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A MINIMAX THEOREM INVOLVING TWO FUNCTIONS

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Abstract. A minimax theorem involving two functions is derived, where the convexity assumptions on two functions are given by mixed conditions, and in which we answer the open question raised by Forgó and Joó in [2]. Simons's upward-downward minimax theorem and Lin-Quan's two functions symmetric minimax theorem are generalized.

Let X and Y be two nonempty sets. Let $f, g : X \times Y \to \mathbb{R}$ with $f \le g$ on $X \times Y$. A minimax theorem implies that, under certain conditions, the following equality holds:

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

A two-function minimax theorem implies that, under certain conditions, the following inequality holds:

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

Throughout this paper, suppose that X is a compact topological space, and $f(\cdot, y)$ and $g(\cdot, y)$ are upper semicontinuous (usc) on X for any $y \in Y$. For any $r \in \mathbb{R}$, any $y \in Y$ and any finite subset A of Y, we denote

$$X_f^r(y) = \{x \in X : f(x,y) \ge r\} \text{ and } X_f^r(A) = \bigcap_{y\in A} X_f^r(y).$$

Theorem A [4]. *Suppose that there exist s, t in $(0,1)$ satisfying*:

(A1) *for any $x_1, x_2 \in X$, there exists $x_0 \in X$ such that*

$$f(x_0,y) \ge s\max\{f(x_1,y), f(x_2,y)\} + (1-s)\min\{f(x_1,y), f(x_2,y)\}$$

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for all $y \in Y$,

(A2) *for any* $y_1, y_2 \in Y$, *there exists* $y_0 \in Y$ *such that*

$f(x, y_0) \leq t \max\{f(x, y_1), f(x, y_2)\} + (1-t) \min\{f(x, y_1), f(x, y_2)\}$

for all $x \in X$.

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

By relaxing the conditions (A1) and (A2), Simons gave an upward-downward minimax theorem which generalizes Theorem A.

Theorem B [6]. *Suppose that*

(B1) *for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any* $x_1, x_2 \in X$, *there exists* $x_0 \in X$ *such that for all* $y \in Y$,

$f(x_0, y) \geq \min\{f(x_1, y), f(x_2, y)\}$

and for all $y \in \{y \in Y : |f(x_1, y) - f(x_2, y)| \geq \epsilon\}$,

$f(x_0, y) \geq \min\{f(x_1, y), f(x_2, y)\} + \delta.$

(B2) *for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any* $y_1, y_2 \in Y$, *there exists* $y_0 \in Y$ *such that for all* $x \in X$,

$f(x, y_0) \leq \max\{f(x, y_1), f(x, y_2)\}$

and for all $x \in \{x \in X : |f(x, y_1) - f(x, y_2)| \geq \epsilon\}$,

$f(x, y_0) \leq \max\{f(x, y_1), f(x, y_2)\} - \delta.$

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

In another direction, Lin and Quan generalize Theorem A into two-function minimax theorem.

Theorem C [5]. *Suppose that there exist* s, t *in* $(0, 1)$ *satisfying* :

(C1) *for any* $x_1, x_2 \in X$, *there exists* $x_0 \in X$ *such that*

$f(x_0, y) \geq s \max\{f(x_1, y), g(x_2, y)\} + (1-s) \min\{f(x_1, y), g(x_2, y)\}$

for all $y \in Y$,

(C2) *for any* $y_1, y_2 \in Y$, *there exists* $y_0 \in Y$ *such that*

$g(x, y_0) \leq t \max\{f(x, y_1), g(x, y_2)\} + (1-t) \min\{f(x, y_1), g(x, y_2)\}$

for all $x \in X$.

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

The aim of this paper is to give the following two-function minimax theorem which generalizes Theorem B and Theorem C.

Theorem. *Suppose that*

(i) *for any $x_1, x_2 \in X$ and any finite subset A of Y, there exists $x_0 \in X$ such that for all $y \in A$,*

$f(x_0, y) \geq \min\{f(x_1, y), g(x_2, y)\}$

and for all $y \in \{y \in A : f(x_1, y) \neq g(x_2, y)\}$,

$f(x_0, y) > \min\{f(x_1, y), g(x_2, y)\}$.

(ii) *for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y_1, y_2 \in Y$, there exists $y_0 \in Y$ such that for all $x \in X$,*

$g(x, y_0) \leq \max\{f(x, y_1), g(x, y_2)\}$

and for all $x \in \{x \in X : |f(x, y_1) - g(x, y_2)| \geq \epsilon\}$,

$g(x, y_0) \leq \max\{f(x, y_1), g(x, y_2)\} - \delta$.

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y).$$

The proof of Theorem is based on the following Propositions 1–3 and Lemma. Proposition 1 originates from the technique that is used in [7] by Simons. We omit the proof.

Proposition 1. *Let $r \in \mathbb{R}$, $\delta > 0$ and $\phi(y) = \max_{x\in X} f(x, y)$. Suppose that $X_f^r(y) \neq \emptyset$ for any $y \in Y$. Then for any $y_1 \in Y$, there exists $z_1 \in Y$ such that*

$$X_f^r(z_1) \subset X_f^r(y_1)$$

and for any $y \in Y$,

$$X_f^r(y) \subset X_f^r(z_1) \text{ implies } \phi(y) > \phi(z_1) - \delta.$$

Proposition 2. *Let $r \in \mathbb{R}$ such that $X_f^r(y) \neq \emptyset$ for any $y \in Y$, let $y_0, z_1, z_2 \in Y$. Suppose that the condition* (i) *of Theorem is satisfied. If $X_f^r(y_0) \subset X_g^r(y_0) \subset X_f^r(z_1) \cup X_g^r(z_2)$, $X_f^r(z_1) \cap X_g^r(z_2) = \emptyset$ and $X_f^r(y_0) \subset X_g^r(z_2)$, then $X_g^r(y_0) \subset X_g^r(z_2)$.*

Proof. Suppose that $X_g^r(y_0) \cap X_f^r(z_1) \neq \emptyset$. Choose $x_2 \in X_g^r(y_0) \cap X_f^r(z_1)$, and $x_1 \in X_f^r(y_0)$ with $f(x_1, z_1) = \max\limits_{x \in X_f^r(y_0)} f(x, z_1)$.

Let $A = \{y_0, z_1, z_2\}$. By (i), there exists $x_0 \in X$ such that for all $y \in A$,

$$f(x_0, y) \geq \min\{f(x_1, y), g(x_2, y)\}$$

and for all $y \in \{y \in A : f(x_1, y) \neq g(x_2, y)\}$,

$$f(x_0, y) > \min\{f(x_1, y), g(x_2, y)\}.$$

It is obvious that $f(x_0, y_0) \geq \min\{f(x_1, y_0), g(x_2, y_0)\} \geq r$, i.e., $x_0 \in X_f^r(y_0)$. Since $g(x_2, z_1) \geq f(x_2, z_1) \geq r > f(x_1, z_1)$, we have

$$f(x_0, z_1) > \min\{f(x_1, z_1), g(x_2, z_1)\} = f(x_1, z_1).$$

This contradicts the maximality of $f(x_1, z_1)$ in $X_f^r(y_0)$. Therefore $X_g^r(y_0) \subset X_g^r(z_2)$. ■

Proposition 3. *Let* $r \in \mathbb{R}$. *Suppose that the condition* (i) *of Theorem is satisfied. Then* $X_g^r(y) \neq \emptyset$ *implies* $X_f^r(y) \neq \emptyset$ *for any* $y \in Y$.

Proof. Suppose that there exists $y_1 \in Y$ such that $X_g^r(y_1) \neq \emptyset$ and $X_f^r(y_1) = \emptyset$. Take $x_2 \in X_g^r(y_1)$, and $x_1 \in X$ with $f(x_1, y_1) = \max\limits_{x \in X} f(x, y_1)$. Since $f(x_1, y_1) < r \leq g(x_2, y_1)$, by (i), $f(x_0, y_1) > \min\{f(x_1, y_1), g(x_2, y_1)\} = f(x_1, y_1)$. This contradicts the maximality of $f(x_1, y_1)$ on X. Therefore $X_g^r(y) \neq \emptyset$ implies $X_f^r(y) \neq \emptyset$ for any $y \in Y$.

Lemma. *Under the conditions of Theorem, for any* $r < \inf\limits_{y \in Y} \sup\limits_{x \in X} f(x, y)$ *and any* $y_1, y_2 \in Y$*, we have*

$$X_f^r(y_1) \cap X_g^r(y_2) \neq \emptyset.$$

Proof. Suppose that there exist $y_1, y_2 \in Y$ and $r < \inf\limits_{y \in Y} \sup\limits_{x \in X} f(x, y)$ such that

$$X_f^r(y_1) \cap X_g^r(y_2) = \emptyset. \tag{1}$$

Let $\epsilon > 0$ with $r < r + \epsilon < \inf\limits_{y \in Y} \sup\limits_{x \in X} f(x, y)$, and $\delta > 0$ as in (ii). By Proposition 1, there exist $z_1, z_2 \in Y$ such that

$$\begin{cases} X_f^r(z_1) \subset X_f^r(y_1) \\ \forall y \in Y, X_f^r(y) \subset X_f^r(z_1) \Longrightarrow \phi(y) > \phi(z_1) - \delta \end{cases} \tag{2}$$

and

$$(3) \qquad \begin{cases} X_g^r(z_2) \subset X_g^r(y_2) \\ \forall y \in Y, X_g^r(y) \subset X_g^r(z_2) \Longrightarrow \psi(y) > \psi(z_2) - \delta, \end{cases}$$

where $\phi(y) = \max_{x \in X} f(x, y)$ and $\psi(y) = \max_{x \in X} g(x, y)$. It follows from (1) that

$$(4) \qquad X_f^r(z_1) \cap X_g^r(z_2) = \emptyset.$$

By (ii), for z_1 and z_2, there exists $y_0 \in Y$ such that for all $x \in X$,

$$(5) \qquad g(x, y_0) \leq \max\{f(x, z_1), g(x, z_2)\}$$

and for all $x \in \{x \in X : |f(x, z_1) - g(x, z_2)| \geq \epsilon\}$,

$$(6) \qquad g(x, y_0) \leq \max\{f(x, z_1), g(x, z_2)\} - \delta.$$

Hence

$$(7) \qquad X_f^r(y_0) \subset X_g^r(y_0) \subset X_f^r(z_1) \cup X_g^r(z_2).$$

Next, we prove

$$(8) \qquad X_f^r(y_0) \cap X_f^r(z_1) \neq \emptyset \neq X_f^r(y_0) \cap X_g^r(z_2).$$

Suppose that $X_f^r(y_0) \cap X_g^r(z_2) = \emptyset$. Then $X_f^r(y_0) \subset X_f^r(z_1)$ by (7). Take $\overline{x} \in X_f^{r+\epsilon}(y_0) \subset X_f^r(y_0) \subset X_f^r(z_1)$. By (5),

$$r + \epsilon \leq f(\overline{x}, y_0) \leq g(\overline{x}, y_0) \leq \max\{f(\overline{x}, z_1), g(\overline{x}, z_2)\}.$$

Since $g(\overline{x}, z_2) < r$ by (4), we have $f(\overline{x}, z_1) \geq r + \epsilon$. By (6),

$$\begin{aligned} f(\overline{x}, y_0) &\leq g(\overline{x}, y_0) \leq \max\{f(\overline{x}, z_1), g(\overline{x}, z_2)\} - \delta \\ &\leq f(\overline{x}, z_1) - \delta \leq \phi(z_1) - \delta. \end{aligned}$$

It follows that $\phi(y_0) \leq \phi(z_1) - \delta$. This contradicts (2).

Suppose that $X_f^r(y_0) \cap X_f^r(z_1) = \emptyset$. Then $X_f^r(y_0) \subset X_g^r(z_2)$ by (7). It follows from (4), (7) and Proposition 2 that $X_g^r(y_0) \subset X_g^r(z_2)$. Take $\overline{x} \in X_g^{r+\epsilon}(y_0) \subset X_g^r(y_0) \subset X_g^r(z_2)$. By (5),

$$r + \epsilon \leq g(\overline{x}, y_0) \leq \max\{f(\overline{x}, z_1), g(\overline{x}, z_2)\}.$$

Since $f(\overline{x}, z_1) < r$ by (4), we have $g(\overline{x}, z_2) \geq r + \epsilon$. By (6),

$$\begin{aligned} g(\overline{x}, y_0) &\leq \max\{f(\overline{x}, z_1), g(\overline{x}, z_2)\} - \delta \\ &\leq g(\overline{x}, z_2) - \delta \leq \psi(z_2) - \delta. \end{aligned}$$

It follows that $\psi(y_0) \leq \psi(z_2) - \delta$. This contradicts (3). Therefore (8) holds.

Since $f(\cdot, y)$ and $g(\cdot, y)$ are usc on the compact space X for any $y \in Y$, there exist $x_1 \in X_f^r(y_0) \cap X_f^r(z_1) =: D_1$ and $x_2 \in X_f^r(y_0) \cap X_g^r(z_2) =: D_2$ such that

$$f(x_1, z_2) = \max_{x \in D_1} f(x, z_2) \tag{9}$$

and

$$f(x_2, z_1) = \max_{x \in D_2} f(x, z_1). \tag{10}$$

Let $A = \{y_0, z_1, z_2\}$. By (i), there exist $x_0 \in X$ such that for all $y \in A$,

$$f(x_0, y) \geq \min\{f(x_1, y), g(x_2, y)\}$$

and for all $y \in \{y \in A : f(x_1, y) \neq g(x_2, y)\}$,

$$f(x_0, y) > \min\{f(x_1, y), g(x_2, y)\}.$$

Thus $f(x_0, y_0) \geq \min\{f(x_1, y_0), g(x_2, y_0)\} \geq r$, this is, $x_0 \in X_f^r(y_0) \subset X_f^r(z_1) \cup X_g^r(z_2)$.

Suppose that $x_0 \in D_1$. Since $g(x_2, z_2) \geq r > g(x_1, z_2) \geq f(x_1, z_2)$, we have $f(x_0, z_2) > \min\{f(x_1, z_2), g(x_2, z_2)\} = f(x_1, z_2)$. This contradicts (9).

Suppose that $x_0 \in D_2$. By (4), $x_0 \notin X_f^r(z_1)$ and so $r > f(x_0, z_1) \geq \min\{f(x_1, z_1), g(x_2, z_1)\}$. Since $f(x_1, z_1) \geq r$, it follows that $g(x_2, z_1) < r$ and so $f(x_0, z_1) > \min\{f(x_1, z_1),\, g(x_2, z_1)\} = g(x_2, z_1) \geq f(x_2, z_1)$. This contradicts (10).

This completes the proof of Lemma. ■

Proof of Theorem. Since X is compact and $g(\cdot, y)$ is usc on X for any $y \in Y$, it is sufficient to prove that for any finite subset E of Y and any $r < \inf_{y \in Y} \sup_{x \in X} f(x, y)$,

$$X_g^r(E) \neq \emptyset.$$

When Card $E = 1$, $X_g^r(y) \supset X_f^r(y) \neq \emptyset$.

When Card $E = 2$, let $E = \{y_1, y_2\}$. By Lemma,

$$X_g^r(E) = X_g^r(y_1) \cap X_g^r(y_2) \supset X_f^r(y_1) \cap X_g^r(y_2) \neq \emptyset.$$

Suppose that $X_g^r(E) \neq \emptyset$ for any subset E with Card $E \leq n$ and any $r < \inf_{y \in Y} \sup_{x \in X} f(x, y)$, and $X_g^{r_0}(E_0) = \emptyset$ for some subset E_0 with Card $E_0 = n + 1$ and some $r_0 < \inf_{y \in Y} \sup_{x \in X} f(x, y)$. Let $E_0 = F \cup \{\overline{y}\}$ and $\overline{y} \notin F$. Hence Card

$F = n$. We denote $\overline{X} = X_f^{r_0}(\overline{y})$. Then $\overline{X}$ is compact, the functions $f\mid_{\overline{X}\times Y}$ and $g\mid_{\overline{X}\times Y}$ satisfy the conditions of Theorem, and $\overline{X}_g^{r_0}(F) = \bigcap_{y\in F}\{x \in \overline{X} : g(x,y) \geq r_0\} = \overline{X} \cap X_g^{r_0}(F) = \emptyset$. By the hypothsis of induction, it follows that $\inf\limits_{y\in Y}\sup\limits_{x\in\overline{X}} f(x,y) \leq r_0$.

Let $\epsilon > 0$ with $r_0 < r_0 + \epsilon < \inf\limits_{y\in Y}\sup\limits_{x\in X} f(x,y)$. Then $\inf\limits_{y\in Y}\sup\limits_{x\in\overline{X}} f(x,y) < r_0 + \epsilon$, and so there exists $\tilde{y} \in Y$ such that $\overline{X}_f^{r_0+\epsilon}(\tilde{y}) = \{x \in \overline{X} : f(x,\tilde{y}) \geq r_0+\epsilon\} = \emptyset$. By Proposition 3, $\overline{X}_g^{r_0+\epsilon}(\tilde{y}) = \emptyset$. Thus $X_f^{r_0+\epsilon}(\overline{y}) \cap X_g^{r_0+\epsilon}(\tilde{y}) \subset X_f^{r_0}(\overline{y}) \cap X_g^{r_0+\epsilon}(\tilde{y}) = \overline{X}_g^{r_0+\epsilon}(\tilde{y}) = \emptyset$. This contradicts Lemma. This complete the proof of Theorem. ■

Remark 1) From Proposition 3, it is easy to see that the equality $\inf\limits_{y\in Y}\sup\limits_{x\in X} f(x,y) = \inf\limits_{y\in Y}\sup\limits_{x\in X} g(x,y)$ holds. In other words, the conclusion of Theorem can be written "$\inf\limits_{y\in Y}\sup\limits_{x\in X} g(x,y) = \sup\limits_{x\in X}\inf\limits_{y\in Y} g(x,y)$" which is a minimax theorem on g. This is surprising since it is difficult to remark this from the conditions of Theorem.

2) Since the condition (C1) implies the condition (i) and the condition (C2) implies the condition (ii) by taking $\delta = (1-t)\epsilon$, Theorem generalizes Theorem C. When $f = g$, similarly Theorem generalizes Theorem B.

3) In [1], we divide two-function minimax theorems into three types. If the "lower" function f is "concave" and the "upper" function g is "convex", we call these kind of two-function minimax theorems type (A); If the "lower" function f is "convex" and the "upper" function g is "concave", we call these kind of two-function minimax theorems type (B); For other two-function minimax theorems like Theorem C or Theorem in which the "concave-convex" conditions are involved both f and g, we call them mixed-type. In [3], an example is given to show that Theorem B cannot be extended to two-function minimax theorems of type B. Specifically, let $X = Y = [-1,1]$. There exist functions $f, g\colon X \times Y \to \mathbb{R}$ with $f \leq g$ on $X \times Y$, f is upward on Y (i.e., (B2) is satisfied) and g is downward on X (i.e., (B1) is satisfied when f is replaced by g) but $\inf\limits_{y\in Y}\sup\limits_{x\in X} f(x,y) = 1$ and $\sup\limits_{x\in X}\inf\limits_{y\in Y} g(x,y) = -1$. In [1], it is pointed out that this example also shows that Theorem A cannot be extended to two-function minimax theorems of Type B. In [Remark (iii), 2], the question is raised how to extend Theorem B (Simons's upward-downward minimax theorem) to a two-function minimax theorem of mixed type. Theorem in this paper provides an affirmative answer to the question.

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