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A NOTE ON THE CONTINUITY OF OPERATORS INTERTWINING WITH CONVOLUTION OPERATORS

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Abstract. Let G be a MAP group, let μ be a bounded complex-valued Borel measure on G, and let T_μ be the corresponding convolution operator on $L^1(G)$. Let X be a Banach space and let S be a continuous linear operator on X. We show that every linear operator $\Phi : X \to L^1(G)$ such that $\Phi S = T_\mu \Phi$ is continuous if, and only if, the pair (S, T_μ) has no critical eigenvalue.

1. Introduction

Let X and Y be Banach spaces and let S and T be continuous linear operators on X and Y, respectively. An operator $\Phi : X \to Y$ is said to *intertwine* the pair (S, T) if $S\Phi = \Phi T$. Typical results on automatic continuity theory deals with conditions on S and/or T which imply the continuity of Φ. A classical reference in this context is [5]. As a matter of fact, it is well known [5, Lemma 3.2] that the existence of a critical eigenvalue implies the existence of a discontinuous intertwining operator. Let us recall that $\lambda \in \mathbb{C}$ is a critical eigenvalue of the pair (S, T) if λ is an eigenvalue of T and $(\lambda\mathbf{I} - S)(X)$ has infinite codimension.

K. B. Laursen and M. M. Neumann asked in [1, Open Problem 6.3.3] the following question: suppose that μ and ν are bounded complex-valued Borel measures on a locally compact abelian group G such that the corresponding pair (T_ν, T_μ) has no critical eigenvalue. Is it true that every linear operator $\Phi : L^1(G) \to L^1(G)$ such that $T_\nu \Phi = \Phi T_\mu$ is continuous? It should be pointed out that the special case $\nu = \mu$ dates back to B. E. Johnson [3].

Recently, C. Aparicio and the third-named author [1, Theorem 2.1] solved positively the above mentioned question. The aim of this short note is to extend this

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result to a well known class of locally compact groups that includes abelian groups and compact groups as well: the class of MAP groups.

Theorem. *Let G be a MAP group, let μ be a bounded complex-valued Borel measure on G, and let T_μ be the corresponding convolution operator $f \mapsto \mu * f$ on $L^1(G)$. Let X be a Banach space and let S be a continuous linear operator on X. Suppose that (S, T_μ) has no critical eigenvalue. Then every linear operator $\Phi : X \to L^1(G)$ such that $\Phi S = T_\mu \Phi$ is continuous.*

Let us recall that a locally compact group G is said to be a MAP group if the set of finite-dimensional irreducible unitary representations, $\widehat{G}_{FIN}$, separates the elements of G. Every unitary representation π of G on a Hilbert space H lifts to a $*$-homomorphism from the Banach algebra $M(G)$ of all bounded complex-valued regular Borel measures on G into $B(H)$, which is defined by

$$\pi(\mu) = \int_G \pi(t)\,d\mu(t)$$

for each $\mu \in M(G)$. It is important to note here that $\widehat{G}_{FIN}$ also separates the elements of $L^1(G)$ (see [2, Theorem 3.2]).

Proof. On account of the closed graph theorem, we are reduced to show that the separating space of Φ,

$$\mathfrak{S}(\Phi) = \{f \in L^1(G) : \text{ there exists } (x_n) \to 0 \text{ in } X \text{ with } (\Phi(x_n)) \to f\},$$

is $\{0\}$.

We claim that there exists a non-zero polynomial p such that $p(T_\mu)(\mathfrak{S}(\Phi)) = \{0\}$. To obtain a contradiction, suppose that $p(T_\mu)(\mathfrak{S}(\Phi)) \neq \{0\}$ for each non-zero polynomial p. This entails that $\mathfrak{S}(\Phi) \neq \{0\}$ and, therefore, that there exists $\pi_1 \in \widehat{G}_{FIN}$ such that $\pi_1(\mathfrak{S}(\Phi)) \neq \{0\}$. Let p_1 be the characteristic polynomial of $\pi_1(\mu)$. We can choose inductively $\pi_n \in \widehat{G}_{FIN}$ with the property that

$$\pi_{n+1}((p_1 \dots p_n)(T_\mu)(\mathfrak{S}(\Phi))) \neq \{0\}$$

for each $n \in \mathbb{N}$, where p_k stands for the characteristic polynomial of $\pi_k(\mu)$ for $k = 1, \dots, n$.

Since $p_n(S)\Phi = \Phi p_n(T_\mu)$, for each $n \in \mathbb{N}$, the stability lemma (see [5, Lemma 1.6]) gives $N \in \mathbb{N}$ such that

$$\overline{(p_1 \dots p_N)(T_\mu)(\mathfrak{S}(\Phi))} = \overline{(p_1 \dots p_{N+1})(T_\mu)(\mathfrak{S}(\Phi))}.$$

Observe that if $f \in (p_1 \ldots p_N)(T_\mu)(\mathfrak{S}(\Phi))$, then π_{N+1} vanishes on f. Indeed, f can be written $f = (p_1 \ldots p_{N+1})((T_\mu)(g))$ for some $g \in \mathfrak{S}(\Phi)$ and

$$\begin{aligned}\pi_{N+1}(f) &= \pi_{N+1}((p_1 \ldots p_N)(T_\mu)(g)) \\ &= p_1(\pi_{N+1}(\mu)) \ldots p_{N+1}(\pi_{N+1}(\mu))\pi_{N+1}(g) = 0\end{aligned}$$

since $p_{N+1}(\pi_{N+1}(\mu)) = 0$. Therefore,

$$\pi_{N+1}((p_1 \ldots p_N)(T_\mu)(\mathfrak{S}(\Phi))) = \{0\}$$

which contradicts the choice of π_{N+1}.

Since our claim holds, there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ such that

$$(\alpha_1 \mathbf{I} - T_\mu) \ldots (\alpha_k \mathbf{I} - T_\mu)\mathfrak{S}(\Phi) = \{0\}.$$

The last step of the proof follows by the same method as in [1, Theorem 2.1]. We can assume that $\alpha_1, \ldots \alpha_k$ are eigenvalues of T_μ and, since (S, T_μ) does not have critical eigenvalues, it follows that $\operatorname{codim}(\alpha_i \mathbf{I} - S) < \infty$ for $i = 1, \ldots, k$. Let us consider the finite-codimensional subspace $M = (\alpha_1 \mathbf{I} - S) \ldots (\alpha_k \mathbf{I} - S)(X)$ of X. M is the range of the continuous linear operator $R = (\alpha_1 \mathbf{I} - S) \ldots (\alpha_k \mathbf{I} - S)$ and, by [5, Lemma 3.3], it is closed. Let us denote by Ψ the restriction of Φ to M. Now, [5, Lemma 1.3] shows that ΨR is continuous since

$$\mathfrak{S}(\Psi R) = (\alpha_1 \mathbf{I} - T_\mu) \ldots (\alpha_k \mathbf{I} - T_\mu)(\mathfrak{S}(\Phi)) = \{0\}.$$

Let us check that Ψ is continuous. Let (y_n) be a sequence in M with $\lim y_n = 0$. Since R is open by the open mapping theorem, there exists a sequence (x_n) in X such that $\lim x_n = 0$ and $R(x_n) = y_n$ for each $n \in \mathbb{N}$. Using that ΨR is continuous, it follows that $\lim \Psi(y_n) = \lim \Psi R(x_n)$, and therefore Ψ is continuous. Finally, note that Φ is continuous on a complemented finite-codimensional subspace and this implies that Φ is continuous. ■

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