TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 565-574, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

C^*-CROSSED PRODUCTS OF C^*-ALGEBRAS WITH THE WEAK BANACH-SAKS PROPERTY BY COACTIONS

Masaharu Kusuda

Abstract. Let A be a C^*-algebra and let δ be a nondegenerate coaction of a locally compact group G on A. Suppose that δ is pointwise unitary and that $\widehat{A}$ is the Hausdorff spectrum of A. Then it is shown that A has the weak Banach-Saks property and G is discrete if and only if the crossed product $A \times_\delta G$ by δ has the weak Banach-Saks property.

1. Introduction

In [1], Banach and Saks showed that every bounded sequence in $L^p([0,1])$ with $1 < p < \infty$ has a subsequence whose arithmetic means converge in the norm topology. More generally, if every bounded sequence in a Banach space X has a subsequence whose arithmetic means converge in the norm topology, we say that X has the *Banach-Saks property*. It is known that Banach spaces with the Banach-Saks property are reflexive. It hence follows that $L^1([0,1])$ cannot have the Banach-Saks property.

Let X be a Banach space. If given any weakly null sequence $\{x_n\}$ in X, one can extract a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty} \frac{1}{k} \|x_{n(1)} + \cdots + x_{n(k)}\| = 0,$$

we say that X has the *weak Banach-Saks property*. It was shown by Szlenk [11] that $L^1([0,1])$ has the weak Banach-Saks property.

Recently, Chu [2] has studied C^*-algebras with the weak Banach-Saks property in detail as a noncommutative extension of characterizations of the Banach space,

Received April 26, 2000; revised July 26, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: Primary 46L55; Secondary 46L05.
Key words and phrases: C^*-crossed product, coaction, the weak Banach-Saks property, dual C^*-algebra, fixed point algebra.

of complex continuous functioins on a compact Hausdorff space, with the weak Banach-Saks property. Actually he has obtained the following characterization of C^*-algebras with the weak Banach-Saks property.

Theorem A [2, Theorem 2]. *Let A be a C^*-algebra. Then the following conditions are equivalent*:

(1) *A has the weak Banach-Saks property.*

(2) *A is scattered and $c_0(A)$ does not contain an isometric copy of $C_0(\omega^\omega)$, where ω^ω denotes the set $[0, \omega^\omega)$ of ordinals preceding ω^ω with the order topology.*

(3) *A is scattered and does not contain an isometric copy of $C_0(\omega^\omega)$.*

(4) *A is of type I and $\widehat{A}^{(k)}$ is empty for some natural number k, where $\widehat{A}^{(0)} = \widehat{A}$, the spectrum of A, and $\widehat{A}^{(n)}$ is the nth derived set of $\widehat{A}$, consisting of the accumulation points of $\widehat{A}^{(n-1)}$.*

(5) *There exists some natural number k such that $\sigma(a)^{(k)}$ is empty for every self-adjoint $a \in A$, where $\sigma(a)$ denotes the spectrum of a.*

Given a C^*-dynamical system (A, G, α), the author has recently studied when the C^*-crossed product $A \times_\alpha G$ by the action α of G has the weak Banach-Saks property, under the condition that A should have the weak Banach-Saks property (see Theorems B and C in §2 below). In this paper, we treat C^*-algebras A with coactions δ of locally compact groups G, and we consider when the crossed product $A \times_\delta G$ by a coaction δ has the weak Banach-Saks property, under the condition that A should have the weak Banach-Saks property. We shall show some results, for crossed products by coactions, similar to the results obtained for crossed products by actions.

2. Results

For a C^*-algebra A, we denote again by $\widehat{A}$ the spectrum of A, that is, the set of (unitary) equivalence classes $[\pi]$ of nonzero irreducible representations π of A equipped with the Jacobson topology. We note that $\widehat{A}$ is a locally compact space, not necessarily a Hausdorff space. However, we will pay our attension later to the case where $\widehat{A}$ is a Hausdorff space.

First we state results on C^*-crossed products by actions without the proofs (see the references cited below for the proofs). Let (A, G, α) be a C^*-dynamical system. By a *C^*-dynamical system*, we mean a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and a group homomorphism α from G into the automorphism group of A such that $G \ni t \mapsto \alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $A \times_\alpha G$ (resp. $A \times_{\alpha r} G$) the C^*-crossed product of A by G (resp. the reduced C^*-crossed product of A by G).

Given a C^*-dynamical system (A, G, α), α induces the natural action of G on $\widehat{A}$ which is defined by

$$(t, [\pi]) \in G \times \widehat{A} \mapsto [\pi \circ \alpha_{t^{-1}}] \in \widehat{A}.$$

This map makes G into a topological transformation group acting on $\widehat{A}$. For $[\pi] \in \widehat{A}$, we denote by $S_{[\pi]}$ the stability group at $[\pi]$, which is defined by $S_{[\pi]} = \{t \in G | [\pi \circ \alpha_{t^{-1}}] = [\pi]\}$. If all stability groups are trivial, i.e., $S_{[\pi]}$ consists only of the identity of G at every $[\pi] \in \widehat{A}$, it is said that G acts *freely* on $\widehat{A}$. In this situation, the author has shown the following result:

Theorem B [6, Theorem 2.6]. *Let (A, G, α) be a C^*-dynamical system. Suppose that G acts freely on $\widehat{A}$. Then the following conditions are equivalent*:

(1) *A has the weak Banach-Saks property.*

(2) *G is discrete and $A \times_\alpha G$ has the weak Banach-Saks property.*

(3) *G is discrete and $A \times_{\alpha r} G$ has the weak Banach-Saks property.*

In [6, Theorem 2.6], the equivalence of (1) and (2) was shown. Since every quotient of a C^*-algebra with the weak Banach-Saks property has the weak Banach-Saks property [2, p. 6], Condition (2) obviously implies Condition (3). If G is discrete, A is embedded into $A \times_{\alpha r} G$. Hence Condition (3) then implies Condition (1).

For a C^*-dynamical system (A, G, α), we say that α is *pointwise unitary* if for every irreducible representation (π, H_π) of A, there exists a strongly continuous unitary representation u of G on H_π such that

$$\pi(\alpha_t(a)) = u_t \pi(a) u_t^*$$

for all $a \in A$ and $t \in G$.

In Theorem B above, the group which acts on A as an automorphism group is discrete and $S_{[\pi]}$ consists only of the identity of the group at every $[\pi] \in \widehat{A}$. Hence, given a C^*-dynamical system (A, G, α), the situation opposite to that of Theorem B is that G is compact and $S_{[\pi]} = G$ at every $[\pi] \in \widehat{A}$. A result in such a situation is the following:

Theorem C [6, Theorem 3.2]. *Let (A, G, α) be a C^*-dynamical system and let G be compact. We denote by A^α the fixed point algebra of A under α. Consider the following conditions*:

(1) *A has the weak Banach-Saks property.*

(2) *$A \times_\alpha G$ has the weak Banach-Saks property.*

(3) *A^α has the weak Banach-Saks property.*

Then we have $(1) \Longrightarrow (2) \Longrightarrow (3)$.

Furthermore, we suppose that G is (compact) abelian. Then the implication $(3) \Longrightarrow (2)$ *holds. If A is of type I and if α is pointwise unitary, the implication* $(2) \Longrightarrow (1)$ *holds.*

Now we turn to crossed products by coactions, and we briefly review their definition. Let G be a locally compact group. We denote by λ the left regular representation of G on $L^2(G)$. We define the representation $\tilde{\lambda}$ of $L^1(G)$ on $L^2(G)$ by $\tilde{\lambda}(f) = \int_G f(t)\lambda_t dt$ for $f \in L^1(G)$. Then the reduced group C^*-algebra $C_r^*(G)$ of G is defined as the norm closure of $\tilde{\lambda}(L^1(G))$ in the set of all bounded linear operators on $L^2(G)$.

Let A be a C^*-algebra and denote by $M(A \otimes_{\min} C_r^*(G))$ the multiplier algebra of the injective C^*-tensor product $A \otimes_{\min} C_r^*(G)$. We then define the C^*-subalgebra $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ of $M(A \otimes_{\min} C_r^*(G))$ by

$$\begin{aligned}\widetilde{M}(A \otimes_{\min} C_r^*(G)) = \{m \in M(A \otimes_{\min} C_r^*(G)) | m(1 \otimes x), (1 \otimes x)m \\ \in A \otimes_{\min} C_r^*(G) \text{ for all } x \in C_r^*(G)\}.\end{aligned}$$

We denote by W_G the unitary operator on $L^2(G \times G)$ defined by

$$(W_G\xi)(s,t) = \xi(s, s^{-1}t) \text{ for } \xi \in L^2(G \times G) \text{ and } s, t \in G.$$

Define the homomorphism δ_G from $C_r^*(G)$ into $\widetilde{M}(C_r^*(G) \otimes_{\min} C_r^*(G))$ by

$$\delta_G(\tilde{\lambda}(f)) = W_G(\tilde{\lambda}(f) \otimes 1)W_G^* \quad \text{for} \quad f \in L^1(G).$$

We say that an injective homomorphism δ from A into $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ is a *coaction* of a locally compact group G on A if δ satisfies:

(a) there is an approximate identity $\{e_i\}$ for A such that $\delta(e_i) \to 1$ strictly in $\widetilde{M}(A \otimes_{\min} C_r^*(G))$,

(b) $(\delta \otimes id)(\delta(a)) = (id \otimes \delta_G)(\delta(a))$ for all $a \in A$.

Furthermore, the coaction δ is said to be *nondegenerate* if it satisfies the additional condition:

(c) for every nonzero $\varphi \in A^*$, there exists $\psi \in C_r^*(G)^*$ such that $(\varphi \otimes \psi) \circ \delta \neq 0$.

(In (b) and (c), we implicitly extended δ to $M(A \otimes_{\min} C_r^*(G))$, which is ensured by Condition (a).)

Let δ be a coaction of a locally compact group G on A and let $C_0(G)$ be the set of all continuous functions on G vanishing at infinity. We denote by M_f the multiplication operator on $L^2(G)$ given by $f \in C_0(G)$, which is defined by $(M_f\xi)(t) = f(t)\xi(t)$ for all $\xi \in L^2(G)$. Then the *crossed product* $A \times_\delta G$ of A by

δ is the C^*-subalgebra of $\widetilde{M}(A\otimes_{\min}C_r^*(G))$ generated by the set $\{\delta(a)(1\otimes M_f)|a\in A, f\in C_0(G)\}$.

Suppose that δ is nondegenerate. If, for every irreducible representation π of A, there exists a unitary $W\in M(\pi(A)\otimes_{\min}C_r^*(G))$ such that

$$(\pi\otimes id)(\delta(a)) = W(\pi(a)\otimes 1)W^*$$

for $a\in A$, then δ is called *pointwise unitary*.

Theorem 2.1. *Let A be a C^*-algebra and let δ be a nondegenerate coaction of a locally compact group G on A. Consider the following conditions*:

(1) *A has the weak Banach-Saks property and G is discrete.*

(2) *$A\times_\delta G$ has the weak Banach-Saks property.*

Then we have $(1)\Longrightarrow(2)$. *If δ is pointwise unitary and if the spectrum $\widehat{A}$ is a Hausdorff space*, *then the implication* $(2)\Longrightarrow(1)$ *holds.*

Proof. $(1)\Longrightarrow(2)$. It follows from [5] that there exists the dual action $\widehat{\delta}$ of G such that we have

$$(A\times_\delta G)\times_{\widehat{\delta}} G\cong A\otimes C(L^2(G)),$$

where $C(L^2(G))$ denotes the C^*-algebra of all compact operators on $L^2(G)$. We note here that A is of type I if and only if $A\otimes C(L^2(G))$ is, and that $\widehat{A}$ is homeomorphic to $(A\otimes C(L^2(G)))\widehat{\ }$. Thus we see that A has the weak Banach-Saks property if and only if $A\otimes C(L^2(G))$ does. Hence $(A\times_\delta G)\times_{\widehat{\delta}} G$ has the weak Banach-Saks property. Since G is discrete, $C_r^*(G)$ is unital. Since $(A\times_\delta G)\times_{\widehat{\delta}} G$ is generated by $\tilde{\pi}(A\times_\delta G)(1\otimes 1\otimes C_r^*(G))$, where $\tilde{\pi}$ is some faithful representation of $A\times_\delta G$ (see [5] or [9, p. 768]), we see that $A\times_\delta G$ is embedded into $(A\times_\delta G)\times_{\widehat{\delta}} G$ as a C^*-subalgebra. Since every C^*-subalgebra of a C^*-algebra with the weak Banach-Saks property has also the weak Banach-Saks property [2], we conclude that $A\times_\delta G$ has the weak Banach-Saks property.

$(2)\Longrightarrow(1)$. Since δ is pointwise unitary and $\widehat{A}$ is a Hausdorff space, it follows from [9, Theorem 5.5] that the action of G induced by the dual action $\widehat{\delta}$ is free on $(A\times_\delta G)\widehat{\ }$. Hence we see from Theorem B above that G is discrete and that $(A\times_\delta G)\times_{\widehat{\delta}} G$ has the weak Banach-Saks property. Since $(A\times_\delta G)\times_{\widehat{\delta}} G$ is isomorphic to $A\otimes C(L^2(G))$, $A\otimes C(L^2(G))$ also has the weak Banach-Saks property. Thus we see that A has the weak Banach-Saks property. ■

The following Lemma is obtained as an application of Morita equivalence (see [8, Lemma 2.1]). But we give a direct elementary proof.

Lemma 2.2. *Let A be a C^*-algebra and let B be a hereditary C^*-subalgebra of A. We denote by $I(B)$ the closed ideal of A generated by B. Then B has the weak Banach-Saks property if and only if $I(B)$ has the weak Banach-Saks property.*

Proof. Suppose that B has the weak Banach-Saks property. For every nonzero irreducible representation (π, H) of $I(B)$, the restriction of π to B is not zero. Hence the map $\pi \mapsto \pi|_B$ induces a homeomorphism from $\widehat{I(B)}$ onto $\widehat{B}$. Hence, if $\widehat{B}^{(k)}$ is empty for some natural number k, then $\widehat{I(B)}^{(k)}$ is empty. On the other hand, since B is postliminal, there exists x in B such that $\pi|_B(x)$ is compact and nonzero. Hence $\pi(x)$ is compact and nonzero on H. Thus we conclude that $I(B)$ is postliminal or, equivalently, of type I. Thus we see that $I(B)$ has the weak Banach-Saks property. Since the reverse implication is trivial, we complete the proof. ∎

Let X be a topological space. Then the nth derived set $X^{(n)}$ of X is defined as follows: Put $X^{(0)} = X$, and define $X^{(n)}$ as the set of all accumulation points of $X^{(n-1)}$. We need the following Lemma to show Theorem 2.3 below.

Lemma D [6, Lemma 3.1]. *Let X be a topological space and let $\{\mathcal{O}_i\}_{i\in I}$ be a family of open subsets in X. Suppose that $X = \bigcup_{i\in I} \mathcal{O}_i$. Then we have $X^{(k)} = \bigcup_{i\in I} \mathcal{O}_i^{(k)}$ for each natural number $k \in \mathbb{N}$.*

Let A be a C^*-algebra and let δ be a nondegenerate coaction of a *discrete* group G on A. Then we denote by A^δ the fixed point algebra of A under δ, which is defined by

$$A^\delta = \{a \in A \mid \delta(a) = a \otimes 1\}.$$

Then A^δ is not zero [10, Remark 2.2]. Now we denote by M_e the multiplication operator defined by

$$(M_e\xi)(t) = \xi(e) \quad \text{for} \quad \xi \in l^2(G),$$

where e denotes the identity of G.

Theorem 2.3. *Let A be a C^*-algebra and let δ be a nondegenerate coaction of a discrete group G on A. Consider the following conditions*:

(1) *$A \times_\delta G$ has the weak Banach-Saks property.*

(2) *A^δ has the weak Banach-Saks property.*

Then we have $(1) \Longrightarrow (2)$. *Conversely, if G is* (*discrete*) *amenable*, *then the implication* $(2) \Longrightarrow (1)$ *holds.*

Proof. $(1) \Longrightarrow (2)$. Since $\{a \otimes 1 | a \in A^\delta\}$ is a C^*-subalgebra of $A \times_\delta G$ which is isomorphic to A^δ, A^δ has the weak Banach-Saks property.

(2) $\Longrightarrow$ (1). Since A^δ is of type I, so is $A \times_\delta G$ (see [10, Theorem 3.6]). Hence we have only to show that the kth derived set $(\widehat{A \times_\delta G})^{(k)}$ is empty for some natural number k.

Let $p = 1 \otimes M_e \in M(A \times_\delta G)$. Then p is a projection, and it follows from the proof of [10, Proposition 2.4] that the hereditary C^*-subalgebra $p(A \times_\delta G)p$ coincides with $\{(a \otimes 1)(1 \otimes M_e) | a \in A^\delta\} = \{a \otimes M_e | a \in A^\delta\}$. Since A^δ is isomorphic to $\{a \otimes M_e | a \in A^\delta\} \subset A \times_\delta G$ [10, Proposition 2,4], it follows from Lemma 2.2 that the closed ideal J of $A \times_\delta G$ generated by $\{a \otimes M_e | a \in A^\delta\}$ has the weak Banach-Saks property. By Theorem A, we can conclude that $\widehat{J}^{(k)}$ is empty for some natural number k. Let $\{I_j\}$ be the family of closed ideals of $A \times_\delta G$ such that $\widehat{I_j}^{(k)}$ is empty for every j and let I be the closed ideal of $A \times_\delta G$ generated by $\bigcup I_j$. Since we have $\widehat{I} = \bigcup \widehat{I_j}$ and since the spectrum of every closed ideal is an open subset, it follows from Lemma D that $\widehat{I}^{(k)} = \bigcup \widehat{I_j}^{(k)}$is empty. Thus, applying Lemma D again, we easily see that I is the largest ideal of $A \times_\delta G$ among all closed ideals whose kth derived sets of the spectra are empty. Hence an easy application of lemma D shows that I is invariant under every automorphism of $A \times_\delta G$; in particular, I is $\widehat{\delta}$-invariant for the dual action $\widehat{\delta}$ of δ. Thus we see that there exists a δ-invariant ideal B of A such that $I = B \times_\delta G$ (see [4, 3.7]). Then we conclude from the proof of [10, Theorem 3.6] that $A = B$, that is, $I = A \times_\delta G$. Hence $(\widehat{A \times_\delta G})^{(k)} (= \widehat{I}^{(k)})$ is empty. Thus we complete the the proof. ■

Note that every coaction of an amenable group is always nondegenerate [9, Remarks 2.2].

Proposition 2.4. *Let A be a C^*-algebra and let δ be a coaction of a compact group G on A. If $A \times_\delta G$ has the weak Banach-Saks property, then A also has the same property.*

Proof. Since G is compact, $C_0(G)$ has the identity. Since $A \times_\delta G$ is generated by $\{\delta(a)(1 \otimes M_f) | a \in A, f \in C_0(G)\}$, A is regarded as a C^*-subalgebra of $A \times_\delta G$. Hence A has the weak Banach-Saks property. ■

We obtain the following corollary from Theorem 2.1, Theorem 2.3 and Proposition 2.4.

Corollary 2.5. *Let A be a C^*-algebra and let δ be a coaction of a finite group G on A. Then the following conditions are equivalent*:

(1) *A has the weak Banach-Saks property.*

(2) *$A \times_\delta G$ has the weak Banach-Saks property.*

(3) *A^δ has the weak Banach-Saks property.*

3. Remarks on Crossed Products of Dual C^*-algebras by Coactions

In this section, we give results for dual C^*-algebras similar to Theorem 2.1, Theorem 2.3, Proposition 2.4 and Corollary 2.5. We recall that a C^*-algebra A is called *dual* if and only if it is isomorphic to a C^*-subalgebra of the C^*-algebra of compact operators on some Hilbert space or, equivalently, every maximal abelian subalgebra of A is generated by minimal projections [3, 4.7.20]. Since the C^*-algebra of compact operators on some Hilbert space has the weak Banach-Saks property, it follows that every dual C^*-algebra has the weak Banach-Saks property. As is easily seen, A is a type-I C^*-algebra with discrete spectrum $\widehat{A}$ if and only if it is a c_0-sum of C^*-algebras of compact operators. Thus the C^*-algebra A is dual if and only if it is a type-I C^*-algebra with discrete spectrum $\widehat{A}$. Note that dual C^*-algebras play an essential role in the study of C^*-algebras with the weak Banach-Saks property. In fact, at the end of [2], Chu has shown that a C^*-algebras A has the weak Banach-Saks property if and only if there are closed ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset A$ such that I_1 and all the successive quotients are dual C^*-algebras. For dual C^*-algebras, we have the result similar to Thoerem B as follows.

Theorem E [6, Theorem 2.3]. *Let (A, G, α) be a C^*-dynamical system. Suppose that G acts freely on $\widehat{A}$. Then the following conditions are equivalent*:

(1) *A is a dual C^*-algebra.*

(2) *G is discrete and $A \times_\alpha G$ is a dual C^*-algebra.*

(3) *G is discrete and $A \times_{\alpha r} G$ is a dual C^*-algebra.*

In fact, the equivalence of (1) and (2) was shown in [6, Theorem 2.3], and (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are obvious.

We remark that by definition, every C^*-subalgebra of a dual C^*-algebra is dual. As we easily see that A is a dual C^*-algebra if and only if $A \otimes C(L^2(G))$ is a dual C^*-algebra, employing Theorem E now we can show the following result for dual C^*-algebras similar to Theorem 2.1. In fact, the proof of Theorem 3.1 proceeds along lines similar to those of the proof of Theorem 2.1. Hence we will leave the details of the proof to the reader.

Theorem 3.1. *Let A be a C^*-algebra and let δ be a nondegenerate coaction of a locally compact group G on A. Consider the following conditions*:

(1) *A is a dual C^*-algebra and G is discrete.*

(2) *$A \times_\delta G$ is a dual C^*-algebra.*

Then we have $(1) \Longrightarrow (2)$. If δ is pointwise unitary and if the spectrum $\widehat{A}$ is a Hausdorff space, then the implication $(2) \Longrightarrow (1)$ holds.

Note that a C^*-algebra A is dual if and only if A is of type I and $\widehat{A}^{(1)}$ is empty. Hence we obtain the result on dual C^*-algebras, which is similar to Lemma 2.2, that a hereditary C^*-subalgebra B of A is dual if and only if the closed ideal $I(B)$ of A generated by B is dual. Thus, using Lemma D, the following theorem can be shown by modifying the proof of Theorem 2.3.

Theorem 3.2. *Let A be a C^*-algebra and let δ be a nondegenerate coaction of a discrete group G on A. Consider the following conditions*:

(1) *$A \times_\delta G$ is a dual C^*-algebra.*

(2) *A^δ is a dual C^*-algebra.*

Then we have $(1) \Longrightarrow (2)$. *Conversely, if G is* (*discrete*) *amenable, then the implication* $(2) \Longrightarrow (1)$ *holds.*

Proposition 3.3. *Let A be a C^*-algebra and let δ be a coaction of a compact group G on A. If $A \times_\delta G$ is a dual C^*-algebra, then A is also a dual C^*-algebra.*

The proof of Proposition 2.4 is valid for Proposition 3.3. Hence we will leave the details to the reader.

Corollary 3.4. *Let A be a C^*-algebra and let δ be a coaction of a finite group G on A. Then the following conditions are equivalent*:

(1) *A is a dual C^*-algebra.*

(2) *$A \times_\delta G$ is a dual C^*-algebra.*

(3) *A^δ is a dual C^*-algebra.*

This corollary follows from Theorem 3.1, Theorem 3.2 and Proposition 3.3.

Acknowkedgements

This work was financially supported by Kansai University, Japan in 1999, and it was done while the author was visiting Mathematisches Institut, Westfälische Wilhelms-Universität Münster in 1999. He would like to thank Kansai University for their financial support, and to express his gratitude to J. Cuntz and S. Echterhoff for their hospitality.

Added in proof: The reference [7] will appear as §3 in [6].

References

1. S. Banach and S. Saks, Sur la convergence forte dans les champs L^p, *Studia Math.* **2** (1930), 51-57.

2. C.-H. Chu, The weak Banach-Saks property in C^*-algebras, *J. Funct. Anal.* **121** (1994), 1-14.

3. J. Dixmier, *C^*-algebras*, North Holland, New York, 1982.

4. E. C. Gootman and A. J. Lazar, Applications of non-commutative duality to crossed product C^*-algebras determined by actions or coactions, *Proc. London Math. Soc.* **59** (1989), 593-624.

5. Y. Katayama, Takesaki's duality for a nondegenerate coaction, *Math. Scand.* **55** (1985), 141-151.

6. M. Kusuda, C^*-crossed products of C^*-algegbras with the weak Banach-Saks property, to appear.

7. M. Kusuda, C^*-crossed products of C^*-algegbras with the weak Banach-Saks property by compact actions, to appear.

8. M. Kusuda, Morita equivalence for C^*-algegbras with the weak Banach-Saks property, to appear.

9. M. B. Landstad, J. Phillips, I. Raeburn and C. E. Sutherland, Representations of crossed products by coactions and principal bundles, *Trans. Amer. Math. Soc.* **299** (1987), 747-784.

10. C.-K. Ng, Discrete coactions on C^*-algebras, *J. Austral. Math. Soc. Ser. A* **60** (1996), 118-127.

11. W. Szlenk, Sur les suites faiblementes convergentes dans l'espace L, *Studia Math.* **25** (1965), 337-341.

Department of Mathematics, Faculty of Engineering, Kansai University
Yamate-cho 3-3-35, Suita, Osaka 564-8680, Japan
E-mail: kusuda@ipcku.kansai-u.ac.jp