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ASYMPTOTIC BEHAVIOR OF SOME WAVELET SERIES

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Abstract. In this paper, the asymptotic behavior of wavelet series at a neighborhood of a point of divergence is investigated. Our results extend the works of Reyes [8, 9].

1. Introduction

Let $0 < s \leq 1$ and $C^s(\mathbb{R})$ be the Hölder space of all bounded, continuous functions on $\mathbb{R}$ such that $|f(x) - f(y)| \leq C|x - y|^s$ for some constant C. The wavelet series in question is of the form

$$F(x) = \sum_{j=1}^{\infty} c_j \psi(2^{n_j} x - k_j), \tag{1.1}$$

where $\{c_j\}_{j=1}^{\infty}$ is a bounded complex sequence, and ψ, n_j, k_j satisfy conditions $(i) - (iv)$, stated below:

(i) $\psi \in C^s(\mathbb{R})$ and there exist $C > 0, N > 0$ such that

$$|\psi(x)| \leq C(1 + |x|)^{-N} \qquad (x \in \mathbb{R}),$$

(ii) $n_j \in \mathbb{N}$ and $k_j \in \mathbb{Z}$ such that $n_1 < n_2 < \cdots$ and

$$\sup_{j \in \mathbb{N}} \quad (n_{j+1} - n_j) < \infty,$$

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(iii) there exists $x_0 \in \mathbb{R}$ for which

$$\theta_j := 2^{n_j} x_0 - k_j \longrightarrow \theta^* \in \mathbb{R} \qquad (\text{as} \quad j \to \infty),$$

(iv) the sequence $\{j n_j{}^{-1}\}_{j=1}^{\infty}$ converges to a real number q^*.

By an elementary arguement, we can easily see that (ii) is equivalent to (ii^*):

(ii^*) $n_j \in \mathbb{N}$ and $k_j \in \mathbb{Z}$ such that $n_1 < n_2 < \cdots$ and $\{n_j\}_{j=1}^{\infty}$ is relatively dense in $\mathbb{N}$ in the sense that for some $M \in \mathbb{N}$,

$$\{l+1, \cdots, l+M\} \cap \{n_1, n_2, \cdots\} \neq \phi \qquad \text{for every integer} \quad l \geq 0.$$

In [8, 9], Reyes investigated the pointwise asymptotic behavior of F near x_0 for the case that ψ has a bounded derivative and $\{c_j\}_{j=1}^{\infty}$ is nonnegative. The purpose of this paper is to generalize Reyes' results in the following three directions. First, we extend ψ with a bounded derivative to $\psi \in C^s(\mathbb{R})$. Second, we relax $\{c_j\}_{j=1}^{\infty}$ from nonnegative sequences to complex sequences. The last one is to establish the following two formulas for the case that $j^{\alpha} c_j \to A \in \mathbb{C}$:

$$F(x_0 + \delta) \sim A C_{\alpha} \psi(\theta^*) (\log(|\delta|^{-1}))^{1-\alpha} \qquad (0 \leq \alpha < 1), \tag{1.2}$$

$$F(x_0 + \delta) \sim A \psi(\theta^*) \log\log(|\delta|^{-1}) \qquad (\alpha = 1), \tag{1.3}$$

where $C_{\alpha} = (1-\alpha)^{-1} (q^* / \log 2)^{1-\alpha}$. For $c_j = j^{-\alpha}$, (1.2)-(1.3) take the form

$$\sum_{j=1}^{\infty} j^{-\alpha} \psi(\theta_j + 2^{n_j} \delta) \sim C_{\alpha} \psi(\theta^*) (\log(|\delta|^{-1}))^{1-\alpha} \quad (0 \leq \alpha < 1), \tag{1.4}$$

$$\sum_{j=1}^{\infty} j^{-1} \psi(\theta_j + 2^{n_j} \delta) \sim \psi(\theta^*) \log\log(|\delta|^{-1}). \tag{1.5}$$

They are analogous to the ones established in [1, 3, 4, 6, 7, 10] for trigonometric series. The details will be thoroughly discussed in §2 and §3

2. Main Results

Theorem 2.1. *Assume that* $\{c_j\}_{j=1}^{\infty} \in \ell^{\infty}, \sum_{j=1}^{\infty} |c_j| = \infty$, *and* $(i) - (iii)$ *are satisfied. Then*

$$F(x_0 + \delta) = \psi(\theta^*) \sum_{j=1}^{r(\delta)} c_j + o\left(\sum_{j=1}^{r(\delta)} |c_j| \right) \quad (\text{as} \quad \delta \to 0), \tag{2.1}$$

where $r(\delta) := \min\{r \in \mathbb{N} : 2^{n_r}|\delta| \geq 1\}$.

The symbol ℓ^∞ denotes the space consisting of all bounded sequences. Obviously, if $c_j \geq 0$ for all j, then (2.1) is the same as

$$\lim_{\delta\to 0}\left(\sum_{j=1}^{r(\delta)} c_j\right)^{-1} F(x_0+\delta) = \psi(\theta^*). \tag{2.2}$$

Hence, Theorem 2.1 generalizes [9, Theorem 1]. As shown in the proof of [9, Theorem 2], under (iv), we have

$$\lim_{\delta\to 0}\frac{r(\delta)}{s(\delta)} = 1, \tag{2.3}$$

where $s(\delta) := [q^*(\log 2)^{-1}\log(|\delta|^{-1})]$. This leads us to the following result.

Theorem 2.2. *Assume that* $\{c_j\}_{j=1}^\infty \in \ell^\infty, \sum_{j=1}^\infty |c_j| = \infty$, *and there exists a constant* K *such that*

$$|c_{n+j}| \leq K|c_n| \qquad (1 \leq j \leq n). \tag{2.4}$$

If $(i) - (iv)$ *are satisfied, then*

$$F(x_0+\delta) = \psi(\theta^*)\sum_{j=1}^{s(\delta)} c_j + o\left(\sum_{j=1}^{s(\delta)} |c_j|\right) \qquad (\text{as} \quad \delta \to 0). \tag{2.5}$$

For $c_j \geq 0$, (2.5) can be restated in the form

$$\lim_{\delta\to 0}\left(\sum_{j=1}^{s(\delta)} c_j\right)^{-1} F(x_0+\delta) = \psi(\theta^*). \tag{2.6}$$

It is obvious that (2.4) is satisfied by those nonnegative sequences $\{c_j\}_{j=1}^\infty$ with c_j/R_j decreasing for some nondecreasing sequence $\{R_j\}_{j=1}^\infty$ of positive numbers subject to the condition: $\sup_{j\geq 1} R_{2j}/R_j < \infty$. Any of such $\{c_j\}_{j=1}^\infty$ is said to be an O-regularly varying quasimonotone sequence ($cf.$ [2]). In particular, nonincreasing null sequences belong to such a class. Thus, Theorem 2.2 generalizes [9, Theorem 2]. For $\delta \to 0$, we have

$$\sum_{j=1}^{s(\delta)} \frac{1}{j^\alpha} \sim \begin{cases} (1-\alpha)^{-1}[q^*(\log 2)^{-1}\log(|\delta|^{-1})]^{1-\alpha} & (0 \leq \alpha < 1), \\ \log\log(|\delta|^{-1}) & (\alpha = 1). \end{cases}$$

Applying Theorem 2.2 to the case $c_j = j^{-\alpha}$, $(1.4) - (1.5)$ will be derived. The next theorem allows us to extend them to $(1.2) - (1.3)$ for the case:

$$j^\alpha c_j \to A \qquad \text{as} \quad j \to \infty. \tag{2.7}$$

Theorem 2.3. *Let* $A \in \mathbb{C}, 0 \le \alpha \le 1$, *and* $\{c_j\}_{j=1}^{\infty} \in \ell^{\infty}$. *If* $(i) - (iv)$ *and* (2.7) *are satisfied, then* $(1.2) - (1.3)$ *hold.*

It is clear that formula (1.2) with $\alpha = 0$ reduces to

$$\lim_{\delta \to 0} \frac{F(x_0 + \delta)}{\log(|\delta|^{-1})} = \lambda_0 \psi(\theta^*), \tag{2.8}$$

where $\lambda_0 = (\log 2)^{-1} q^*(\lim_{j\to\infty} c_j)$. Hence, Theorem 2.3 generalizes [8, Theorem 1]. Consider the case $c_j = j^{-s}$, where $s > 1$. We have $\lambda_0 = 0$. Thus, applying [8, Theorem 1] (i.e. (2.8)) to such a case, we only obtain

$$\lim_{\delta \to 0} \frac{F(x_0 + \delta)}{\log(|\delta|^{-1})} = 0. \tag{2.9}$$

In contrast, $jc_j \to 0 = A$, so Theorem 2.3 will lead us to

$$\lim_{\delta \to 0} \frac{F(x_0 + \delta)}{\log\log(|\delta|^{-1})} = 0,$$

which is better than (2.9). The same example also satisfies $\sum_{j=1}^{\infty} |c_j| < \infty$. Therefore, Theorem 2.2 cannot apply to such a case. This differs Theorem 2.3 from Theorem 2.2.

3. Proofs

Proof of Theorem 2.1. Let $|\delta| \le 1$. Set

$$S(\delta) = \sum_{j=1}^{r(\delta)} c_j \{\psi(\theta_j + 2^{n_j}\delta) - \psi(\theta^*)\},$$
$$R(\delta) = \sum_{j > r(\delta)} c_j \psi(\theta_j + 2^{n_j}\delta).$$

Then

$$F(x_0 + \delta) = \psi(\theta^*) \sum_{j=1}^{r(\delta)} c_j + S(\delta) + R(\delta). \tag{3.1}$$

As proved in [9, Theorem 1],

$$|R(\delta)| \le \left(\sup_{j > r(\delta)} |c_j|\right) \left\{ j_0 \|\psi\|_{\infty} + \frac{2^N C}{1 - 2^{-N}} \right\} = o\left(\sum_{j=1}^{r(\delta)} |c_j|\right), \tag{3.2}$$

where j_0 is a positive integer with $|\theta_j| \le 2^{j_0-1}$ for all $j \ge 1$. We have assumed that $\psi \in C^s(\mathbb{R})$. Thus, there exists $P > 0$ such that $|\psi(x)-\psi(y)| \le P|x-y|^s$ for all $x, y \in \mathbb{R}$. This implies

$$\begin{aligned} |S(\delta)| &\le P\sum_{j=1}^{r(\delta)}|c_j||\theta_j + 2^{n_j}\delta - \theta^*|^s \\ &\le 2^sP\Bigg(\sum_{j=1}^{r(\delta)}|c_j||\theta_j - \theta^*|^s\Bigg) + 2^sP\Bigg(|\delta|^s\sum_{j=1}^{r(\delta)}2^{n_js}|c_j|\Bigg) \\ &= S_1(\delta)+S_2(\delta), \text{say.} \end{aligned} \tag{3.3}$$

Since $\sum_{j=1}^{\infty}|c_j| = \infty$, it follows from [5, Theorem 12] that the method $(\bar{N}, p_n)$ with $p_n = |c_n|$ is regular. We have $|\theta_j - \theta^*|^s \to 0$ as $j \to \infty$, so

$$S_1(\delta) = o\Bigg(\sum_{j=1}^{r(\delta)}|c_j|\Bigg) \qquad (\text{as} \quad \delta \to 0). \tag{3.4}$$

The definition of $r(\delta)$ gives $|\delta| \le 2^{-n_{r(\delta)-1}} \le Q2^{-n_{r(\delta)}}$, where $\log_2 Q = \sup_j(n_{j+1} - n_j)$. Hence,

$$\begin{aligned} |S_2(\delta)| &\le 2^sPQ^s\Big(\sup\nolimits_{j\ge1}|c_j|\Big)2^{-n_{r(\delta)}s}\sum_{j=1}^{r(\delta)}2^{n_js} \\ &\le \frac{2^sPQ^s}{s\ln 2}\Big(\sup_{j\ge1}|c_j|\Big) = o\Bigg(\sum_{j=1}^{r(\delta)}|c_j|\Bigg). \end{aligned} \tag{3.5}$$

Putting (3.1) – (3.5) together yields (2.1). This is what we want. ■

Proof of Theorem 2.2. Rewrite (2.1) into the form

$$F(x_0+\delta) = \psi(\theta^*)\sum_{j=1}^{s(\delta)}c_j + \text{error terms.}$$

To compare such a form with (2.5), we see that it suffices to show

$$\sum_{j=m(\delta)+1}^{M(\delta)}|c_j| = o\Bigg(\sum_{j=1}^{m(\delta)}|c_j|\Bigg) \qquad \text{as} \quad \delta \to 0, \tag{3.6}$$

where $m(\delta) = \min\{r(\delta), s(\delta)\}$ and $M(\delta) = \max\{r(\delta), s(\delta)\}$. From (2.3), we see $M(\delta)/m(\delta) \to 1$ as $\delta \to 0$. This indicates that $M(\delta) \le 2m(\delta)$ as δ is small enough. Thus, (2.4) implies

$$\sum_{j=m(\delta)+1}^{M(\delta)} |c_j| \leq K(M(\delta)-m(\delta))|c_{m(\delta)}|$$

and

$$\sum_{j=1}^{m(\delta)} |c_j| \geq \frac{m(\delta)}{2K}|c_{m(\delta)}|.$$

Putting these together yields

$$\begin{aligned}\sum_{j=m(\delta)+1}^{M(\delta)} |c_j| &\leq 2K^2\left(\frac{M(\delta)}{m(\delta)}-1\right)\sum_{j=1}^{m(\delta)}|c_j| \\ &= o\left(\sum_{j=1}^{m(\delta)}|c_j|\right) \qquad (\text{as} \quad \delta \to 0),\end{aligned}$$

and so the desired result follows. ∎

Proof of Theorem 2.3. First, consider the case $0 \leq \alpha < 1$. Then Theorem 2.2 ensures the validity of (1.4). Let $0 < |\delta| < 1$. We have

$$\begin{aligned}&\left|F(x_0+\delta) - A\sum_{j=1}^{\infty} j^{-\alpha}\psi(\theta_j + 2^{n_j}\delta)\right| \\ &\leq \left|\sum_{j\leq r(\delta)} (c_j - Aj^{-\alpha})\psi(\theta_j+2^{n_j}\delta)\right| \\ &\quad +\left(\sup_j |c_j - Aj^{-\alpha}|\right)\sum_{j>r(\delta)} |\psi(\theta_j+2^{n_j}\delta)| \\ &= S_1(\delta)+S_2(\delta), \quad \text{say,}\end{aligned} \tag{3.7}$$

where $r(\delta) := \min\{r \in \mathbb{N} : 2^{n_r}|\delta| \geq 1\}$. As (3.2) indicated,

$$\begin{aligned}|S_2(\delta)| &\leq \left(\sup_{j\in\mathbb{N}} |c_j| + |A|\right)\left\{j_0\|\psi\|_\infty + \tfrac{2^N C}{1-2^{-N}}\right\} \\ &= o(\log(|\delta|^{-1}))^{1-\alpha} \qquad \text{as} \quad \delta \to 0.\end{aligned} \tag{3.8}$$

On the other hand, for $1 \le M \le r(\delta)$, we have

$$\begin{aligned}|S_1(\delta)| &\le M\|\psi\|_\infty\left(\sup\nolimits_{1\le j\le M}\frac{|j^\alpha c_j - A|}{j^\alpha}\right)\\ &\quad+\|\psi\|_\infty\left(\sup\nolimits_{M<j\le r(\delta)}|j^\alpha c_j - A|\right)\sum_{j=M+1}^{r(\delta)} j^{-\alpha}\\ &\le M\|\psi\|_\infty\left(\sup\nolimits_{j\ge 1}|j^\alpha c_j - A|\right)+\frac{\|\psi\|_\infty(r(\delta))^{1-\alpha}}{1-\alpha}\left(\sup_{j>M}|j^\alpha c_j - A|\right).\end{aligned}$$

It follows from (2.3) that $r(\delta) \sim q^*(\log 2)^{-1}\log(|\delta|^{-1})$, as $\delta \to 0$. This guarantees the existence of a constant K_α such that

$$|S_1(\delta)| \le \|\psi\|_\infty\left\{M\left(\sup_{j\ge 1}|j^\alpha c_j - A|\right)+K_\alpha(\log(|\delta|^{-1}))^{1-\alpha}\left(\sup_{j>M}|j^\alpha c_j - A|\right)\right\}.$$

By (2.7), we can choose M so large that $\sup_{j>M}|j^\alpha c_j - A|$ is as small as possible. Therefore,

$$|S_1(\delta)| = o(\log(|\delta|^{-1}))^{1-\alpha} \qquad \text{as} \quad \delta \to 0. \tag{3.9}$$

Putting (1.4) and (3.7)–(3.9) together yields (1.2). To replace (1.4) by (1.5) and to change $(\log(|\delta|^{-1}))^{1-\alpha}$ to $\log\log(|\delta|^{-1})$ for each occurrence, we see that the above proof still works for the case $\alpha = 1$. This means that (1.3) holds and the proof is complete. ■

REFERENCES

1. R. Bojanić and J. Karamata, On slowly varying functions and asymptotic relations Math. Research Center Tech. Report 432, Madison, Wis. 1963.
2. C.-P. Chen and G.-B. Chen, Uniform convergence of double trigonometric series *Studia Math*. **118** (1996), 245-259.
3. G. H. Hardy, A theorem concerning trigonometrical series, *J. London Math. Soc.* **3** (1928), 12-13.
4. ———, Some theorems concerning trigonometrical series of a special type, *Proc. London Math. Soc.* (2) **32** (1931), 441-448.
5. ———, *Divergent Series*, Oxford University Press, Oxford, 1949.
6. G. H. Hardy and W. W. Rogosinski, Notes on Fourier series (III): asymptotic formulae for the sums of certain trigonometric series, *Quart. J. Math. Oxford Ser.* (2) **16** (1945), 49-58.
7. J. R. Nurcombe, On trigonometric series with quasimonotone coefficients, *J. Math. Anal. Appl.* **178** (1993), 63-69.

8. N. Reyes, On the pointwise behavior of some lacunary wavelet series, *Acta Math. Vietnam.* **21** (1996), 147-153.

9. ———, An asymptotic formula for some wavelet series, *J. Approx. Theory* **89** (1997), 89-95.

10. A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1968, p. 186.

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