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SOME GENERALIZATIONS OF OPIAL'S INEQUALITIES ON TIME SCALES

Fu-Hsiang Wong, Wei-Cheng Lian, Shiueh-Ling Yu and Cheh-Chih Yeh

Abstract. The Opial inequality is of great interest in differential and difference equations, and other areas of mathematics. The purpose of this paper is to generalize the Opial inequality to some time scale versions. One of these results says:

$$\int_a^b h(x)\{|g(x)|^p|f^{\Delta^n}(x)|^q+|f(x)|^p|g^{\Delta^n}(x)|^q\}\Delta x$$
$$\le \frac{2q}{p+q}[(\frac{b-a}{2})^p]^n\int_a^b h(x)\{|f^{\Delta^n}(x)|^{p+q}+|g^{\Delta^n}(x)|^{p+q}\}\Delta x,$$

if $p\ge 1$, $q\ge 1$ and $f,\ g\in C_{rd}([a,b],\mathbb{R})$ satisfy some suitable conditions.

1. Introduction

The Opial inequality [11] is of great interest in differential and difference equations, and other areas of mathematics. The original Opial inequality is as follows:

Theorem 1.A. *Let* $a>0$. *If* $f\in C^1[0,a]$ *with* $f(0)=f(a)=0$ *and* $f(t)>0$ *on* $(0,a)$. *Then*

$$\int_0^a |f(x)f'(x)|dx\le \frac{a}{4}\int_0^a |f'(x)|^2dx.$$

There are many authors dealing with this renowned inequality, see, for example, Agarwal etc [1, 2, 3, 4]. *Bessack* [5]*, Das* [7]*, He* [8]*, Mallows* [10]*, Pachpatte* [12]*, Willett* [13] *and Yang* [14].

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In 2001, Agarwal, Bohner and Peterson [4] *extended Theorem 1.A on a time scale and obtained the following*

Theorem 1.B. *For delta differentiable* $x : [0, h] \to R$ *with* $x(0) = 0$*, we have*

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le h \int_0^h |x^\Delta|^2(t)\Delta t.$$

The purpose of this paper is to generalize the Opial inequality to more general cases on time scales.

To do this, we briefly introduce the time scales theory and refer to Bohner and Peterson [6] *and Kaymakcalan* [9] *for further details.*

Theorem 1.C. *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of the the set* $\mathbb{R}$ *of all real numbers. Let* $\mathbb{T}$ *have the topology that it inherits from the standard topology on* $\mathbb{R}$*. For* $t \in \mathbb{T}$*, if* $t < \sup \mathbb{T}$*, we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

while if $t > \inf \mathbb{T}$*, the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\} \in \mathbb{T}.$$

If $\sigma(t) > t$*, we say* t *is right scattered, while if* $\rho(t) < t$*, we say* t *is left scattered. If* $\sigma(t) = t$*, we say* t *is right dense, while if* $\rho(t) = t$*, we say* t *is left dense.*

Throughout this paper, we suppose that

(a) $\mathbb{R} = (-\infty, +\infty)$*;*

(b) $\mathbb{T}$ *is a time scale;*

(c) *an interval means the intersection of a real interval with the given time scale.*

Theorem 1.D. *If* $f : \mathbb{T} \to \mathbb{R}$*, then* $f^\sigma : \mathbb{T} \to \mathbb{R}$ *is defined by*

$$f^\sigma(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$.

Theorem 1.E. *A mapping* $f : \mathbb{T} \to \mathbb{R}$ *is called rd-continuous if it satisfies:*

(A) f *is continuous at each right-dense point or maximal element of* $\mathbb{T}$*,*

(B) *the left-sided limit* $\lim\limits_{s \to t^-} f(s) = f(t^-)$ *exists at each left-dense point* t *of* $\mathbb{T}$.

Let

$$C_{rd}(\mathbb{T},\mathbb{R}) := \{f \mid f : \mathbb{T} \to \mathbb{R} \ \textit{ is a rd-continuous function}\}$$

and

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\}, \ \textit{if } \mathbb{T} \textit{ has a left-scattered maximal point } m, \\ \mathbb{T}, \ \textit{otherwise.} \end{cases}$$

Theorem 1.F. *Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^k$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that*

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the **delta derivative** *of $x(t)$ at t.*

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Theorem 1.G. *A function $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, we define the integral of f by*

$$\int_s^t f(\tau)\ \Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}$.

It follows from Theorem 1.74 of Bohner and Peterson [6] that every rd-continuous function has an antiderivative.

2. Main Results

We now in a position to extend Theorem A of He [8] (see, also, Yang [14]) to a time scale version.

Theorem 2.1. *Let $f : [a, b] \to R$ with $f(a) = 0$ be delta differentiable, $h \in C_{rd}([a, b], [1, \infty)) := \{f \mid f : [a, b] \to [1, \infty) \ \textit{ is a rd-continuous function}\}$, $p \geq 0$ and $q \geq 1$. Then,*

$$\int_a^b h(x)|f(x)|^p|f^{\Delta}(x)|^q\Delta x \leq \frac{q}{p+q}(b-a)^p \int_a^b h(x)|f^{\Delta}(x)|^{p+q}\Delta x.$$

Proof. Without loss of generality, we assume that $a = 0$. Let

$$F(t) := \frac{q}{p+q}\{t^p \int_0^t h(x)|f^\Delta(x)|^{p+q}\Delta x\} - \int_0^t h(x)|f(x)|^p|f^\Delta(x)|^q\Delta x.$$

It follows from the property of the delta derivative (Theorems 1.20 and 1.24 in Bohner and Peterson [6]) that

$$\begin{aligned} F^\Delta(t) =& \frac{q}{p+q}\Big\{\Big(\sum_{v=0}^{p-1}[\big(\sigma(t)^v\big)t^{p-1-v}]\Big)\int_0^t h(x)|f^\Delta(x)|^{p+q}\Delta x\Big\} \\ &+\frac{q}{p+q}\big\{(\sigma(t))^p h(t)|f^\Delta(t)|^{p+q}\big\} \\ &-h(t)|f(t)^p||f^\Delta(t)|^q. \end{aligned} \tag{1}$$

Since

$$\begin{aligned} |f(t)| &= \left|\int_0^t f^\Delta(x)\Delta x\right| \le \int_0^t |f^\Delta(x)|\Delta x \\ &\le \Big(\int_0^t 1^r\Delta x\Big)^{\frac{1}{r}}\Big(\int_0^t |f^\Delta(x)|^{p+q}\Delta x\Big)^{\frac{1}{p+q}} \\ &\le \Big(\int_0^t 1^r\Delta x\Big)^{\frac{1}{r}}\Big(\int_0^t h(x)|f^\Delta(x)|^{p+q}\Delta x\Big)^{\frac{1}{p+q}}, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{p+q} = 1$, we have

$$|f(t)|^{p+q} \le t^{p+q-1}\Big(\int_0^t h(x)|f^\Delta(x)|^{p+q}\Delta x\Big),$$

which implies

$$\frac{|f(t)|^{p+q}}{t^{p+q-1}} \le \int_0^t h(x)|f^\Delta(x)|^{p+q}\Delta x. \tag{2}$$

It follows from (1) and (2) that

$$\begin{aligned} F^\Delta(t) \ge& \frac{1}{(p+q)t^{p+q-1}}\{q|f(t)|^{p+q}(\sum_{v=0}^{p-1}[(\sigma(t))^v t^{p-1-v}]) \\ &+qt^{p+q-1}h(t)(\sigma(t))^p|f^\Delta(t)|^{p+q} \\ &-(p+q)t^{p+q-1}h(t)|f(t)|^p|f^\Delta(t)|^q\} \\ \ge& \frac{1}{(p+q)t^{p+q-1}}\{|f(t)|^{p+q}(\sum_{v=0}^{p-1}[(\sigma(t))^v t^{p-1-v}]) \\ &+qt^{p+q-1}h(t)(\sigma(t))^p|f^\Delta(t)|^{p+q} \\ &-(p+q)t^{p+q-1}h(t)|f(t)|^p|f^\Delta(t)|^q\}. \end{aligned}$$

Set

$$\begin{aligned} A(t) &:= |f(t)|^{p+q} \geq 0 \\ B(t) &:= h(t)(t|f^{\Delta}(t)|)^{p+q} \geq 0 \\ \alpha &= \frac{p}{p+q} \\ \beta &= \frac{q}{p+q}. \end{aligned}$$

Then

$$\begin{aligned} F^{\Delta}(t) \geq\ & \frac{1}{t^q}\Big\{\frac{\sum_{v=0}^{p-1}(\frac{\sigma(t)}{t})^v}{p+q}A(t) + \frac{q}{p+q}(\frac{\sigma(t)}{t})^p B(t) \\ & -(h(t))^{1-\frac{q}{p+q}}(A(t))^{\frac{p}{p+q}}(B(t))^{\frac{q}{p+q}}\Big\} \\ \geq\ & \frac{1}{t^q}\Big\{\frac{p}{p+q}A(t) + \frac{q}{p+q}B(t) - \Big(h(t))^{1-\frac{q}{p+q}}(A(t))^{\frac{p}{p+q}}(B(t)\Big)^{\frac{q}{p+q}}\Big\} \\ \geq\ & \frac{1}{t^q}\Big\{\alpha A(t) + \beta B(t) - (A(t))^{\alpha}(B(t))^{\beta}\Big\} \text{ (cf: } h(t) \geq 1) \\ \geq\ & 0, \end{aligned}$$

i.e.,

$$F(b) = \int_0^b F^{\Delta}(t)\Delta t \geq 0.$$

Therefore,

$$\frac{q}{p+q}\Big\{b^p \int_0^b h(x)|f^{\Delta}(x)|^{p+q}\Delta x\Big\} - \int_0^b h(x)|f(x)|^p|f^{\Delta}(x)|^q\Delta x \geq 0,$$

and hence we obtain the desired result.

Theorem 2.2. *Let $f : [a, b] \to R$ be n-times delta differentiable with $f(a) = f^{\Delta}(a) = ... = f^{\Delta^{n-1}}(a) = 0$ and $h \in C_{rd}([a, b], [1, \infty))$. If $p \geq 0$ and $q \geq 1$, then*

$$\int_a^b h(x)|f(x)|^p|f^{\Delta^n}(x)|^q\Delta x \leq \frac{q}{p+q}\Big[(b-a)^p\Big]^n \int_a^b h(x)|f^{\Delta^n}(x)|^{p+q}\Delta x.$$

Proof. Let

$$g(t) := \int_a^t \int_a^{t_{n-1}} ... \int_a^{t_1} |f^{\Delta^n}(x)|\Delta x\Delta t_1...\Delta t_{n-1}.$$

Then

$$g^{\Delta}(t), ..., g^{\Delta^{n-1}}(t) \geq 0,\ g^{\Delta^n}(t) = |f^{\Delta^n}(t)| \geq 0$$

and

$$\begin{aligned} g(t) &\geq |f(t)|, \\ g^{\Delta^i}(t) &= \int_a^t g^{\Delta^{i+1}}(x)\Delta x \\ &\leq g^{\Delta^{i+1}}(t)\int_a^t (1)\Delta x \\ &\leq (t-a)g^{\Delta^{i+1}}(t) \ \text{ for all } i = 0, 1, ..., n-2. \end{aligned}$$

By Theorem 2.1,

$$\begin{aligned} \int_a^b h(x)|f(x)|^p|f^{\Delta^n}(x)|^q\Delta x &\leq \int_a^b h(x)(g(x))^p(g^{\Delta^n}(x))^q\Delta x \\ &\leq \int_a^b h(x)[(x-a)g^{\Delta}(x)]^p(g^{\Delta^n}(x))^q\Delta x \\ &\leq \int_a^b h(x)[(x-a)^2g^{\Delta^2}(x)]^p(g^{\Delta^n}(x))^q\Delta x \\ &\ \ \vdots \\ &\leq \int_a^b h(x)[(x-a)^{n-1}g^{\Delta^{n-1}}(x)]^p(g^{\Delta^n}(x))^q\Delta x \\ &\leq [(b-a)^p]^{n-1}\int_a^b h(x)(g^{\Delta^{n-1}}(x))^p(g^{\Delta^n}(x))^q\Delta x \\ &\leq [(b-a)^p]^{n-1}\frac{q}{p+q}(b-a)^p\int_a^b h(x)|g^{\Delta^n}(x)|^{p+q}\Delta x \\ &= \frac{q}{p+q}[(b-a)^p]^n\int_a^b h(x)|g^{\Delta^n}(x)|^{p+q}\Delta x \\ &= \frac{q}{p+q}[(b-a)^p]^n\int_a^b h(x)|f^{\Delta^n}(x)|^{p+q}\Delta x. \end{aligned}$$

Theorem 2.3. *Let $f, g : [a, b] \to R$ be n-times delta differentiable with*

$$f(a) = f^{\Delta}(a) = f^{\Delta^{n-1}}(a) = 0 \text{ and } g(a) = g^{\Delta}(a) = ... = g^{\Delta^{n-1}}(a) = 0$$

and $h \in C_{rd}([a, b], [1, \infty))$. If $p \geq 0$ and $q \geq 1$, then

$$\begin{aligned} &\int_a^b h(x)\{|g(x)|^p|f^{\Delta^n}(x)|^q + |f(x)|^p|g^{\Delta^n}(x)|^q\}\Delta x \\ &\quad \leq \frac{2q}{p+q}[(b-a)^p]^n\int_a^b h(x)\{|f^{\Delta^n}(x)|^{p+q} + |g^{\Delta^n}(x)|^{p+q}\}\Delta x. \end{aligned}$$

Proof. Define

$$K(t) = \int_a^t\int_a^{t_{n-1}}...\int_a^{t_1}\{|f^{\Delta^n}(x)|^{p+q} + |g^{\Delta^n}(x)|^{p+q}\}^{\frac{1}{p+q}}\Delta x\Delta t_1...\Delta t_{n-1}.$$

Then

$$K^{\Delta^n}(t) = \{|f^{\Delta^n}(t)|^{p+q} + |g^{\Delta^n}(t)|^{p+q}\}^{\frac{1}{p+q}} \geq max\{|f^{\Delta^n}(t)|, |g^{\Delta^n}(t)|\}$$

and

$$\begin{aligned} K(t) &\geq \int_a^t ... \int_a^{t_1} \{|f^{\Delta^n}(x)|^{p+q}\}^{\frac{1}{p+q}} \Delta x \Delta t_1 ... \Delta t_{n-1} \\ &\geq |\int_a^t ... \int_a^{t_1} f^{\Delta^n}(x) \Delta x \Delta t_1 ... \Delta t_{n-1}| \\ &= |f(t)|. \end{aligned}$$

Similarly,

$$K(t) \geq |g(t)|.$$

By Theorem 2.2,

$$\begin{aligned} &\int_a^b h(x)\{|g(x)|^p|f^{\Delta^n}(x)|^q + |f(x)|^p|g^{\Delta^n}(x)|^q\}\Delta x \\ &\leq \int_a^b h(x)|K(x)|^p\{|f^{\Delta^n}(x)|^q + |g^{\Delta^n}(x)|^q\}\Delta x \\ &\leq \int_a^b h(x)|K(x)|^p \cdot 2 \cdot |K^{\Delta^n}(x)|^q \Delta x \\ &\leq \frac{2q}{p+q}[(b-a)^p]^n \int_a^b h(x)|K^{\Delta^n}(x)|^{p+q}\Delta x \\ &= \frac{2q}{p+q}[(b-a)^p]^n \int_a^b h(x)\{|f^{\Delta^n}(x)|^{p+q} + |g^{\Delta^n}(x)|^{p+q}\}\Delta x. \end{aligned}$$

Theorem 2.4. *Let $f, g : [a, b] \to R$ be n-times delta differentiable with*

$$f(a) = f^{\Delta}(a) = f^{\Delta^{n-1}}(a) = 0, f(b) = f^{\Delta}(b) = f^{\Delta^{n-1}}(b) = 0,$$

$$g(a) = g^{\Delta}(a) = ... = g^{\Delta^{n-1}}(a) = 0 \textit{ and } g(b) = g^{\Delta}(b) = ... = g^{\Delta^{n-1}}(b) = 0$$

and $h \in C_{rd}([a, b], [1, \infty))$. If $\frac{a+b}{2} \in [a, b], p \geq 0$ and $q \geq 1$, then

$$\begin{aligned} &\int_a^b h(x)\{|g(x)|^p|f^{\Delta^n}(x)|^q + |f(x)|^p|g^{\Delta^n}(x)|^q\}\Delta x \\ &\leq \frac{2q}{p+q}[(\frac{b-a}{2})^p]^n \int_a^b h(x)\{|f^{\Delta^n}(x)|^{p+q} + |g^{\Delta^n}(x)|^{p+q}\}\Delta x. \end{aligned}$$

Proof. Define

$$f_1(t) = f(t),\ g_1(t) = g(t),\ f_2(t) = f(a+b-t) \text{ and } g_2(t) = g(a+b-t)$$

on $[a, \frac{a+b}{2}]$. It is clear that f_i and g_i satisfy the conditions of Theorem 2.3, $i = 1, 2$. Therefore, we can apply Theorem 2.3 to the pairs of functions (f_1, g_1) and (f_2, g_2) on the interval $[a, \frac{a+b}{2}]$. Adding them and hence we obtain the desired results.

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Fu-Hsiang Wong
Department of Mathematics,
National Taipei University of Education,
Taipei 10659, Taiwan
E-mail: wong@tea.ntue.edu.tw

Wei-Cheng Lian
Department of Information Management,
National Kaohsiung Marine University,
Kaohsiung, Taiwan,
Republic of China
E-mail: wclian@mail.nkmu.edu.tw

Shiueh-Ling Yu
Holistic Education Center,
St. John's and St. Mary's Institute of Technology,
Tamsui, Taipei, Taiwan,
Republic of China E-mail: slyu@mail.sjsmit.edu.tw

Cheh-Chih Yeh
Department of Information Management,
Lunghwa University of Science and Technology,
Kueishan, Taoyuan 33306, Taiwan,
Republic of China
E-mail: ccyeh@mail.lhu.edu.tw