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EXISTENCE OF SOLUTIONS OF THE g-NAVIER-STOKES EQUATIONS

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Abstract. The g-Navier-Stokes equations in spatial dimension 2 are the following equations introuduced in [3]

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f},$$

with the continuity equation

$$\frac{1}{g}\nabla\cdot(g\mathbf{u}) = 0.$$

Here, we show the existence and uniqueness of solutions of g-Navier-Stokes equations on $\mathbf{R}^n$ for $n = 2,3$.

1. Introduction

The governing equations for the fluid are the well-known incompressible Navier–Stokes equations of the form

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.1}$$

$$(\nabla\cdot\mathbf{u}) = 0, \tag{1.2}$$

with some initial and boundary conditions. Here, ν and f are given and the velocity $\mathbf{u}$ and the pressure p are the unknowns. The first equations are called the momentum equations and the second one continuity equation. For the analysis on the Navier–Stokes equations, refer to [1], [2], [4] and [5].

Consider the Navier–Stokes equations (1.1) and (1.2) on the spatial domain $_g := {}_2 \times [0, g]$, where $_2$ is a bounded region in the plane and $g = g(x_1, x_2)$ is

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a smooth function defined on $\ _2$ with $0 < m \cdot \ g(x_1, x_2) \cdot \ M$, for $(x_1, x_2) \in \ _2$. The 2D g-Navier-Stokes equations have been drived in [3] from the 3D Navier-Stokes equations on $\ _g$:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.3}$$

$$\frac{1}{g}\big(\nabla \cdot (g\mathbf{u})\big) = \frac{\nabla g}{g} \cdot \mathbf{u} + \nabla \cdot \mathbf{u} = 0 \tag{1.4}$$

in $\ _2$. Equation (1.3) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\nu}{g}(\nabla \cdot (g\nabla))\mathbf{u} + \nu(\frac{\nabla g}{g} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}.$$

Roh [3] proved the existence of solutions for periodic boundary conditions as well as Dirichlet boundary conditions on bounded domains. Global attractors are also discussed for suitable g. For these results, we need the smoothness of g and the smallness of $\|\nabla g\|_\infty$. Refer to [3] for the details on g-Navier–Stokes equations.

In this paper, we prove the existence of the solutions for the g-Navier-Stokes equation (1.3)-(1.4) on the whole domain $\mathbf{R}^n$.

In section 2, we give a short introduction for the g-Navier-Stokes equations. In section 3, we review the solution space for the equations. In section 4, we consider the nonlinear term and perturbation term. In section 5, we review the compactness theorem in [5]. In section 6, we prove our main result about the existence. In section 7, we show the solution obtained in section 6 is unique.

2. Short Introduction of g-Navier-stokes Equations

Let $\ _3 = \ _2 \times [0, 1]$. Let $\mathbf{U}$, $\mathbf{V}$ be functions of $y = (y_1, y_2, y_3) \in \ _g$ where $(y_1, y_2) \in \ _2$ and $0 \cdot \ y_3 \cdot \ g(y_1, y_2)$. Then the change of variables

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3 g(x_1, x_2) \tag{2.1}$$

maps $\ _3$ onto $\ _g$. The standard $3D$ Navier-Stokes equations have the form

$$\frac{\partial \mathbf{U}}{\partial t} - \nu \Delta \mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla \Phi = \mathbf{F}$$
$$\nabla \cdot \mathbf{U} = 0$$

on $\ _g$. We assume that $\mathbf{U}$ satisfy the boundary condition

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial_{top} \ _g \cup \partial_{bottom} \ _g \tag{2.2}$$

where

$$\partial_{top} \ {}_g = \{(y_1, y_2, y_3) \in \ {}_g \ : \ y_3 = g(y_1, y_2)\},$$

$$\partial_{bottom} \ {}_g = \{(y_1, y_2, y_3) \in \ {}_g \ : \ y_3 = 0\}.$$

Let $\mathbf{u}(x_1, x_2, x_3) = \mathbf{U}(y_1, y_2, y_3)$, where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ satisfy (2.1).

Now we define $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ as

$$\mathbf{v}_i = \mathbf{v}_i(x_1, x_2) = \int_0^1 \mathbf{u}_i(x_1, x_2, x_3)\ dx_3 = \frac{1}{g(y_1, y_2)} \int_0^{g(y_1, y_2)} \mathbf{u}_i(y_1, y_2, y_3)\ dy_3,$$

for $i = 1, 2$ and we get the following proposition.

Proposition 2.1. *Assume that* $\nabla \cdot \mathbf{U} = 0$ *in* $\ {}_g$ *and that* (**??**) *is valid. Then one has*

$$\nabla_2 \cdot (g\mathbf{v}) = \frac{\partial(g\mathbf{v}_1)}{\partial x_1} + \frac{\partial(g\mathbf{v}_2)}{\partial x_2} = \nabla g \cdot \mathbf{v} + g\ (\nabla_2 \cdot \mathbf{v}) = 0,$$

where $\nabla_2 = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $\nabla g = (\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2})$.

Proof. See Roh [3]

Next, we need the following assumption.

Assumption 1. $g(\mathbf{x}) \in C^2(\mathbf{R}^n)$ *and* $0 < m \cdot \ g(\mathbf{x}) \cdot \ M$*, for all* $\mathbf{x} \in \mathbf{R}^n$*, where* $m = m(g)$ *and* $M = M(g)$*. We also assume*

$$\|\nabla g\|_\infty = \sup_{(x,y) \in \mathbf{R}^n} |\nabla g(x, y)| < +\infty.$$

3. Functional Spaces

We consider the physical domain $\ = \mathbf{R}^n$ for $n = 2, 3$. We denote by $L^2(\ , g)$ the space with the scalar product and the norm given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_g = \int (\mathbf{u} \cdot \mathbf{v})\ g\ d\mathbf{x} \quad \text{and} \quad |\mathbf{u}|^{\mathbf{2}} = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}},$$

where $\mathbf{x} = (x_1, \cdots, x_n)$. Similarly, we will use the space $H^1(\ , g)$ with the norm by

$$\|\mathbf{u}\|_{H^1(\ ,g)} = \left[\langle \mathbf{u}, \mathbf{u} \rangle_g + \sum_{i=1}^n \langle \partial_i \mathbf{u}, \partial_i \mathbf{u} \rangle_g \right]^{\frac{1}{2}},$$

where $\frac{\partial \mathbf{u}}{\partial x_i} = \partial_i \mathbf{u}$.

Remark 1. Since $0 < m \cdot \ g(\mathbf{x}) \cdot \ M$ for all $\mathbf{x} \in \mathbf{R}^n$, and g is smooth, $|\mathbf{u}|_{L^2(\mathbf{R}^n)}$ is equivalent to $|\mathbf{u}|_g$ as well as $\|\mathbf{u}\|_{H^1(\mathbf{R}^n)}$ is equivalent to $\|\mathbf{u}\|_{H^1(\mathbf{R}^n,g)}$.

Let $\mathcal{D}(\mathbf{R}^n)$ be the space of C^∞ functions with compact support contained in $\mathbf{R}^n$. The closure of $\mathcal{D}(\mathbf{R}^n)$ in $W^{m,p}(\mathbf{R}^n)$ is denoted by $W_0^{m,p}(\mathbf{R}^n)(H_0^m(\mathbf{R}^n)$ when $p = 2)$.

For the mathematical setting, we define the spaces as the followings,

$$\begin{aligned} \mathcal{V} &= \{\mathbf{u} \in \mathcal{D}(\mathbf{R}^n) : \ \nabla \cdot (g\mathbf{u}) = 0\} \\ H_g &= \text{the closure of } \ \mathcal{V} \ \text{ in } \ L^2(\mathbf{R}^n) \\ V_g &= \text{the closure of } \ \mathcal{V} \ \text{ in } \ H_0^1(\mathbf{R}^n), \end{aligned}$$

where H_g are endowed with the scalar product and the norm in $L^2(\mathbf{R}^n, g)$, and V_g are endowed with the scalar product and the norm in $H^1(\mathbf{R}^n, g)$. The space V_g is contained in H_g, is dense in H_g, and the injection is continuous. Let H_g' and V_g' denote the dual spaces of H_g and V_g, and let i denote the injection mapping from V_g into H_g. The adjoint operator i' is linear continuous from H' into V_g', and is one to one since $i(V_g) = V_g$ is dense in H_g and $i'(H_g')$ is dense in V_g' since i is one to one . Therefore H_g' can be indentified with a dense subspace of V_g'. Moreover, by the Riesz representation theorem, we can identify H_g and H_g', and we arrive at the inclusions

$$V_g \subset H_g = H_g' \subset V_g',$$

where each space is dense in the following one and the injections are continuous. So we note that the scalar product in H_g of $\mathbf{f} \in H_g$ and $\mathbf{u} \in V_g$ is the same as the scalar product of $\mathbf{f}$ and $\mathbf{u}$ in the duality between V_g' and V_g,

$$\langle \mathbf{f}, \mathbf{u} \rangle_g = (\mathbf{f}, \mathbf{u}), \quad \forall \mathbf{f} \in H_g, \quad \forall \mathbf{u} \in V_g. \tag{3.1}$$

For each $\mathbf{u}$ in V_g, the form

$$\mathbf{v} \in V_g \to ((\mathbf{u}, \mathbf{v}))_g \in \mathbf{R}$$

is linear and continuous on V_g; therefore, there exist an element of V_g' which we denote by $A\mathbf{u}$ such that

$$\langle A\mathbf{u}, \mathbf{v} \rangle_g = ((\mathbf{u}, \mathbf{v}))_g, \quad \forall \mathbf{v} \in V_g, \tag{3.2}$$

where

$$((\mathbf{u}, \mathbf{v}))_g = \sum_{i=1}^n \langle D_i \mathbf{u}, D_i \mathbf{v} \rangle_g.$$

Also, we denote

$$\|\mathbf{u}\|^2 = ((\mathbf{u}, \mathbf{u}))_g = \sum_{i=1}^n \langle D_i \mathbf{u}, D_i \mathbf{u} \rangle_g.$$

Therefore, we have

$$\|\mathbf{u}\|_{V_g}^2 = |\mathbf{u}|^2 + \|\mathbf{u}\|^2,$$

where $\|\mathbf{u}\|_{H_g} = |\mathbf{u}|$.

Problem 1. Given $\mathbf{f} \in L^2(0,T;V_g')$ and $\mathbf{u}_0 \in H_g$, to find $\mathbf{u}$ satisfying

$$\begin{aligned}
&\mathbf{u} \in L^2(0,T;V_g), \quad \mathbf{u}' \in L^2(0,T;V_g'),\\
&\mathbf{u}' + \nu A\mathbf{u} = \mathbf{f}, \quad \text{on } (0,T),\\
&\mathbf{u}(0) = \mathbf{u}_0.
\end{aligned}$$

Lemma 3.1. *Problem 1 has unique solution* $\mathbf{u}$ *and moreover* $\mathbf{u} \in C([0,T];H_g)$.

Proof. One can prove by similar method in Chapter 3, [5]. ■

Remark 2. Assuming that $\mathbf{f}$, $\mathbf{u}_0$ are sufficiently smooth, we can obtain as much regularity as desired for $\mathbf{u}$ and p. For given $\mathbf{f} \in L^2(0,T;H_g)$ and $\mathbf{u}_0 \in V_g$, one can obtain that

$$\begin{aligned}
&\mathbf{u} \in L^2(0,T;H^2(\ \ )),\\
&\mathbf{u}' \in L^2(0,T;H_g), \text{ and } p \in L^2(0,T;H^1(\ \ )).
\end{aligned}$$

For our problem, one should note that

$$-\frac{1}{g}(\nabla \cdot g\nabla)\mathbf{u} = -\Delta\mathbf{u} - (\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}.$$

Therefore, one obtains

$$\langle -\Delta\mathbf{u}, \mathbf{v}\rangle_g = ((\mathbf{u},\mathbf{v}))_g + \langle (\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}, \mathbf{v}\rangle_g = \langle A\mathbf{u}, \mathbf{v}\rangle_g + \langle (\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}, \mathbf{v}\rangle_g,$$

for $\mathbf{u}, \mathbf{v} \in V_g$.

4. Nonlinear and Perturbation Terms

We define the trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} \int_{R^n} \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{w}_j g dx,$$

where $\mathbf{u},\mathbf{v},\mathbf{w}$ lie in appropriate subspaces of $L^2(R^n, g)$ and $D_i = \frac{\partial}{\partial x_i}$. Since $\nabla \cdot g\mathbf{u} = \sum_i D_i(g\mathbf{u}_i) = 0$, for $\mathbf{u} \in H_g$, one obtains

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \sum_{i,j=1}^{n} \int_{R^n} \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{w}_j g dx \\ &= -\sum_{i,j=1}^{n} \int_{R^n} D_i (g\mathbf{u}_i) \mathbf{v}_j \mathbf{w}_j dx - \sum_{i,j=1}^{n} \int_{R^n} g\mathbf{u}_i \mathbf{v}_j (D_i \mathbf{w}_j) dx \\ &= -\sum_{i,j=1}^{n} \int_{R^n} g\mathbf{u}_i \mathbf{v}_j (D_i \mathbf{w}_j) dx = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \end{aligned}$$

for sufficient smooth functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H_g$. Therefore $b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v})$ and $b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$, for smooth functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H_g$.

For $\mathbf{u}$, $\mathbf{v}$ in V_g, we denote by $B(\mathbf{u},\mathbf{v})$ the element of V_g' defined by

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle_g = b(\mathbf{u},\mathbf{v},\mathbf{w}), \quad \forall \mathbf{w} \in V_g,$$

and we set

$$B(\mathbf{u}) = B(\mathbf{u},\mathbf{u}) \in V_g', \quad \forall \mathbf{u} \in V_g.$$

Before we estimate the nonlinear term $B(\mathbf{u})$, let us look at the useful inequalities.

Lemma 4.1 *If $n = 2$, then we have*

$$\|\mathbf{u}\|_{L^4(\mathbf{R}^2,g)} \cdot \; c|\mathbf{u}|^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{1}{2}}, \quad \forall \mathbf{u} \in H^1(\mathbf{R}^2, g).$$

and if $n = 3$, then we have

$$\|\mathbf{u}\|_{L^4(\mathbf{R}^3,g)} \cdot \; c|\mathbf{u}|^{\frac{1}{4}} \|\mathbf{u}\|^{\frac{3}{4}}, \quad \forall \mathbf{u} \in H^1(\mathbf{R}^3, g). \tag{4.1}$$

Proof. One can easily see by the equivalence of the norms. ∎

Lemma 4.2 *We assume that $\mathbf{u} \in L^2(0,T;V_g)$. Then the function $B\mathbf{u}$ defined by*

$$\langle B\mathbf{u}(t), \mathbf{v}\rangle_g = b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}), \ \forall \ \mathbf{u} \in V_g, \ a.e. \ t \in [0,T],$$

belongs to $L^1(0,T;V_g')$. Moreover, the function $C\mathbf{u}$ defined by

$$\langle C\mathbf{u}(t), \mathbf{v}\rangle_g = \langle (\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}, \mathbf{v}\rangle_g = \sum_{i,j=1}^{2} \int_{R^n} \frac{D_i g}{g} (D_i \mathbf{u}_j) \mathbf{v}_j g \, d\mathbf{x} = b(\frac{\nabla g}{g}, \mathbf{u}, \mathbf{v}),$$

for all $\mathbf{v} \in V_g$, belong to $L^2(0,T;H_g)$, and hence belong to $L^2(0,T;V_g')$.

Proof. One can easily check by the previous lemma that for almost all t, $B\mathbf{u}(t) \in V_g'$. For $\mathbf{u}, \mathbf{v} \in V_g$, one has

$$\begin{aligned}
|\langle B(\mathbf{u}), \mathbf{v}\rangle_g| &= |b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \\
&= |\int_{\mathbf{R}^n} \sum_{i,j=1}^{n} \mathbf{u}_i (D_i \mathbf{u}_j) \mathbf{v}_j g \; d\mathbf{x}| \\
&= |\int_{\mathbf{R}^n} \sum_{i,j=1}^{n} \mathbf{u}_i (D_i \mathbf{v}_j) \mathbf{u}_j g \; d\mathbf{x}| \\
&\cdot \; c\|\mathbf{v}\| \|\mathbf{u}\|^2_{L^4(\mathbf{R}^n, g)} \cdot \; c\|\mathbf{v}\|_{V_g} \|\mathbf{u}\|^2_{L^4(\mathbf{R}^n, g)}.
\end{aligned}$$

So, if $n = 2$, then

$$\|B(\mathbf{u})\|_{V_g'} \cdot \; c\|\mathbf{u}\|^2_{L^4(\mathbf{R}^2, g)} \cdot \; c|\mathbf{u}| \; \|\mathbf{u}\|.$$

Also, if $n = 3$, then

$$\|B(\mathbf{u})\|_{V_g'} \cdot \; c\|\mathbf{u}\|^2_{L^4(\mathbf{R}^2, g)} \cdot \; c|\mathbf{u}|^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{3}{2}}.$$

Hence, for $n = 2, 3$, one has that

$$\|B\mathbf{u}\|_{V_g'} \cdot \; c\|\mathbf{u}\|^2_{V_g}, \; \forall \; \mathbf{u} \in V_g, \tag{4.2}$$

for some constant c. Hence, we obtain

$$\int_0^T \|B\mathbf{u}\|_{V_g'} \; dt \cdot \quad c \int_0^T \|\mathbf{u}(t)\|^2_{V_g} \; dt < +\infty$$

which implies that $B\mathbf{u}$ belong to $L^1(0, T; V_g')$.

Next, for the estimate of $C\mathbf{u}$, we have

$$\begin{aligned}
|\langle C\mathbf{u}, \mathbf{v}\rangle| &= |\textstyle\sum_{i,j=1}^{n} \int_{\mathbf{R}^n} \frac{D_i g}{g} (D_i \mathbf{u}_j) \mathbf{v}_j g \; d\mathbf{x}| \\
&\cdot \; c\|\nabla g\|_\infty \|\mathbf{u}\| \; |\mathbf{v}|.
\end{aligned} \tag{4.3}$$

So, one obtains

$$|C\mathbf{u}(t)| \cdot \; c\|\nabla g\|_\infty \|\mathbf{u}\|. \tag{4.4}$$

Hence, we have

$$\int_0^T |C\mathbf{u}(t)|^2 dt \cdot \; c\|\nabla g\|^2_\infty \int_0^T \|\mathbf{u}\|^2 dt \cdot \; c\|\nabla g\|^2_\infty \int_0^T \|\mathbf{u}\|^2_{V_g} dt < +\infty$$

which implies that $C\mathbf{u}(t)$ belong to $L^2(0, T; H_g)$. ■

5. Compactness

The following two propositions are stated in [5].

Proposition 5.1. *Let X_0, X and X_1 be three Banach spaces such that*

$$X_0 \subset X \subset X_1,$$

the injection of X into X_1 being continuous, and the injection of X_0 into X is compact. Then for every $\eta > 0$, there exist some constant c_η depending on η (and on the spaces X_0, X, X_1) such that:

$$\|\mathbf{v}\|_X \cdot \ \eta\|\mathbf{v}\|_{X_0} + c_\eta\|\mathbf{v}\|_{X_1}, \quad \forall \mathbf{v} \in X_0.$$

Now, we assume that X_0, X, X_1, are Hilbert spaces with

$$X_0 \subset X \subset X_1, \tag{5.1}$$

the injections being continuous and

$$\text{the injection of } X_0 \text{ into } X \text{ is compact.} \tag{5.2}$$

If $\mathbf{v}$ is a function from $\mathbf{R}$ into X_1, we denote by $\hat{\mathbf{v}}$ its Fourier transform

$$\hat{\mathbf{v}}(\tau) = \int_{-\infty}^{\infty} e^{-2i\pi t\tau}\mathbf{v}(t)\ dt.$$

The derivative in t of order γ of $\mathbf{v}$ is the inverse Fourier transform of $(2i\pi\tau)^\gamma\hat{\mathbf{v}}$ or

$$\widehat{D_t^\gamma \mathbf{v}}(\tau) = (2i\pi\tau)^\gamma\hat{\mathbf{v}}(\tau).$$

For given $\gamma > 0$, we define the space

$$\mathcal{H}^\gamma(\mathbf{R}; X_0, X_1) = \{\mathbf{v} \in L^2(\mathbf{R}; X_0), D_t^\gamma\mathbf{v} \in L^2(\mathbf{R}; X_1)\}.$$

This is a Hilbert space for the norm,

$$\|\mathbf{v}\|_{\mathcal{H}^\gamma(\mathbf{R}, X_0, X_1)} = \{\|\mathbf{v}\|^2_{L^2(\mathbf{R}; X_0)} + \||\tau|^\gamma\hat{\mathbf{v}}\|^2_{L^2(\mathbf{R}; X_1)}\}^{\frac{1}{2}}.$$

We also define the subspace $\mathcal{H}^\gamma_K$ of $\mathcal{H}^\gamma$, for any set $K \subset \mathbf{R}$, as

$$\mathcal{H}^\gamma_K(\mathbf{R}; X_0, X_1) = \{\mathbf{u} \in \mathcal{H}^\gamma(\mathbf{R}; X_0, X_1), \text{support } \mathbf{u} \subset K\}.$$

Proposition 5.2. *Let us assume that X_0, X, X_1 are Hilbert spaces which satisfy (5.1) and (5.2).*

Then for any bounded set K and any $\gamma > 0$, the injection of $H^\gamma_K(\mathbf{R}; X_0, X_1)$ into $L^2(\mathbf{R}, X)$ is compact.

Remark 3. Let us recall the mathematical spaces for our problem. For the mathematical setting, we defined the spaces as the followings,

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{u} \in \mathcal{D}(\mathbf{R}^n),\ \nabla \cdot (g\mathbf{u}) = 0\} \\
H_g &= \text{ the closure of } \mathcal{V} \text{ in } L^2(\mathbf{R}^n) \\
V_g &= \text{ the closure of } \mathcal{V} \text{ in } H_0^1(\mathbf{R}^n),
\end{aligned}$$

where H_g are endowed with the scalar product and the norm in $L^2(\mathbf{R}^n, g)$, and V_g are endowed with the scalar product and the norm in $H^1(\mathbf{R}^n, g)$. The space V_g is contained in H_g, is dense in H_g, and the injection is continuous. But, the injection is not compact. So, we can not use the previous compactness theorem. Hence, to use the previous compactness theorem, we consider a bounded ball $\mathcal{Q}$ in $\mathbf{R}^n$ instead of $\mathbf{R}^n$ and

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{u} \in \mathcal{D}(\mathcal{Q}),\ \nabla \cdot (g\mathbf{u}) = 0\} \\
H_g(\mathcal{Q}) &= \text{ the closure of } \mathcal{V} \text{ in } L^2(\mathcal{Q}) \\
V_g(\mathcal{Q}) &= \text{ the closure of } \mathcal{V} \text{ in } H_0^1(\mathcal{Q}).
\end{aligned}$$

Then the space $V_g(\mathcal{Q})$ is contained in $H_g(\mathcal{Q})$, is dense in $H_g(\mathcal{Q})$, and the injection being continuous is compact. Therefore, we can use the previous compactness theorem and we have the following lemma.

Lemma 5.3. *If $\mathbf{u}_k$ converges to $\mathbf{u}$ in $L^2(0,T;V_g(\mathcal{Q}))$ weakly and $L^2(0,T;H_g(\mathcal{Q}))$ strongly, then for any vector function $\mathbf{w}$ with components in $C_0^1(\mathcal{Q})$,*

$$\int_0^T b(\mathbf{u}_k(t), \mathbf{u}_k(t), \mathbf{w}(t))dt \to \int_0^T b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{w}(t))dt.$$

Proof. We note that

$$\begin{aligned}
\int_0^T b(\mathbf{u}_k, \mathbf{u}_k, \mathbf{w})dt &= -\int_0^T b(\mathbf{u}_k, \mathbf{w}, \mathbf{u}_k) \\
&= -\sum_{i,j=1}^n \int_0^T \int (\mathbf{u}_k)_i (D_i \mathbf{w}_j)(\mathbf{u}_k)_j \ g \ d\mathbf{x}dt.
\end{aligned}$$

These integrals converge to

$$-\sum_{i,j=1}^n \int_0^T \int \mathbf{u}_i (D_i \mathbf{w}_j) \mathbf{u}_j \ g \ d\mathbf{x}dt = -\int_0^T b(\mathbf{u}, \mathbf{w}, \mathbf{u})dt = \int_0^T b_g(\mathbf{u}, \mathbf{u}, \mathbf{w})dt,$$

because g is bounded function on $\mathbf{R}^n$ and $\mathbf{w} \in C_0^1(\mathcal{Q})$. ■

6. Proof of Existence

The initial value problem of the g-Navier-Stokes equations is to find suitable vector function $\mathbf{u}$ and scalar function p such that

$$\mathbf{u}: \quad \times [0,T] \to \mathbf{R}^n, \quad p: \quad \times [0,T] \to \mathbf{R}$$

satisfying

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \sum_{i=1}^{n} \mathbf{u}_i D_i \mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in} \quad \times (0,T), \\ \frac{1}{g}(\nabla \cdot (g\mathbf{u})) = \nabla \cdot \mathbf{u} + (\frac{\nabla g}{g} \cdot \mathbf{u}) = 0 \quad & \text{in} \quad \times (0,T), \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad & \text{in} \quad . \end{aligned}$$

Problem 2. For $\mathbf{f} \in L^2(0,T;V_g')$ and $\mathbf{u}_0 \in H_g$, to find $\mathbf{u} \in L^2(0,T;V_g)$ satisfying

(6.1)
$$\begin{aligned} \frac{d}{dt}(\mathbf{u},\mathbf{v})_g + \nu((\mathbf{u},\mathbf{v}))_g \quad & + b(\mathbf{u},\mathbf{u},\mathbf{v}) \\ & = \langle \mathbf{f}, \mathbf{v} \rangle_g - \langle (\tfrac{\nabla g}{g} \cdot \nabla)\mathbf{u}, \mathbf{v} \rangle_g \quad \forall \mathbf{v} \in V_g \end{aligned}$$

and

(6.2)
$$\mathbf{u}(0) = \mathbf{u}_0.$$

If $\mathbf{u} \in L^2(0,T;V_g)$ satisfies the equation (6.1), then by (3.1), (3.2) and lemma 4.2, one can write the equation (6.1) as

$$\frac{d}{dt}\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f} - \nu A\mathbf{u} - B\mathbf{u} - C\mathbf{u}, \mathbf{v} \rangle, \ \forall \ \mathbf{v} \in V_g.$$

One note that since $A\mathbf{u}$ belong to $L^2(0,T;V_g')$, the function $\mathbf{f} - \nu A\mathbf{u} - B\mathbf{u} - C\mathbf{u}$ belong to $L^1(0,T;V_g')$.

Theorem 6.1. *Assume that* $\mathbf{f} \in L^2(0,T;V_g')$ *and* $\mathbf{u}_0 \in H_g$. *Then there exist at least one solution* $\mathbf{u}$ *of problem 2. Moreover,*

$$\mathbf{u} \in L^\infty(0,T;H_g)$$

and $\mathbf{u}$ *is weakly continuous from* $[0,T]$ *into* H_g.

Proof. We apply the Galerkin procedure. Since V_g is seperable and $\mathcal{V}$ is dense in V_g, there exists a sequence $\mathbf{w}_1, ..., \mathbf{w}_m, ...$ of elelments of $\mathcal{V}$, which is free and total in V_g. For each m we define an approximate solution $\mathbf{u}_m$ of equation (6.1) as

$$\mathbf{u}_m = \sum_{i=1}^{m} \phi_{im}(t)\mathbf{w}_i$$

which satisfies

$$(\mathbf{u}_m'(t), \mathbf{w}_j) + \nu((\mathbf{u}_m(t), \mathbf{w}_j))_g - b(\frac{\nabla g}{g}, \mathbf{u}_m(t), \mathbf{w}_j) \\ + b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{w}_j) = \langle \mathbf{f}(t), \mathbf{w}_j \rangle_g, \tag{6.3}$$

for $t \in [0, T]$, $j = 1, ..., m$, and $\mathbf{u}_m(0) = \mathbf{u}_{0m}$, where $\mathbf{u}_{0m}$ is the orthogonal projection in H_g of $\mathbf{u}_0$ onto the space spanned by $\mathbf{w}_1, ..., \mathbf{w}_m$. Then one can get

$$\sum_{i=1}^{m} (\mathbf{w}_i, \mathbf{w}_j)\phi_{im}'(t) + \nu \sum_{i=1}^{m} ((\mathbf{w}_i, \mathbf{w}_j))_g \phi_{im}(t) \\ + \sum_{i}^{m} b(\frac{\nabla g}{g}, \mathbf{w}_i, \mathbf{w}_j)\phi_{im}(t) + \sum_{i,l=1}^{m} b(\mathbf{w}_i, \mathbf{w}_l, \mathbf{w}_j)\phi_{im}(t)\phi_{lm}(t) \\ = \langle \mathbf{f}(t), \mathbf{w}_j \rangle_g.$$

Inverting the nonsingular matrix with elements $\langle \mathbf{w}_i, \mathbf{w}_j \rangle_g$, $1 \leq i, j \leq m$, we can write the differential equations in the usual form

$$\phi_{im}'(t) + \sum_{j=1}^{m} \alpha_{ij}\phi_{jm}(t) + \sum_{j,k=1}^{m} \alpha_{ijk}\phi_{jm}(t)\phi_{km}(t) = \sum_{j=1}^{m} \beta_{ij}\langle \mathbf{f}(t), \mathbf{w}_j \rangle_g, \tag{6.4}$$

where α_{ij}, α_{ijk}, $\beta_{ij} \in \mathbf{R}$. Let

$$\phi_{im}(0) = \text{the } i^{th} \text{ component of } \mathbf{u}_{0m}. \tag{6.5}$$

The nonlinear ordinary differential system (6.4) with the initial condition (6.5) has a maximal solution defined on some interval $[0, t_m]$. If $t_m < T$, then $|\mathbf{u}_m(t)|$ must tend to $+\infty$ as $t \to t_m$; the a priori estimates we shall prove later show that this does not happen and therefore $t_m = T$. To do that, we need several estimates.

(i) We multiply (6.3) by $\phi_{jm}(t)$ and add these equations for $j = 1, ..., m$ to get

$$(\mathbf{u}_m'(t), \mathbf{u}_m(t)) + \nu\|\mathbf{u}_m(t)\|^2 = \langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_g - b(\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}_m(t), \mathbf{u}_m(t)).$$

Then we write

$$\frac{d}{dt}|\mathbf{u}_m(t)|^2 + 2\nu\|\mathbf{u}_m(t)\|^2 = 2\langle \mathbf{f}(t), \mathbf{u}_m(t) \rangle_g + 2b(\frac{\nabla g}{g} \cdot \nabla)\mathbf{u}_m(t), \mathbf{u}_m(t)) \\ \leq 2\|\mathbf{f}(t)\|_{V'} \|\mathbf{u}_m(t)\|_V + \tfrac{2}{m}|\nabla g|_\infty |\mathbf{u}_m(t)| \|\mathbf{u}_m(t)\| + \|\mathbf{u}_m\|^2, \\ \leq \nu\|\mathbf{u}_m(t)\|^2 + \tfrac{8}{\nu}\|\mathbf{f}(t)\|_{V'}^2 + \tfrac{2}{\nu m^2}|\nabla g|_\infty^2 |\mathbf{u}_m(t)|^2 + \nu|\mathbf{u}_m|^2,$$

so that

$$\frac{d}{dt}|\mathbf{u}_m(t)|^2 + \nu\|\mathbf{u}_m(t)\|^2 \leq \frac{8}{\nu}\|\mathbf{f}(t)\|_{V'}^2 + \alpha|\mathbf{u}_m(t)|^2, \tag{6.6}$$

where $\alpha = \frac{2}{\nu m^2}|\nabla g|_\infty^2 + \nu$.

Hence, one obtains

$$\frac{d}{dt}|\mathbf{u}_m(t)|^2 \cdot \ \alpha|\mathbf{u}_m(t)|^2 + \frac{8}{\nu}\|\mathbf{f}(t)\|_{V'}^2,$$

where $\alpha = \frac{2}{\nu m^2}|\nabla g|_\infty^2$. So, by the usual method of the Gronwall inequality, we have

$$|\mathbf{u}_m(t)|^2 \cdot \ e^{\alpha t}(|\mathbf{u}_m(0)|^2 + \frac{8}{\nu}\int_0^t |\mathbf{f}(s)|_{V_g'}^2 ds).$$

By the assumption the right side of the above inequality is uniformly bounded for $s \in [0, T]$ and m.

Hence

$$\sup_{s\in[0,T]} |\mathbf{u}_m(s)|^2 \cdot \ e^{\alpha T}(|\mathbf{u}_m(0)|^2 + \frac{8}{\nu}\int_0^T |\mathbf{f}(s)|_{V_g'}^2 ds)$$

which implies that

(6.7) the sequence $\mathbf{u}_m$ remains in a bounded set of $L^\infty(0, T; H_g)$.

(ii) For the convenience, let us define

$$K(T) = e^{\alpha T}(|\mathbf{u}_m(0)|^2 + \frac{8}{\nu}\int_0^T |\mathbf{f}(s)|_{V_g'}^2 ds).$$

Now, we integrate (6.6) from 0 to T to get

$$\begin{aligned}
&|\mathbf{u}_m(T)|^2 + \nu\int_0^T \|\mathbf{u}_m(t)\|^2\ dt \\
&\quad \cdot \ |\mathbf{u}_{0m}|^2 + \frac{8}{\nu}\int_0^T \|\mathbf{f}(t)\|_{V'}^2\ dt + \alpha\int_0^T |\mathbf{u}_m(t)|^2 dt \\
&\quad \cdot \ |\mathbf{u}_0|^2 + \frac{8}{\nu}\int_0^T \|\mathbf{f}(t)\|_{V'}^2\ dt + \alpha\ K(T)\ T.
\end{aligned}$$

Therefore,

(6.8) the sequence $\mathbf{u}_m$ remains in a bounded set of $L^2(0, T; V_g)$.

(iii) Let $\tilde{\mathbf{u}}_m$ denote the function from $\mathbf{R}$ into V_g, which is equal to $\mathbf{u}_m$ on $[0, T]$ and to 0 on the complitement of this interval. The Fourier transform of $\tilde{\mathbf{u}}_m$ is denoted by $\hat{\mathbf{u}}_m$. Then, we want to show that there exist a positive constant c and γ such that

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma}|\hat{\mathbf{u}}_m(\tau)|^2\ d\tau \cdot \ c. \tag{6.9}$$

So, since the sequence $\mathbf{u}_m$ remains in a bounded set of $L^2(0,T;V_g)$,

(6.10) the sequence $\tilde{\mathbf{u}}_m$ remains in a bounded set of $\mathcal{H}^\gamma(\mathbf{R};V_g,H_g)$.

It is classical that since $\tilde{\mathbf{u}}_m$ has two discontinuities, at 0 and T, the distribution derivative of $\tilde{\mathbf{u}}_m$ is given by

$$\frac{d}{dt}\tilde{\mathbf{u}}_m = \tilde{\phi}_m + \mathbf{u}_m(0)\delta_0 - \mathbf{u}_m(T)\delta_T,$$

where δ_0 and δ_T are the Dirac distributions at 0 and T, and $\phi_m = \mathbf{u}_m'$ is the derivative of $\mathbf{u}_m$ on $[0,T]$. Therefore by (6.3), one obtains that

$$\frac{d}{dt}\langle\tilde{\mathbf{u}}_m,\mathbf{w}_j\rangle_g = \langle\tilde{\mathbf{f}}_m,\mathbf{w}_j\rangle_g + \langle\mathbf{u}_{0m},\mathbf{w}_j\rangle_g\delta_0 - \langle\mathbf{u}_m(T),\mathbf{w}_j\rangle_g\delta_T, \tag{6.11}$$

for $j = 1,...,m$, where δ_0, δ_T are Dirac distributions at 0 and T, $\mathbf{f}_m = \mathbf{f} - \nu A\mathbf{u}_m - B\mathbf{u}_m - C\mathbf{u}_m$, and $\tilde{\mathbf{f}}_m = \mathbf{f}_m$ on $[0,T]$, 0 outside this interval. By the Fourier transform, (6.11) gives

$$\begin{aligned} 2i\pi\tau\langle\hat{\mathbf{u}}_m,\mathbf{w}_j\rangle_g &= \langle\hat{\mathbf{f}}_m,\mathbf{w}_j\rangle_g + \langle\mathbf{u}_{0m},\mathbf{w}_j\rangle_g \\ &\quad -\langle\mathbf{u}_m(T),\mathbf{w}_j\rangle_g\exp(-2i\pi T\tau), \end{aligned} \tag{6.12}$$

$\hat{\mathbf{u}}_m$ and $\hat{\mathbf{f}}_m$ denoting the Fourier transforms of $\tilde{\mathbf{u}}_m$ and $\tilde{\mathbf{f}}_m$ respectively. We multiply (6.12) by $\hat{\phi}_{jm}(\tau)$(=Fourier transform of $\tilde{\phi}_{jm}$) and add the resulting equations for $j = 1,...,m$; we get:

$$\begin{aligned} 2i\pi\tau|\hat{\mathbf{u}}_m(\tau)|^2 &= \langle\hat{\mathbf{f}}_m(\tau),\hat{\mathbf{u}}_m(\tau)\rangle_g + \langle\mathbf{u}_{0m},\hat{\mathbf{u}}_m(\tau)\rangle_g \\ &\quad - \langle\mathbf{u}_m(T),\hat{\mathbf{u}}_m(\tau)\rangle_g\exp(-2i\pi T\tau). \end{aligned}$$

Because of inequality (3.2), (4.2), (4.3) and (4.4) one obtains

$$\int_0^T \|\mathbf{f}_m(t)\|_{V_g'}\,dt \cdot \int_0^T (\|\mathbf{f}(t)\|_{V_g'} + \nu\|\mathbf{u}_m(t)\| + c\|\nabla g\|_\infty\,\|\mathbf{u}_m\| + c\|\mathbf{u}_m(t)\|_{V_g}^2)\,dt.$$

Therefore, $\mathbf{f}_m(t)$ belong to a bounded set in the space $L^1(0,T;V_g')$. Hence,

$$\sup_{\tau\in R}\|\hat{\mathbf{f}}_m(\tau)\|_{V_g'} \cdot \ constant, \ \ \forall m.$$

So, by using

$$|\mathbf{u}_m(0)| \cdot \ K(T), \ \ |\mathbf{u}_m(T)| \cdot \ K(T),$$

we deduce from (6.12) that

$$|\tau||\hat{\mathbf{u}}_m(\tau)|^2 \cdot \ c_2\|\hat{\mathbf{u}}_m(\tau)\|_{V_g} + c_3|\hat{\mathbf{u}}_m(\tau)|$$

or

$$|\tau||\hat{\mathbf{u}}_m(\tau)|^2 \cdot \ \ c_4\|\hat{\mathbf{u}}_m(\tau)\|_{V_g}. \tag{6.13}$$

For γ fixed, $\gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \cdot \ \ c_5(\gamma)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \quad \forall \tau \in R.$$

Thus, by (6.13), we have

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{\mathbf{u}}_m(\tau)|^2\ d\tau \cdot \ \ & c_5(\gamma)\int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}\ |\hat{\mathbf{u}}_m(\tau)|^2\ d\tau \\ \cdot \ \ & c_6\int_{-\infty}^{+\infty} \frac{\|\hat{\mathbf{u}}_m(\tau)\|_{V_g}}{1+|\tau|^{1-2\gamma}}\ d\tau + c_7\int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(\tau)\|_{V_g}^2\ d\tau.\end{aligned}$$

Since $\mathbf{u}_m \in L^2(0,T;V_g)$, by the Parseval equality

$$\int_{-\infty}^{+\infty} \|\hat{\mathbf{u}}_m(\tau)\|_{V_g}^2\ d\tau \ < \ constant.$$

Also, by the Schwarz inequality and the Parseval equality, one obtains

$$\int_{-\infty}^{+\infty} \frac{\|\hat{\mathbf{u}}_m(\tau)\|_{V_g}}{1+|\tau|^{1-2\gamma}}\ d\tau \ \cdot \ \ \left(\int_{-\infty}^{+\infty} \frac{1}{(1+|\tau|^{1-2\gamma})^2}d\tau\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\mathbf{u}_m(t)\|_{V_g}^2 dt\right)^{\frac{1}{2}},$$

which is finite since $\gamma < \frac{1}{4}$. So, the proof of (6.9) is achieved and $\mathbf{u}_m \in \mathcal{H}^\gamma(\mathbf{R}; V_g, H_g)$.

Therefore, so far, we obtained that $\mathbf{u}_m$ remains in a bounded set of $L^\infty(0,T;H_g)$, $L^2(0,T;V_g)$ and $\mathcal{H}^\gamma(\mathbf{R}; V_g, H_g)$.

The estimates (6.7) and (6.8) enable us to assert the existence of an element $\mathbf{u} \in L^2(0,T;V_g) \cap L^\infty(0,T;H_g)$ and a sub-sequence $\mathbf{u}_{m'}$ such that

$$\mathbf{u}_{m'} \to \mathbf{u} \ \text{ in } L^2(0,T;V_g) \ \text{ weakly} \tag{6.14}$$

and

$$\mathbf{u}_{m'} \to \mathbf{u} \ \text{ in } L^\infty(0,T;H_g) \ \text{ weak-star} \tag{6.15}$$

as $m' \to \infty$. For any ball $\mathcal{Q}$ included in $\mathbf{R}^n$, the injection of $V_g(\mathcal{Q})$ into $H_g(\mathcal{Q})$ is compact and (6.10) shows that $\mathbf{u}_m|_{\mathcal{Q}}$ belong to a bounded set of $\mathcal{H}^\gamma(\mathbf{R}; V_g(\mathcal{Q}), H_g(\mathcal{Q}))$. Then, proposition 5.2 implies that

$$\mathbf{u}_{m'}|_{\mathcal{Q}} \to \mathbf{u}|_{\mathcal{Q}} \text{ in } L^2(0,T;H_g(\mathcal{Q})), \text{ strongly } \forall\ \mathcal{Q}.$$

Similarly, for any support $\mathcal{Q}_j$ of $\mathbf{w}_j$, we have

(6.16) $$\mathbf{u}_{m'}|_{\mathcal{Q}_j} \to \mathbf{u}|_{\mathcal{Q}_j} \text{ in } L^2(0,T;H_g(\mathcal{Q}_j)), \text{ strongly.}$$

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply (6.3) by $\psi(t)$, and then integrate by parts. This leads to the equation

$$\begin{aligned}
&-\int_0^T (\mathbf{u}_m(t), \psi'(t)\mathbf{w}_j)dt + \nu \int_0^T ((\mathbf{u}_m(t), \mathbf{w}_j\psi(t)))dt \\
&+\int_0^T b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{w}_j\psi(t))dt + \int_0^T b(\frac{\nabla g}{g}, \mathbf{u}_m(t), \mathbf{w}_j\psi(t))dt \\
&= (\mathbf{u}_{0m}, \mathbf{w}_j)\psi(0) + \int_0^T \langle \mathbf{f}(t), \mathbf{w}_j\psi(t)\rangle_g dt.
\end{aligned}$$

One should note that each term is same value when we replace $\mathbf{u}_m$ by $\mathbf{u}_m|_{\mathcal{Q}_j}$. Therefore, by passing to the limit with the sequence m', one obtains from (6.14), (6.15) and (6.16) that

(6.17) $$\begin{aligned}
&-\int_0^T (\mathbf{u}(t), \mathbf{v}\psi'(t))dt + \nu \int_0^T ((\mathbf{u}(t), \mathbf{v}\psi(t)))dt \\
&+\int_0^T b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}\psi(t))dt + \int_0^T b(\frac{\nabla g}{g}, \mathbf{u}(t), \mathbf{v}\psi(t))dt \\
&= (\mathbf{u}_0, \mathbf{v})\psi(0) + \int_0^T \langle \mathbf{f}(t), \mathbf{v}\psi(t)\rangle_g dt.
\end{aligned}$$

Also, we note that the limit holds for $\mathbf{v} = \mathbf{w}_1, \mathbf{w}_2, \cdots$; by linearity this equation holds for $\mathbf{v}$ = any finite linear combination of the $\mathbf{w}_j$, and by a continuity argument (6.17) is still true for any $\mathbf{v} \in V_g$. Now writing, in particular, (6.17) with $\psi = \phi \in \mathcal{D}((0,T))$, we see that $\mathbf{u}$ satisfies (6.1) in the distribution sense.

Finally, it remains to prove that $\mathbf{u}$ satisfies (6.2). For this we multiply (6.1) by ψ, and integrate. After integrating the first term by parts, we get

$$\begin{aligned}
&-\int_0^T (\mathbf{u}(t), \mathbf{v}\psi'(t))dt + \nu \int_0^T ((\mathbf{u}(t), \mathbf{v}\psi(t)))dt \\
&+\int_0^T b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}\psi(t))dt + \int_0^T b(\frac{\nabla g}{g}, \mathbf{u}(t), \mathbf{v}\psi(t))dt \\
&= (\mathbf{u}(0), \mathbf{v})\psi(0) + \int_0^T \langle \mathbf{f}(t), \mathbf{v}\psi(t)\rangle_g dt.
\end{aligned}$$

By comparison with (6.17),

$$(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v})\psi(0) = 0.$$

We can choose ψ with $\psi(0) = 1$; thus

$$(\mathbf{u}(0) - \mathbf{u}_0, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in V_g,$$

which implies (6.2). The proof of continuity comes from usual continuity lemma. ■

7. Uniqueness of Solutions of Problems 2

Lemma 7.1. *If $n = 2$, then we have*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \cdot \ c|\mathbf{u}|^{\frac{1}{2}} \ \|\mathbf{u}\|^{\frac{1}{2}} \ \|\mathbf{v}\| \ |\mathbf{w}|^{\frac{1}{2}} \ \|\mathbf{w}\|^{\frac{1}{2}}, \tag{7.1}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\mathbf{R}^n)$. Also, if $\mathbf{u}$ belong to $L^2(0,T;V) \cap L^\infty(0,T;V_g)$, then $B\mathbf{u}$ belong to $L^2(0,T;V_g')$ and

$$\|B\mathbf{u}\|_{L^2(0,T;V')} \cdot \ 2^{\frac{1}{2}}|\mathbf{u}|_{L^\infty(0,T;H)}\|\mathbf{u}\|_{L^2(0,T;V)}. \tag{7.2}$$

If $n = 3$, we have

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \cdot \ c\|\mathbf{u}\|_{L^4(\mathbf{R}^3)} \ \|\mathbf{u}\| \ \|\mathbf{v}\|_{L^4(\mathbf{R}^3)} \tag{7.3}$$

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| \cdot \ c|\mathbf{u}|^{\frac{1}{4}} \ \|\mathbf{u}\|^{\frac{7}{4}} \ \|\mathbf{v}\|_{L^4(\mathbf{R}^3)}. \tag{7.4}$$

Proof. (7.1) and (7.3) come from (2.1) and (4.1), respectively. And (7.2) is from (7.1). ■

Teorem 7.2. *If $n = 2$ then the solution of problem 2 given by theorem 1 is unique.*

Proof. Let us assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of problem 2, and let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then, we have

$$\mathbf{u}' + \nu A\mathbf{u} + C\mathbf{u} = -B\mathbf{u}_1 + B\mathbf{u}_2,$$
$$\mathbf{u}(0) = 0.$$

We take a.e. in t the scalar product of (7.1) with $\mathbf{u}(t)$ in the duality between V_g and V_g'. Then one obtains

$$\begin{aligned} \frac{d}{dt}|\mathbf{u}(t)|^2 \quad &+2\nu\|\mathbf{u}(t)\|^2 + 2b(\tfrac{\nabla g}{g}, \mathbf{u}(t), \mathbf{u}(t)) \\ &= 2b(\mathbf{u}_2(t), \mathbf{u}_2(t), \mathbf{u}(t)) - 2b(\mathbf{u}_1(t), \mathbf{u}_1(t), \mathbf{u}(t)) \\ &= -2b(\mathbf{u}(t), \mathbf{u}_2(t), \mathbf{u})) \end{aligned} \tag{7.5}$$

Also, by (7.1), we have

$$\begin{aligned} |2b(\mathbf{u}(t), \mathbf{u}_2(t), \mathbf{u}))| \ &\cdot \ c|\mathbf{u}(t)| \ \|\mathbf{u}(t)\| \ \|\mathbf{u}_2(t)\| \\ &\cdot \ \nu\|\mathbf{u}(t)\|^2 + \tfrac{c^2}{\nu}|\mathbf{u}(t)|^2\|\mathbf{u}_2(t)\|^2, \end{aligned}$$

and

$$
\begin{aligned}
|2b(\tfrac{\nabla g}{g}, \mathbf{u}(t), \mathbf{u}(t))| &\cdot \; 2c\|\nabla g\|_\infty \|\mathbf{u}\| \; |\mathbf{u}| \\
&\cdot \; \nu\|\mathbf{u}(t)\|^2 + \tfrac{c^2}{\nu}\|\nabla g\|_\infty^2 |\mathbf{u}(t)|^2.
\end{aligned} \tag{7.6}
$$

Therefore, we have

$$\frac{d}{dt}|\mathbf{u}(t)|^2 \cdot \left(\frac{c^2}{\nu}\|\mathbf{u}_2(t)\|^2 + \frac{c^2}{\nu}\|\nabla g\|_\infty^2\right)|\mathbf{u}(t)|^2.$$

So, one has

$$\frac{d}{dt}\Box Exp\left(\int_0^t (\frac{8}{\nu}\|\mathbf{u}_2(t)\|^2 + \frac{c^2}{\nu}\|\nabla g\|_\infty^2)ds\right) \cdot |\mathbf{u}(t)|^2\Big] \cdot \; 0.$$

Hence, we get

$$|\mathbf{u}(t)|^2 \cdot \; 0, \quad \forall\, t \in [0, T].$$

Thus, $\mathbf{u}_1 = \mathbf{u}_2$.

For the case $n = 3$, we have different theory.

Theorem 7.3. *If $n = 3$, then there is at most one solution of problem 2 such that*

$$\mathbf{u} \in L^2(0, T; V_g) \cap L^\infty(0, T; H_g), \tag{7.7}$$

$$\mathbf{u} \in L^8(0, T; L^4(\mathbf{R}^3)). \tag{7.8}$$

Proof. Let us assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of problem 2 which satisfies (7.7) and (7.8) and let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then, by (7.5)

$$
\begin{aligned}
&\frac{d}{dt}|\mathbf{u}(t)|^2 + 2\nu\|\mathbf{u}(t)\|^2 + 2b(\frac{\nabla g}{g}, \mathbf{u}(t), \mathbf{u}(t)) \\
&= -2b(\mathbf{u}(t), \mathbf{u}_2(t), \mathbf{u})) = 2b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{u}_2(t)).
\end{aligned} \tag{7.9}
$$

Now, we have from (7.4) and young inequality ($p = \frac{8}{7}, q = 8, \epsilon = \nu$) that

$$|2b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{u}_2(t))| \cdot \; \nu\|\mathbf{u}(t)\|^2 + \frac{c^8}{\nu^7}|\mathbf{u}(t)|^2 \; \|\mathbf{u}_2(t)\|_{L^4(\mathbf{R}^3)}^8. \tag{7.10}$$

So, by (7.6), (7.9) and (7.10), one obtains that

$$\frac{d}{dt}|\mathbf{u}(t)|^2 \cdot \left(\frac{c^8}{\nu^7}\;\|\mathbf{u}_2(t)\|_{L^4(\mathbf{R}^3)}^8 + \frac{c^2}{\nu}\|\nabla g\|_\infty^2\right)|\mathbf{u}(t)|^2.$$

So, one has

$$\frac{d}{dt}\left[Exp\left(\int_0^t \left(\frac{c^8}{\nu^7}|\mathbf{u}(t)|^2 \ \|\mathbf{u}_2(t)\|^8_{L^4(\mathbf{R}^3)} + \frac{c^2}{\nu}\|\nabla g\|^2_\infty\right)ds\right)\cdot|\mathbf{u}(t)|^2\right] \cdot \ 0.$$

Hence, we get

$$|\mathbf{u}(t)|^2 \cdot \ 0, \quad \forall\, t \in [0,T].$$

Thus, $\mathbf{u}_1 = \mathbf{u}_2$.

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