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THE NEW $k-n$ TYPE NEUBERG-PEDOE INEQUALITIES

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Abstract. In this paper, a class of geometric inequalities for the volumes of two $n-$simplexes and their $k-$subsimplexes are established. The results are generalizations to several dimensions of the well-known Neuberg-Pedoe inequality of two triangles.

1. Introduction

Geometric inequalities for simplices which are the simplest and the most useful polytopes have been a very attractive subject for a long time. Mitrinovic, Pecaric, Volenec [9], Ali [1], Gerber [4], Petty, Waterman [11] and other authors [7,8] have obtained a great number of elegant results. Specially, the quantity relations involving two simplices have been studied extensively. The well-known Neuberg-Pedoe inequality is the first inequality for the edge-lengths and areas involving two triangles [10].

The Neuberg-Pedoe inequality is as follows.

Let $a_i, b_i, c_i (i=1,2)$ be the edge-lengths of the triangle with area $\triangle_i$, then

$$H_2 = a_1^2(-a_2^2+b_2^2+c_2^2)+b_1^2(a_2^2-b_2^2+c_2^2)+c_1^2(a_2^2+b_2^2-c_2^2) \geq 16\triangle_1\triangle_2, \tag{1.1}$$

with equality holds if and only if two triangles are similar.

Following Neuberg-Pedoe, a number of inequalities for two simplexes have been established.

In 1984, P. Chia-Kuei proved the following sharpening of the Neuberg-Pedoe inequality [3].

$$bH_2 \geq 8\left(\frac{a_2^2+b_2^2+c_2^2}{a_1^2+b_1^2+c_1^2}\triangle_1^2+\frac{a_1^2+b_1^2+c_1^2}{a_2^2+b_2^2+c_2^2}\triangle_2^2\right), \tag{1.2}$$

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with equality holds if and only if two triangles are similar.

In 1981, Yang Lu and Zhang Jingzhong got a generalization to several dimensions of Neuberg-Pedoe inequality (1.1) [13]. Su Huaming [12], Chen Ji and Ma Yuan[2] also gave the generalization of Neuberg-Pedoe inequality (1.1) for the edge lengths and volumes of two $n-$simplexes. In 1997, Leng Gangsong and Tang Lihua ([5,6]) obtained the generalization of the inequality (1.2) for the edge lengths and facet areas and volumes of two $n-$simplexes.

In this paper, except for the introduction, is divided into two sections. In Section 2, we shall extend inequalities (1.1) and (1.2) to $n-$dimensional Euclidean space E^n, and establish a class of geometric inequalities for the volumes of two $n-$simplexes and their $k-$subsimplexes. Moreover, we shall get the generalizations and strengthening of the results in [2, 5, 6, 12], which different from the result of Yang and zhang[13]. In Section 3, we introduce a class of lemmas which contains the inequalities concerning mass-point systems for two simplexes. Further, we prove our main results by applying these lemmas.

2. Main Results

Our main results are the following three theorems and five corollaries.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $n-$simplexes in E^n with the $n-$dimensional volumes V_A and V_B respectively. Let $S_i(k)$ denote the $k-$dimensional volumes of $k-$dimensional subsimplexes spanned by $k+1$ vertexes $A_{i_1}, A_{i_2}, \cdots, A_{i_{k+1}}$ of $\mathcal{A}$, and $S = \sum_{i=1}^{m} S_i^{\theta}(k)$, where $m = \binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!(n-k)!}$, and $F_i(k)$ denote the $k-$dimensional volumes of $k-$dimensional subsimplexes spanned by $k+1$ vertexes $B_{i_1}, B_{i_2}, \cdots, B_{i_{k+1}}$ of $\mathcal{B}$, and $F = \sum_{i=1}^{m} F_i^{\theta}(k)$. If $\alpha, \beta \in (0,1], \gamma \in [0, n+1-k], n \geq 3$, and $a_i, b_i \in R^+$ $(i = 1, 2)$ (here a_1, b_1 are any $k\alpha$ degree geometric quantities, and a_2, b_2 are any $k\beta$ degree). Then*

$$
\begin{aligned}
&\sum_{i=1}^{m}\left(a_1 S_i^{\alpha}(k) + a_2 S_i^{\beta}(k)\right)\\
&\left[\sum_{j=1}^{m}\left(b_1 F_j^{\alpha}(k) + b_2 F_j^{\beta}(k)\right) - \gamma\left(b_1 F_i^{\alpha}(k) + b_2 F_i^{\beta}(k)\right)\right]\\
&\geq \frac{1}{2} m(m-\gamma)\left[\frac{b_1 F_\alpha + b_2 F_\beta}{a_1 S_\alpha + a_2 S_\beta}\left(a_1 \mu_{n,k}^{\alpha} V_A^{\frac{k\alpha}{n}} + a_2 \mu_{n,k}^{\beta} V_A^{\frac{k\beta}{n}}\right)^2\right.\\
&\left.+\frac{a_1 S_\alpha + a_2 S_\beta}{b_1 F_\alpha + b_2 F_\beta}\left(b_1 \mu_{n,k}^{\alpha} V_B^{\frac{k\alpha}{n}} + b_2 \mu_{n,k}^{\beta} V_B^{\frac{k\beta}{n}}\right)^2\right] + R_1,
\end{aligned}
\tag{2.1}
$$

with equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular, where $\mu_{n,k} = \frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}$,

$$(2.1)' \qquad \begin{aligned} R_1 = &\frac{\gamma}{2(a_1S_\alpha + a_2S_\beta)(b_1F_\alpha + b_2F_\beta)} \\ &\sum_{i=1}^{m}\Big[(b_1F_\alpha + b_2F_\beta)(a_1S_i^\alpha(k) + a_2S_i^\beta(k)) \\ &-(a_1S_\alpha + a_2S_\beta)(b_1F_i^\alpha(k) + b_2F_i^\beta(k))\Big]^2 \geq 0. \end{aligned}$$

Theorem 2.2. *Under the hypotheses in theorem* 2.1*, we have*

$$(2.2) \qquad \begin{aligned} &\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}S_i^\alpha(k)S_j^\beta(k)}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}F_i^\alpha(k)F_j^\beta(k)} \\ &-\gamma\sum_{i=1}^{m}S_i^{\frac{\alpha+\beta}{2}}(k)F_i^{\frac{\alpha+\beta}{2}}(k) \\ \geq &\frac{1}{2}m(m-\gamma)\mu_{n,k}^{\alpha+\beta}\left[\frac{\sqrt{F_\alpha F_\beta}}{\sqrt{S_\alpha S_\beta}}V_A^{\frac{k(\alpha+\beta)}{n}} + \frac{\sqrt{S_\alpha S_\beta}}{\sqrt{F_\alpha F_\beta}}V_B^{\frac{k(\alpha+\beta)}{n}}\right] + R_2, \end{aligned}$$

with equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular, where

$$(2.2)' \quad R_2 = \frac{\gamma}{2\sqrt{S_\alpha S_\beta F_\alpha F_\beta}}\sum_{i=1}^{m}\left(\sqrt{F_\alpha F_\beta}S_i^{\frac{\alpha+\beta}{2}}(k) - \sqrt{S_\alpha S_\beta}F_i^{\frac{\alpha+\beta}{2}}(k)\right)^2 \geq 0.$$

Theorem 2.3. *Under the hypotheses in theorem* 2.1*, we have*

$$(2.3) \qquad \begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{m}S_i^\alpha(k)F_j^\beta(k)\left(\sum_{u=1}^{m}\sum_{v=1}^{m}S_u^\beta(k)F_v^\alpha(k) - \gamma^2S_i^\beta(k)F_j^\alpha(k)\right) \\ \geq &\frac{1}{2}m^2(m^2-\gamma^2)\mu_{n,k}^{2(\alpha+\beta)}\left(\frac{F_\alpha F_\beta}{S_\alpha S_\beta}V_A^{\frac{2k(\alpha+\beta)}{n}} + \frac{S_\alpha S_\beta}{F_\alpha F_\beta}V_B^{\frac{2k(\alpha+\beta)}{n}}\right) + R_3, \end{aligned}$$

with equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular, where

$$(2.3)' \qquad R_3 = \frac{\gamma^2}{2S_\alpha S_\beta F_\alpha F_\beta}(F_\alpha F_\beta S_{\alpha+\beta} - S_\alpha S_\beta F_{\alpha+\beta})^2 \geq 0.$$

By applying the arithmetic mean-geometric inequality in (2.1), (2.2), (2.3), we get the following corollaries respectively.

Corollary 2.1. *Under the hypotheses in theorem* 2.1*, we have*

$$\begin{aligned}
&\sum_{i=1}^{m}\left(a_1 S_i^{\alpha}(k)+a_2 S_i^{\beta}(k)\right)\\
&\left[\sum_{j=1}^{m}\left(b_1 F_j^{\alpha}(k)+b_2 F_j^{\beta}(k)\right)-\gamma\left(b_1 F_i^{\alpha}(k)+b_2 F_i^{\beta}(k)\right)\right]\\
&\geq 2m(m-\gamma)\mu_{n,k}^{\alpha+\beta}\left[a_1 a_2\frac{F_\alpha+F_\beta}{S_\alpha+S_\beta}V_A^{\frac{k(\alpha+\beta)}{n}}\right.\\
&\left.+b_1 b_2\frac{S_\alpha+S_\beta}{F_\alpha+F_\beta}V_B^{\frac{k(\alpha+\beta)}{n}}\right]+R_1,\\
&\geq 4m(m-\gamma)\sqrt{a_1 a_2 b_1 b_2}\mu_{n,k}^{\alpha+\beta}\left(V_A V_B\right)^{\frac{k(\alpha+\beta)}{2n}}+R_1,
\end{aligned}\tag{2.4}$$

with equality holds if and only if $\mathcal{A}$ *and* $\mathcal{B}$ *are regular.*

Corollary 2.2. *Under the hypotheses in Theorem* 2.2*, we have*

$$\begin{aligned}
&\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}S_i^{\alpha}(k)S_j^{\beta}(k)}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}F_i^{\alpha}(k)F_j^{\beta}(k)}\\
&-\gamma\sum_{i=1}^{m}S_i^{\frac{\alpha+\beta}{2}}(k)F_i^{\frac{\alpha+\beta}{2}}(k)\geq m(m-\gamma)\mu_{n,k}^{\alpha+\beta}\left(V_A V_B\right)^{\frac{k(\alpha+\beta)}{2n}}+R_2,
\end{aligned}\tag{2.5}$$

with equality holds if and only if $\mathcal{A}$ *and* $\mathcal{B}$ *are regular.*

Corollary 2.3. *Under the hypotheses in Theorem* 2.3*, we have*

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{m}S_i^{\alpha}(k)F_j^{\beta}(k)\left(\sum_{u=1}^{m}\sum_{v=1}^{m}S_u^{\beta}(k)F_v^{\alpha}(k)-\gamma^2 S_i^{\beta}(k)F_i^{\alpha}(k)\right)\\
&\geq m^2(m^2-\gamma^2)\mu_{n,k}^{2(\alpha+\beta)}\left(V_A V_B\right)^{\frac{k(\alpha+\beta)}{n}}+R_3,
\end{aligned}\tag{2.6}$$

with equality holds if and only if $\mathcal{A}$ *and* $\mathcal{B}$ *are regular.*

Put $\alpha=\beta$ and $a_1=a_2, b_1=b_2$ in inequalities (2.1), (2.4) and (2.2), (2.5), we obtain following corollary.

Corollary 2.4. *Under the hypotheses in theorem* 2.1*, we have*

$$\begin{aligned}
&\sum_{i=1}^{m} S_i^{\alpha}(k)\left(\sum_{j=1}^{m} F_j^{\alpha}(k)-\gamma F_i^{\alpha}(k)\right)\\
&\geq \frac{1}{2}m(m-\gamma)\mu_{n,k}^{2\alpha}\left(\frac{F_\alpha}{S_\alpha}V_A^{\frac{2k\alpha}{n}}+\frac{S_\alpha}{F_\alpha}V_B^{\frac{2k\alpha}{n}}\right)+R_4\\
&\geq m(m-\gamma)\mu_{n,k}^{2\alpha}\left(V_AV_B\right)^{\frac{k\alpha}{n}}+R_4,
\end{aligned} \tag{2.7}$$

with equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular, where

$$R_4=\frac{\gamma}{2S_\alpha F_\alpha}\sum_{i=1}^{m}\left(F_\alpha S_i^{\alpha}(k)-S_\alpha F_i^{\alpha}(k)\right)^2\geq 0. \tag{2.7$'$}$$

Put $\alpha=\beta$ in (2.3), (2.6), we obtain following corollary.

Corollary 2.5. *Under the hypotheses in theorem* 2.1*, we have*

$$\begin{aligned}
&\left(\sum_{i=1}^{m} S_i^{\alpha}(k)\right)^2\left(\sum_{i=1}^{m} F_i^{\alpha}(k)\right)^2-\gamma^2\left(\sum_{i=1}^{m} S_i^{2\alpha}(k)\right)\left(\sum_{i=1}^{m} F_i^{2\alpha}(k)\right)\\
&\geq \frac{1}{2}m^2(m^2-\gamma^2)\mu_{n,k}^{4\alpha}\left(\frac{F_\alpha^2}{S_\alpha^2}V_A^{\frac{4k\alpha}{n}}+\frac{S_\alpha^2}{F_\alpha^2}V_B^{\frac{4k\alpha}{n}}\right)+R_5\\
&\geq \frac{1}{2}m^2(m^2-\gamma^2)\mu_{n,k}^{4\alpha}\left(V_AV_B\right)^{\frac{2k\alpha}{n}}+R_5,
\end{aligned} \tag{2.8}$$

with equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular, where

$$R_5=\frac{\gamma^2}{2S_\alpha^2F_\alpha^2}\left(F_\alpha^2S_{2\alpha}-S_\alpha^2F_{2\alpha}\right)^2\geq 0. \tag{2.8$'$}$$

3. Proofs of the Theorems

To prove the theorems in Section 2, we establish a number of lemmas as follows.

Lemma 3.1. ([5]) *Let $\mathcal{A}$ an $n-$simplexes in E^n with the $n-$dimensional volumes V_A, and S_i denote the $(n-1)-$dimensional volumes of $(n-1)-$dimensional subsimplexes spanned by $n-1$ vertexes $A_1,\cdots,A_{i-1},A_{i+1}\cdots,A_{n+1}$ of $\mathcal{A}(i=1,2,\cdots,n+1)$. Put $\lambda_i\in R^+,\ \theta\in(0,1]$. Then*

$$\left(\sum_{i=1}^{n+1}\lambda_i S_i^{2\theta}\right)^n\geq (n+1)^{(n-1)(1-\theta)}\left(\frac{n^{3n}}{n!^2}\right)^{\theta}\left(\sum_{i=1}^{n+1}\prod_{\substack{j=1\\ j\neq i}}^{n+1}\lambda_j\right)V_A^{2(n-1)\theta}, \tag{3.1}$$

with equality holds if $\mathcal{A}$ is regular and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$.

Lemma 3.2. ([5]) *Under the hypotheses in lemma* 3.1*, put* $\alpha_i = \dfrac{\sum_{j=1}^{n+1} S_j^{\theta} - 2S_i^{\theta}}{S_i^{\theta}}$ $(i = 1, 2, \cdots, n+1)$. *Then*

$$\sum_{i=1}^{n+1} \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \alpha_j \geq (n+1)(n-1)^n, \tag{3.2}$$

with equality holds if and only if $S_1 = S_2 = \cdots = S_{n+1}$.

Lemma 3.3. *Under the hypotheses in lemma* 3.1*, put* $\alpha, \beta \in (0, 1]$*, then*

$$\begin{aligned} \sum_{i=1}^{n+1} S_i^{\alpha} \left(\sum_{j=1}^{n+1} S_j^{\beta} - 2S_i^{\beta} \right) &= \sum_{1\leq i<j\leq n+1} (S_i^{\alpha} S_j^{\beta} + S_i^{\beta} S_j^{\alpha}) - \sum_{i=1}^{n+1} S_i^{(\alpha+\beta)} \\ &\geq (n^2-1) \left[\frac{n^3}{n+1} \left(\frac{\sqrt{n+1}}{n!} \right)^{\frac{2}{n}} \right]^{\frac{\alpha+\beta}{2}} V_A^{\frac{(n-1)(\alpha+\beta)}{n}}, \end{aligned} \tag{3.3}$$

with equality holds if $\mathcal{A}$ is regular and $S_1 = S_2 = \cdots = S_{n+1}$.

Proof. Put $\lambda_i = \dfrac{\sum_{j=1}^{n+1} S_j^{\beta} - 2S_i^{\beta}}{S_i^{\beta}}$ $(i = 1, 2, \cdots, n+1)$, according to Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} \sum_{i=1}^{n+1} S_i^{\alpha} \left(\sum_{j=1}^{n+1} S_j^{\beta} - 2S_i^{\beta} \right) &= \sum_{i=1}^{n+1} \lambda_i S_i^{(\alpha+\beta)} \\ &\geq (n+1)^{\frac{n-1}{n}(1-\frac{\alpha+\beta}{2})} \left(\frac{n^{3n}}{n!^2} \right)^{\frac{\alpha+\beta}{2n}} \left(\sum_{i=1}^{n+1} \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j \right)^{\frac{1}{n}} V_A^{\frac{(n-1)(\alpha+\beta)}{n}} \\ &\geq (n+1)^{\frac{n-1}{n}(1-\frac{\alpha+\beta}{2})} \left(\frac{n^{3n}}{n!^2} \right)^{\frac{\alpha+\beta}{2n}} (n+1)^{\frac{1}{n}} (n-1) V_A^{\frac{(n-1)(\alpha+\beta)}{n}} \\ &= (n^2-1) \left[\frac{n^3}{n+1} \left(\frac{\sqrt{n+1}}{n!} \right)^{\frac{2}{n}} \right]^{\frac{\alpha+\beta}{2}} V_A^{\frac{(n-1)(\alpha+\beta)}{n}}. \end{aligned}$$

■

Lemma 3.4. *Under the hypotheses in theorem* 2.1*, then*

$$\left(\prod_{i=1}^{m} S_i(k)\right)^{\frac{1}{m}} \geq \mu_{n,k} V_A^{\frac{k}{n}}, \tag{3.4}$$

with equality holds if and only if $\mathcal{A}$ is regular.

Proof. Applying the result in [14] or [15], we obtain

$$\left(\prod_{i=1}^{n+1} S_i\right)^{\frac{2n}{n+1}} \geq \frac{1}{(n+1)^{n-1}} \left(\frac{n^{3n}}{n!^2}\right) V_A^{2(n-1)}, \tag{3.5}$$

Using induction on k, the inequality (3.5) yields (3.4). ∎

Lemma 3.5. *Under the hypotheses in theorem* 2.1*, then*

$$\begin{aligned} S_\alpha S_\beta - \gamma \sum_{i=1}^{m} S_i^{\alpha+\beta}(k) &= \sum_{i=1}^{m} S_i^{\alpha}(k)(\sum_{j=1}^{m} S_j^{\beta}(k) - \gamma S_i^{\beta}(k)) \\ &\geq m(m-\gamma)\mu_{n,k}^{\alpha+\beta} V_A^{\frac{k(\alpha+\beta)}{n}}, \end{aligned} \tag{3.6}$$

with equality holds if and only if $\mathcal{A}$ is regular.

Proof. For convenience, we employ $R(\alpha,\beta,\gamma)$ to denote the left side of the inequality (3.6), then

$$\begin{aligned} R(\alpha,\beta,\gamma) &= \left[\sum_{1\leq i<j\leq m} (S_i^{\alpha}(k)S_j^{\beta}(k) + S_i^{\beta}(k)S_j^{\alpha}(k)) - (n-k)\sum_{i=1}^{m} S_i^{\alpha+\beta}(k)\right] \\ &+(n+1-k-\gamma)\sum_{i=1}^{m} S_i^{\alpha+\beta}(k) = I_1 + I_2, \end{aligned}$$

where

$$I_1 = \sum_{(i_1,i_2,\cdots,i_{k+2})\in T} \left[\sum_{1\leq r<t\leq k+2} (S_{i_r}^{\alpha}(k)S_{i_t}^{\beta}(k) + S_{i_r}^{\beta}(k)S_{i_t}^{\alpha}(k)) - \sum_{r=1}^{k+2} S_{i_r}^{\alpha+\beta}(k)\right],$$

$$I_2 = \sum_{(i_r,i_t)\in Q} \left(S_{i_r}^{\alpha}(k)S_{i_t}^{\beta}(k) + S_{i_r}^{\beta}(k)S_{i_t}^{\alpha}(k)\right) - (n+1-k-\gamma)\sum_{i=1}^{m} S_i^{\alpha+\beta}(k),$$

and

$T = \big\{(i_1,i_2,\cdots,i_{k+2})|$There exists a $(k+1)-$ subsimplex $\mathcal{A}_{(i_1,i_2,\cdots,i_{k+2})}$ of $\mathcal{A}$, such that its $k+2$ side facet volumes are $S_{i_1}(k), S_{i_2}(k),\cdots,S_{i_{k+2}}(k), (1\leq i_1,i_2,\cdots,i_{k+2}\leq n+1)$ $\big\}$

$$Q = \left\{(i_r, i_t) | \text{There is not a } (k+1)- \text{ subsimplex } \mathcal{A}_{(i_1, i_2, \cdots, i_{k+2})} \text{ of } \mathcal{A}, \right.$$
$$\left. \text{such that its two side facet volumes are } S_{i_r}(k), S_{i_t}(k) \right\}.$$

Obviously, we easily get

$$|T| = \binom{n+1}{k+2}, \; |Q| = \binom{m}{2} - \binom{n+1}{k+2}\binom{k+2}{2} = \frac{1}{2}m[m-(n-k)(k+1)-1],$$

If $(i_1, i_2, \cdots, i_{k+2}) \in T$, we use $S_i(k+1)$ to denote the volume of $(k+1)-$subsimplex with side facet volumes $S_{i_1}(k), S_{i_2}(k), \cdots, S_{i_{k+2}}(k)$, and put $m' = |T| = \binom{n+1}{k+2} = \frac{n-k}{k+2}m$. Combining Lemma (3.3) with Lemma (3.4),and applying arithmetic-geometric mean inequality, we infer that

$$\begin{aligned}
I_1 &\geq \sum_{i=1}^{m'} k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{\sqrt{k+2}}{(k+1)!}\right)^{\frac{2}{k+1}}\right]^{\frac{\alpha+\beta}{2}} (S_i(k+1))^{\frac{k(\alpha+\beta)}{k+1}} \\
&\geq k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{\sqrt{k+2}}{(k+1)!}\right)^{\frac{2}{k+1}}\right]^{\frac{\alpha+\beta}{2}} m' (S_i(k+1))^{\frac{k(\alpha+\beta)}{m'(k+1)}} \\
&\geq k(k+2)\left[\frac{(k+1)^3}{k+2}\left(\frac{\sqrt{k+2}}{(k+1)!}\right)^{\frac{2}{k+1}}\right]^{\frac{\alpha+\beta}{2}} \\
&\quad \frac{n-k}{k+2}m\left[\frac{\sqrt{k+2}}{(k+1)!}\left(\frac{n!}{\sqrt{n+1}}V_A\right)^{\frac{k+1}{n}}\right]^{\frac{k(\alpha+\beta)}{k+1}} \\
&= mk(n-k)\left[\frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{\alpha+\beta} V_A^{\frac{k(\alpha+\beta)}{n}} \\
&= mk(n-k)\mu_{n,k}^{\alpha+\beta} V_A^{\frac{k(\alpha+\beta)}{n}}, \\
I_2 &\geq m[m-(n-k)(k+1)-1]\left(\prod_{i=1}^{m} S_i(k)\right)^{\frac{\alpha+\beta}{m}} \\
&\quad + m(n+1-k-\gamma)\left(\prod_{i=1}^{m} S_i(k)\right)^{\frac{\alpha+\beta}{m}} \\
&= m[m-(n-k)k-\gamma]\left(\prod_{i=1}^{m} S_i(k)\right)^{\frac{\alpha+\beta}{m}} \\
&\geq m[m-(n-k)k-\gamma]\left[\frac{\sqrt{k+1}}{k!}\left(\frac{n!}{\sqrt{n+1}}\right)^{\frac{k}{n}}\right]^{\alpha+\beta} V_A^{\frac{k(\alpha+\beta)}{n}} \\
&= m[m-(n-k)k-\gamma]\mu_{n,k}^{\alpha+\beta} V_A^{\frac{k(\alpha+\beta)}{n}},
\end{aligned}$$

Hence

$$R(\alpha,\beta,\gamma)=I_1+I_2\geq m(m-\gamma)\mu_{n,k}^{\alpha+\beta}V_A^{\frac{k(\alpha+\beta)}{n}}.$$

Thus inequality (3.6) valid with equality holds if and only if $\mathcal{A}$ is regular. ■

Lemma 3.6. *Under the hypotheses in theorem* 2.1*, then*

$$\begin{aligned}&(a_1S_\alpha+a_2S_\beta)^2-\gamma\sum_{i=1}^m(a_1S_i^\alpha(k)+a_2S_i^\beta(k))^2\\ &=\sum_{i=1}^m\left(a_1S_i^\alpha(k)+a_2S_i^\beta(k)\right)\\ &\left[\sum_{j=1}^m\left(a_1S_j^\alpha(k)+a_2S_j^\beta(k)\right)-\gamma\left(a_1S_i^\alpha(k)+a_2S_i^\beta(k)\right)\right]\\ &\geq m(m-\gamma)\left(a_1\mu_{n,k}^\alpha V_A^{\frac{\alpha k}{n}}+a_2\mu_{n,k}^\beta V_A^{\frac{\beta k}{n}}\right)^2,\end{aligned}\tag{3.7}$$

with equality holds if and only if $\mathcal{A}$ is regular.

Proof. We denote the left side of inequality (3.7) by $P(\alpha,\beta,\gamma)$, then

$$\begin{aligned}P(\alpha,\beta,\gamma)&=(a_1S_\alpha+a_2S_\beta)^2-\gamma\sum_{i=1}^m(a_1^2S_i^{2\alpha}(k)\\ &\quad+2a_1a_2S_i^{\alpha+a_2\beta}(k)+a_2^2S_i^{2\beta}(k))\\ &=a_1^2\left(S_\alpha^2-\gamma\sum_{i=1}^mS_i^{2\alpha}(k)\right)+2a_1a_2\left(S_\alpha S_\beta-\gamma\sum_{i=1}^mS_i^{\alpha+\beta}(k)\right)\\ &\quad+a_2^2\left(S_\beta^2-\gamma\sum_{i=1}^mS_i^{2\beta}(k)\right).\end{aligned}\tag{3.8}$$

Employing Lemma 3.5, we infer inequality (3.7) from (3.8), and equality holds if and only if $\mathcal{A}$ is regular. ■

Lemma 3.7. *Under the hypotheses in theorem* 2.1*, then*

$$\begin{aligned}&S_\alpha^2S_\beta^2-\gamma^2(\sum_{i=1}^mS_i^{\alpha+\beta}(k))^2=\sum_{i=1}^m\sum_{j=1}^mS_i^\alpha(k)S_j^\beta(k)\\ &\left[\sum_{u=1}^m\sum_{v=1}^mS_u^\beta(k)S_v^\alpha(k)-\gamma^2S_i^\beta(k)S_j^\alpha(k)\right]\\ &\geq m^2(m^2-\gamma^2)\mu_{n,k}^{2(\alpha+\beta)}V_A^{\frac{2k(\alpha+\beta)}{n}},\end{aligned}\tag{3.9}$$

with equality holds if and only if $\mathcal{A}$ is regular.

Proof. We denote the left side of inequality (3.9) by $W(\alpha,\beta,\gamma)$. Combining Lemma 3.4 and Lemma 3.5 and applying arithmetic-geometric mean inequality, we deduce

$$\begin{aligned}
&W(\alpha,\beta,\gamma)\\
&=\left(\sum_{i=1}^{m} S_i^{\alpha}(k)\right)^2\left(\sum_{i=1}^{m} S_i^{\beta}(k)\right)^2-\gamma\left(\sum_{i=1}^{m}(S_i^{\alpha+\beta}(k)\right)^2\\
&=\sum_{i=1}^{m} S_i^{\alpha}(k)\left(\sum_{j=1}^{m} S_j^{\beta}(k)-\gamma S_i^{\beta}(k)\right)\cdot\sum_{i=1}^{m} S_i^{\alpha}(k)\left(\sum_{j=1}^{m} S_j^{\beta}(k)+\gamma S_i^{\beta}(k)\right)\\
&\geq m(m-\gamma)\mu_{n,k}^{\alpha+\beta}V_A^{\frac{k(\alpha+\beta)}{n}}\cdot(m^2+\gamma m)\left(\prod_{i=1}^{m} S_i^{\alpha+\beta}(k)\right)^{\frac{1}{m}}\\
&\geq m^2(m^2-\gamma^2)\mu_{n,k}^{\alpha+\beta}V_A^{\frac{k(\alpha+\beta)}{n}}\cdot\mu_{n,k}^{\alpha+\beta}V_A^{\frac{k(\alpha+\beta)}{n}}\\
&=m^2(m^2-\gamma^2)\mu_{n,k}^{2(\alpha+\beta)}V_A^{\frac{2k(\alpha+\beta)}{n}}.
\end{aligned}$$

Thus inequality (3.9) valid with equality holds if and only if $\mathcal{A}$ is regular. ■

Further, applying above lemmas, we can prove three theorems in Section 2.

Proof of the Theorem 2.1. Note

$$\begin{aligned}
H_{AB}&=\sum_{i=1}^{m}(a_1S_i^{\alpha}(k)+a_2S_i^{\beta}(k))\\
&\quad\left[\sum_{j=1}^{m}(b_1F_j^{\alpha}(k)+b_2F_j^{\beta}(k))-\gamma(b_1F_i^{\alpha}(k)+b_2F_i^{\beta}(k))\right]\\
&=(a_1S_\alpha+a_2S_\beta)(b_1F_\alpha+b_2F_\beta)\\
&\quad-\gamma\sum_{i=1}^{m}(a_1S_i^{\alpha}(k)+a_2S_i^{\beta}(k))(b_1F_i^{\alpha}(k)+b_2F_i^{\beta}(k)),\\
H_A&=(a_1S_\alpha+a_2S_\beta)^2-\gamma\sum_{i=1}^{m}(a_1S_i^{\alpha}(k)+a_2S_i^{\beta}(k))^2,\\
H_B&=(b_1F_\alpha+b_2F_\beta)^2-\gamma\sum_{i=1}^{m}(b_1F_i^{\alpha}(k)+b_2F_i^{\beta}(k))^2,
\end{aligned}$$

By computation, we easily deduce the following inequality.

$$H_{AB} = \frac{1}{2}\left(\frac{b_1F_\alpha + b_2F_\beta}{a_1S_\alpha + a_2S_\beta}H_A + \frac{a_1S_\alpha + a_2S_\beta}{b_1F_\alpha + b_2F_\beta}H_B\right) + R_1. \tag{3.10}$$

Substituting (3.7) to (3.10), we infer inequality (2.1). Moreover, by Lemma 3.6, we know the equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular. Therefore We complete the proof of Theorem 2.1. ■

Proof of the Theorem 2.2. Note

$$\begin{aligned} G_{AB} &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} S_i^\alpha(k)S_j^\beta(k)}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} F_i^\alpha(k)F_j^\beta(k)} - \gamma\sum_{i=1}^{m} S_i^{\frac{\alpha+\beta}{2}}(k)F_i^{\frac{\alpha+\beta}{2}}(k) \\ &= \sqrt{S_\alpha S_\beta}\sqrt{F_\alpha F_\beta} - \gamma\sum_{i=1}^{m} S_i^{\frac{\alpha+\beta}{2}}(k)F_i^{\frac{\alpha+\beta}{2}}(k) \\ G_A &= S_\alpha S_\beta - \gamma\sum_{i=1}^{m} S_i^{\alpha+\beta}(k), G_B = F_\alpha F_\beta - \gamma\sum_{i=1}^{m} F_i^{\alpha+\beta}(k). \end{aligned}$$

By computation, we easily deduce the following inequality.

$$G_{AB} = \frac{1}{2}\left(\frac{\sqrt{F_\alpha F_\beta}}{\sqrt{S_\alpha S_\beta}}G_A + \frac{\sqrt{S_\alpha S_\beta}}{\sqrt{F_\alpha F_\beta}}G_B\right) + R_2. \tag{3.11}$$

Substituting (3.6) to (3.11), we infer inequality (2.2). Moreover, by Lemma 3.5, we know the equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular. The Theorem 2.2 is proved. ■

Proof of the Theorem 2.3.

Note

$$\begin{aligned} Q_{AB} &= \sum_{i=1}^{m}\sum_{j=1}^{m} S_i^\alpha(k)F_j^\beta(k)\left(\sum_{u=1}^{m}\sum_{v=1}^{m} S_u^\beta(k)F_v^\alpha(k) - \gamma^2 S_i^\beta(k)F_i^\beta(k)\right) \\ &= S_\alpha S_\beta F_\alpha F_\beta - \gamma^2\sum_{i=1}^{m}\sum_{j=1}^{m}(S_i(k)F_j(k))^{\alpha+\beta} \\ &= S_\alpha S_\beta F_\alpha F_\beta - \gamma^2 S_{\alpha+\beta}F_{\alpha+\beta} \end{aligned}$$

$$\begin{aligned} Q_A &= (S_\alpha S_\beta)^2 - \gamma^2\left(\sum_{i=1}^{m} S_i^{\alpha+\beta}(k)\right)^2 = (S_\alpha S_\beta)^2 - \gamma^2 S_{\alpha+\beta}^2, \\ Q_B &= (F_\alpha F_\beta)^2 - \gamma^2\left(\sum_{i=1}^{m} F_i^{\alpha+\beta}(k)\right)^2 = (F_\alpha F_\beta)^2 - \gamma^2 F_{\alpha+\beta}^2, \end{aligned}$$

By computation, we easily deduce the following inequality.

$$Q_{AB} = \frac{1}{2}\left(\frac{F_\alpha F_\beta}{S_\alpha S_\beta} Q_A + \frac{S_\alpha S_\beta}{F_\alpha F_\beta} Q_B\right) + R_3. \tag{3.12}$$

Substituting (3.9) to (3.12), we infer inequality (2.3). Moreover, by Lemma 3.7, we know the equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular. Therefore We complete the proof of Theorem 2.3. ∎

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