TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 5, pp. 1157-1167, September 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

PARTIAL LATIN SQUARES AND THEIR GENERALIZED QUOTIENTS

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Abstract. A (partial) Latin square is a table of multiplication of a (partial) quasigroup. Multiplication of a (partial) quasigroup may be considered as a set of triples. We give a necessary and sufficient condition for a set of triples to be a quotient of a (partial) latin square.

1. Introduction

Generalized quotient of quasigroup is a quotient with respect to an equivalence relation which is not a congruence. Such a quotient is neither a quasigroup nor an algebraic system. It may be thought as a multivalued algebraical system or a set of triples. The authors of [3, 4] found useful the notion of generalized quotient in their investigations of approximations of algebraic systems. The aim of the work is to study in more details combinatorial structures using in [3, 4]. We also formulate some open questions (Conjectures 1, 2, 3). The positive answer on Conjecture 1 will essentially simplify the measure-theoretical part of the proof of the main theorem in [4]. The construction corresponding to generalized quotient is not new. It is known in combinatoric by the name “amalgamation” (see 2, 5, 7) but we prefer here more algebraic terms.

Theorem 1 (A. J. W. Hilton) gives necessary and sufficient conditions for a set of triples to be a generalized quotient of a quasigroup. Here we extend it to generalized quotients of partial quasigroups. Investigation of generalized quotients leads to a more general objects – 3-indexed matrices.

The article is organized as follows. In Section 2 we formulate the main results (Theorem 2 and Theorem 3) about 3-indexed matrices. In Section 3 we define generalized quotient partial quasigroups (GQPQ) and generalized uniformly quotient partial quasigroups (GUQPQ), interpret results about matrices on this language, and

Received September 1, 2004; accepted September 23, 2004.
Communicated by V. Afraimovich.
2000 *Mathematics Subject Classification*: 05B15; 05D15; 05B20; 05C65.
Key words and phrases: Latin squares, Quotients, Amalgamation, Quasigroups.

discuss some connection with the theory of hypergraphs. We give an example of a GQPQ which is not a GUQPQ and formulate some conjectures. Section 4 is devoted to the proof of Theorem 2.

2. Formulation of the Main Results

We shall deal with 2- and 3-indexed matrices. For positive integers $n_1, \dots, n_k$, a k-indexed $n_1 \times \cdots \times n_k$-matrix M is a function $M : (n_1) \times \cdots \times (n_k) \to \mathbb{R}$, where $(n) = \{1, \dots, n\}$. Through this text we shall use the notation $M(X) = \sum_{x \in X} M(x)$, where $X \subseteq (n_1) \times \cdots \times (n_k)$.

We shall denote by $T(n_1, \dots, n_k)$ the set of all k-indexed $n_1 \times \cdots \times n_k$-matrices with entries being nonnegative integers.

We call an $n_1 \times \cdots \times n_k$-line any set $l \subset (n_1) \times \cdots \times (n_k)$ such that in all k-tuples in l, $k-1$ indexes are fixed and the other, say the i-th index, runs over all (n_i). A line of a $n_1 \times \cdots \times n_k$-matrix is the restriction of this matrix on a $n_1 \times \cdots \times n_k$-line. If l is a line of M, $M(l)$ is its line sum.

For $n_1 \times n_2$-lines we shall use the following names and notations.

$$l_a^1 = \{(x, a) : x \in (n_1)\} \text{ (column)}$$

$$l_a^2 = \{(a, x) : x \in (n_2)\} \text{ (row)}$$

Similarly, for $n_1 \times n_2 \times n_3$-lines we have

$$l_{ab}^1 = \{(x, a, b) : x \in (n_1)\} \text{ (horizontal line)}$$

$$l_{ab}^2 = \{(a, x, b) : x \in (n_2)\} \text{ (transversal line)}$$

$$l_{ab}^3 = \{(a, b, x) : x \in (n_3)\} \text{ (vertical line)}$$

For a function $f : (n) \to (n)$, *the graph of* f is the matrix $\Gamma \in T(n, n)$ such that

$$\Gamma(i, j) = \begin{cases} 1, & \text{if} \quad f(i) = j, \\ 0, & \text{if} \quad f(i) \neq j. \end{cases}$$

It is easy to see that the following proposition holds.

Proposition 1. *A matrix $M \in T(n, n)$ is the graph of a permutation if and only if every line sum of M equals 1. We shall call such a matrix to be a permutation matrix.*

An analogue of this proposition for 3-indexed matrices leads to quasigroups and Latin squares.

Definition 1. A quasigroup $(Q, \star)$ is an algebraic system Q with a binary operation $\star$ such that

i) equation $x \star a = b$ has a unique solution with respect to x for all $a, b \in Q$,

ii) equation $a \star x = b$ has a unique solution with respect to x for all $a, b \in Q$.

This definition implies immediately the following

Proposition 2. *A matrix $M \in T(n, n, n)$ is the graph of a quasigroup operation $\star$ on (n) ($M(i, j, k) = 1$, if $i \star j = k$ and $M(i, j, k) = 0$ otherwise) if and only if every line sum of M equals 1. We shall call such a matrix to be a Latin square.*

For 2-indexed matrices the following lemma is well-known (e.g.[6]).

Lemma 1. *Let $M \in T(n, n)$. Let each line sum of M equals k, where $k > 0$. Then*

$$\text{supp}(M) \supseteq \text{supp}(P)$$

for a permutation matrix P.

This Lemma easily implies

Lemma 2. *Let $M \in T(n, n)$. Let each line sum of M equals k, where $k > 0$. Then*

$$M = P_1 + P_2 + \cdots + P_k,$$

where each P_i is a permutation matrix.

One may formulate the following "generalization" of Lemma 1

(A) *Let $M \in T(n, n, n)$. Let each line sum of M equals k, with $k > 0$. Then*

$$\text{supp}(M) \supseteq \text{supp}(L),$$

for a Latin square L.

Statement (A) is not true. Indeed, consider the $3 \times 3 \times 3$ matrix M, see Fig. 1. Every line sum of M equals 2. The existence of the odd cycle $\mathcal{C}$ in M (marked bold) implies that supp$(M) \not\supseteq$ supp(L) for any Latin square L. Indeed, let supp$(M) \supseteq$ supp(L) for some Latin square L. Then supp(L) has to contain only one dot marked of every line of $\mathcal{C}$. But this is impossible because $\mathcal{C}$ is an odd cycle.

Remark. A set of triples, as a hypergraph may not have a (dual) König property. On the contrary, any set of pairs is a balanced hypergraph and satisfies a (dual) König property, i.e. $\rho = \bar{\alpha}$; see Section 3 and [1]. Nevertheless, there is some connection of matrices described in statement (A) with Latin squares through quotients.

M =

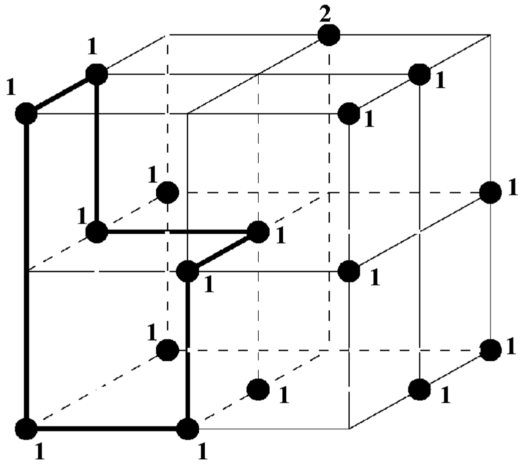

Fig. 1.

Let $M : (n_1) \times \cdots \times (n_k) \to \mathbb{R}$ be a k-indexed matrix and $\sigma = \{P_1, \ldots, P_r\}$ be an (ordered) partition of (n_i). We define the quotient matrix

$$M \circ_i \sigma : (n_1) \times \cdots \times (n_{i-1}) \times (r) \times (n_{i+1}) \times \cdots \times (n_k) \to \mathbb{R},$$

by the formula

$$M \circ_i \sigma(x_1, \ldots, x_i, \ldots, x_k) = M(\{x_1\} \times \cdots \times P_{x_i} \times \cdots \times \{x_k\}).$$

Example. Let

$$M = \begin{pmatrix} 0 & 3 & 3 & 1 \\ 5 & 2 & 4 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & 5 & 0 \end{pmatrix}$$

and $\sigma = \{\{1,2\},\{3,4\}\}$. Then

$$M \circ_1 \sigma = \begin{pmatrix} 5 & 5 & 7 & 1 \\ 3 & 4 & 5 & 1 \end{pmatrix}, \quad M \circ_2 \sigma = \begin{pmatrix} 3 & 4 \\ 7 & 4 \\ 2 & 1 \\ 5 & 5 \end{pmatrix}$$

and

$$(M \circ_1 \sigma) \circ_2 \sigma = (M \circ_2 \sigma) \circ_1 \sigma = \begin{pmatrix} 10 & 8 \\ 7 & 6 \end{pmatrix}.$$

Let $L \in T(n,n,n)$ be a Latin square, $\sigma = \{P_1, P_2, \ldots, P_k\}$ be a partition of (n) and $M = ((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma \in T(k,k,k)$. Then it is easy to check that

$$M(l_{ij}^t) = |P_i| \cdot |P_j|,$$

for every $k \times k \times k$-line l_{ij}^t, $t = 1, 2, 3$ and $i, j \in (k)$. It was proved by Hilton [5, 2] that the inverse statement is also true.

Theorem 1. [A. J. W. Hilton] *Let $M \in T(k, k, k)$ and $r_1, r_2, \ldots, r_k$ be positive integers such that*

$$M(l_{ij}^t) = r_i r_j$$

for $t = 1, 2, 3$ and $i, j \in (k)$. Then $M = ((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma$ for a Latin square $L \in T(n, n, n)$ and a partition $\sigma = \{P_1, \ldots, P_k\}$ of (n) such that $|P_i| = r_i$ and $n = \sum_i r_i$.

In the paper we generalize this theorem for partial Latin squares (and partial quasigroups).

Definition 2. Let Q be a finite set and $S \subseteq Q \times Q$. A partial S-quasigroup on Q is a partial binary operation $\star$ on Q such that

i) $S \subseteq Dom(\star)$.

ii) equation $x \star a = b$ has at most one solution with respect to x for all $a, b \in Q$.

iii) equation $a \star x = b$ has at most one solution with respect to x for all $a, b \in Q$.

It is known that any partial quasigroup Q can be extended to a quasigroup $Q' \supset Q$, $|Q'| \leqslant 2|Q|$.

Proposition 3. *A matrix $M \in T(n, n, n)$ is the graph of a partial S-quasigroup operation on (n) if and only if every line sum of M is no more than 1 and*

$$M(l_{ij}^3) = 1$$

for every $(i, j) \in S$. We shall call such a matrix to be a partial S-Latin square.

Let $L \in T(n, n, n)$ be a partial S-Latin square, $\sigma = \{P_1, \ldots, P_k\}$ be a partition of (n), and $M = ((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma \in T(k, k, k)$. Then it is easy to verify that

$$M(l_{ij}^t) \leqslant |P_i| \cdot |P_j|$$

for every $k \times k \times k$-line l_{ij}^t, $t = 1, 2, 3$, $i, j \in (k)$ and

$$M(l_{ij}^3) = |P_i| \cdot |P_j|,$$

if $P_i \times P_j \subseteq S$.

Theorem 2. *Let $M \in T(k, k, k)$, $S \subseteq (k) \times (k)$ and $r_1, \ldots, r_k$ be positive integers such that*

$$M(l_{ij}^t) \leqslant r_i r_j$$

for $t = 1, 2, 3$*, and*

$$M(l_{ij}^3) = r_i r_j,$$

for $(i, j) \in S$.
Then $M = ((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma$ *for a partial* S'*-Latin square* $L \in T(n, n, n)$*, a partition* $\sigma = \{P_1, \ldots, P_k\}$ *of* (n) *such that* $|P_i| = r_i$*,* $n = \sum_i r_i$*, and*

$$S' = \bigcup_{(i,j)\in S} P_i \times P_j.$$

If we substitute $S = (k) \times (k)$ in Theorem 2 we get Theorem 1. (If for all vertical lines one has equalities then one has equalities for all lines.) The uniform partial case of Theorem 2, where $r_1 = \cdots = r_k = r$, may be generalized for real-valued matrices.

Theorem 3. *Let* $M : (k) \times (k) \times (k) \to \mathbb{R}^+$*,* $\beta \in \mathbb{R}^+$ *and* $S \subseteq (k) \times (k)$ *such that*

$$M(l_{ij}^t) \leqslant \beta$$

for $t = 1, 2, 3$ *and* $i, j \in (k)$*, and*

$$M(l_{ij}^3) = \beta,$$

for $(i, j) \in S$.
Then $\mathrm{supp}(M) = \mathrm{supp}(((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma)$ *for a partial* S'*-Latin square* $L \in T(n, n, n)$*, a partition* $\sigma = \{P_1, \ldots, P_k\}$ *of* (n) *such that* $|P_i| = |P_j|$ *for every* $i \neq j$*, and*

$$S' = \bigcup_{(i,j)\in S} P_i \times P_j.$$

Proof. Let M and β satisfy the conditions of the theorem. We can write equalities and strict inequalities separately. Consider non-zero elements of M and β as variables. Then this system of equalities and (strict) inequalities has a rational solution. Multiplying this solution by a proper integer, we construct a matrix $M' \in T(k, k, k)$, $\mathrm{supp}(M') = \mathrm{supp}(M)$, satisfying the conditions of Theorem 2. ∎

As we see, the crucial step in the proof of Theorem 3 is to show the existence of a rational solution. For the non-uniform case these equations will be nonlinear (quadratic). So, general consideration cannot prove that the existence of a real solution implies the existence of a rational one. We don't know so far if a non-uniform version of Theorem 3 is valid:

Conjecture 1. Let $M : (k) \times (k) \times (k) \to \mathbb{R}^+$, $\beta_1, ..., \beta_k \in \mathbb{R}^+$ and $S \subseteq (k) \times (k)$ such that

$$M(l_{ij}^t) \leqslant \beta_i \beta_j$$

for $t = 1, 2, 3$ and $i, j \in (k)$, and

$$M(l_{ij}^3) = \beta_i \beta_j,$$

for $(i, j) \in S$.
Then $\operatorname{supp}(M) = \operatorname{supp}(((L \circ_1 \sigma) \circ_2 \sigma) \circ_3 \sigma)$ for a partial S'-Latin square $L \in T(n, n, n)$, a partition $\sigma = \{P_1, \ldots, P_k\}$ of (n) such that

$$S' = \bigcup_{(i,j) \in S} P_i \times P_j.$$

In fact, we don't know if it is true for $S = (k) \times (k)$.

3. Generalized Quotient Quasigroup

Let Q be a finite set and σ an equivalence relation on Q which we shall identify with the partition of Q by equivalence classes. So, $\sigma = \{Q_1, Q_2, ..., Q_k\}$. Let $X \subseteq Q^r$. Define weak ($X/^w\sigma \subseteq \{1, 2, ...k\}^r$) and strong ($X/^s\sigma \subseteq \{1, 2, ...k\}^r$) quotient of X:

$$X/^w\sigma = \{< i_1, i_2, ..., i_k >: \ Q_{i_1} \times Q_{i_2} \times \cdots \times Q_{i_k} \cap X \neq \emptyset\}$$

$$X/^s\sigma = \{< i_1, i_2, ..., i_k >: \ Q_{i_1} \times Q_{i_2} \times \cdots \times Q_{i_k} \subseteq X\}$$

For example, if $\star \subseteq Q^3$ is a quasigroup operation on Q and σ – a congruence relation (i.e. it preserves the operation $\star$) then $\star/^w\sigma = \star/^s\sigma = \star/\sigma$ is a quotient quasigroup operation.

Definition 3. Let $\sigma = \{Q_1, Q_2, ..., Q_k\}$ be an equivalence relation on Q. We shall call σ to be uniform iff all Q_i have the same cardinality.

Let $(Q, \star)$ be a quasigroup. A set $\star/^w\sigma$ will be called a generalized quotient quasigroup (GQQ). For uniform σ a set $\star/^w\sigma$ will be called a generalized uniformly quotient quasigroup (GUQQ).

Let $(Q, \star)$ be an S-quasigroup ($S \subseteq Q^2$) and σ be an equivalence relation on Q (not necessarily a congruence). A set $\star/^w\sigma$ will be called a $S/^s\sigma$-generalized quotient partial quasigroup ($S/^s\sigma$-GQPQ) or a $S/^s\sigma$-generalized uniform quotient partial quasigroup ($S/^s\sigma$-GQUPQ) in the case of uniform σ.

Theorem 2 and Theorem 3 have the following obvious interpretation:
$H \subseteq (k)^3$ *is a S-GQPQ (S-GUQPQ) if and only if there exists a matrix M,* $\operatorname{supp}(M) = H$, *satisfying conditions of Theorem 2* (*Theorem* 3).

Now we give an interpretation of our results on the language of hypergraphs. A set of triples $H \subseteq (k)^3$ has a natural structure of a hypergraph if we consider lines as edges. Precisely, with H we associate hypergraph (V, E) with $V = H$ and $E = \{l \cap H \ : \ l - \text{line}\}$. Several useful numeric characteristic of hypergraphs are known. We are interested in 3 of them: the covering number ρ, the independent number $\bar{\alpha}$ and the fractional independent number α^*, for the general definition; see [1]. For the case of a set of triples $H \subseteq (k)^3$ these numbers have the following meaning:
$\rho(H)$ is the minimum number of lines, covering H, i.e. $\rho(H) = \min\{|R| \ : \ R \text{ is a set of lines, and } H \subseteq \cup R\}$;
$\bar{\alpha}(H) = \max\{|X| \ : \ X \subseteq H \text{ and } |X \cap l| \leqslant 1 \text{ for every line } l\}$
$\alpha^*(H) = \max\{M(H) \ : \ M : (k) \times (k) \times (k) \to \mathbb{R}^+, \ \text{supp}(M) \subseteq H \text{ and } M(l) \leqslant 1 \text{ for any line } l\}$.
From the theory of hypergraphs (see [1]), it follows that $\bar{\alpha}(H) \leqslant \alpha^*(H) \leqslant \rho(H)$. One immediately verifies the following

Proposition 4. *$H \subseteq (k)^3$ contains an S-quasigroup if and only if $\bar{\alpha}(H \cap (S \times (k))) = |S|$.*

Without loss of generality one can put $\beta = 1$ in Theorem 3. This implies

Proposition 5. *$H \subseteq (k)^3$ contains an S-GUQPQ if and only if $\alpha^*(H \cap (S \times (k))) = |S|$. Matrix M on Fig. 2 is an example of a GQQ which does not contain a GUQQ.*

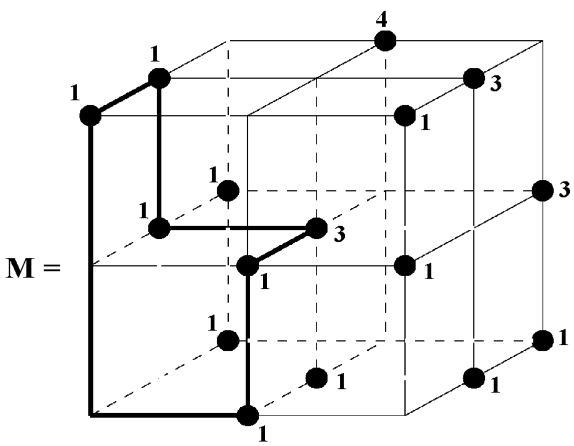

Fig. 2.

The numbers show that the set marked by black dots is a GQQ if we put $r_1 = 1$, $r_2 = 2$ and $r_3 = 2$ in Theorem 1. On the other hand if the set were a GUQQ one could put numbers, such that sums along every line are the same. To see that it is impossible one can try to put numbers along the odd cycle $\mathcal{C}$ (marked bold).

We wonder if the following conjectures are true.

Conjecture 2. If $H \subseteq (k)^3$ contains an S-GQPQ then $\rho(H \cap S \times (k)) = |S|$.

Conjecture 3. If $\rho(H \cap S \times (k)) = |S|$ then $H \subseteq (k)^3$ contains an S-GQPQ.

4. Proof of Theorem 2

We shall need the following proposition which is a reformulation of De Werra's theorem on balanced edge-coloring of a finite bipartite graph.

Proposition 6. *Let* $M \in T(n,m)$ *and* $k \in \mathbb{N}$. *Then* $M = M_1 + M_2 + \cdots + M_k$, *such that*

- $M_i \in T(n,m)$,
- $\forall i,j,k,r \ |M_i(k,r) - M_j(k,r)| \leqslant 1$,
- $\forall i,j,k \ |\sum_{r=1}^m M_i(k,r) - \sum_{r=1}^m M_j(k,r)| \leqslant 1$, *and* $|\sum\limits_{r=1}^{n} M_i(r,k) - \sum\limits_{r=1}^{n} M_j(r,k)| \leqslant 1$.

Proof. A proof of De Werra's theorem may be found in [2]. To obtain our reformulation one may associate a matrix M to a bipartite graph: $\{1,2,\ldots,n\}$ – the vertexes of one part, $\{1,2,\ldots,m\}$ – the vertexes of the other part, $M(ij)$ is the number of edges from i to j. ■

Let $M \in T(m,n)$. Let the sum of row i of M be denoted by r_i and let the sum of column j of M be denoted by s_j. We call the vector

$$R = (r_1, \ldots, r_m)$$

the *row sum vector* and the vector

$$S = (s_1, \ldots, s_n)$$

the *column sum vector* of M.
The vectors R and S determine the class

$$\mathcal{C}(R,S)$$

consisting of all matrices of size m by n, whose entries are nonnegative integers, with row sum vector R and column sum vector S.

Let $R = (r_1, r_2, \cdots, r_m)$, $S = (s_1, s_2, \cdots s_n)$, $I \subseteq (n)$. Denote by $\mathcal{C}'_I(R, S)$ the union of all $\mathcal{C}(R', S')$ such that $R' \leqslant R$, $S' \leqslant S$ and $s_i = s'_i$ for $i \in I$.

Lemma 3. *Let $M \in \mathcal{C}'_I(kR, kS)$ such that $|R| = |S|$. Then*

$$M = Q'_1 + Q'_2 + \cdots + Q'_k,$$

where $Q'_i \in \mathcal{C}'_I(R, S)$ for every $i = 1, 2, \ldots, k$.

Proof. It follows immediately from Proposition 6. ∎

Let $M \in T(k, k, k)$, $S \subseteq (k) \times (k)$ and $r_1, \ldots, r_k$ be positive integers such that $M(l^t_{ij}) \leqslant r_i r_j$ for $t = 1, 2, 3$, and $M(l^3_{ij}) = r_i r_j$ for $(i, j) \in S$. Take $n = r_1 + r_2 + \cdots + r_k$ and a partition $\sigma = \{P_1, P_2, \ldots, P_k\}$ of (n) such that $|P_i| = r_i$. We shall consecuently construct $M_1 \in T(k, k, n), M_2 \in T(k, n, n)$ and $M_3 \in T(n, n, n)$ such that

i) $M = M_1 \circ_3 \sigma,\ M_1 = M_2 \circ_2 \sigma,\ M_2 = M_3 \circ_1 \sigma$;

ii) $M_1(l^3_{ij}) = r_i r_j$ if $(i, j) \in S$, $M_1(l^3_{ij}) \leqslant r_i r_j$, and $M_1(l^t_{ij}) \leqslant r_i$ for $t = 1, 2$;

iii) $M_2(l^3_{ij}) = r_i$ for $(i, j) \in \bigcup_{(i,k) \in S} \{i\} \times P_k$, $M_2(l^3_{ij}) \leqslant r_i$, $M_2(l^2_{ij}) \leqslant r_i$, and $M_2(l^1_{ij}) \leqslant 1$;

iv) $M_3(l^t_{ij}) \leqslant 1$ for $t = 1, 2, 3$, and $M_3(l^3_{ij}) = 1$ if $(i, j) \in \bigcup_{(i,j) \in S} P_i \times P_j$.

Construction of M_1.

For every c fixed, $M'_c = M(\cdot, \cdot, c) \in T(k, k)$ such that $M'_c(l^t_i) \leqslant r_i r_c$ for $i \in (k)$ and $t = 1, 2$. So, by Lemma 3 we can write

$$M(\cdot, \cdot, c) = Q_{\alpha_1} + \cdots + Q_{\alpha_{r_c}},$$

such that $Q_{\alpha_m} \in T(k, k)$ and $Q_{\alpha_m}(l^t_i) \leqslant r_i$ for $i \in (k)$ and $t = 1, 2$. One can choose α_i such that $\{\alpha_1, \alpha_2, \ldots, \alpha_{r_c}\} = P_c$. Doing the same for all c, we shall have matrices $Q_1, \ldots, Q_n \in T(k, k)$. Let $M_1(a, b, c) = Q_c(a, b)$.
Construction of M_2.

For every c fixed, $M'_c = M_1(\cdot, c, \cdot) \in T(k, n)$ such that $M'_c(l^1_i) \leqslant r_i r_c$, $M'_c(l^2_i) \leqslant r_c$, and $M'_c(l^1_i) = r_i r_c$ for $(i, c) \in S$. By Lemma 3, we can write

$$M_1(\cdot, c, \cdot) = Q_{\alpha_1} + \cdots + Q_{\alpha_{r_c}},$$

such that $Q_{\alpha_m} \in T(k,n)$, $Q_{\alpha_m}(l_i^2) \leqslant 1$, $Q_{\alpha_m}(l_i^1) \leqslant r_i$, and $Q_{\alpha_m}(l_i^1) = r_i$ if $(i,c) \in S$.

One can choose α_i such that $\{\alpha_1, \alpha_2, \ldots, \alpha_{r_c}\} = P_c$. Doing the same for all c, we shall have matrices $Q_1, \ldots, Q_n \in T(k,n)$. Let $M_2(a,c,b) = Q_c(a,b)$.

Construction of M_3 is similar. It is clear that $L = M_3$ satisfies the theorem.

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