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COMPACTNESS AND NORM CONTINUITY OF THE DIFFERENCE OF TWO COSINE FUNCTIONS

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Abstract. We show that the compactness of the difference of two cosine functions can be characterized by that of the difference of their resolvents. We also make some comparisons between such results for C_0-semigroups and cosine functions.

1. Introduction

Many linear distributed parameter control systems can be put into the form

$$v'(t) = Av(t) + Bu(t),\ v(0) = v_0,\ t \in \mathbb{R}_+, \tag{1.1}$$

where A generates a C_0-semigroup in the state Hilbert or Banach space E and B is the control operator from control space to the state space. When one design a feedback control $u(t) = Fv(t)$ for some feedback operator from the state space to the control space, the closed-loop system takes the form

$$v'(t) = (A + BF)v(t),\ v(0) = v_0,\ t \in \mathbb{R}_+. \tag{1.2}$$

In the context of stabilization theory, one wants to select a feedback operator F to force the closed-loop system to possess stability properties that are not enjoyed by the original system. One important class in physical applications is that of operators F such that BF is compact in the state space. When BF is compact, it was first proved in [16] that the difference of the semigroup $e^{(A+BF)t}$ generated by $A + BF$ and e^{At} generated by A is compact for any positive t. Hence

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$$\omega_{ess}(A) = \omega_{ess}(A + BF), \tag{1.3}$$

where ω_{ess} stands for the essential growth rate of the associated semigroup. Property (1.3) holds true for any two C_0-semigroups whenever their difference is compact for some $t > 0$ (see Theorem 3.52 in [7]). This is the basis of "compact method" that was used to study the stabilization of elastic systems (see [10]) and spectral property of neutron transport equation (see [14]). Compact method was first formulated in [13] for Hilbert spaces and later generalized to Banach spaces in [5], which says that the compact perturbation can not make the system which is asymptotically but not exponentially stable to be exponentially stable.

This gives rise to the general study of necessary and sufficient conditions for compactness of the difference of two C_0-semigroups. A recent result in [6] says that $e^{At} - e^{Bt}$ is compact for $t > 0$ if and only if $R(\lambda, A) - R(\lambda, B)$ is compact under the norm continuity assumption on $e^{At} - e^{Bt}$.

On the other hand, most of controlled hyperbolic systems are more convenient to be written as the second order instead of the first order evolution equation in abstract space (see [11], [4]):

$$v''(t) = Av(t) + Bu(t),\ v(0) = v_0,\ v'(0) = v_1,\ t \in \mathbb{R}_+, \tag{1.4}$$

Same problem also occurs for system (1.4). Certainly, one can transfer system (1.4) into the first order equation (1.1); however, the difficulty with this approach is that the transformed first order equation does not necessarily produce a C_0-semigroup (see [12]) on $E \times E$ space. It turns out that it is sometimes more convenient to treat second order system directly by introducing the cosine function, which plays the same role to the second order system as does C_0-semigroup to the first order system (see [4]). By definition, a family of bounded linear operators $\{C(t), t \in \mathbb{R}\}$ in a Banach space E is called a cosine function if

(a) $C(0) = I$;

(b) $C(t + s) + C(t - s) = 2C(t)C(s)$ for $t, s \in \mathbb{R}$;

(c) the function $C(\cdot)x$ is continuous on $\mathbb{R}$ for any $x \in E$.

The infinitesimal generator of cosine function $C(\cdot)$ could be defined as $Ax = \lim_{t\to 0} 2(C(t) - I)x/t^2$ for such x for which the limit exists.

In terms of a cosine function $C(t)$ generated by A, the solution of the second order equation

$$v''(t) = Av(t),\ v(0) = v_0,\ v'(0) = v_1,\ t \in \mathbb{R}, \tag{1.5}$$

can be written as

$$(1.6)\qquad \begin{pmatrix} v(t) \\ v'(t) \end{pmatrix} = \exp(t\mathcal{A}) \begin{pmatrix} v(0) \\ v'(0) \end{pmatrix} = \begin{pmatrix} C(t)v_0 + S(t)v_1 \\ AS(t)v_0 + C(t)v_1 \end{pmatrix},\ t \in \mathbb{R},$$

in the energy space $\mathcal{E} = E^1 \times E$, where $S(t)$ is the sine function associated with cosine $C(t)$ by the formula $S(t) = \int_0^t C(s)ds$ and E^1 is the Kisynskii space defined as

$$(1.7)\qquad \begin{cases} E^1 = \{x \in E : C(\cdot)x \in C^1(\mathbb{R}; E)\}, \\ \|x\|_{E^1} = \|x\|_E + \sup_{0\le t\le 1} \|C'(t)x\|_E \quad \forall x \in E^1 \end{cases}$$

(see [12], [2]). If the Cauchy problem (1.5) is well-posed, then operator $\mathcal{A}$ generates C_0-group $\exp(\cdot\mathcal{A})$ on $\mathcal{E}$ space.

In this paper, we are concerned with the general condition on the difference of two cosine and sine functions. To distinguish different cosine functions, we use $C(t, A)$ and $S(t, A)$ to denote the cosine and sine functions generated by A, respectively.

2. Compactness and Norm Continuity

Let $C(t, A)$ and $C(t, B)$ be the cosine functions on a Banach space E generated by A and B, respectively, and satisfying $\|C(t, A)\|$, $\|C(t, B)\| \le Me^{w|t|}$, $t \in \mathbb{R}$, for some constants M, $w \ge 0$. Let us denote also by $\Delta_{A,B}(t) = C(t, A) - C(t, B)$ for all $t \in \mathbb{R}$ and by $B(E)$ the space of all bounded linear operators on a Banach space E.

Theorem 2.1. *Let $\Delta_{A,B}(t)$ be norm continuous for $t > 0$. Then for all $\lambda > w^2$, the operator $R(\lambda, A) - R(\lambda, B)$ is compact iff $\Delta_{A,B}(t)$ is compact for $t \ge 0$.*

Proof. The sufficiency follows from the integral representation

$$\lambda\Big(R(\lambda^2, A) - R(\lambda^2, B)\Big) = \int_0^\infty e^{-\lambda t}(C(t, A) - C(t, B))\, dt$$

and norm continuity (see also [15]).

The proof of the necessity falls naturally into 3 steps. First we show that

$$(2.1)\qquad \lim_{\lambda\to\infty} \|\lambda R(\lambda, A)C(t, A) - \lambda R(\lambda, B)C(t, B) - \Delta_{A,B}(t)\| = 0 \quad \text{for } t > 0.$$

To this end consider the identity

$$\begin{aligned}
&\lambda^2 R(\lambda^2, A)C(t,A) - \lambda^2 R(\lambda^2, B)C(t,B) - \Delta_{A,B}(t) \\
&= \lambda \int_0^\infty e^{-\lambda s}[C(t,A)C(s,A) - C(t,B)C(s,B)]ds - \Delta_{A,B}(t) \\
&= \lambda \int_0^\infty e^{-\lambda s}\Big[\frac{1}{2}C(t+s,A) + \frac{1}{2}C(t-s,A) - \frac{1}{2}C(t+s,B) \\
&\quad -\frac{1}{2}C(t-s,B) - \Delta_{A,B}(t)\Big]ds \\
&= \lambda \int_0^\infty e^{-\lambda s}\Big(\frac{1}{2}\Delta_{A,B}(t+s) + \frac{1}{2}\Delta_{A,B}(t-s) - \Delta_{A,B}(t)\Big)ds.
\end{aligned}$$

Now let $\lambda > w$. Then for every $\delta > 0$ we have

$$\begin{aligned}
&\|\lambda^2 R(\lambda^2, A)C(t,A) - \lambda^2 R(\lambda^2, B)C(t,B) - \Delta_{A,B}(t)\| \\
&\le \int_0^\delta \lambda e^{-\lambda s}\|\frac{1}{2}\Delta_{A,B}(t+s) + \frac{1}{2}\Delta_{A,B}(t-s) - \Delta_{A,B}(t)\|ds \\
&\quad + \int_\delta^\infty \lambda e^{-\lambda s}\|\frac{1}{2}\Delta_{A,B}(t+s) + \frac{1}{2}\Delta_{A,B}(t-s) - \Delta_{A,B}(t)\|ds \\
&\le \sup_{0\le s\le\delta} \|\frac{1}{2}\Delta_{A,B}(t+s) + \frac{1}{2}\Delta_{A,B}(t-s) - \Delta_{A,B}(t)\| \\
&\quad + M\frac{5\lambda}{\lambda - w}\, e^{w(t+\delta)}e^{-\lambda\delta}.
\end{aligned}$$

Since $\Delta_{A,B}(t)$ is norm continuous in t, the first term on the right is smaller than any ϵ if δ is small enough. Now take λ so large that the second term is smaller than ϵ too. Since ϵ is arbitrary our statment (2.1) holds.

Second we show that

$$\lim_{\lambda\to\infty} \|\lambda^2 R(\lambda, A)^2 C(t,A) - \lambda^2 R(\lambda, B)^2 C(t,B) - \Delta_{A,B}(t)\| = 0. \tag{2.2}$$

By (2.1), we only need to show

$$\begin{aligned}
\lim_{\lambda\to\infty} &\|[\lambda^2 R(\lambda, A)^2 C(t,A) - \lambda^2 R(\lambda, B)^2 C(t,B)] \\
&-[\lambda R(\lambda, A)C(t,A) - \lambda R(\lambda, B)C(t,B)]\| = 0.
\end{aligned} \tag{2.3}$$

Since for $\lambda > \omega$,

$$\begin{aligned}
&\lambda^4 R(\lambda^2, A)^2 C(t, A) \\
&= \lambda^2 R(\lambda^2, A)\lambda \int_0^\infty e^{-\lambda s}\Big(\frac{1}{2}C(t+s, A) + \frac{1}{2}C(t-s, A)\Big)ds \\
&= \lambda \int_0^\infty e^{-\lambda r} C(r, A)\Big[\lambda \int_0^\infty e^{-\lambda s}\Big(\frac{1}{2}C(t+s, A) + \frac{1}{2}C(t-s, A)\Big)ds\Big]dr \\
&= \lambda \int_0^\infty e^{-\lambda r}\Big[\lambda \int_0^\infty e^{-\lambda s}\Big(\frac{1}{4}C(t+s+r, A) + \frac{1}{4}C(t+s-r, A) \\
&\quad + \frac{1}{4}C(t-s+r, A) + \frac{1}{4}C(t-s-r, A)\Big)ds\Big]dr,
\end{aligned}$$

so we have

$$\begin{aligned}
&\Big[\lambda^4 R(\lambda^2, A)^2 C(t, A) - \lambda^4 R(\lambda^2, B)^2 C(t, B)\Big] \\
&- \Big[\lambda^2 R(\lambda^2, A) C(t, A) - \lambda^2 R(\lambda^2, B) C(t, B)\Big] \\
&= \lambda \int_0^\infty e^{-\lambda r}\Big[\lambda \int_0^\infty e^{-\lambda s}\Big(\frac{1}{4}\Delta_{A,B}(t+s+r) + \frac{1}{4}\Delta_{A,B}(t+s-r) \\
&\quad + \frac{1}{4}\Delta_{A,B}(t-s+r) + \frac{1}{4}\Delta_{A,B}(t-s-r) - \frac{1}{2}\Delta_{A,B}(t+s) \\
&\quad - \frac{1}{2}\Delta_{A,B}(t-s)\Big)ds\Big]dr.
\end{aligned}$$

A similar argument as in the first step show that (2.3) holds.

To make up the third step let us note that for $\sigma > w$, $x \in D(A)$, one can write ([3], p. 42)

$$C(t, A)x = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} \lambda e^{\lambda t} R(\lambda^2, A)x\, d\lambda. \tag{2.4}$$

To finish the proof we are going to show that $\lambda^2 R(\lambda, A)^2 C(t, A) - \lambda^2 R(\lambda, B)^2 C(t, B)$ is compact for λ large enough. To do this fix $x \in E$ and $\sigma > w$. Using (2.4) one gets

$$\begin{aligned}
&\lambda^2 R(\lambda, A)^2 C(t, A)x - \lambda^2 R(\lambda, B)^2 C(t, B)x \\
&= \frac{\lambda^2}{2\pi}\lim_{n\to\infty}\int_{-n}^{n}(\sigma + ir)e^{t(\sigma+ir)}\Big[R((\sigma+ir)^2, A)R(\lambda, A)^2 \\
&\quad - R((\sigma+ir)^2, B)R(\lambda, B)^2\Big]x\, dr.
\end{aligned}$$

Let

$$S_n x := \int_{-n}^{n} (\sigma + ir) e^{t(\sigma+ir)} \Big[R((\sigma+ir)^2, A) R(\lambda, A)^2 \\ - R((\sigma+ir)^2, B) R(\lambda, B)^2 \Big] x dr,$$

then for $n > m$, by integration by parts we have

$$\begin{aligned}(S_n - S_m)x =& \left(\int_m^n + \int_{-n}^{-m} \right) (\sigma+ir) e^{t(\sigma+ir)} \\ & \cdot \Big[R((\sigma+ir)^2, A) R(\lambda, A)^2 - R((\sigma+ir)^2, B) R(\lambda, B)^2 \Big] x \, dr \\ =& \frac{e^{t(\sigma+ir)}}{it} (\sigma+ir) \Big[R((\sigma+ir)^2, A) R(\lambda, A)^2 \\ & - R((\sigma+ir)^2, B) R(\lambda, B)^2 \Big] x \Big(\Big|_m^n + \Big|_{-n}^{-m} \Big) \\ & + \Big(\int_m^n + \int_{-n}^{-m} \Big) \frac{e^{t(\sigma+ir)}}{t} \Big\{ 2(\sigma+ir)^2 \Big[R((\sigma+ir)^2, A)^2 R(\lambda, A)^2 \\ & - R((\sigma+ir)^2, B)^2 R(\lambda, B)^2 \Big] \\ & - \Big[R((\sigma+ir)^2, A) R(\lambda, A)^2 - R((\sigma+ir)^2, B) R(\lambda, B)^2 \Big] \Big\} x dr.\end{aligned}$$

Since

$$R((\sigma+ir)^2, A) R(\lambda, A) = \frac{1}{(\sigma+ir)^2 - \lambda} [R(\lambda, A) - R((\sigma+ir)^2, A)]$$

and

$$\|(\sigma+ir) R((\sigma+ir)^2, A)\| \le \frac{M}{\sigma - w},$$

we have for $|r| \to \infty$

$$\begin{aligned}&(\sigma+ir)^2 R((\sigma+ir)^2, A)^2 R(\lambda, A)^2 \\ =& \Big(\frac{\sigma+ir}{(\sigma+ir)^2 - \lambda} \Big)^2 \Big[R((\sigma+ir)^2, A) - R(\lambda, A) \Big]^2 = O(|r|^{-2})\end{aligned}$$

and

$$\begin{aligned}&R((\sigma+ir)^2, A) R(\lambda, A)^2 \\ =& \frac{-1}{(\sigma+ir)^2 - \lambda} \cdot \Big[R((\sigma+ir)^2, A) - R(\lambda, A) \Big] R(\lambda, A) = O(|r|^{-2}).\end{aligned}$$

Since the same holds for A replaced by B, one obtains

$$\|S_n - S_m\| \to 0, \qquad n, m \to \infty.$$

On the other hand, for any $\lambda, \mu > w^2, \sigma$, we get that

$$\begin{aligned}
&R(\mu, A)R(\lambda, A)^2 - R(\mu, B)R(\lambda, B)^2 \\
&= \Big[R(\mu, A) - R(\mu, B)\Big]R(\lambda, A)^2 + R(\mu, B)\Big[R(\lambda, A)^2 - R(\lambda, B)^2\Big] \\
&= \Big[R(\mu, A) - R(\mu, B)\Big]R(\lambda, A)^2 + R(\mu, B)\Big[R(\lambda, A)(R(\lambda, A) - R(\lambda, B)) \\
&\quad +(R(\lambda, A) - R(\lambda, B))R(\lambda, B)\Big]
\end{aligned}$$

is compact. So the uniform limit of operators S_n is equals to

$$\lim_{N\to\infty} \int_{-N}^{N} (\sigma+ir)e^{t(\sigma+ir)}\Big(R((\sigma+ir)^2, A)R(\lambda, A)^2 - R((\sigma+ir)^2, B)R(\lambda, B)^2\Big)\, dr$$

and it is compact. Therefore from (2.2) it follows that $\Delta_{A,B}(t)$ is compact. ■

We can characterize the norm continuity in Hilbert space analogous to Theorem 2.5 in [6] and the proof is just a simple modification.

Proposition 2.1. *Let A and B generate cosine functions $C(t, A)$ and $C(t, B)$, respectively, on a Hilbert space H, and $\|C(t, A)\|$, $\|C(t, B)\| \le Me^{\omega t}$ for some constants $M \ge 1$, $\omega \in \mathbb{R}$. Then $C(t, A) - C(t, B)$ is norm continuous for $t > 0$ if and only if for every $\sigma > \omega$,*

$$\lim_{|r|\to\infty} \|(\sigma + ir)[R((\sigma + ir)^2, A) - R((\sigma + ir)^2, B)]\| = 0$$

and

$$\lim_{n\to\infty} \int_{n}^{\infty} \|(\sigma \pm ir)[R((\sigma \pm ir)^2, A) - R((\sigma \pm ir)^2, B)]x\|^2 dr = 0,$$

$$\lim_{n\to\infty} \int_{n}^{\infty} \|(\sigma \pm ir)[R((\sigma \pm ir)^2, A^*) - R((\sigma \pm ir)^2, B^*)]y\|^2 dr = 0$$

uniformly for $x \in H, y \in H^*$ with $\|x\|, \|y\| \le 1$.

The case for sine functions is easier to treat.

Theorem 2.2. *Let $S(t, A)$ and $S(t, B)$ be the corresponding sine functions of $C(t, A)$ and $C(t, B)$, respectively. Then $S(t, A) - S(t, B)$ is compact for $t > 0$ if and only if $R(\lambda, A) - R(\lambda, B)$ is compact for $\lambda > w^2$.*

Proof. Since sine functions are norm continuous, the identity

$$R(\lambda^2, A) - R(\lambda^2, B) = \int_0^\infty e^{-\lambda t}\Big(S(t,A) - S(t,B)\Big)\,dt$$

is valid in the uniform topology, and so the compactness of the resolvents follows from that of $S(t,A) - S(t,B)$.

Conversely, since $S(t,A) - S(t,B)$ is norm continuous, similarly as in the proof of Theorem 2.1, we can show that the compactness of $R(\lambda, A) - R(\lambda, B)$ implies that of $S(t,A) - S(t,B)$. ■

Now we show a similar result for cosine as in [6], Proposition 2.7.

Proposition 2.2. *Suppose that* $\Delta_{A,B}(t)$ *is compact for* $t > 0$ *and norm continuous at* $t = 0$. *Then*

$$\lim_{h\to 0} \|\Delta_{A,B}(t+h) - 2\Delta_{A,B}(t) + \Delta_{A,B}(t-h)\| = 0 \ for\ any\ t \geq 0. \tag{2.5}$$

Proof. We have

$$\begin{aligned}
&\Delta_{A,B}(t+h) + \Delta_{A,B}(t-h) - 2\Delta_{A,B}(t)\\
&= \Big(C(t+h,A) + C(t-h,A)\Big) - \Big(C(t+h,B) + C(t-h,B)\Big) - 2\Delta_{A,B}(t)\\
&= 2C(t,A)C(h,A) - 2C(t,B)C(h,B) - 2\Delta_{A,B}(t)\\
&= 2\Big(C(t,A)C(h,A) - C(h,A)C(t,B)\Big)\\
&\quad +2\Big(C(h,A)C(t,B) - C(t,B)C(h,B)\Big) - 2\Delta_{A,B}(t)\\
&= 2C(h,A)\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B) - 2\Delta_{A,B}(t)\\
&= 2[C(h,A) - I]\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B)\\
&\to\ 0 \qquad \text{as } h \to 0.
\end{aligned}$$

■

Remark 2.1. In the proof of Theorem 2.1 we used actually (2.5), but not the norm continuity of $\Delta_{A,B}(\cdot)$.

Theorem 2.3. *Let* $\Delta_{A,B}(t)$ *be norm continuous in* t *at* 0. *Then* $\Delta_{A,B}(t)$ *is compact for* $t > 0$ *if and only if* $R(\lambda, A) - R(\lambda, B)$ *is compact for* $\lambda > w^2$ *and* (2.5) *holds.*

Now we can deal with unbounded perturbations. For $B \in B(E)$ define the function $F(t) = (\lambda^2 - A)\int_0^t S(s,A)B\,ds$ and also

$$SV(F(\cdot),t) := \sup\Big\{\Big\|\sum_{j=1}^n [F(t_j) - F(t_{j-1})]x_j\Big\|;\ x_j \in X, \|x_j\| \leq 1\Big\} < \infty,$$

where the supremum is taken over all subdivisions of $[0, t]$.

Theorem 2.4. [9] *Let $C(t, A)$ be the cosine function and $SV(F(t), t) \to 0$ as $t \to 0$ for some $\lambda > \omega$. Then operator $A(I + B)$ generates the cosine function and*

$$\|C(t, A) - C(t, A(I + B))\| \leq c\, SV(F(t), t) \ as\ t \to 0.$$

Theorem 2.5. [9] *Let $C(t, A)$ be the cosine function and*

$$\delta_B(t) := \sup \Big\{ \int_0^t \|BS(t, A)Ax\| ds : x \in D(A), \|x\| \leq 1 \Big\} \to 0 \ as\ t \to 0.$$

Then $(I + B)A$ generates a cosine function and

$$\|C(t, A) - C(t, (I + B)A)\| \leq c\, \delta_B(t) \ as\ t \to 0.$$

Most of all known perturbation's Theorems could be obtained [9] as a consequence of Theorem 2.4 and Theorem 2.5 including M. Watanabe unbounded perturbation $P \in B(E^1, E)$, which is exactly the case of $\|C(t, A) - C(t, A + P)\| = O(t)$ as $t \to 0$.

Proposition 2.3. *Suppose that the assumptions of Theorem* 2.4 (*resp. Theorem* 2.5) *are satisfies. Then $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) is compact for $t > 0$ if and only if $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) satisfies* (2.5) *and $R(\lambda, A) - R(\lambda, A(I + B))$ (resp. $R(\lambda, A) - R(\lambda, (I + B)A)$) is compact for λ large enough.*

3. Comparison of the First and the Second Order Equations

First we consider bounded perturbations. It is well known that when A generates a C_0-semigroup e^{tA}, then $A + B$, $B \in B(E)$, also generates a C_0-semigroup $e^{t(A+B)}$. It was shown in [17] that $e^{t(A+B)} - e^{tA}$ is norm continuous for $t > 0$ if it is compact for $t > 0$. As for the cosine case, the compact hypotheses can be removed.

Theorem 3.1 *Let A be the generator of the cosine function $C(t, A)$, $B \in B(E)$. Then $\Delta_{A+B,A}(t)$ is norm continuous in $t \in \mathbb{R}$.*

Proof. Note that for all $t \geq 0$ and $x \in E$, we have

$$\Delta_{A+B,A}(t)x = C(t, A + B)x - C(t, A)x = -\int_0^t S(t - s, A + B)BC(s, A)xds,$$

so

$$\Delta_{A+B,A}(t) - \Delta_{A+B,A}(s) = -\int_0^s [S(t-r, A+B) - S(s-r, A+B)]BC(r, A)dr$$
$$- \int_s^t S(t-r, A+B)BC(r, A)dr$$

and the first term on the right converges to 0 in norm as $t \to s$ since $S(t, A+B)$ is norm continuous. The convergence of the second term to 0 is due to the boundedness of all operators under the integral. ■

Combining Theorems 2.1 and 3.1, we have

Theorem 3.2. *Let $B \in B(E)$ and A generate a cosine function. Then $\Delta_{A+B,A}(t)$ is compact for $t > 0$ if and only if $R(\lambda, A+B) - R(\lambda, A)$ is compact for λ large enough.*

We give a sufficient condition under which the compactness of the difference of two C_0-semigroups implies that the perturbating operator is a compact one.

Proposition 3.1. *Suppose that the C_0-semigroups e^{tA} and e^{tB} commute and $D(B) \subseteq D(A)$. Assume also that e^{tB} is a C_0-group. If $\Theta(t) := e^{tA} - e^{tB}$ is compact for all $t > 0$, then $A = B + K$, where the operator K is compact.*

Proof. One can write $e^{-tB}\Theta(t) = e^{-tB}e^{tA} - I$, $t \in \mathbb{R}_+$. By assumption the operator $e^{-tB}e^{tA} - I$ is compact for any $t > 0$ and, moreover, $e^{-tB}e^{tA}$ is a C_0-semigroup with generator $A - B$. From [1] it follows that operator $A - B$ is compact. ■

For cosines with bounded generators, we also have

Proposition 3.2. *Let operator $B \in B(E)$. Then $\Delta_{A,B}(t)$ is compact for $t > 0$ iff $A - B$ is compact.*

Proof. Compactness of $\Delta_{A,B}(t)$ for any $t > 0$ implies (see [15]) that $R(\mu, A) - R(\mu, B)$ is compact for some μ. In such a case the operator $I - (\mu - B)R(\mu, A)$ is compact. This means that $(\mu - B)R(\mu, A)$ is Fredholm operator with index 0, i.e. it has closed range $\mathcal{R}((\mu - B)R(\mu, A)) = E$. Since $\mu - B$ is one to one on E we get that $\mathcal{R}(R(\mu, A)) = E$. By Closed Graph Theorem $\mu - A$ is bounded. Since the operator A is bounded we have $\|\frac{2}{t^2}\int_0^t S(s, A)\, ds - I\| \to 0$ as $t \to 0$. Hence operator $\int_0^t S(s, A)\, ds$ is invertible. If $\Delta_{A,B}(t)$ is compact, then $B\int_0^t (S(s, B) - S(s, A))\, ds$ is also compact. Now from

$$\Delta_{A,B}(t) = (A - B)\int_0^t S(s, A)\, ds - B\int_0^t (S(s, B) - S(s, A))\, ds$$

it follows that the difference $A - B$ is a compact operator.

Conversely, if $A - B$ is a compact operator, then A is bounded and

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(B - A)R(\lambda, B)$$

is compact, so the compactness of $\Delta_{A,B}(t)$ follows from Theorem 2.1. ■

In the following example both A and B generate C_0-semigroup and cosine function. The operator $e^{tA} - e^{tB}$ is compact for all $t > 0$, but $C(t, A) - C(t, B)$ is not compact.

Example 3.1. Let $E = l^1$ and $\{e_n\}$ be the standard basis for it, i.e. $e_n = (0, ..., 0, 1, 0, ..., 0, ...)$, where 1 is in the n^{th} coordinate. Let

$$Ax := \sum_{n=1}^{\infty} -n(x, e_n)e_n, \quad Bx := \sum_{n=1}^{\infty} -(n + n^2)(x, e_n)e_n,$$

where $x = (x^1, x^2, ..., x^n, ...)$, $(x, e_n) = x^n$, $\|x\|_{l^1} = \sum_{i=1}^{\infty} |x^i|$. Then the C_0-semigroups generated by them are

$$e^{tA}x = \sum_{n=1}^{\infty} e^{-nt}(x, e_n)e_n, \quad e^{tB}x = \sum_{n=1}^{\infty} e^{-(n+n^2)t}(x, e_n)e_n.$$

And the cosine functions are given by the formula

$$C(t, A)x = \sum_{n=1}^{\infty} \cos(nt)\ (x, e_n)e_n, \quad C(t, B)x = \sum_{n=1}^{\infty} \cos(n + n^2)t\ (x, e_n)e_n.$$

Since $e^{-nt} - e^{-(n+n^2)t} \to 0$ as $n \to \infty$, the operator $e^{tA} - e^{tB}$ can be approximated in norm by a sequence of operators with ranges of finite dimension $S_N(t)x = \sum_{n=1}^{N} \left(e^{-nt} - e^{-(n+n^2)t}\right)(x, e_n)e_n$. So the operator $e^{tA} - e^{tB}$ is compact for $t \geq 0$. However, $C(t, A) - C(t, B)$ is not compact. Indeed, take $t = \pi/2$ and choose $l^1 \supset \{y_k\} := \{(\delta_n^{2k+1})\}$, where δ_i^j is the Kronecker delta. Now for $n = 2k + 1$ one gets $nt = k\pi + \pi/2$ and $(n + n^2)t = 2k^2\pi + 3k\pi + \pi$. So we have $\cos(nt) - \cos((n + n^2)t) = \pm 1$, when k can be divided exactly by 2 we choose $+$; otherwise choose $-$. Thus

$$(C(\pi/2, A) - C(\pi/2, B))y_k = \Big([\cos(nt) - \cos(n + n^2)t]\delta_n^{2k+1}\Big) = \Big(\pm\, \delta_n^{2k+1}\Big),$$

which means $\|[C(\pi/2, A) - C(\pi/2, B)](y_k - y_m)\|_{l^1} = 2$ for $k \neq m$. So we cannot choose convergent subsequence from the sequence $\{[C(\pi/2, A) - C(\pi/2, B)]y_k\}$.

Also we can see that $C(t,A)-C(t,B)$ is not norm continuous in t. In fact, for each $t>0$, let us define $s_k = t + \frac{1}{k+k^2}$. Then $s_k \to t$ as $k\to\infty$ and

$$\begin{aligned}
&\Big\|[C(t,A)-C(t,B)]-[C(s_k,A)-C(s_k,B)]\Big\|_{B(l^1)}\\
=&\Big\|\{[\cos(nt)-\cos(ns_k)]-[\cos(n+n^2)t-\cos(n+n^2)s_k]\}_{n=1}^{\infty}\Big\|_{l^\infty}\\
\geq& 2\Big|\sin\Big(\frac{k}{2}(s_k+t)\Big)\sin\Big(\frac{k}{2}(s_k-t)\Big)-\sin\Big(\frac{k+k^2}{2}(s_k+t)\Big)\sin\Big(\frac{k+k^2}{2}(s_k-t)\Big)\Big|\\
=&2\Big|\sin\Big(\frac{k}{2}(s_k+t)\Big)\sin\Big(\frac{1}{2}(1+k)\Big)-\sin((k+k^2)t+1/2))\sin\frac{1}{2}\Big|.
\end{aligned}$$

It is clear that $\sin\frac{1}{2(1+k)}\to 0$ as $k\to\infty$. But $\sin((k+k^2)t+1/2))$ does not converge to 0 as $k\to\infty$ for every $t>0$! To prove this suppose contrary to our claim that $\sin((k+k^2)t+1/2))\to 0$ as $k\to\infty$. Then $\sin((k+1+(k+1)^2)t+1/2)\to 0$ $k\to\infty$. Now, since

$$\begin{aligned}
&\sin((k+1+(k+1)^2)t+1/2)=\sin(((k+k^2)t+1/2)+2(k+1)t)\\
=&\sin((k+k^2)t+1/2))\cos(2(k+1)t)+\cos((k+k^2)t+1/2))\sin(2(k+1)t),
\end{aligned}$$

we obtain that $\sin(2(k+1)t)\to 0$ as $k\to\infty$ because $\cos((k+k^2)t+1/2)$ can not converge to 0 according to the relation $\sin^2 x+\cos^2 x = 1$. Hence $\sin(2(k+1+1)t)\to 0$ as $k\to\infty$. Thus from $\sin(2(k+1+1)t)=\sin(2(k+1)t)\cos(2t)+\cos(2(k+1)t)\sin(2t)$ we obtain $\sin(2t)\to 0$ as $k\to\infty$. So we have that $t=n\pi/2$ for some $n\in\mathbb{N}$. But for such t one finds that $\sin((k+k^2)t+1/2))=\pm\sin(1/2)$, which contradicts our assumption of convergence to 0. This means that $C(t,A)-C(t,B)$ is not norm continuous.

The converse does not happen, we have

Proposition 3.3. *Suppose that A and B generate cosine functions $C(t,A)$ and $C(t,B)$. If $C(t,A)-C(t,B)$ is compact for $t>0$, then $e^{tA}-e^{tB}$ is compact. Moreover, if $C(t,A)-C(t,B)$ is norm continuous for $t>0$, then $C(t,A)-C(t,B)$ is compact for $t>0$ iff $e^{tA}-e^{tB}$ is compact.*

Proof. Since $C(t,A)-C(t,B)$ is compact, it follows from [15] that $S(t,A)-S(t,B)=\int_0^t(C(s,A)-C(s,B))ds$ is also compact, which implies $R(\lambda,A)-R(\lambda,B)$ is compact by Theorem 2.2. Moreover, since both e^{tA} and e^{tB} are analytic, $e^{tA}-e^{tB}$ is norm continuous, and compactness of $e^{tA}-e^{tB}$ follows from Theorem 2.3 of [6]. If, in addition, $C(t,A)-C(t,B)$ is norm continuous, then the compactness of $e^{tA}-e^{tB}$ implies that $R(\lambda,A)-R(\lambda,B)$ is compact. Now the compactness of $C(t,A)-C(t,B)$ follows from Theorem 2.1. ∎

The following extends Proposition 2.7 in [6].

Proposition 3.4. *Suppose that* $D(A) \subseteq D(B)$, *where* A *generates an analytic semigroup* e^{tA} *and* B *is the generator of* C_0*-semigroup* e^{tB}. *If* $\Theta(t) := e^{tA} - e^{tB}$ *is compact for* $t > 0$, *then* $\Theta(t)$ *is norm continuous for* $t \geq 0$.

Proof. Since e^{tA} is an analytic semigroup, it maps E into $D(A)$, and thus $(\lambda - A)e^{tA}$ is a bounded operator, and we can write

$$\Theta(h)e^{tA} = \Theta(h)R(\lambda, A)(\lambda - A)e^{tA}.$$

Moreover,

$$\begin{aligned}\Theta(h)R(\lambda, A) &= \int_0^h \frac{d}{ds}(e^{(h-s)B}e^{sA})R(\lambda, A)\, ds \\ &= -\int_0^h e^{(h-s)B}(B - A)R(\lambda, A)e^{sA}\, ds \\ &\to 0 \quad \text{as} \quad h \to 0\end{aligned}$$

since $(B - A)R(\lambda, A)$ is bounded. Therefore, $\|\Theta(h)e^{tA}\| \to 0$ as $h \to 0$. One can write

$$\Theta(t + h) - \Theta(t) = (e^{hB} - I)\Theta(t) + \Theta(h)e^{tA}.$$

Since $\Theta(t)$ is compact and e^{tB} is strongly continuous, we have $\|(e^{hB} - I)\Theta(t)\| \to 0$ as $h \to 0$. Thus $\|\Theta(t + h) - \Theta(t)\| \to 0$ as $h \to 0$ for every $t \geq 0$. ∎

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REFERENCES

1. J. R. Cuthbert, On semi-groups such that $T_t - I$ is compact for some $t > 0$, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **18** (1971), 9-16.
2. H. O. Fattorini, *The Cauchy Problem*, Addison-Wesley, London, 1983.
3. H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North Holland, Amsterdam, 1985.

4. J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, London, 1985.

5. B. Z. Guo, On the exponential stability of C_0-semigroups on Banach spaces with compact perturbations, *Semigroup Forum*, **59**(2) (1999), 190-196.

6. M. Li, X. H. Gu, and F. L. Huang, On unbounded perturbations of semigroups: compactness and norm continuity, *Semigroup Forum*, **65** (2002), 58-70.

7. Z. H. Luo, B. Z. Guo, and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London Berlin Heidelberg, 1999.

8. J. Peng, M. Wang, and G. Zhu, The spectrum of the generator of a class of perturbed semigroups with applications, *Acta Math. Appl. Sinica*, **20** (1997), 107-113. (in Chinese)

9. S. Piskarev and S. Y. Shaw, Perturbation and comparison of cosine operator functions, *Semigroup Forum*, **51** (1995), 225-246.

10. B. P. Rao, Uniform stabilization of a hybrid system of elasticity, *SIAM J. Control & Optim.* **33** (1995), 440-445.

11. D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, *SIAM Review*, **20** (1978), 639-739.

12. C. C. Travis and G. F. Webb, Second order differential equations, in "*Nonlinear Equations in Abstract Spaces*" edited by V. Lakshmikantham, Academic Press, New York, 1978.

13. R. Triggiani, Lack of uniform stabilization for noncontractive semigroups under compact perturbation, *Proc. Amer. Math. Soc.* **105** (1989), 375-383.

14. I. Vidav, Spectral of perturbed semigroups with applications to transport theory, *J. Math. Anal. Appl.* **30** (1970), 264-279.

15. J. Voigt, On the convex compactness property for the strong operator topology, *Note di Matematica*, **12** (1992), 259-269.

16. G. F. Webb, Compactness of bounded trajectories of dynamic systems in infinite dimensional systems, *Proc. of the Royal Soc. of Edinburg*, **84A** (1979), 19-33.

17. Q. C. Zhang and F. L. Huang, On compact perturbations of C_0-semigroups, *J. Sichuan Univ.* **35** (1998), 829-833. (in Chinese)

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