(w)(D_n) \subset E_n$ *for* $w \in D_n$.

(H4-iv) *There exists* $L \geq 0$, *independent of* n, *such that*

$$\|C_n(w)u - C_n(z)u\|_{E_n} \leq h_n L \|w - z\|_{X_n} (\|u\|_{X_n} + \|S_n u\|_{Z_n})$$

for $w, z, u \in D_n$.

The following is a consistency condition.

(H5) *If* $u \in D$ *and* $\{u_n\}$ *is a sequence such that* $u_n \in D_n$ *for* $n \geq 1$, $\lim_{n\to\infty} \|P_{X_n}u - u_n\|_{X_n} = 0$ *and* $\lim_{n\to\infty} \|P_{Z_n}Su - S_n u_n\|_{Z_n} = 0$, *then*

$$\lim_{n\to\infty} \|P_{E_n}u - u_n\|_{E_n} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|P_{E_n}A(u)u - A_n(u_n)u_n\|_{E_n} = 0,$$

where $A_n(w) = (C_n(w) - I)/h_n$ *for* $w \in D_n$ *and* $n \geq 1$.

The main theorem in this paper is given by

Theorem 1. *Assume* (1.1) *through* (1.4) *and* (H1) *through* (H5) *to be satisfied. Let* $u_0 \in D_0$ *and let* $\{u_{0,n}\}$ *be a sequence such that* $u_{0,n} \in D_{0,n}$ *for each* $n \geq 1$ *and* $\lim_{n\to\infty} \|P_{X_{0,n}}u_0 - u_{0,n}\|_{X_{0,n}} = 0$. *Then the following assertions hold.*

(i) *For each* $n \geq 1$, *there exists a sequence* $\{u_{i,n}\}_{i=1}^{K_n}$ *in* D_n *such that* $u_{i,n} = C_n(u_{i-1,n})u_{i-1,n}$ *for* $1 \leq i \leq K_n$.

(ii) *For each* $n \geq 1$*, define a step function* $u_n : [0,T] \to D_n$ *by*

$$u_n(t) = u_{i,n} \quad \textit{for } t \in [ih_n, (i+1)h_n) \cap [0,T] \textit{ and } i = 0,1,2,\ldots,K_n.$$

Then, it holds that

$$\lim_{n\to\infty} (\sup\{\|u_n(t) - P_{X_n}u(t)\|_{X_n}; t \in [0,T]\}) = 0.$$

Remark. The main theorem seems to be new, even if $X_{0,n} = X_n = E_n$ for all $n \geq 1$. This case gives an approximation theorem for Kato's quasilinear evolution equations.

We conclude this section by applying Theorem 1 to an approximation problem of local regularized semigroups by a sequence of discrete semigroups.

Let $C \in B(X)$ be injective and assume that C has the dense range $R(C)$. Let $\tau \in (0,\infty]$. A one parameter family $\{S(t); t \in [0,\tau)\}$ in $B(X)$ is a *local regularized semigroup on* X *with regularizing operator* C if the following conditions are satisfied:

(S1) $S(0) = C$ and $S(t)S(s) = S(t+s)C$ for $t,s \in [0,\tau)$ and $t+s \in [0,\tau)$.

(S2) For each $x \in X$, $S(\cdot)x : [0,\tau) \to X$ is continuous.

Let $\{S(t); t \in [0,\tau)\}$ be a local regularized semigroup on X with regularizing operator C. The operator A in X defined by

$$\begin{cases} Ax = C^{-1}\left(\lim_{h\downarrow 0}(S(h)x - x)/h)\right) & \text{for } x \in D(A) \\ D(A) = \{x \in X; \lim_{h\downarrow 0}(S(h)x - x)/h \text{ exists in } X \text{ and is in } R(C)\} \end{cases}$$

is called the *generator* of $\{S(t); t \in [0,\tau)\}$ and satisfies the following conditions:

(A1) A is a densely defined closed linear operator in X and $C^{-1}AC = A$.

(A2) For $u \in D(A)$, $S(t)u \in D(A)$, $AS(t)u = S(t)Au$ for $t \in [0,\tau)$ and $S(\cdot)u \in C([0,\tau); [D(A)]) \cap C^1([0,\tau); X)$, where $[D(A)]$ is the Banach space $D(A)$ equipped with the graph norm of A.

The definition of generators of regularized semigroups was first given by Da Prato [4]. Several types of characterizations of the generators of local regularized semigroups were given by [25] and [28]. The following approximation theorem of such regularized semigroups is a generalization of the Chernoff product formula [3]. Another type of approximation theorem is found in the paper due to Piskarev et. al. [23].

Theorem 2. *Let A be the generator of a local regularized semigroup $\{S(t); t \in [0,\tau)\}$ on X with regularizing operator C. Assume that X is approximated by a sequence $\{X_n\}$ of Banach spaces in the following sense: There exists a sequence $\{P_{X_n}\}$ such that $P_{X_n} \in B(X, X_n)$ for $n \geq 1$ and*

$$(2.1) \qquad \lim_{n\to\infty} \|P_{X_n} u\|_{X_n} = \|u\|_X \qquad \text{for } u \in X.$$

Assume that there exists a sequence $\{C_n\}$ such that $C_n \in B(X_n)$ is injective for $n \geq 1$ and

$$(2.2) \quad \lim_{n\to\infty} \|x_n - P_{X_n} x\|_{X_n} = 0 \text{ implies that } \lim_{n\to\infty} \|C_n x_n - P_{X_n} C x\|_{X_n} = 0.$$

For each $n \geq 1$, let $F_n \in B(X_n)$ satisfy the following conditions:

(F1) *For each $\sigma \in (0,\tau)$ there exists $M_\sigma > 0$, independent of n, such that*

$$\|F_n^i C_n u\|_{X_n} \leq M_\sigma \|u\|_{X_n} \qquad \text{for } 1 \leq i \leq K_{\sigma,n} := [\sigma/h_n] \text{ and } u \in X_n,$$

where $[a]$ is the integer part of a.

(F2) $C_n F_n = F_n C_n$.

Let $A_n = (F_n - I)/h_n$ for $n \geq 1$. Assume that for each $u \in D(A)$ there exists a sequence $\{u_n\}$ such that $u_n \in X_n$ for $n \geq 1$ and

$$(2.3) \qquad \lim_{n\to\infty} (\|u_n - P_{X_n} u\|_{X_n} + \|A_n u_n - P_{X_n} A u\|_{X_n}) = 0.$$

Then, for each $\sigma \in (0,\tau)$ and $u \in X$,

$$\lim_{n\to\infty} \Big(\sup\{\|F_n^{[t/h_n]} C_n P_{X_n} u - P_{X_n} S(t) u\|_{X_n}; t \in [0,\sigma]\}\Big) = 0.$$

Proof. Let Y be the Banach space $C(D(A))$ equipped with the norm $\|\cdot\|_Y$ defined by $\|u\|_Y = \|u\|_X + \|C^{-1}u\|_X + \|AC^{-1}u\|_X$ for $u \in Y$. Let Z be the Banach space $X \times X$ equipped with the norm $\|(u,v)\|_Z = \|u\|_X + \|v\|_X$ for $(u,v) \in Z$, and define $S \in B(Y,Z)$ by $Su = (C^{-1}u, AC^{-1}u)$ for $u \in Y$. Then, the inequality (1.4) holds for $c_S = 1$. Let E be the Banach space $R(C)$ equipped with the norm $\|\cdot\|_E$ defined by $\|u\|_E = \|u\|_X + \|C^{-1}u\|_X$ for $u \in E$, and let $X_0 = E$. Clearly, $A \in B(Y,E)$ by the definition of the spaces Y and E. Let $D = C(D(A))$ and $D_0 = C^2(D(A))$. Then, the relation (1.3) is satisfied and it is seen by (A2) that the abstract Cauchy problem for A has a unique solution $u \in C([0,\tau); D) \cap C^1([0,\tau); E)$ given by $u(t) = S(t)C^{-1}u_0$ for $t \in [0,\tau)$, for each initial data $u_0 \in D_0$. This means that condition (H1) is satisfied.

For each $n \geq 1$, let Z_n be the Banach space $X_n \times X_n$ equipped with the norm $\|(u,v)\|_{Z_n} = \|u\|_{X_n} + \|v\|_{X_n}$ for $(u,v) \in Z_n$, and define $P_{Z_n} \in B(Z, Z_n)$ by $P_{Z_n}(u,v) = (P_{X_n}u, P_{X_n}v)$ for $(u,v) \in Z$. Then, hypothesis (H2-iv) is clearly checked by (2.1). For each $n \geq 1$, let E_n be the Banach space $R(C_n)$ equipped with the norm $\|u\|_{E_n} = \|u\|_{X_n} + \|C_n^{-1}u\|_{X_n}$ for $u \in E_n$, and define $P_{E_n} \in B(E, E_n)$ by $P_{E_n}u = C_n P_{X_n} C^{-1}u$ for $u \in E$. By (2.2) we have

$$\lim_{n\to\infty} \|C_n P_{X_n} C^{-1}u - P_{X_n}u\|_{X_n} = 0 \tag{2.4}$$

for $u \in E$. This fact together with (2.1) shows that (H2-ii) is satisfied. All the other hypotheses in (H2) are checked by taking $X_{0,n} = E_n$, $D_n = R(C_n)$ and $D_{0,n} = R(C_n)$ for each $n \geq 1$.

For each $n \geq 1$ we consider the operator $S_n \in B(E_n, Z_n)$ defined by $S_n u = (C_n^{-1}u, A_n C_n^{-1}u)$ for $u \in E_n$. For every $u \in C(D(A))$ and every sequence $\{u_n\}$ such that $u_n \in R(C_n)$ for $n \geq 1$, we have

$$\|P_{Z_n}Su - S_n u_n\|_{Z_n} = \|P_{X_n}C^{-1}u - C_n^{-1}u_n\|_{X_n} + \|P_{X_n}AC^{-1}u - A_n C_n^{-1}u_n\|_{X_n}. \tag{2.5}$$

Let $u \in D = C(D(A))$. Then, by (2.3) there exists a sequence $\{v_n\}$ such that $v_n \in X_n$ for $n \geq 1$ and $\lim_{n\to\infty}(\|v_n - P_{X_n}C^{-1}u\|_{X_n} + \|A_n v_n - P_{X_n}AC^{-1}u\|_{X_n}) = 0$. The sequence $\{u_n\}_{n=1,2,\ldots}$, defined by $u_n = C_n v_n \in R(C_n) = D_n$ for $n \geq 1$, satisfies hypothesis (H3-i) by (2.2) and (2.5). To check (H3-ii), notice that

$$\|P_{X_{0,n}}u - u_n\|_{X_{0,n}} = \|C_n P_{X_n}C^{-1}u - u_n\|_{X_n} + \|P_{X_n}C^{-1}u - C_n^{-1}u_n\|_{X_n} \tag{2.6}$$

for every $u \in R(C)$ and every sequence $\{u_n\}$ such that $u_n \in R(C_n)$ for $n \geq 1$, and let $u \in D_0 = C^2(D(A))$. Then, by (2.3) there exists a sequence $\{v_n\}$ such that $v_n \in X_n$ for $n \geq 1$ and

$$\lim_{n\to\infty}(\|v_n - P_{X_n}C^{-2}u\|_{X_n} + \|A_n v_n - P_{X_n}AC^{-2}u\|_{X_n}) = 0. \tag{2.7}$$

Consider the sequence $\{u_n\}$ defined by $u_n = C_n^2 v_n \in D_{0,n}$ for $n \geq 1$. Then, by (2.5) and (2.6) we have $\|P_{X_{0,n}}u - u_n\|_{X_{0,n}} \leq (\|C_n\|_{X_n \to X_n} + 1)\|P_{X_n}C^{-1}u - C_n v_n\|_{X_n}$ and $\|P_{Z_n}Su - S_n u_n\|_{Z_n} = \|P_{X_n}C^{-1}u - C_n v_n\|_{X_n} + \|P_{X_n}AC^{-1}u - A_n C_n v_n\|_{X_n}$ for $n \geq 1$. Notice that by (2.2) that the sequence $\{\|C_n\|_{X_n \to X_n}\}_{n \geq 1}$ is bounded in a way similar to that in [8, Theorem 5.7.1]. Thus, by (2.2), (H3-ii) follows from (2.7), since $C^{-1}AC = A$ (by (A1)) and $A_n C_n = C_n A_n$ for $n \geq 1$ (by (F2)).

To check hypothesis (H4), let $n \geq 1$ and $\sigma \in (0, \tau)$. Let $x_0 \in D_{0,n}$ and set $x_i = F_n^i x_0$ for $i = 1, 2, \ldots, K_{\sigma,n}$. By condition (F2) we have $x_i = C_n F_n^i(C_n^{-1}x_0) \in R(C_n) = D_n$ and $x_i = F_n x_{i-1}$ for $1 \leq i \leq K_{\sigma,n}$. This implies that hypothesis (H4-i) is satisfied. Let $w_0 \in X_{0,n}$ and $\{f_i\}_{i=1}^{K_{\sigma,n}}$ be a sequence in $R(C_n)$. If

$\{w_i\}_{i=1}^{K_{\sigma,n}}$ is a sequence satisfying $w_i = F_n w_{i-1} + h_n f_i$ for $1 \le i \le K_{\sigma,n}$, then $w_i = F_n^i w_0 + h_n \sum_{l=1}^{i} F_n^{i-l} f_l$ for $1 \le i \le K_{\sigma,n}$. We use condition (F1) to find the inequality

$$\|w_i\|_{X_n} \le M_\sigma \|C_n^{-1} w_0\|_{X_n} + h_n \sum_{l=1}^{i} M_\sigma \|C_n^{-1} f_l\|_{X_n}$$

for $1 \le i \le K_{\sigma,n}$. This inequality means that hypothesis (H4-ii) is satisfied. Hypothesis (H4-iii) is checked by condition (F2). Since for every $u \in C(D(A))$ and $u_n \in R(C_n)$, $\|P_{E_n} u - u_n\|_{E_n} \le (\|C_n\|_{X_n \to X_n} + 1)\|P_{X_n} C^{-1} u - C_n^{-1} u_n\|_{X_n}$ (by (2.6) with $X_{0,n} = E_n$) and $\|P_{E_n} Au - A_n u_n\|_{E_n} \le (\|C_n\|_{X_n \to X_n} + 1)\|P_{X_n} AC^{-1} u - A_n C_n^{-1} u_n\|_{X_n}$ (by the definition of P_{E_n} and the norm of E_n), we verify (H5) by (2.5). Therefore, we apply Theorem 1 to prove that

$$\lim_{n\to\infty} \left(\sup\{\|F_n^{[t/h_n]} C_n P_{X_n} C^{-1} u_0 - P_{X_n} S(t) C^{-1} u_0\|_{X_n}; t \in [0,\sigma]\} \right) = 0$$

for every $u_0 \in C^2(D(A))$. Since $C(D(A))$ is dense in X, the theorem is proved by a standard density argument. ■

3. Key Estimate and the Proof of the Main Theorem

Throughout the following lemma, let E, X and X_0 be three real Banach spaces and let D and D_0 be two subsets of X such that they satisfy the following relation:

$$\begin{array}{ccccc} D & \subset & E & \subset & X \\ \cup & & & & \cup \\ D_0 & & \subset & & X_0 \end{array}$$

Let $h > 0$ and let $\{C(w); w \in D\}$ be a family in $B(X)$ satisfying the following conditions:

(C1) *If $x_0 \in D_0$, then there exists a sequence $\{x_i\}_{i=1}^{K}$ in D such that $x_i = C(x_{i-1})x_{i-1}$ for $1 \le i \le K$, where K is the greatest integer such that $Kh \le T$.*

(C2) *There exist $M \ge 1$ and $p \ge 1$ such that if $x_0 \in D_0$, $\{x_i\}_{i=1}^{K}$ is a sequence in D satisfying $x_i = C(x_{i-1})x_{i-1}$ for $1 \le i \le K$, $w_0 \in X_0$, $\{w_i\}_{i=1}^{K}$ is a sequence in X satisfying $w_i = C(x_{i-1})w_{i-1} + hf_i$ for $1 \le i \le K$ and $\{f_i\}_{i=1}^{K}$ is a sequence in E, then the inequality*

$$\|w_i\|_X^p \le M \left(\|w_0\|_{X_0}^p + h \sum_{l=1}^{i} \|f_l\|_E^p \right)$$

holds for $1 \le i \le K$.

Lemma 1. *Let $u_0 \in D_0$ and let $\{u_i\}_{i=1}^K$ be a sequence in D such that*

$$u_i = C(u_{i-1})u_{i-1} \qquad \textit{for } 1 \le i \le K.$$

Define a step function $u : [0,T] \to D$ by

$$u(t) = u_i \quad \textit{for } t \in [ih,(i+1)h) \cap [0,T] \textit{ and } i = 0,1,2,\dots,K.$$

Let $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ be a partition of $[0,T]$ such that

$$h \le \min_{1 \le j \le N}(t_j - t_{j-1}). \tag{3.1}$$

Set $B(w) = (C(w)-I)/h$ for $w \in D$. Let $v_0 \in D_0$ and let $\{v_j\}_{j=1}^N$ be a sequence in D such that

$$(v_j - v_{j-1})/(t_j - t_{j-1}) = B(v_{j-1})v_{j-1} + z_j \qquad \textit{for } 1 \le j \le N, \tag{3.2}$$

where $\{z_j\}_{j=1}^N$ is a sequence in E. Define a step function $v : [0,T] \to D$ by

$$v(t) = \begin{cases} v_{j-1} & \textit{for } t \in [t_{j-1}, t_j) \textit{ and } j = 1,2,\dots,N \\ v_N & \textit{for } t = t_N. \end{cases}$$

Assume that the following conditions are satisfied:

(C3) *$C(w)(D) \subset E$ for $w \in D$.*

(C4) *There exists $L_0 \ge 0$ such that*

$$\max_{0 \le j \le N} \|(C(w) - C(z))v_j\|_E \le hL_0\|w - z\|_X \qquad \textit{for } w, z \in D.$$

Then there exists $c > 0$, depending only on M, p and T, such that

$$\|u(t) - v(t)\|_X^p \le c\exp(cL_0^p t)(\|u_0 - v_0\|_{X_0}^p + \alpha^p + \beta^p + (1 + L_0^p)\gamma^p) \tag{3.3}$$

for $t \in [0,T]$. Here the symbols α, β and γ are defined by

$$\alpha = \max_{1 \le j \le N} \|B(v_j)v_j - B(v_{j-1})v_{j-1}\|_E,$$
$$\beta = \max_{1 \le j \le N} \|z_j\|_E, \qquad \gamma = \max_{1 \le j \le N} \|v_j - v_{j-1}\|_X.$$

Proof. We use the function $w : [0,T] \to \mathrm{co}(D)$ defined by

$$w(t) = v_{j-1} + (t - t_{j-1})(v_j - v_{j-1})/(t_j - t_{j-1})$$

for $t \in [t_{j-1}, t_j]$ and $j = 1,2,\dots,N$, where $\mathrm{co}(D)$ is the convex hull of D. Notice that $Kh \le T$ and define

$$f_i = (w(ih) - w((i-1)h))/h - B(u_{i-1})w((i-1)h)$$

for $i = 1, 2, \ldots, K$. Then, by the definition of $B(w)$ we have

$$w(ih) = C(u_{i-1})w((i-1)h) + hf_i \tag{3.4}$$

for $i = 1, 2, \ldots, K$. Since $w(t) \in \text{co}(D) \subset E$ for $t \in [0, T]$, we have $\{f_i\}_{i=1}^{K} \subset E$ by condition (C3). Since $u_i = C(u_{i-1})u_{i-1}$, we have by (3.4)

$$u_i - w(ih) = C(u_{i-1})(u_{i-1} - w((i-1)h)) - hf_i \tag{3.5}$$

for $1 \leq i \leq K$. Since $u_0 - w(0) = u_0 - v_0 \in D_0 - D_0 \subset X_0$, we apply condition (C2) to the equality (3.5), so that

$$\|u_i - w(ih)\|_X^p \leq M \left(\|u_0 - v_0\|_{X_0}^p + h \sum_{l=1}^{i} \|f_l\|_E^p \right) \tag{3.6}$$

for $0 \leq i \leq K$.

We want to estimate $\sum_{l=1}^{i} \|f_l\|_E^p$ in (3.6), for $1 \leq i \leq K$. For this purpose, let $1 \leq l \leq K$ and $r \in ((l-1)h, lh)$. Then we have

$$u(r) = u_{l-1}. \tag{3.7}$$

Since $(l-1)h \leq (K-1)h < T$, there exists $j \in \{1, 2, \ldots, N\}$ such that

$$(l-1)h \in [t_{j-1}, t_j). \tag{3.8}$$

By the definition of w we have

$$\begin{aligned} w((l-1)h) = {} & ((t_j - (l-1)h)/(t_j - t_{j-1}))v_{j-1} \\ & +(((l-1)h - t_{j-1})/(t_j - t_{j-1}))v_j. \end{aligned} \tag{3.9}$$

Since $B(v_{j-1})v_{j-1} - B(u_{l-1})w((l-1)h)$ is written as

$$\begin{aligned} & B(v_{j-1})v_{j-1} - B(u_{l-1})w((l-1)h) \\ = {} & ((t_j - (l-1)h)/(t_j - t_{j-1}))(B(v_{j-1})v_{j-1} - B(u_{l-1})v_{j-1}) \\ & +(((l-1)h - t_{j-1})/(t_j - t_{j-1}))(B(v_{j-1})v_{j-1} - B(u_{l-1})v_j) \end{aligned}$$

by (3.9), and since

$$B(v_{j-1})v_{j-1} - B(u_{l-1})v_j = (B(v_{j-1})v_{j-1} - B(v_j)v_j) + (B(v_j) - B(u_{l-1}))v_j,$$

we have by (3.7) and condition (C4)

$$\begin{aligned} & \|B(v_{j-1})v_{j-1} - B(u_{l-1})w((l-1)h)\|_E \\ & \quad \leq \alpha + L_0 \max\{\|v_{j-1} - u(r)\|_X, \|v_j - u(r)\|_X\}. \end{aligned} \tag{3.10}$$

By (3.1) and (3.8), we need to consider the following two cases:

(a) $lh \in [t_{j-1}, t_j]$, (b) $lh \in [t_j, t_{j+1}]$ and $1 \le j \le N-1$.

We start with the case (b). Notice that

$$t_{j-1} \le (l-1)h < t_j \le lh \le t_{j+1}. \tag{3.11}$$

By (3.11) we have $r \in (t_{j-1}, t_{j+1})$; hence $v(r) = v_{j-1}$ or v_j. Since

$$\begin{aligned}&(v_{j+1} - v_j)/(t_{j+1} - t_j) - B(u_{l-1})w((l-1)h)\\&\quad = (B(v_j)v_j - B(v_{j-1})v_{j-1}) + (B(v_{j-1})v_{j-1} - B(u_{l-1})w((l-1)h)) + z_{j+1}\end{aligned}$$

by (3.2), we use (3.10) to get

$$\begin{aligned}&\|(v_{j+1} - v_j)/(t_{j+1} - t_j) - B(u_{l-1})w((l-1)h)\|_E\\&\le 2\alpha + \beta + L_0(\|v(r) - u(r)\|_X + \gamma).\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}&(v_j - v_{j-1})/(t_j - t_{j-1}) - B(u_{l-1})w((l-1)h)\\&= B(v_{j-1})v_{j-1} + z_j - B(u_{l-1})w((l-1)h),\end{aligned}$$

we have by (3.10)

$$\begin{aligned}&\|(v_j - v_{j-1})/(t_j - t_{j-1}) - B(u_{l-1})w((l-1)h)\|_E\\&\le \alpha + \beta + L_0(\|v(r) - u(r)\|_X + \gamma).\end{aligned} \tag{3.13}$$

We apply (3.12) and (3.13) to f_l which is written as

$$\begin{aligned}f_l &= ((w(lh) - w(t_j)) + (w(t_j) - w((l-1)h)))/h - B(u_{l-1})w((l-1)h)\\&= \{(lh - t_j)((v_{j+1} - v_j)/(t_{j+1} - t_j) - B(u_{l-1})w((l-1)h))\\&\quad + (t_j - (l-1)h)((v_j - v_{j-1})/(t_j - t_{j-1}) - B(u_{l-1})w((l-1)h))\}/h\end{aligned}$$

by the definition of w and (3.11). This yields

$$\|f_l\|_E \le 2\alpha + \beta + L_0\gamma + L_0\|v(r) - u(r)\|_X. \tag{3.14}$$

In the case of (a), we have $t_{j-1} \le (l-1)h < r < lh \le t_j$ by (3.8), so that $(w(lh) - w((l-1)h))/h = (v_j - v_{j-1})/(t_j - t_{j-1})$. This together with (3.13) implies that (3.14) is also valid in the case of (a). It is thus shown that (3.14) holds for $r \in ((l-1)h, lh)$ and $1 \le l \le K$. It follows that

$$h\|f_l\|_E^p \le c\left((\alpha^p + \beta^p)h + L_0^p\gamma^p h + L_0^p \int_{(l-1)h}^{lh} \|u(r) - v(r)\|_X^p\, dr\right)$$

for $1 \leq l \leq K$. Substituting this inequality into (3.6), we find

$$\begin{aligned} &\|u_i - w(ih)\|_X^p \\ &\leq c\left(\|u_0 - v_0\|_{X_0}^p + (\alpha^p + \beta^p)T + L_0^p \gamma^p T + L_0^p \int_0^{ih} \|u(r) - v(r)\|_X^p \, dr \right) \end{aligned} \tag{3.15}$$

for $0 \leq i \leq K$.

Now, we turn to the proof of (3.3). Let $t \in [0, T]$. There exists $i \in \{0, 1, \ldots, K\}$ such that $t \in [ih, (i+1)h)$, and then $u(t) = u_i$. Since $ih \leq t \leq T$, there exists $j \in \{1, 2, \ldots, N\}$ such that $ih \in [t_{j-1}, t_j]$, and then

$$w(ih) = v_{j-1} + (ih - t_{j-1})(v_j - v_{j-1})/(t_j - t_{j-1}). \tag{3.16}$$

To estimate $\|u(t) - v(t)\|_X$, by (3.15) it suffices to estimate $\|v(t) - w(ih)\|_X$. By (3.1) we need to consider the following three cases:

(i) $t \in [t_{j-1}, t_j)$, (ii) $t \in [t_j, t_{j+1})$ and $j \leq N - 1$, (iii) $t = t_j$ and $j = N$.

In the case of (i), we have $v(t) = v_{j-1}$ and $\|v(t) - w(ih)\|_X \leq \|v_j - v_{j-1}\|_X \leq \gamma$ by (3.16). Next, we consider the cases (ii) and (iii). In both cases, we have $v(t) = v_j$. By (3.16) we have

$$v(t) - w(ih) = ((v_j - v_{j-1})/(t_j - t_{j-1}))((t_j - t_{j-1}) - (ih - t_{j-1}));$$

hence $\|v(t) - w(ih)\|_X \leq \|v_j - v_{j-1}\|_X \leq \gamma$. Combining these estimates and (3.15) and using the fact that $ih \leq t$, we have

$$\|u(t) - v(t)\|_X^p \leq c\left(\|u_0 - v_0\|_{X_0}^p + \alpha^p + \beta^p + \gamma^p + L_0^p \gamma^p + L_0^p \int_0^t \|u(r) - v(r)\|_X^p \, dr \right)$$

for $t \in [0, T]$. An application of Gronwall's inequality gives the desired inequality (3.3). ∎

Proof of Theorem 1. Assertion (i) is a direct consequence of hypothesis (H4-i). To prove that assertion (ii) is true, let $\varepsilon > 0$. Since $u \in C([0, T]; D)$ and A is strongly continuous on D in $B(Y, E)$ (by (1.2)), there exists a partition $\{0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N_\varepsilon}^\varepsilon = T\}$ of $[0, T]$ such that

$$\begin{aligned} &t_j^\varepsilon - t_{j-1}^\varepsilon \leq \varepsilon \quad \text{for } j = 1, 2, \ldots, N_\varepsilon, \\ &\|u(t) - u(t_{j-1}^\varepsilon)\|_X \leq \varepsilon \quad \text{for } t \in [t_{j-1}^\varepsilon, t_j^\varepsilon] \text{ and } j = 1, 2, \ldots, N_\varepsilon, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\|A(u(t))u(t) - A(u(t_{j-1}^\varepsilon))u(t_{j-1}^\varepsilon)\|_E \leq \varepsilon \\ &\text{for } t \in [t_{j-1}^\varepsilon, t_j^\varepsilon] \text{ and } j = 1, 2, \ldots, N_\varepsilon. \end{aligned} \tag{3.18}$$

Set $v_j^\varepsilon = u(t_j^\varepsilon)$ for $j = 0, 1, \ldots, N_\varepsilon$. Since $v_0^\varepsilon = u_0 \in D_0$, there exists a sequence $\{v_{0,n}^\varepsilon\}_{n=1,2,\ldots}$ such that $v_{0,n}^\varepsilon \in D_{0,n}$ for $n \geq 1$ and

$$\begin{aligned}\lim_{n\to\infty}(\|P_{X_{0,n}}v_0^\varepsilon - v_{0,n}^\varepsilon\|_{X_{0,n}} + \|P_{X_n}v_0^\varepsilon - v_{0,n}^\varepsilon\|_{X_n} \\ +\|P_{Z_n}Sv_0^\varepsilon - S_n v_{0,n}^\varepsilon\|_{Z_n}) = 0,\end{aligned} \tag{3.19}$$

by hypothesis (H3-ii). Since $v_j^\varepsilon \in D$ for $1 \leq j \leq N_\varepsilon$, hypothesis (H3-i) ensures that for each $j = 1, 2, \ldots, N_\varepsilon$ there exists a sequence $\{v_{j,n}^\varepsilon\}_{n=1,2,\ldots}$ such that $v_{j,n}^\varepsilon \in D_n$ for $n \geq 1$ and

$$\lim_{n\to\infty}(\|P_{X_n}v_j^\varepsilon - v_{j,n}^\varepsilon\|_{X_n} + \|P_{Z_n}Sv_j^\varepsilon - S_n v_{j,n}^\varepsilon\|_{Z_n}) = 0. \tag{3.20}$$

By (3.19) and (3.20), the consistency condition (H5) implies that $\lim_{n\to\infty}\|P_{E_n}v_j^\varepsilon - v_{j,n}^\varepsilon\|_{E_n} = 0$ and $\lim_{n\to\infty}\|P_{E_n}A(v_j^\varepsilon)v_j^\varepsilon - A_n(v_{j,n}^\varepsilon)v_{j,n}^\varepsilon\|_{E_n} = 0$ for $j = 0, 1, \ldots, N_\varepsilon$. For each $n \geq 1$, the sequence $\{z_{j,n}^\varepsilon\}_{j=1}^{N_\varepsilon}$, defined by

$$z_{j,n}^\varepsilon = (v_{j,n}^\varepsilon - v_{j-1,n}^\varepsilon)/(t_j^\varepsilon - t_{j-1}^\varepsilon) - A_n(v_{j-1,n}^\varepsilon)v_{j-1,n}^\varepsilon$$

for $j = 1, 2, \ldots, N_\varepsilon$, satisfies that $z_{j,n}^\varepsilon \in E_n$ and

$$\lim_{n\to\infty}\|z_{j,n}^\varepsilon\|_{E_n} = \|(v_j^\varepsilon - v_{j-1}^\varepsilon)/(t_j^\varepsilon - t_{j-1}^\varepsilon) - A(v_{j-1}^\varepsilon)v_{j-1}^\varepsilon\|_E \tag{3.21}$$

for $1 \leq j \leq N_\varepsilon$. We shall apply Lemma 1 to estimate the difference between u_n and the step function $v_n^\varepsilon : [0, T] \to D_n$ defined by

$$v_n^\varepsilon(t) = \begin{cases} v_{j-1,n}^\varepsilon & \text{for } t \in [t_{j-1}^\varepsilon, t_j^\varepsilon) \text{ and } j = 1, 2, \ldots, N_\varepsilon, \\ v_{N_\varepsilon,n}^\varepsilon & \text{for } t = T. \end{cases}$$

Let $n_0 \geq 1$ be an integer such that $h_n \leq \min_{1\leq j\leq N_\varepsilon}(t_j^\varepsilon - t_{j-1}^\varepsilon)$ for all $n \geq n_0$. By condition (H4-iv) we have

$$\|C_n(w)v_{j,n}^\varepsilon - C_n(z)v_{j,n}^\varepsilon\|_{E_n} \leq h_n L(\|v_{j,n}^\varepsilon\|_{X_n} + \|S_n v_{j,n}^\varepsilon\|_{Z_n})\|w - z\|_{X_n}$$

for $w, z \in D_n$ and $0 \leq j \leq N_\varepsilon$. Since $\lim_{n\to\infty}\|A_n(v_{j,n}^\varepsilon)v_{j,n}^\varepsilon - A_n(v_{j-1,n}^\varepsilon)v_{j-1,n}^\varepsilon \|_{E_n} = \|A(v_j^\varepsilon)v_j^\varepsilon - A(v_{j-1}^\varepsilon)v_{j-1}^\varepsilon\|_E \leq \varepsilon$ (by 3.18) and $\lim_{n\to\infty}\|v_{j,n}^\varepsilon - v_{j-1,n}^\varepsilon\|_{X_n} = \|v_j^\varepsilon - v_{j-1}^\varepsilon\|_X \leq \varepsilon$ (by 3.17) for $1 \leq j \leq N_\varepsilon$, we apply Lemma 1 to find

$$\begin{aligned}\|u_n(t) - v_n^\varepsilon(t)\|_{X_n}^p &\leq c\exp(cL^p(a_n^\varepsilon)^p T)\{\|u_{0,n} - v_{0,n}^\varepsilon\|_{X_{0,n}}^p \\ &\quad +\varepsilon^p + (b_n^\varepsilon)^p + (1 + L^p(a_n^\varepsilon)^p)\varepsilon^p\}\end{aligned} \tag{3.22}$$

for $t \in [0, T]$ and $n \geq n_0$, where the symbols a_n^ε and b_n^ε are defined by

$$a_n^\varepsilon = \max_{0\leq j\leq N_\varepsilon}(\|v_{j,n}^\varepsilon\|_{X_n} + \|S_n v_{j,n}^\varepsilon\|_{Z_n}), \qquad b_n^\varepsilon = \max_{1\leq j\leq N_\varepsilon}\|z_{j,n}^\varepsilon\|_{E_n}.$$

By (3.19) and (3.20) we have

$$\lim_{n\to\infty} a_n^\varepsilon = \max_{0\le j\le N_\varepsilon}(\|v_j^\varepsilon\|_X + \|Sv_j^\varepsilon\|_Z) \le c_S \max_{0\le j\le N_\varepsilon}\|u(t_j^\varepsilon)\|_Y, \tag{3.23}$$

where we have used (1.4) to obtain the last inequality. By (3.18) and (3.21) we have

$$\lim_{n\to\infty} b_n^\varepsilon \le \varepsilon. \tag{3.24}$$

We employ the step function $v^\varepsilon : [0,T] \to D$ defined by

$$v^\varepsilon(t) = \begin{cases} v_{j-1}^\varepsilon & \text{for } t \in [t_{j-1}^\varepsilon, t_j^\varepsilon) \text{ and } j = 1,2,\ldots,N_\varepsilon, \\ v_{N_\varepsilon}^\varepsilon & \text{for } t = T. \end{cases}$$

By (3.19) and (3.20) we have

$$\lim_{n\to\infty}(\sup\{\|P_{X_n}v^\varepsilon(t) - v_n^\varepsilon(t)\|_{X_n}; t \in [0,T]\}) = 0. \tag{3.25}$$

By (3.17) we have

$$\sup\{\|P_{X_n}u(t) - P_{X_n}v^\varepsilon(t)\|_{X_n}; t \in [0,T]\} \le (\|P_{X_n}\|_{X\to X_n})\varepsilon. \tag{3.26}$$

We use (3.22) through (3.26) to obtain

$$\begin{aligned}
&\limsup_{n\to\infty}(\sup\{\|u_n(t) - P_{X_n}u(t)\|_{X_n}^p; t \in [0,T]\}) \\
&\le c\{(\sup\{\|P_{X_n}\|_{X\to X_n}; n \ge 1\})^p\varepsilon^p \\
&\quad + \exp(cL^p(c_S \sup\{\|u(t)\|_Y; t \in [0,T]\})^p T) \\
&\times(3 + L^p(c_S \sup\{\|u(t)\|_Y; t \in [0,T]\})^p)\varepsilon^p\}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired claim is thus proved. ∎

4. An Approximation of a Degenerate Kirchhoff Equation

This section is devoted to an approximation of the solution of the system

$$\begin{cases} u_t(x,t) = v_x(x,t) & \text{for } (x,t) \in \mathbb{R} \times [0,\infty), \\ v_t(x,t) = \|u(\cdot,t)\|_{L^2}^{2\alpha} u_x(x,t) & \text{for } (x,t) \in \mathbb{R} \times [0,\infty), \end{cases} \tag{4.1}$$

which is obtained by setting $u = w_x$ and $v = w_t$ in the Kirchhoff equation

$$w_{tt}(x,t) = \|w_x(\cdot,t)\|_{L^2}^{2\alpha} w_{xx}(x,t) \qquad \text{for } (x,t) \in \mathbb{R} \times [0,\infty).$$

Here $\alpha \geq 1$ and $\|u\|_{L^2}$ denotes the usual norm in $L^2(\mathbb{R})$.

We are interested in the degenerate case where $u(\cdot, 0) = 0$, which implies that the right-hand side of the second equation of (4.1) is zero when $t = 0$.

Let $\{h_n\}$ and $\{k_n\}$ be two null sequences of positive numbers such that $h_n/k_n = r$, where r is an appropriate positive constant to be determined later. Consider the difference scheme of Lax-Friedrichs type

$$
(4.2) \qquad \begin{cases} (u_{l,i} - (u_{l+1,i-1} + u_{l-1,i-1})/2)/h_n = (v_{l+1,i-1} - v_{l-1,i-1})/(2k_n), \\ (v_{l,i} - (v_{l+1,i-1} + v_{l-1,i-1})/2)/h_n = \|(u_{l,i-1})\|_n^{2\alpha}(u_{l+1,i-1} - u_{l-1,i-1})/(2k_n) \end{cases}
$$

for $l \in \mathbb{Z}$ and $i = 1, 2, \ldots$. Here the symbol $\|\cdot\|_n$ is the norm in $l^2(\mathbb{Z})$ defined by $\|u\|_n = \left(\sum_{l=-\infty}^{\infty} |u_l|^2 k_n\right)^{1/2}$ for $u = (u_l) \in l^2(\mathbb{Z})$.

Theorem 3. *Let $v_0 \in H^3(\mathbb{R})$ and $\partial_x v_0 \neq 0$. Then there exists $T > 0$ such that the following assertions hold:*

(i) *The Cauchy problem for the system* (4.1) *with the initial condition* $u(x, 0) = 0$ *and* $v(x, 0) = v_0(x)$ *has a unique solution* (u, v) *in the class* $C([0, T]; H^2(\mathbb{R}) \times H^2(\mathbb{R})) \cap C^1([0, T]; H^1(\mathbb{R}) \times H^1(\mathbb{R}))$.

(ii) *The solution* (u, v) *of* (4.1) *can be approximated by the solution* (u_i, v_i) *in* $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ *of the system* (4.2) *with the initial condition* $(u_{l,0}) = 0$ *and* $(v_{l,0}) = p_n v_0$ *in the sense that*

$$
\lim_{n\to\infty} (\sup\{\|u_{[t/h_n]} - p_n u(t)\|_n + \|v_{[t/h_n]} - p_n v(t)\|_n; t \in [0, T]\}) = 0,
$$

where $u_i = (u_{l,i})$, $v_i = (v_{l,i})$ *and* p_n *is the operator on* $L^2(\mathbb{R})$ *to* $l^2(\mathbb{Z})$ *defined by*

$$
(4.3) \qquad p_n u = \left(\frac{1}{k_n} \int_{(l-1/2)k_n}^{(l+1/2)k_n} u(x)\, dx \right) \qquad \text{for } u \in L^2(\mathbb{R}).
$$

Let X be the Banach space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ equipped with the norm $\|(u, v)\|_X = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$ for $(u, v) \in X$. Let $E = X_0 = H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $Y = H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $Z = X \times X$. Here Z is equipped with the norm $\|((u, v), (\hat{u}, \hat{v}))\|_Z = (\|(u, v)\|_X^2 + \|(\hat{u}, \hat{v})\|_X^2)^{1/2}$ and $H^k(\mathbb{R}) \times H^k(\mathbb{R})$ is equipped with the norm $\|(u, v)\|_{H^k \times H^k} = (\|u\|_{H^k}^2 + \|v\|_{H^k}^2)^{1/2}$, where $\|w\|_{H^k} = (\sum_{l=0}^{k} \|\partial_x^l w\|_{L^2}^2)^{1/2}$ for $w \in H^k(\mathbb{R})$. Then, the operator $S \in B(Y, Z)$, defined by $S(u, v) = ((u_x, v_x), (u_{xx}, v_{xx}))$ for $(u, v) \in Y$, satisfies condition (1.4) with $c_S = 2$.

Let $v_0 \in H^3(\mathbb{R})$ and $\partial_x v_0 \neq 0$. Let r_0, R_0 and R be positive constants such that $\|\partial_x v_0\|_{L^2} > r_0$, $\|v_0\|_{H^3} \leq R_0$ and $r_0 < R_0 < R$, and define $D = \{(u, v) \in Y; \|u\|_{H^2} \leq R,\ \|v\|_{H^2} \leq R\}$ and $D_0 = \{(0, v_0)\}$. Then, relation (1.3) is satisfied and it is shown [26, Theorem 8.1] that the family $\{A((w,z)); (w,z) \in D\}$ of closed linear operators in X defined by

$$\begin{cases} A((w,z))(u,v) = (v_x,\ (\|w\|_{L^2}^{2\alpha} u)_x) & \text{for } (u,v) \in D((A(w,z))), \\ D(A((w,z))) = \{(u,v) \in X;\ v \in H^1(\mathbb{R}),\ \|w\|_{L^2}^{2\alpha} u \in H^1(\mathbb{R})\} \end{cases}$$

satisfies conditions (1.1), (1.2) and (H1) for sufficiently small $T > 0$. This means that assertion (i) holds.

Let X_n and E_n be the Banach spaces $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ equipped with the norms $\|(u,v)\|_{X_n} = (\|u\|_n^2 + \|v\|_n^2)^{1/2}$ and $\|(u,v)\|_{E_n} = (\|u\|_n^2 + \|\delta_n^- u\|_n^2 + \|v\|_n^2 + \|\delta_n^- v\|_n^2)^{1/2}$, respectively. Here the operator δ_n^- on $l^2(\mathbb{Z})$ is defined by

$$\delta_n^- u = ((u_l - u_{l-1})/k_n) \quad \text{for } u = (u_l) \in l^2(\mathbb{Z}).$$

Let $P_{X_n}(u,v) = (p_n u, p_n v)$ for $(u,v) \in X$, and let $P_{E_n}(u,v) = (p_n u, p_n v)$ for $(u,v) \in E$ and $X_{0,n} = E_n$. Let Z_n be the Banach space $X_n \times X_n$ with the norm $\|((u,v),(\hat{u},\hat{v}))\|_{Z_n} = (\|(u,v)\|_{X_n}^2 + \|(\hat{u},\hat{v})\|_{X_n}^2)^{1/2}$, and let $P_{Z_n}((u,v),(\hat{u},\hat{v})) = (P_{X_n}(u,v), P_{X_n}(\hat{u},\hat{v}))$ for $((u,v),(\hat{u},\hat{v})) \in Z$. Let D_n be the set of all $(u,v) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ such that $\|u\|_n^2 + \|\delta_n^- u\|_n^2 + \|\delta_n^+ \delta_n^- u\|_n^2 \le R^2$ and $\|v\|_n^2 + \|\delta_n^- v\|_n^2 + \|\delta_n^+ \delta_n^- v\|_n^2 \le R^2$, where δ_n^+ is the operator on $l^2(\mathbb{Z})$ defined by

$$\delta_n^+ u = ((u_{l+1} - u_l)/k_n) \qquad \text{for } u = (u_l) \in l^2(\mathbb{Z}).$$

Let $D_{0,n} = \{(0, p_n v_0)\}$. Then, we have $D_{0,n} \subset D_n$ by Lemma 4 (ii) in Appendix, since $\|v_0\|_{H^2} \le \|v_0\|_{H^3} \le R_0 \le R$. All the other hypotheses in (H2) are easily shown to be satisfied by (5.1) and Lemma 4 (i), (ii).

To check (H3) we employ $S_n \in B(E_n, Z_n)$ defined by

$$S_n(u,v) = ((\delta_n^- u, \delta_n^- v), (\delta_n^+ \delta_n^- u, \delta_n^+ \delta_n^- v))$$

for $(u,v) \in E_n$. For $(u,v) \in D$, the sequence $((p_n u, p_n v))$ in $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ is a desired one satisfying (H3-i) by Lemma 4 (i), since $(p_n u, p_n v) \in D_n$ (by (5.1) and Lemma 4 (ii)) and

$$\begin{aligned} &\|P_{Z_n} S(u,v) - S_n(u_n, v_n)\|_{Z_n}^2 \\ &= \|p_n u_x - \delta_n^- u_n\|_n^2 + \|p_n v_x - \delta_n^- v_n\|_n^2 \\ &\quad + \|p_n u_{xx} - \delta_n^+ \delta_n^- u_n\|_n^2 + \|p_n v_{xx} - \delta_n^+ \delta_n^- v_n\|_n^2 \end{aligned} \tag{4.4}$$

for $(u,v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $(u_n, v_n) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$. For $(u,v) \in D_0$, the sequence $((p_n u, p_n v))$ in $l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$ is a desired one satisfying (H3-ii), since $p_n u = 0$ and

$$\begin{aligned} \|P_{X_{0,n}}(u,v) - (u_n, v_n)\|_{X_{0,n}}^2 &= \|p_n u - u_n\|_n^2 + \|\delta_n^- (p_n u - u_n)\|_n^2 \\ &\quad + \|p_n v - v_n\|_n^2 + \|\delta_n^- (p_n v - v_n)\|_n^2 \end{aligned} \tag{4.5}$$

for $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and $(u_n, v_n) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$. Hypothesis (H3) is thus checked.

For each $n \geq 1$, consider the family $\{C_n((w, z)); (w, z) \in D_n\}$ in $B(X_n)$ defined in the following way: $C_n((w, z))(u, v) = (f, g)$ if and only if

$$\tag{4.6} \begin{cases} f = (\tau^+ u + \tau^- u)/2 + (h_n/k_n)(\tau^+ v - \tau^- v)/2, \\ g = (\tau^+ v + \tau^- v)/2 + (h_n/k_n)\|w\|_n^{2\alpha}(\tau^+ u - \tau^- u)/2, \end{cases}$$

where $\tau^+ u = (u_{l+1})$ and $\tau^- u = (u_{l-1})$ for $u = (u_l) \in l^2(\mathbb{Z})$. Since $(w_n, z_n) := A_n((u_n, v_n))(u_n, v_n)$ is written as

$$w_n = (1/r)(\delta_n^+ u_n - \delta_n^- u_n)/2 + (\delta_n^+ v_n + \delta_n^- v_n)/2,$$
$$z_n = (1/r)(\delta_n^+ v_n - \delta_n^- v_n)/2 + \|u_n\|_n^{2\alpha}(\delta_n^+ u_n + \delta_n^- u_n)/2$$

for $(u_n, v_n) \in D_n$ and $P_{E_n} A((u, v))(u, v) = (p_n v_x, \|u\|_{L^2}^{2\alpha} p_n u_x)$ for $(u, v) \in D$, we use (4.4) and (4.5) with $X_{0,n} = E_n$ to show that the family $\{C_n((w, z)); (w, z) \in D_n\}$ satisfies the consistency condition (H5) by (5.1) and Lemma 4.

We want to show that the family $\{C_n((w, z)); (w, z) \in D_n\}$ satisfies the stability condition (H4). For this purpose, we need the following lemma.

Lemma 2. *Let $h > 0, k > 0$ and $r = h/k$. Let $a \geq 0$ and assume that $(a + 1)r^2 \leq 1$. Let f, g, ξ, η, w and z in $l^2(\mathbb{Z})$ satisfy the system*

$$\begin{cases} f = (\tau^+ w + \tau^- w)/2 + r(\tau^+ z - \tau^- z)/2 \\ g = (\tau^+ z + \tau^- z)/2 + ar(\tau^+ w - \tau^- w)/2 \\ \eta = \xi + r(\tau^+ w - \tau^- w)/2. \end{cases}$$

Then it holds that

$$\tag{4.7} \begin{aligned} &(a + 1)\|f\|^2 + \|g\|^2 + 2\langle g, \eta\rangle - a\|\eta\|^2 \\ &\quad \leq (a + 1)\|w\|^2 + \|z\|^2 + 2\langle(\tau^+ z + \tau^- z)/2, \xi\rangle - a\|\xi\|^2. \end{aligned}$$

Here $\|u\| = \left(\sum_{l=-\infty}^{\infty} |u_l|^2 k\right)^{1/2}$ and $\langle u, v\rangle = \sum_{l=-\infty}^{\infty} u_l v_l k$ for $u = (u_l), v = (v_l) \in l^2(\mathbb{Z})$.

Proof. The left-hand side of (4.7) is written as

$$\tag{4.8} \begin{aligned} &(a + 1)\|f\|^2 + \|g + \eta\|^2 - (a + 1)\|\eta\|^2 \\ &= (a + 1)\{\|(\tau^+ w + \tau^- w)/2\|^2 \\ &\quad +2r\langle(\tau^+ w+\tau^- w)/2, (\tau^+ z-\tau^- z)/2\rangle + r^2\|(\tau^+ z-\tau^- z)/2\|^2\} \\ &\quad +\|(\tau^+ z + \tau^- z)/2 + (a + 1)r(\tau^+ w - \tau^- w)/2 + \xi\|^2 \\ &\quad -(a + 1)\{\|\xi\|^2 + 2r\langle\xi, (\tau^+ w-\tau^- w)/2\rangle + r^2\|(\tau^+ w-\tau^- w)/2\|^2\}. \end{aligned}$$

The second term on the right-hand side of (4.8) is equal to

$$\begin{aligned}&\|(\tau^+z+\tau^-z)/2\|^2+(a+1)^2r^2\|(\tau^+w-\tau^-w)/2\|^2+\|\xi\|^2\\ &+2(a+1)r\langle(\tau^+z+\tau^-z)/2,(\tau^+w-\tau^-w)/2\rangle+2\langle(\tau^+z+\tau^-z)/2,\xi\rangle\\ &+2(a+1)r\langle(\tau^+w-\tau^-w)/2,\xi\rangle.\end{aligned}\tag{4.9}$$

Substituting (4.9) into (4.8) and using the following two equalities

$$\|(\tau^+u+\tau^-u)/2\|^2+\|(\tau^+u-\tau^-u)/2\|^2=\|u\|^2\quad\text{for } u\in l^2(\mathbb{Z})$$

and $\langle\tau^+w,\tau^+z\rangle=\langle\tau^-w,\tau^-z\rangle$ for $w,z\in l^2(\mathbb{Z})$, we obtain the desired inequality (4.7), by the condition that $(a+1)r^2\le 1$. ∎

Lemma 3. *Let $h>0,k>0$ and $r=h/k$. Let $K\ge 1$ be an integer such that $Kh\le T$. Let $M\ge 0$ and $L\ge 0$. Let $\{a_i\}_{i=0}^{K-1}$ be a sequence such that $0\le a_i\le M$ and $0\le a_i-a_{i-1}\le Lh$ for $0\le i\le K-1$, where $a_{-1}=a_0$. Let $\{f_i\}_{i=1}^{K}$ and $\{g_i\}_{i=1}^{K}$ be two sequences in $l^2(\mathbb{Z})$. Let $\{w_i\}_{i=0}^{K}$ and $\{z_i\}_{i=0}^{K}$ be two sequences in $l^2(\mathbb{Z})$ satisfying the system*

$$\begin{cases} w_i=(\tau^+w_{i-1}+\tau^-w_{i-1})/2+r(\tau^+z_{i-1}-\tau^-z_{i-1})/2+hf_i\\ z_i=(\tau^+z_{i-1}+\tau^-z_{i-1})/2+a_{i-1}r(\tau^+w_{i-1}-\tau^-w_{i-1})/2+hg_i\end{cases}\tag{4.10}$$

for $1\le i\le K$. Assume that $(M+1)r^2\le 1$ and $(M+1)h\le 1/2$. Then it holds that

$$\|w_i\|^2+\|z_i\|^2\le\exp((2(M+1)+L+T+1)T)((M+1)M_1+M_2+TM_3)$$

for $0\le i\le K$. Here M_1,M_2 and M_3 are defined by

$$\begin{aligned}&M_1=\|w_0\|^2+h\textstyle\sum_{i=1}^K\|f_i\|^2, \qquad M_2=\|z_0\|^2+h\sum_{i=1}^K\|g_i\|^2\\ &M_3=\|\delta^-z_0\|^2+h\textstyle\sum_{i=1}^K\|\delta^-g_i\|^2,\end{aligned}$$

where $\delta^-u=((u_l-u_{l-1})/k)$ for $u=(u_l)\in l^2(\mathbb{Z})$.

Proof. Let $1\le j\le K$. To use Lemma 2, we employ the sequence $\{\xi_i\}_{i=0}^{j}$ in $l^2(\mathbb{Z})$ defined inductively by $\xi_j=0$ and

$$\xi_{i-1}=(\tau^+\xi_i+\tau^-\xi_i)/2-r(\tau^+w_{i-1}-\tau^-w_{i-1})/2\tag{4.11}$$

for $1\le i\le j$. Since $r(\tau^+w_{i-1}-\tau^-w_{i-1})/2=h(\delta^+w_{i-1}+\delta^-w_{i-1})/2$, we have

$$\xi_i+h\sum_{p=i}^{j-1}(2^{-1}(\tau^++\tau^-))^{p-i}(2^{-1}(\delta^++\delta^-)w_p)=0\tag{4.12}$$

for $0 \le i \le j$. Consider the sequence $\{E_i\}_{i=0}^{j}$ in $\mathbb{R}$ defined by

$$E_i = (a_{i-1}+1)\|w_i\|^2 + \|z_i\|^2 + 2\langle z_i, (\tau^+\xi_i + \tau^-\xi_i)/2\rangle - a_{i-1}\|\xi_i\|^2$$

for $0 \le i \le j$. To obtain the recursive inequality (4.14) for $\{E_i\}_{i=0}^{j}$, let $1 \le i \le j$. Since $(a_{i-1}+1)r^2 \le (M+1)r^2 \le 1$, we apply Lemma 2 with $(a, f, g, \xi, \eta, w, z) = (a_{i-1}, w_i - hf_i, z_i - hg_i, \xi_{i-1}, (\tau^+\xi_i + \tau^-\xi_i)/2, w_{i-1}, z_{i-1})$ to the system (4.10) and (4.11), so that

$$\begin{aligned}
&(a_{i-1}+1)\|w_i - hf_i\|^2 + \|z_i - hg_i\|^2 \\
&+2\langle z_i - hg_i, (\tau^+\xi_i + \tau^-\xi_i)/2\rangle - a_{i-1}\|(\tau^+\xi_i + \tau^-\xi_i)/2\|^2 \\
\le\ &(a_{i-1}+1)\|w_{i-1}\|^2 + \|z_{i-1}\|^2 \\
&+2\langle(\tau^+ z_{i-1} + \tau^- z_{i-1})/2, \xi_{i-1}\rangle - a_{i-1}\|\xi_{i-1}\|^2.
\end{aligned}$$

Since $a_{i-2} \le a_{i-1}$ and $\|(\tau^+\xi_i + \tau^-\xi_i)/2\| \le \|\xi_i\|$, we find

$$\begin{aligned}
E_i \le\ &E_{i-1} + (a_{i-1} - a_{i-2})\|w_{i-1}\|^2 + 2h(a_{i-1}+1)\langle w_i, f_i\rangle + 2h\langle z_i, g_i\rangle \\
&+2h\langle g_i, (\tau^+\xi_i + \tau^-\xi_i)/2\rangle.
\end{aligned} \tag{4.13}$$

Since $\langle\delta^+ u, v\rangle + \langle u, \delta^- v\rangle = 0$ for $u, v \in l^2(\mathbb{Z})$, we see by (4.12) that the last term on the right-hand side of (4.13) is equal to

$$2h^2 \sum_{p=1}^{j-1} \langle 2^{-1}(\delta^+ + \delta^-)g_i, (2^{-1}(\tau^+ + \tau^-))^{p-i+1} w_p\rangle.$$

Since $2\langle u, v\rangle \le \|u\|^2 + \|v\|^2$ for $u, v \in l^2(\mathbb{Z})$, it follows that

$$\begin{aligned}
E_i \le\ &E_{i-1} + Lh\|w_{i-1}\|^2 + (M+1)h(\|w_i\|^2 + \|f_i\|^2) \\
&+h(\|z_i\|^2 + \|g_i\|^2) + (j-1)h^2\|\delta^- g_i\|^2 + h^2 \sum_{p=1}^{j-1}\|w_p\|^2.
\end{aligned} \tag{4.14}$$

Since $\xi_j = 0$ and $a_{j-1} \ge 0$, we have $\|w_j\|^2 + \|z_j\|^2 \le E_j$. Adding (4.14) from $i = 1$ to $i = j$, we find

$$\begin{aligned}
\|w_j\|^2 + \|z_j\|^2 \le\ &(M+1)M_1 + M_2 + 2\langle z_0, 2^{-1}(\tau^+ + \tau^-)\xi_0\rangle \\
&+Lh\sum_{i=0}^{j-1}\|w_i\|^2 + (M+1)h\sum_{i=1}^{j}\|w_i\|^2 \\
&+h\sum_{i=1}^{j}\|z_i\|^2 + Th\sum_{i=1}^{j}\|\delta^- g_i\|^2 + Th\sum_{p=1}^{j-1}\|w_p\|^2.
\end{aligned} \tag{4.15}$$

The third term on the right-hand side of (4.15) is estimated by $hj\|\delta^- z_0\|^2 + h\sum_{p=0}^{j-1}\|w_p\|^2$, since it is written as $2h\sum_{p=0}^{j-1}\langle (2^{-1}(\tau^+ + \tau^-))^{p+1}(2^{-1}(\delta^+ + \delta^-)z_0), w_p\rangle$ by (4.12). It follows that

$$\begin{aligned}\|w_j\|^2 + \|z_j\|^2 \quad &\le (M+1)M_1 + M_2 + TM_3 + (L+T+1)h\textstyle\sum_{p=0}^{j-1}\|w_p\|^2 \\ &\qquad + (M+1)h\textstyle\sum_{i=1}^{j}\|w_i\|^2 + h\sum_{i=1}^{j}\|z_i\|^2\end{aligned}$$

for $0 \le j \le K$. By A_j we denote the right-hand side. Then, we have $\|w_j\|^2 + \|z_j\|^2 \le A_j$ for $0 \le j \le K$ and $A_j - A_{j-1} \le (L+T+1)hA_{j-1} + (M+1)hA_j$ for $1 \le j \le K$. Since $1 + t \le \exp(t)$ for $t \ge 0$ and $(1-t)^{-1} \le \exp(2t)$ for $0 \le t \le 1/2$, we have $A_j \le \exp((2(M+1) + L + T + 1)h)A_{j-1}$ for $1 \le j \le K$. The desired inequality is obtained by solving this inequality and using the fact that $A_0 = (M+1)M_1 + M_2 + TM_3$. ■

Now, we show that for each $n \ge 1$, the family $\{C_n((w,z)); (w,z) \in D_n\}$ defined by (4.6) satisfies the stability condition (H4). Since $(d/d\theta)\|\theta w + (1-\theta)\hat{w}\|_n^{2\alpha} = 2\alpha\|\theta w + (1-\theta)\hat{w}\|_n^{2(\alpha-1)}\langle \theta w + (1-\theta)\hat{w}, w - \hat{w}\rangle_n$, where $\langle u, v\rangle_n = \sum_{l=-\infty}^{\infty} u_l v_l k_n$ for $u = (u_l), v = (v_l) \in l^2(\mathbb{Z})$, we have

$$\big|\|w\|_n^{2\alpha} - \|\hat{w}\|_n^{2\alpha}\big| \le 2\alpha \max(\|w\|_n, \|\hat{w}\|_n)^{2\alpha-1}\|w - \hat{w}\|_n \tag{4.16}$$

for $w, \hat{w} \in l^2(\mathbb{Z})$. Since $r(\tau_n^+ u - \tau_n^- u)/2 = h_n(\delta_n^+ u + \delta_n^- u)/2$, (H4-iv) follows from (4.16). Hypothesis (H4-iii) is automatically satisfied, since $E_n = l^2(\mathbb{Z}) \times l^2(\mathbb{Z})$.

To check (H4-i), let $M = R^{2\alpha}$, $0 < r \le 1/(M+1)^{1/2}$ and $L = 2\alpha(1/r+1)R^{2\alpha}$. Let $(w_0, z_0) \in D_{0,n}$. Since $\lim_{n\to\infty}\|2^{-1}(\delta_n^+ + \delta_n^-)(p_n v_0)\|_n = \|\partial_x v_0\|_{L^2} > r_0$ (by Lemma 4 (i)), $z_0 = p_n v_0$ and $\lim_{n\to\infty} h_n = 0$, there exists an integer $n_0 \ge 1$ such that $(M+1)h_n \le 1/2$,

$$r_0 \le \|2^{-1}(\delta_n^+ + \delta_n^-)z_0\|_n, \tag{4.17}$$

$$\|z_0\|_n^2 + \|\delta_n^- z_0\|_n^2 + \|\delta_n^+\delta_n^- z_0\|_n^2 + \|\delta_n^-\delta_n^+\delta_n^- z_0\|_n^2 \le R_0^2 \tag{4.18}$$

for $n \ge n_0$. Here (4.18) follows from Lemma 4 (ii), since $\|v_0\|_{H^3} \le R_0$. Let $n \ge n_0$. Then it will be proved that there exists $T > 0$ such that hypothesis (H4-i) is satisfied, by showing inductively that the sequence $\{(w_i, z_i)\}_{i=1}^{K_n}$, defined by

$$(w_i, z_i) = C_n((w_{i-1}, z_{i-1}))(w_{i-1}, z_{i-1}) \tag{4.19}$$

for $1 \le i \le K_n$, satisfies the following conditions:

$$0 \le \|w_j\|_n^{2\alpha} \le M \qquad \text{for } 0 \le j \le K_n, \tag{4.20}$$

$$(4.21)\qquad 0 \le \|w_j\|_n^{2\alpha} - \|w_{j-1}\|_n^{2\alpha} \le Lh_n \qquad \text{for } 0 \le j \le K_n,$$

$$(4.22)\qquad (w_j, z_j) \in D_n \qquad \text{for } 0 \le j \le K_n,$$

where $w_{-1} = w_0$. Since $w_0 = 0$ and $D_{0,n} \subset D_n$, (4.20) through (4.22) are clearly true for $j = 0$. Assume that (4.20) through (4.22) hold for $0 \le j \le i-1$. Then, (w_i, z_i) is well-defined by (4.19). By (4.6), the sequence $\{(w_j, z_j)\}_{j=0}^{i}$ satisfies the system

$$(4.23)\qquad \begin{cases} w_j = (\tau^+ w_{j-1} + \tau^- w_{j-1})/2 + r(\tau^+ z_{j-1} - \tau^- z_{j-1})/2, \\ z_j = (\tau^+ z_{j-1} + \tau^- z_{j-1})/2 + r\|w_{j-1}\|_n^{2\alpha}(\tau^+ w_{j-1} - \tau^- w_{j-1})/2 \end{cases}$$

for $1 \le j \le i$. Since (4.20) and (4.21) hold for $1 \le j \le i-1$, we apply Lemma 3 with $K = i$, $f_j = 0$, $g_j = 0$ and $a_j = \|w_j\|_n^{2\alpha}$ to find the inequality

$$(4.24)\qquad \|w_i\|_n^2 + \|z_i\|_n^2 \le \exp((2M + L + T + 3)T)(\|z_0\|_n^2 + T\|\delta_n^- z_0\|_n^2),$$

since $w_0 = 0$ by the definition of $D_{0,n}$. Since the two sequences $\{(\delta_n^- w_j, \delta_n^- z_j)\}_{j=0}^{i}$ and $\{(\delta_n^+\delta_n^- w_j, \delta_n^+\delta_n^- z_j)\}_{j=0}^{i}$ satisfy the systems similar to (4.23), we have by Lemma 3

$$(4.25)\qquad \|\delta_n^- w_i\|_n^2 + \|\delta_n^- z_i\|_n^2 \le \exp((2M+L+T+3)T)(\|\delta_n^- z_0\|_n^2 + T\|\delta_n^+\delta_n^- z_0\|_n^2)$$

and

$$(4.26)\qquad \begin{aligned} &\|\delta_n^+\delta_n^- w_i\|_n^2 + \|\delta_n^+\delta_n^- z_i\|_n^2 \\ &\quad \le \exp((2M + L + T + 3)T)(\|\delta_n^+\delta_n^- z_0\|_n^2 + T\|\delta_n^-\delta_n^+\delta_n^- z_0\|_n^2). \end{aligned}$$

If $T > 0$ is chosen so that $\exp(\alpha(2M + L + T + 3)T)(1 + T)^\alpha R_0^{2\alpha} \le M$, then the inequality (4.20) is true for $j = i$ by (4.24) combined with (4.18). If $T > 0$ is chosen so that $\exp((2M + L + T + 3)T)(1 + T)R_0^2 \le R^2$, then condition (4.22) is satisfied for $j = i$, by (4.18), (4.24), (4.25) and (4.26). By (4.23) with $j = i$ we have

$$w_i - w_{i-1} = h_n((k_n/h_n)2^{-1}(\delta_n^+ - \delta_n^-)w_{i-1} + 2^{-1}(\delta_n^+ + \delta_n^-)z_{i-1}).$$

Hence $\|w_i - w_{i-1}\|_n \le h_n((1/r)R + R)$. By (4.16) we have

$$\big|\|w_i\|_n^{2\alpha} - \|w_{i-1}\|_n^{2\alpha}\big| \le 2\alpha R^{2\alpha} h_n(1/r + 1).$$

Since $\|w_i\|_n^{2\alpha} - \|w_{i-1}\|_n^{2\alpha} \ge 2\alpha\|w_{i-1}\|_n^{2(\alpha-1)}\langle w_{i-1}, w_i - w_{i-1}\rangle_n$ by convexity, the desired inequality (4.21) will be proved, if $T > 0$ is chosen so that $\langle w_{i-1}, w_i - w_{i-1}\rangle_n \ge 0$. Since $w_0 = 0$ and

$$(4.27)\qquad w_j - w_{j-1} = h_n^2(k_n/h_n)^2 2^{-1}\delta_n^+\delta_n^- w_{j-1} + h_n 2^{-1}(\delta_n^+ + \delta_n^-)z_{j-1}$$

for $1 \le j \le i$, we have

$$w_{i-1} = h_n^2 (k_n/h_n)^2 \sum_{j=1}^{i-1} 2^{-1}\delta_n^+ \delta_n^- w_{j-1} + h_n \sum_{j=1}^{i-1} 2^{-1}(\delta_n^+ + \delta_n^-) z_{j-1}. \tag{4.28}$$

Similarly, we have

$$z_j = z_0 + \sum_{p=1}^{j} \big(h_n (k_n/h_n) 2^{-1}(\delta_n^+ - \delta_n^-) z_{p-1} + h_n \|w_{p-1}\|_n^{2\alpha} 2^{-1}(\delta_n^+ + \delta_n^-) w_{p-1} \big) \tag{4.29}$$

for $0 \le j \le i$. Substituting (4.29) into (4.28) and estimating the resulting equality, we find

$$\begin{aligned} &\|w_{i-1} - h_n(i-1)2^{-1}(\delta_n^+ + \delta_n^-) z_0\|_n \\ &\quad \le h_n^2(k_n/h_n)^2(i-1)2^{-1}R + h_n^2(i-1)^2\{2^{-1}(k_n/h_n)R + R^{2\alpha+1}\}. \end{aligned} \tag{4.30}$$

By (4.27) and (4.29) we have, in a way similar to the derivation of (4.30),

$$\begin{aligned} &\|w_i - w_{i-1} - 2^{-1}h_n(\delta_n^+ + \delta_n^-) z_0\|_n \\ &\quad \le 2^{-1}h_n^2(k_n/h_n)^2 R + h_n^2(i-1)\{2^{-1}(k_n/h_n)R + R^{2\alpha+1}\}. \end{aligned} \tag{4.31}$$

Combining (4.30) and (4.31), we find

$$\begin{aligned} &\langle w_{i-1}, w_i - w_{i-1}\rangle_n \\ &\ge h_n^2(i-1)\{\|2^{-1}(\delta_n^+ + \delta_n^-) z_0\|_n^2 \\ &\quad -2R_0(2^{-1}h_n(k_n/h_n)^2 R + h_n(i-1)(2^{-1}(k_n/h_n)R + R^{2\alpha+1})) \\ &\quad -(2^{-1}h_n(k_n/h_n)^2 R + h_n(i-1)(2^{-1}(k_n/h_n)R + R^{2\alpha+1}))^2\} \\ &\ge h_n^2(i-1)\{r_0^2 - RT(R/r^2 + 2R^{2\alpha+1}) - (T(R/(2r^2) + R^{2\alpha+1}))^2\}. \end{aligned}$$

Here we have used (4.17), (4.18) and the fact that $r^2 \le r$. Since $r_0 > 0$ it is possible to choose $T > 0$ independently of n, i such that $\langle w_{i-1}, w_i - w_{i-1}\rangle_n \ge 0$ for all $n \ge n_0$. Hypothesis (H4-i) is thus shown to be satisfied. Since the sequence $\{(w_i, z_i)\}_{i=1}^{K_n}$ defined by (4.19) satisfies (4.20) and (4.21), Hypothesis (H4-ii) is checked by Lemma 3.

5. Appendix

In this section we study some properties of the operator p_n from $L^2(\mathbb{R})$ into $l^2(\mathbb{Z})$ defined by (4.3). It is known [27] that

$$\|p_n u\|_n \le \|u\|_{L^2} \quad \text{and} \quad \lim_{n\to\infty} \|p_n u\|_n = \|u\|_{L^2} \quad \text{for } u \in L^2(\mathbb{R}).$$

Lemma 4. *The following assertions hold:*

(i) $\lim_{n\to\infty} \|p_n(\partial_x^i u) - (\delta_n^-)^i p_n u\|_n = 0$ *for* $u \in H^i(\mathbb{R})$ *and* $i \geq 0$.

(ii) $\|(\delta_n^-)^i p_n u\|_n \leq \|\partial_x^i u\|_{L^2}$ *for* $u \in H^i(\mathbb{R})$ *and* $i \geq 0$.

(iii) $\lim_{n\to\infty} \|\tau^+(p_n u) - p_n u\|_n = 0$ *for* $u \in L^2(\mathbb{R})$.

Proof. We employ the operator ∇_n on $L^2(\mathbb{R})$ defined by

$$(\nabla_n w)(x) = k_n^{-1}(w(x) - w(x - k_n)) \qquad \text{for } w \in L^2(\mathbb{R}).$$

Since $(\nabla_n w)(x) = \int_0^1 (\partial_x w)(x+(\theta-1)k_n)d\theta$ for $w \in H^1(\mathbb{R})$, we have $\|\nabla_n w\|_{L^2} \leq \|\partial_x w\|_{L^2}$ and $\lim_{n\to\infty} \|\nabla_n w - \partial_x w\|_{L^2} = 0$ for $w \in H^1(\mathbb{R})$, by the Riemann-Lebesgue theorem.

Let $k \geq 1$ and $u \in H^k(\mathbb{R})$. Assume that (i) and (ii) hold for $0 \leq i \leq k-1$. Since

$$\delta_n^-(p_n u) = p_n(\nabla_n u), \tag{5.2}$$

we have by (ii) with $i = k-1$

$$\begin{aligned} &\|(\delta_n^-)^{k-1}(p_n(\partial_x u) - \delta_n^-(p_n u))\|_n \\ &\quad \leq \|\partial_x^{k-1}(\partial_x u - \nabla_n u)\|_{L^2} = \|\partial_x(\partial_x^{k-1}u) - \nabla_n(\partial_x^{k-1}u)\|_{L^2} \end{aligned}$$

and the right-hand side vanishes as $n \to \infty$ by the first part of the proof. This fact and (i) with $i = k-1$ and u replaced by $\partial_x u$ together imply that (i) holds for $i = k$. By (5.2) and the first part of the proof, we show that (ii) is true for $i = k$ in the way that

$$\begin{aligned} \|(\delta_n^-)^k p_n u\|_n &= \|(\delta_n^-)^{k-1} p_n(\nabla_n u)\|_n \\ &\leq \|\partial_x^{k-1}(\nabla_n u)\|_{L^2} = \|\nabla_n(\partial_x^{k-1}u)\|_{L^2} \leq \|\partial_x(\partial_x^{k-1}u)\|_{L^2}. \end{aligned}$$

Since $\tau^+(p_n u) - p_n u = p_n(\tau_{k_n} u - u)$ where $(\tau_{k_n} w)(x) = w(x + k_n)$, we have $\|\tau^+ p_n u - p_n u\|_n \leq \|\tau_{k_n} u - u\|_{L^2}$ by (ii) with $i = 0$. Assertion (iii) is a direct consequence of the Riemann-Lebesgue theorem. ∎

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