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STRONGLY APPROXIMATIVE SIMILARITY OF OPERATORS

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Abstract. For the bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$, we introduce a binary relation $\sim_{sas}$ called strongly approximative similarity. It lies between the similarity and the essential similarity. For a class of biquasitriangular operators and a class of quasitriangular operators, this relation is characterized respectively. As a result, the relation $\sim_{sas}$ is an equivalent relation in this two cases.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of the bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the ideal of the compact operators on $\mathcal{H}$. If dim $\mathcal{H} < \infty$, the Jordan canonical form provides a complete set of similarity invariants of operators. Certainly, one also hopes to obtain a complete set of similarity invariants of operators on infinite-dimensional spaces. For the normal operators, a complete set of similarity invariants has been given in terms of measure theory by Hellinger's multiplicity theory and R. G. Douglas' work (see [4]).

To continue our discussion, let us briefly mention some notations and terminologies (see [10]).

Recall that $T \in \mathcal{L}(\mathcal{H})$ is called a semi-Fredholm operator if ran T is closed and either nul $T < \infty$ or nul $T^* < \infty$, where nul $T = \dim \ker T$; in this case, we define the index of T by

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

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For $T \in \mathcal{L}(\mathcal{H})$, $\rho_{sF}(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is semi-Fredholm operator}\}$ is called the semi-Fredholm domain of T. $\sigma(T)$, $\sigma_r(T)$, $\sigma_e(T)$ and $\sigma_{lre}(T)$ denote the spectrum, the right spectrum, the essential spectrum and the Wolf spectrum of T, respectively. It is known that $\rho_{sF}(T) = \mathbb{C}\backslash\sigma_{lre}(T)$. We also write $\sigma_W(T)$ for the Weyl spectrum of T, i.e.,

$$\sigma_W(T) = \sigma_{lre}(T) \cup \{\lambda \in \rho_{sF} : \text{ind}(T-\lambda) \neq 0\}.$$

The spectral picture of $T \in \mathcal{L}(\mathcal{H})$, denoted by $\Lambda(T)$, is the compact set $\sigma_{lre}(T)$, plus the data corresponding to the indices of $T-\lambda$ for λ in the holes of $\sigma_{lre}(T)$. The set $\sigma_0(T)$ will be used for the normal eigenvalues of T; that is, any isolated point λ of $\sigma(T)$ for which the corresponding Riesz spectral subspace $\mathcal{H}(\lambda; T)$ is finite dimensional.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called strongly irreducible if it does not commute with any nontrivial idempotents (see [6, 13]). Z. J. Jiang [13] conjectured that the strongly irreducible operator is a suitable analogue of the Jordan block acting on finite-dimensional space. Hence, we should first describe a complete set of similarity invariants of strongly irreducible operators. Let us consider the following example.

Example 1.1. Let $S_i\,(i=1,2)$ be the unilateral weighted shift with weight sequence $\{(\frac{n+2}{n+1})^i\}_{n=0}^{+\infty}$, that is, $S_i e_n = (\frac{n+2}{n+1})^i\, e_{n+1}$, where $\{e_n\}_{n=0}^{+\infty}$ is an orthonormal basis (abb. ONB) of $\mathcal{H}$. Then we have the followings:

(i) S_i is strongly irreducible [16, p.63, Cor. 2];

(ii) $\sigma(S_1) = \sigma(S_2) = \overline{\mathbb{D}}$, the closure of the open unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$;

(iii) $\Lambda(S_1) = \Lambda(S_2)$ and $\sigma_0(S_1) = \sigma_0(S_2) = \emptyset$;

(iv) $S_i^* \in \mathcal{B}_1(\mathbb{D})$, where S_i^* is the adjoint of S_i and $\mathcal{B}_1(\mathbb{D})$ denotes the set of Cowen-Douglas operators of index 1 in $\mathbb{D}$ (see [5]);

(v) $\mathcal{A}'(S_i)$, the commutant of S_i, coincides with the WOT-closed subalgebra generated by S_i and the identity operator I. Hence, $\mathcal{A}'(S_i)$ is strongly strictly cyclic [16, p.99, Example 1];

(vi) S_i is reflexive [15, Prop. 37];

(vii) S_i is essentially normal, i.e., $S_i^*S_i - S_iS_i^* \in \mathcal{K}(\mathcal{H})$.

By Brown-Douglas-Fillmore Theorem [3], S_1 unitarily equivalent to some compact perturbation of S_2. However, S_1 is not similar to S_2. In fact, if there is an operator X such that $XS_1 = S_2X$, then $X = 0$ [16, Prop. 5].

Example 1.1 demonstrates that any complete set of similarity invariants would be so complicated that, maybe, it could not be described in terms of operator theory itself. On the other hand, for some purpose, so meticulous a classification is not necessary. For these reasons, one should study invariants for weakened notions of similarity.

For $A, B \in \mathcal{L}(\mathcal{H})$, A and B are said to be approximately similar, denoted by $A \sim_a B$, if $\overline{\mathcal{S}(A)} = \overline{\mathcal{S}(B)}$, where $\mathcal{S}(A) = \{XAX^{-1} : X \in \mathcal{L}(\mathcal{H}) \, \text{invertible}\}$ denotes the similarity orbit of A and $\overline{\mathcal{S}(A)}$ is the norm closure of $\mathcal{S}(A)$. The similarity orbit theorem [1, Thm. 9.2] provides a complete set of this weakened similarity invariants. But, it emploies much complicated terminologies. We may also consider another weakened similarity. $A \sim_{aw} B$ means that, for given $\epsilon > 0$, there exist A_1, $B_1 \in \mathcal{L}(\mathcal{H})$ with $\|A_1\| < \epsilon$ and $\|B_1\| < \epsilon$ such that $A + A_1 \sim B + B_1$, where $\sim$ denotes the similarity relation. However, the following example shows that this sort of weakened similarity is too extensive.

Example 1.2. Let $A = M_t$ be the multiplication operator on $L^2[0, 1]$ and $B = -M_t$. Then easy to see that $\sigma(A) = \sigma_e(A) = [0, 1]$ and $\sigma(B) = \sigma_e(B) = [-1, 0]$. By [11], there exists a quasinilpotent operator Q such that both A and B are in $\overline{\mathcal{S}(Q)}$. Thus $A \sim_{aw} B$.

Now, we consider the third weakened similarity. $A \sim_{ak} B$ means that, for given $\epsilon > 0$, there exist $K_i \in \mathcal{K}(\mathcal{H})$ with $\|K_i\| < \epsilon \, (i = 1, 2)$ such that $A+K_1 \sim B+K_2$. Clearly, if $A \sim_{ak} B$ then $\Lambda(A) = \Lambda(B)$. However, we don't know, in general, whether $\sigma_0(A)$ coincides with $\sigma_0(B)$ when $A \sim_{ak} B$. First of all, the relation is not transitive.

Notation. A and B are said to be strongly approximate similar, denoted by $A \sim_{sas} B$, if

(i) $A \sim_{ak} B$,

(ii) $\sigma_0(A)$=$\sigma_0(B)$ and $\dim \mathcal{H}(\lambda; A) = \dim\, \mathcal{H}(\lambda; B)$ for each $\lambda \in \sigma_0(A)$.

If $\dim\, \mathcal{H} < \infty$, it is not hard to prove that $A \sim_{sas} B$ if and only if they have the same characteristic polynomials.

So, A and B may have different Jordan canonical forms when $A \sim_{sas} B$. Thus, $\sim_{sas}$ is quite weak from the finite dimensional viewpoint. But, this classification is a dequate to some purpose.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be quasitriangular if there exists an increasing sequence $\{P_n\}_n$ of finite-rank projections in $\mathcal{L}(\mathcal{H})$ such that $P_n \to I$ strongly and $\|(I - P_n)TP_n\| \to 0$ as $n \to \infty$. T is biquasitriangular if both T and T^* are quasitriangular. We write (QT) and (BQT) for the class of the quasitriangular operators and the class of the biquasitriangular operators respectively.

Operators A and B are essentially similar if A is similar to some compact perturbation of B.

Our main results are the following theorems.

Theorem 1. *Let $A, B \in (BQT), \sigma_0(A) = \sigma_0(B) = \emptyset$ and $\sigma_e(A)$ connected. Then $A \sim_{sas} B$ if and only if A and B are essentially similar.*

Theorem 2. *Let $A, B \in (QT)$ satisfy the followings*:

(i) $\sigma_0(A) = \sigma_0(B) = \emptyset$,

(ii) *both* $\sigma_W(A)$ *and* $\Omega := \sigma_W(A) \backslash \sigma_{lre}(A)$ *are connected*,

(iii) $\mathrm{ind}\,(A - \lambda) = n < \infty$ *for each* $\lambda \in \Omega$ *and*

(iv) $\Omega = \mathrm{int}\,\overline{\Omega}$.

Then $A \sim_{sas} B$ if and only if A and B are essentially similar.

2. The BiQuasitriangular Case

In what follows, $P_{\mathcal{M}}$ always denotes the orthogonal projection onto a subspace $\mathcal{M}$ of $\mathcal{H}$.

Proposition 2.1. *Let $A, B \in \mathcal{L}(\mathcal{H})$ be such that $\sigma(A) = \sigma(B)$ containing only a single point. Then $A \sim_{sas} B$ if and only if there exist an invertible operator X and a compact operator K such that $XBX^{-1} = A + K$.*

Proof. Only the sufficiency need be proved. Without loss of generality, assume that $\sigma_{lre}(A) = \sigma(A) = \{0\}$. It follows that A is quasitriangular (see [8). Thus, given $\epsilon > 0$, there exists a compact operator C_1 with $\|C_1\| < \frac{\epsilon}{2}$ such that

$$A + C_1 = \begin{bmatrix} 0 & * & * & \cdots \\ & 0 & * & \cdots \\ & & 0 & \cdots \\ & & & \ddots \end{bmatrix}$$

with respect to a suitable ONB $\{e_n\}_{n=1}^{+\infty}$ of $\mathcal{H}$.

Set $C_2 = \sum\limits_{n=1}^{+\infty} \lambda_n e_n \otimes e_n$, where $\lambda_n = \frac{\epsilon}{2(n+1)\,\|X\|\,\|X^{-1}\|}$, and $K_1 = C_1 + C_2$. Then K_1 is compact, $\|K_1\| < \epsilon$ and $XBX^{-1} = (A + K_1) + (K - K_1)$. By the upper semicontinuity of the spectrum, there is a positive number $\delta < \frac{\epsilon}{2}$ such that $\sigma(C) \subset \sigma(B + W) \subset \{\lambda \in \mathbb{C} : |\lambda| < \frac{\epsilon}{8\|X\|\,\|X^{-1}\|}\}$, when $\|W\| < \delta$. Set $P_n = P_{\mathcal{H}_n}$, where $\mathcal{H}_n = \bigvee\{e_i; 1 \le i \le n\}$. Then there exists a positive integer n_0 such that

$$\|P_{n_0}(K - K_1)P_{n_0} - (K - K_1)\| < \frac{\delta}{\|X\|\|X^{-1}\|}.$$

Write $C_3 = X^{-1}[P_{n_0}(K-K_1)P_{n_0} - (K-K_1)]X$. Then C_3 is compact, $\|C_3\| < \frac{\epsilon}{2}$ and

$$X(B+C_3)X^{-1} = \begin{bmatrix} C & D \\ 0 & A_1 \end{bmatrix} \begin{array}{l} P_{n_0}\mathcal{H} \\ (P_{n_0}\mathcal{H})^\perp \end{array} .$$

Moreover, we can write

$$A + K_1 = \begin{bmatrix} A_0 & D \\ 0 & A_1 \end{bmatrix} \begin{array}{l} P_{n_0}\mathcal{H} \\ (P_{n_0}\mathcal{H})^\perp \end{array}$$

It is clear that $\sigma(C) \subset \sigma(B+C_3) \subset \{\lambda \in \mathbb{C} : |\lambda| < \frac{\epsilon}{8\|X\|\,\|X^{-1}\|}\}$. Hence, there exists an operator E acting on $P_{n_0}\mathcal{H}$ with $\|E\| < \frac{\epsilon}{2\|X\|\|X^{-1}\|}$ such that $C+E \sim A_0$. Set $C_4 = X^{-1}(E \oplus 0)X$. Then C_4 is compact, $\|C_4\| < \frac{\epsilon}{2}$ and

$$X(B+C_3+C_4)X^{-1} = \begin{bmatrix} C+E & D \\ 0 & A_1 \end{bmatrix} \sim \begin{bmatrix} A_0 & \overline{D} \\ 0 & A_1 \end{bmatrix}.$$

Claim. $\sigma(A_0) \bigcap \sigma(A_1) = \emptyset$.

Since $\sigma_{lre}(A_1) = \sigma_{lre}(A+K_1) = \sigma_{lre}(A) = \{0\}$, it follows that $\lambda_n \in rho_{sF}(A_1)$ and $\operatorname{ind}(A_1 - \lambda_n) = 0$ for $1 \le n \le n_0$. Since $\operatorname{nul}(A_1 - \lambda_n)^* = 0$, we have that $\lambda_n \notin \sigma(A_1)$. This proves the claim.

Now, it follows from [10, Cor. 3.22] that $\begin{bmatrix} A_0 & \overline{D} \\ 0 & A_1 \end{bmatrix} \sim A + K_1$. This completes the proof. ∎

Consider a class of invertible operators on $\mathcal{H}$. Set

$$(\mathcal{I}+\mathcal{K})(\mathcal{H}) = \{X \in \mathcal{L}(\mathcal{H}) : X \text{ is invertible and } X = I + K,\ K \in \mathcal{K}(\mathcal{H})\}.$$

Clearly, if $X \in (\mathcal{I}+\mathcal{K})(\mathcal{H})$ then $X^{-1} \in (\mathcal{I}+\mathcal{K})(\mathcal{H})$.

The $(\mathcal{I}+\mathcal{K})$ -orbit of an operator $T \in \mathcal{L}(\mathcal{H})$ is defined as

$$(\mathcal{I}+\mathcal{K})(T) = \{XTX^{-1} : X \in (\mathcal{I}+\mathcal{K})(\mathcal{H})\}$$

and $A \sim_{i+k} T$ means that $A \in (\mathcal{I}+\mathcal{K})(T)$. The notion of $(\mathcal{I}+\mathcal{K})$-orbit was introduced by P. S. Guinand and L. Marcoux [7], and the latter has studied it in several subsequent papers.

Lemma 2.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy the followings*:

(i) $B = A + K_0, K_0 \in \mathcal{K}(\mathcal{H})$;

(ii) $\sigma(A)$ *is a connected infinite set and* $\sigma(A) = \sigma(B)$;

(iii) *there exists a denumerable dense subset* $\Gamma = \{\lambda_n\}_{n=1}^{+\infty}$ *of* $\sigma(A)$ *such that* $\Gamma \subset \sigma_p(A)$ (*here* $\sigma_p(A)$ *denotes the set of the eigenvalues of* A), $\bigvee\{\ker(A-\lambda);\, \lambda \in \Gamma\} = \mathcal{H}$ *and* $\operatorname{nul}(A-\lambda) = 1$ *for all* $\lambda \in \Gamma$.

Then, given $\epsilon > 0$, *there exists a compact operator* K *with* $\|K\| < \epsilon$ *such that* $A \sim_{i+k} B + K$.

Proof. By condition (iii), A can be written as

$$A = \begin{pmatrix} \lambda_1 & & & * \\ & \ddots & & \\ & & \lambda_n & \\ 0 & & & \ddots \end{pmatrix}.$$

with respect to a suitable ONB $\{e_n\}_{n=1}^{+\infty}$ of $\mathcal{H}$.

As in the proof of Proposition 2.1, there exists a positive integer n_0 such that $K_1 := P_{n_0}K_0P_{n_0} - K_0$ is compact and $\|K_1\| < \frac{\epsilon}{4}$. Moreover,

$$B + K_1 = A + P_{n_0}K_0P_{n_0} = \begin{bmatrix} C & D \\ 0 & A_1 \end{bmatrix} \begin{matrix} P_{n_0}\mathcal{H} \\ (P_{n_0}\mathcal{H})^{\perp}, \end{matrix}$$

where $A = \begin{bmatrix} A_0 & D \\ 0 & A_1 \end{bmatrix}$ and $\sigma(C) \subset \sigma(A)_{\frac{\epsilon}{4}}$. Hence, there exists an operator E on $P_{n_0}\mathcal{H}$ with $\|E\| < \frac{\epsilon}{4}$ such that

$$C + E \sim \begin{bmatrix} \mu_1 & \cdots & * \\ & \ddots & \vdots \\ & & \mu_{n_0} \end{bmatrix} \begin{matrix} e_1 \\ \vdots \\ e_{n_0} \end{matrix} \triangleq C_1$$

and that $\mu_i \in \sigma(A)$ for $1 \le i \le n_0$. Thus, we can find an $X \in (\mathcal{I} + \mathcal{K})(\mathcal{H})$ and a compact K_2 with $\|K_2\| < \frac{\epsilon}{4}$ such that

$$X_1(B + K_1 + K_2)X_1^{-1} = \begin{bmatrix} C_1 & * \\ 0 & A_1 \end{bmatrix}.$$

By conditions (ii) and (iii), a subset $\{\lambda_{i_j} : 1 \le i \le n_0,\, 1 \le j \le k(i) < \infty\}$ of Γ can be chosen such that each of $|\mu_i - \lambda_{i_1}|$, $|\lambda_{i_j} - \lambda_{i_{j+1}}|$ and $|\lambda_{i_{k(i)}} - \lambda_i|$ is properly smaller than $\frac{\epsilon}{4\|X_1\|\,\|X_1^{-1}\|}$ for $1 \le i \le n_0$ and $1 \le j \le k(i)$.

Now, set

$$X_1K_3X_1^{-1} = \sum_{i=1}^{n_0}\Bigg[(\lambda_{i_1} - \mu_i)e_i \otimes e_i + \sum_{j=1}^{k(i)-1}(\lambda_{i_{j+1}} - \lambda_{i_j})e_{i_j} \otimes e_{i_j} + (\lambda_i - \lambda_{i_{k(i)}})e_{i_{k(i)}} \otimes e_{i_{k(i)}})\Bigg].$$

Then K_3 is compact and $\|K_3\| < \frac{\epsilon}{4}$. Moreover,

$$X_1\left(B+\sum_{i=1}^{3}K_i\right)X_1^{-1} = \begin{bmatrix} C_2 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} P_{n_1}\mathcal{H} \\ (P_{n_1}\mathcal{H})^\perp. \end{matrix}$$

for some n_1, where $A_2 = (I-P_{n_1})A|_{(P_{n_1}\mathcal{H})^\perp}$.

Note that $\sigma(C_2) = \{\lambda_i; 1 \le i \le n_1\}$. It follows that $\overline{A_0} := (A|_{P_1\mathcal{H}}) \sim C_2$. So, there exists an operator $X_2 \in (\mathcal{I}+\mathcal{K})(\mathcal{H})$ such that

$$X_2X_1\left(B+\sum_{i=1}^{3}K_i\right)X_1^{-1}X_2^{-1} = \begin{bmatrix} \overline{A_0} & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} P_{n_1}\mathcal{H} \\ (P_{n_1}\mathcal{H})^\perp \end{matrix} \triangleq \overline{A}.$$

To complete the proof, it suffices to show the following

Claim. There exists a compact operator $\overline{K_4}$ with $\|\overline{K_4}\| < \delta, \delta = \frac{\epsilon}{4\|X_2X_1\|\,\|X_1^{-1}X_2^{-1}\|}$, such that $\overline{A}+\overline{K_4} \sim_{i+k} A$.

By using induction, we prove this only for the case $n_1 = 1$. Let

$$\overline{A} = \begin{bmatrix} \lambda_1 & e_1\otimes f \\ 0 & A_2 \end{bmatrix} \begin{matrix} e_1 \\ \{e_1\}^\perp \end{matrix} \quad\text{and}\quad \begin{bmatrix} \lambda_1 & e_1\otimes g \\ 0 & A_2 \end{bmatrix} \begin{matrix} e_1 \\ \{e_1\}^\perp \end{matrix}.$$

It follows from condition (iii) that $\overline{\operatorname{ran}(A-\lambda_1)^*} = \{e_1\}^\perp$. Hence, there exists an $f_0 \in \{e_1\}^\perp$ with $\|f_0\| < \frac{\delta}{2}$ such that $f+f_0 = (A-\lambda_1)^*h$ for some h in $\mathcal{H}$. Let $h = \alpha e_1 + h_1, h_1 \in \{e_1\}^\perp$. Then $f+f_0 = \alpha g + (A_2-\lambda_1)^*h_1$. Thus,

$$\begin{bmatrix} 1 & 0 \\ h_1 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ f+f_0 & (A_2-\lambda_1)^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -h_1 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha g & (A_2-\lambda_1)^* \end{bmatrix}.$$

This implies that we can find an $X_3 \in (\mathcal{I}+\mathcal{K})(\mathcal{H})$ such that

$$X_3(\overline{A}+F_0)X_3^{-1} = \begin{bmatrix} \lambda_1 & \alpha e_1\otimes g \\ 0 & A_2 \end{bmatrix},$$

where $F_0 = \begin{bmatrix} 0 & e_1\otimes f_0 \\ 0 & 0 \end{bmatrix}$.

If $\alpha \ne 0$, then it is clear that

$$\begin{bmatrix} \lambda_1 & \alpha e_1\otimes g \\ 0 & A_2 \end{bmatrix} \sim_{i+k} \begin{bmatrix} \lambda_1 & e_1\otimes g \\ 0 & A_2 \end{bmatrix}.$$

If $\alpha = 0$, set $F_1 = X_3^{-1}\begin{bmatrix} 0 & \eta e_1\otimes g \\ & 0 \end{bmatrix}X_3$ with a small $\eta > 0$ such that $\|F_1\| < \frac{\delta}{2}$. Thus,

$$X_3(\overline{A}+F_0+F_1)X_3^{-1} = \begin{bmatrix} \lambda_1 & \eta e_1\otimes g \\ 0 & A_2 \end{bmatrix} \sim_{i+k} \begin{bmatrix} \lambda_1 & e_1\otimes g \\ 0 & A_2 \end{bmatrix}.$$

The proof is completed. ∎

Lemma 2.3 [14]. *Let $\sigma(A) = \sigma_{lre}(A)$ be perfect and $\epsilon > 0$. Then there exists a compact operator K with $\|K\| < \epsilon$ such that*

(i) $\Gamma := \sigma_p(A+K) = (\sigma_p(A^* + K^*))^*$ *is a denumerable dense subset of* $\sigma(A)$,

(ii) $\mathrm{nul}\,(A+K-\lambda) = \mathrm{nul}\,(A+K-\lambda)^* = 1$ *for each* $\lambda \in \Gamma$, *and*

(iii) $\bigvee\{\ker(A+K-\lambda); \lambda \in \Gamma\} = \bigvee\{\ker(A+K-\lambda)^*; \lambda \in \Gamma\} = \mathcal{H}$.

Now, we can prove Theorem 1.

Proof of Theorem 1. If $A \sim_{sas} B$, then there exist two compact operators K_1 and K_2 and an invertible operator X such that $A + K_1 = X(B + K_2)X^{-1}$. Write $K = K_2 - X^{-1}K_1X$; then K is compact and $A = X(B+K)X^{-1}$. So A and B are essentially similar. Now we are going to prove the sufficiency. By [10, Thm. 3.48] and Proposition 2.1, we may assume that $\sigma(A) = \sigma_{lre}(A) = \sigma(B) = \sigma_{lre}(B)$ and that they contain more than one point. Given $\epsilon > 0$, it follows from Lemma 2.3 that there exists a compact K with $\|K\| < \epsilon$ such that $A + K$ satisfies (i), (ii) and (iii) of Lemma 2.3. Write $XBX^{-1} = A + K_0$, X invertible and $K_0 \in \mathcal{K}(\mathcal{H})$. Then $XBX^{-1} = A + K + K_0 - K$. By Weyl's Theorem [9, Problem 143], $\sigma(A+K) \subset \sigma(A) \bigcup \sigma_p(A+K) = \sigma(A) \bigcup \Gamma = \sigma(A)$. Thus, $\sigma(A+K) = \sigma(A) = \sigma(B)$. It follows from Lemma 2.2 that there exists a compact $\overline{K_1}$ with $\|\overline{K_1}\| < \frac{\epsilon}{\|X\|\|X^{-1}\|}$ such that $XBX^{-1} + \overline{K_1} \sim A + K$. Set $K_1 = X^{-1}\overline{K_1}X$. We obtain that $\|K_1\| < \epsilon$ and $B + K_1 \sim A + K$. ∎

3. The Quasitriangular Case

Proposition 3.1. *Let $B \in \mathcal{B}_1(\Omega)$ and let $A = K_0 + \bigoplus_{i=1}^{n} B$, K_0 compact and $\sigma(A) = \sigma(B) = \overline{\Omega}$. Then, for given $\epsilon > 0$, there exist an $X \in (\mathcal{I} + \mathcal{K})(\mathcal{H})$ and a compact K with $\|K\| < \epsilon$ such that $X(A+K)X^{-1} = \bigoplus_{i=1}^{n} B$.*

Proof. Without loss of generality, assume that $0 \in \Omega$. We shall proceed by induction on the positive number n. Thus, begin with considering the case $n = 1$. This is an immediate consequence of Lemma 2.2. Now, asumme that the conclusion is true for $n-1$. Set $P_k = \bigoplus_{i=1}^{n} P_{\ker B^k}$. Then there exists a natural number k_0 such that $K_1 = P_{k_0}K_0P_{k_0} - K_0$ is compact and $\|K_1\| < \frac{\epsilon}{5}$. Thus,

$$\begin{aligned} A + K_1 &= \left(\bigoplus_{i=1}^{n} B\right) + P_{k_0}K_0P_{k_0} \\ &= \begin{bmatrix} C_0 & * \\ 0 & \bigoplus_{i=1}^{n} B_1 \end{bmatrix} \begin{matrix} \operatorname{ran} P_{k_0} \\ (\operatorname{ran} P_{k_0})^\perp \end{matrix} = \begin{bmatrix} \begin{bmatrix} C_0 & * \\ 0 & B_1 \end{bmatrix} & \begin{matrix} * \\ 0 \end{matrix} \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix}, \end{aligned}$$

where $B = \begin{bmatrix} B_0 & * \\ 0 & B_1 \end{bmatrix} \begin{matrix} \ker B^{k_0} \\ (\ker B^{k_0})^\perp \end{matrix}$ and C_0 is a compact perturbation of $\bigoplus_{i=1}^{n} B_0$.

By the upper semicontinuity of the spectrum, using the technique used in Lemma 2.2, we may assume that $\sigma(C_0) \subset \Omega$. Let $\lambda_1, \lambda_2, \cdots, \lambda_{(n-1)k_0}$ be pairwise distinct numbers in Ω. Let C be an operator on $\mathbb{C}^l \oplus \mathcal{H}$ whose adjoint can be written as

$$C^* = \begin{bmatrix} \begin{bmatrix} \overline{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \overline{\lambda_l} \end{bmatrix} & 0 \\ \begin{matrix} f_1 & \cdots & f_l \end{matrix} & B^* \end{bmatrix},$$

where $l = (n-1)k_0$ and $f_i \notin \operatorname{ran}(B - \lambda_i)^*$ for $1 \le i \le l$.

Claim. $C \sim B$.

Consider $C_l^* := \begin{bmatrix} \overline{\lambda_l} & 0 \\ f_l & B^* \end{bmatrix}$. We may assume that $\lambda_l = 0$. It is not hard to prove that $0 \notin \sigma_r(C_l)$ and $\mathbb{C} \oplus \{0\} = \ker C_l$. Define $X \in \mathcal{L}(\mathbb{C} \oplus \mathcal{H}, \mathcal{H})$ by $Xy = C_l^* y$, $y \in \mathbb{C} \oplus \mathcal{H}$. Then X is invertible and $X^{-1}B^*X = C_l^*$. That is, $C_l \sim B$.

Note that $C - \begin{bmatrix} C_0 & * \\ 0 & B_1 \end{bmatrix}$ is compact. It follows from Lemma 2.2 that there exists a compact $\overline{K_2}$ with $\|\overline{K_2}\| < \frac{\epsilon}{5}$ such that

$$\begin{bmatrix} C_0 & * \\ 0 & B_1 \end{bmatrix} \sim_{i+k} C.$$

Thus, we can find an $X_1 \in (\mathcal{I} + \mathcal{K})$ and a compact K_2 with $\|K_2\| < \frac{\epsilon}{5}$ such that

$$X_1(A + K_1 + K_2)X_1^{-1} = \begin{bmatrix} C & E_1 \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix},$$

where E_1 is compact.

Since $B_1 \sim B$ and $C \sim B$, it follows from [5] that there are no nonzero compact operators K satisfying $(\bigoplus_{i=1}^{n-1} B_1)K - KC = 0$. Thus, by [15, Lem.1.10], there exist compact operators E_2 and E_3 with $\|E_3\| < \frac{\epsilon}{5\|X_1\|\,\|X_1^{-1}\|}$ such that $E_1 + E_3 = CE_2 - E_2(\bigoplus_{i=1}^{n-1} B_1)$. As a result, $X_2 = \begin{bmatrix} I & E_2 \\ 0 & I \end{bmatrix} \in (\mathcal{I} + \mathcal{K})$, $K_3 = \begin{bmatrix} 0 & E_3 \\ 0 & 0 \end{bmatrix}$ is compact and

$$\begin{aligned} X_2 X_1 \left(A + \sum_{i=1}^{3} K_i\right) X_1^{-1} X_2^{-1} &= X_2 \begin{bmatrix} C & E_1 + E_3 \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix} X_2^{-1} \\ &= \begin{bmatrix} C & 0 \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix} = \begin{bmatrix} \bigoplus_{i=1}^{n-1} \overline{B_i} & E_4 \\ 0 & B \end{bmatrix}, \end{aligned}$$

where

$$\overline{B_i} = \begin{bmatrix} \lambda_{(i-1)k_0+1} & & & 0 \\ & \ddots & & \\ & & \lambda_{ik_0} & \\ 0 & & & B_1 \end{bmatrix}$$

and E_4 is compact.

Since $\bigoplus_{i=1}^{n-1} \overline{B_i}$ is a compact perturbation of $\bigoplus_{i=1}^{n-1} B$, it follows from our induction assumption that there exist an $X_3 \in (\mathcal{I} + \mathcal{K})$ and a compact K_4 with $\|K_4\| < \frac{\epsilon}{5}$ such that

$$X_3 X_2 X_1 \left(A + \sum_{i=1}^{4} K_i\right) X_1^{-1} X_2^{-1} X_3^{-1} = \begin{bmatrix} \bigoplus_{i=1}^{n-1} B & E_5 \\ 0 & B \end{bmatrix}$$

and still, E_5 is compact.

By using a similar method as above, we conclude that there exist an $X_4 \in (\mathcal{I} + \mathcal{K})$ and a compact K_5 with $\|K_5\| < \frac{\epsilon}{5}$ such that

$$X_4 X_3 X_2 X_1 \left(A + \sum_{i=1}^{5} K_i\right) X_1^{-1} X_2^{-1} X_3^{-1} X_4^{-1} = \begin{bmatrix} \bigoplus_{i=1}^{n-1} B & 0 \\ 0 & B \end{bmatrix} = \bigoplus_{i=1}^{n} B.$$

That completes the proof. ∎

Lemma 3.2. *Let $B \in \mathcal{B}_1(\Omega), \sigma(B) = \overline{\Omega}$ and $\sigma_p(B) = \Omega$, and let*

$$S = \begin{bmatrix} A & R \\ 0 & \bigoplus_{i=1}^{n} B \end{bmatrix} \begin{matrix} \mathcal{K} \\ \bigoplus_{i=1}^{n} \mathcal{H} \end{matrix}$$

satisfy the following conditions:

(i) $\Gamma := \sigma_p(A) = (\sigma_p(A^*))^*$ *is a denumerable dense subset of* $\sigma(A)$;

(ii) $\text{nul}\,(A-\lambda) = \text{nul}\,(A-\lambda)^* = 1$ *for each* $\lambda \in \Gamma$;

(iii) $\bigvee\{\ker\,(A-\lambda); \lambda \in \Gamma\} = \mathcal{K}$;

(iv) $\sigma(A) = \sigma_{lre}(A) = \sigma_{lre}(S) = \sigma(S)\backslash\Omega$ *perfect*;

(v) $\sigma(S) = \sigma_W(S)$ *is connected and* $\Omega \subset \sigma(S)$.

If T *is a compact perturbation of* S *and* $\sigma_0(T) = \emptyset$*, then, given* $\epsilon > 0$*, there exists a compact* K *with* $\|K\| < \epsilon$ *such that* $S \sim_{i+k} T + K$.

Proof. By [10, Thm.3.48], we may assume that $\sigma(T) = \sigma_W(T)\,(= \sigma(S))$. Without loss of generality, we may also assume that $0 \in \Omega$.

Set $P_k = I_{\mathcal{K}} \bigoplus (\bigoplus_{i=1}^{n} P_{\ker B^k})$, as in the proof of Proposition 3.1, there exist a positive integer k_0 and a compact K_1 with $\|K_1\| < \frac{\epsilon}{4}$ such that B can be writen as

$$B = \begin{bmatrix} B_0 & * \\ 0 & B_1 \end{bmatrix} \begin{matrix} P_{\ker B^{k_0}} \\ (P_{\ker B^{k_0}})^{\perp} \end{matrix}$$

and

$$T + K_1 = \begin{bmatrix} C & * \\ 0 & \bigoplus_{i=1}^{n} B_1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} C & F \\ 0 & B_1 \end{bmatrix} & \begin{matrix} * \\ 0 \end{matrix} \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix},$$

where C is a compact perturbation of $\begin{bmatrix} A & E \\ 0 & \bigoplus_{i=1}^{n} B_0 \end{bmatrix}$ and $\sigma\left(\begin{bmatrix} C & F \\ 0 & B_1 \end{bmatrix}\right) \subset (\sigma(S))_{\frac{\epsilon}{8}}$.

Let $D = \begin{bmatrix} C & F \\ 0 & B_1 \end{bmatrix}$. Note that $B_1 \sim B$, and it can be shown that $\sigma_{lre}(D) = \sigma_{lre}(S)$. If $\lambda \in \sigma_0(D)$, then $\text{ind}\,(D-\lambda) = 0$. This implies that $\lambda \notin \Omega$. Thus, $\lambda \notin \sigma(S)$. By [10, Thm. 3.48], we may assume that $\sigma(D) = \sigma_W(D) = \sigma_{lre}(S) \bigcup \Omega = \sigma(S)$. Set

$$G = \begin{bmatrix} A & E & F_1 \\ & J & f \otimes e \\ & & B_1 \end{bmatrix},$$

where $F_1 = P_{\mathcal{K}} F$, J is nk_0-order Jordan block, $e \in \ker B_1$ with $\|e\| = 1$ and $f \in \ker J^*$ with $\|f\| = 1$. Then G is a compact perturbation of D and $\sigma(G) = \sigma(S) = \sigma(D)$.

Now, we can verify that G satisfies the conditions (i), (ii) and (iii) of Lemma 2.2. Thus, an $X_1 \in (\mathcal{I}+\mathcal{K})$ and a compact K_2 with $\|K_2\| < \frac{\epsilon}{4}$ can be found such that

$$X_1(T+K_1+K_2)X_1^{-1} = \begin{bmatrix} G & * \\ 0 & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix} = \begin{bmatrix} A & * \\ 0 & L \end{bmatrix},$$

where

$$L = \begin{bmatrix} J & f \otimes e & * \\ & B_1 & 0 \\ & & \bigoplus_{i=1}^{n-1} B_1 \end{bmatrix}$$

is a compact perturbation of $\bigoplus_{i=1}^{n} B_1$.

Since $\sigma(L) = \sigma(B_1) = \overline{\Omega}$, it follows from Proposition 3.1 that there exist an $X_2 \in (\mathcal{I}+\mathcal{K})$ and a compact K_3 with $\|K_3\| < \frac{\epsilon}{4}$ such that

$$X_2 X_1 \left(T + \sum_{i=1}^{3} K_i \right) X_1^{-1} X_2^{-1} = \begin{bmatrix} A & R_0 \\ 0 & \bigoplus_{i=1}^{n} B \end{bmatrix}.$$

Note that $X_1, X_2 \in (\mathcal{I}+\mathcal{K})$, and $R - R_0$ is compact. It follows from [12, Lemma 2] that if $(\bigoplus_{i=1}^{n} B)X - XA = 0$ and X is compact then $X = 0$. Thus, imitating the proof of Proposition 3.1, we can find an $X_3 \in (\mathcal{I}+\mathcal{K})$ and a compact K_4 with $\|K_4\| < \frac{\epsilon}{4}$ such that

$$X_3 X_2 X_1 \left(T + \sum_{i=1}^{4} K_i \right) X_1^{-1} X_2^{-1} X_3^{-1} = S. \qquad \blacksquare$$

Proof of Theorem 2. We need only prove that if A and B are essentially similar, then $A \sim_{sas} B$.

As in the proof of Lemma 3.2, assume that $0 \in \Omega$ and $\sigma(A) = \sigma_W(A)$. It follows from AFV Theorem [2] that there exists a compact K_1 with $\|K_1\| < \frac{\epsilon}{3}$ such that

$$A + K_1 = \begin{bmatrix} A_0 & * \\ 0 & N \end{bmatrix},$$

where N is a diagonal normal operator of uniform infinite multiplicity and $\sigma(N) = \sigma_e(N) = \partial\Omega$; $\sigma(A_0) = \sigma(A)$, $\sigma_{lre}(A_0) = \sigma_{lre}(A)$ and $\mathrm{ind}\,(A_0 - \lambda) = \mathrm{ind}\,(A - \lambda)$ for each $\lambda \in \rho_{sF}(A)$.

Let $B(\Omega)$ denote the Bergmann operator on $L_a^2(\Omega)$ defined by $(B(\Omega)f)(z) = z\,f(z)$. It follows from [10, p.105] that $B(\Omega^*)^* \in \mathcal{B}_1(\Omega)$, where $\Omega^* = \{\overline{z} : z \in \Omega\}$.

Set $B_n = \bigoplus_{i=1}^{n} B(\Omega^*)^*$. Then by BDF Theorem [3], we can find a unitary U_0 and a compact K_0 such that

$$U_0 N U_0 = K_0 + \begin{bmatrix} \bigoplus_{i=1}^{n} B(\Omega) & 0 \\ 0 & B_n \end{bmatrix}.$$

Set $P_m = (\bigoplus_{i=1}^{n} I) \bigoplus P_{\ker B_n^m}$. Then there exist a positive integer m_0 and a compact $\overline{K_0}$ with $\|\overline{K_o}\| < \frac{\epsilon}{6}$ such that

$$U_0(N + \overline{K_0})U_0^* = \begin{bmatrix} \overline{B_0} & * \\ 0 & B_1 \end{bmatrix} \begin{matrix} \text{ran} P_{m_0} \\ (\text{ran} P_{m_0})^\perp \end{matrix},$$

where $\sigma(N + \overline{K_0}) \subset \sigma(N)_{\frac{\epsilon}{12}}$, $\sigma_W(\overline{B_0}) = \overline{\Omega}$ and $\text{ind}\,(\overline{B_0} - \lambda) = -n$ for each $\lambda \in \Omega$. Thus $B_1 \sim B_n$. This implies that $\sigma(\overline{B_0}) \subset (\overline{\Omega})_{\frac{\epsilon}{12}}$. By [10, Thm.3.48], there exists a compact $\overline{K_1}$ with $\|\overline{K_1}\| < \frac{\epsilon}{6}$ such that $\sigma(\overline{B_0} + \overline{K_1}) = \sigma_W(\overline{B_0}) = \overline{\Omega}$. Set $C_0 = \overline{B_0} + \overline{K_1}$. Then we have that

$$U(A + K_1 + K_2)U^* = \begin{bmatrix} A_0 & * & * \\ & C_0 & * \\ & & B_1 \end{bmatrix}$$

for some unitary U and some compact K_2 with $\|K_2\| < \frac{\epsilon}{3}$.

Set $C_1 = \begin{bmatrix} A_0 & * \\ 0 & C_0 \end{bmatrix}$. It is easy to show that $\sigma_e(C_1) = \sigma(A) \backslash \Omega$ is perfect and that $\sigma(C_1) = \sigma(A)$. Thus, $C_1 \in (BQT)$. By [10, Thm.3.48], assume that $\sigma(C_1) = \sigma_W(C_1) = \sigma_e(C_1)$. Now, it follows from Lemma 2.3 that there exists a compact $\overline{K_2}$ with $\|\overline{K_2}\| < \frac{\epsilon}{3}$ such that $A_1 := C_1 + \overline{K_2}$ has the following properties:

(i) $\Gamma := \sigma_p(A_1) = (\sigma_p(A_1^*))^*$ is a denumerable dense subset of $\sigma(C_1)$;

(ii) $\text{nul}\,(A_1 - \lambda) = \text{nul}\,(A_1 - \lambda)^* = 1$ for each $\lambda \in \Gamma$;

(iii) $\bigvee\{\ker(A_1 - \lambda);\, \lambda \in \Gamma\}$ coincides with the space on which A_1 acts.

By Weyl's Theorem and (i), we can also obtain

(iv) $\sigma(A_1) = \sigma(C_1) = \sigma_e(C_1) = \sigma_e(A_1)$.

Thus, we can find a compact K_3 with $\|K_3\| < \frac{\epsilon}{3}$ such that

$$S := U\left(A + \sum_{i=1}^{3} K_i\right)U^* = \begin{bmatrix} A_1 & * \\ 0 & B_1 \end{bmatrix}$$

Clearly, $\sigma_W(S) = \sigma_W(A) = \sigma(A) = \sigma(C_1)$. If $\lambda \notin \sigma_W(S)$, then it follows from (iv) that $\lambda \notin \sigma(A_1)$. Hence, $\lambda \notin \sigma(S)$. This implies that

(v) $\sigma(S) = \sigma_W(S)$ is connected.

It is also obvious that $\sigma_{lre}(S) = \sigma_{lre}(A) = \sigma(A)\backslash\Omega$ is perfect.

Note that $UXBX^{-1}U^* - S$ is compact. Using Lemma 3.2, we can find an invertible Y and a compact K_4 with $\|K_4\| < \epsilon$ such that

$$YUX(B+K_4)X^{-1}U^*Y^{-1} = S = U\left(A + \sum_{i=1}^{3} K_i\right)U^*.$$

Now, we complete the proof. ∎

Remark. Theorems 1 and 2 show that the relation $\sim_{sas}$ is transitive for the classes of operators in the two theorems.

To conclude this paper, we pose the following

Problem. Is relation $\sim_{sas}$ always transitive?

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