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MEAN STABILITY OF SEMIGROUPS

Shmuel Kantorovitz and Serguei Piskarev

Abstract. Let $T(\cdot)$ be a bounded C_0-semigroup on the Banach space X, with generator A. It is shown that the denseness of range A is necessary and sufficient for the semigroup's mean stability with respect to suitable weights. Analogous results are valid for power bounded operators, tensor product semigroups, and cosine operator functions.

1. Statement of Results

1.1. Let X be a complex Banach space, and let $B(X)$ denote the Banach algebra of all bounded linear operators from X into itself. A C_0-semigroup $T(\cdot) : [0, \infty) \to B(X)$ is (strongly) *stable* if $T(t) \to 0$, as $t \to \infty$, in the strong operator topology. By the uniform boundedness theorem, a necessary condition for stability is the *boundedness* of $\|T(\cdot)\|$, from which it follows in particular that the spectrum $\sigma(A)$ of the generator A is contained in the closed left half plane. A well-known *sufficient* condition for stability is that $\sigma(A) \cap i\mathbb{R}$ be countable and that the residual spectrum $R\sigma(A)$ do not meet the imaginary axis.

We consider here the weaker concept of *mean stability* of bounded semigroups with respect to a suitable family of weight functions. A known (folklore) result of this type is that if $f \in L^1(\mathbb{R}^+)$ has nonzero integral over $\mathbb{R}^+$, then the "averages" $\int_0^\infty t^{-1} f(s/t) T(s)x\, ds$ converge strongly to zero as $t \to \infty$ for all $x \in X$ if and only if A has dense range (here is a proof: since the averages are uniformly bounded by $M \|x\| \|f\|_1$, where $M := \sup \|T(\cdot)\|$, the density of $C_c^1(\mathbb{R}^+)$ in $L^1(\mathbb{R}^+)$ and of range A in X imply that the averages converge strongly to zero for all $f \in L^1(\mathbb{R}^+)$ and $x \in X$ if this is true for all $f \in C_c^1(\mathbb{R}^+)$, the C^1-functions with compact support in $\mathbb{R}^+ := (0, \infty)$, and for all $x = Ay$ with $y \in D(A)$, the domain of A.

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In this special case, since $T(s)x = \frac{d}{ds}T(s)y$, an integration by parts shows that the averages equal $-\int_0^\infty t^{-2}f'(s/t)T(s)y\,ds$, and are therefore norm-bounded by

$$M\|y\|\int_0^\infty (1/t)|f'(s/t)|d(s/t) \le M\|y\|t^{-1}\sup|f'| \times |\operatorname{supp} f|,$$

where $|\operatorname{supp} f|$ denotes the linear Lebesgue measure of the support of f.

On the other hand, if range A is not dense in X, there exists $0 \ne x^* \in X^*$ such that $\langle x^*, Ay\rangle = 0$ for all $y \in D(A)$. Therefore $\langle x^*, T(s)y\rangle = \langle x^*, y\rangle$, and consequently $\langle x^*, \int_0^\infty (1/t)f(s/t)T(s)y\,ds\rangle = \langle x^*, y\rangle \int_0^\infty f(u)du$. Since f has nonzero integral, if the averages converge (even weakly) to zero, we have necessarily $\langle x^*, y\rangle = 0$ for all $y \in D(A)$, and the contradiction $x^* = 0$ follows from the density of $D(A)$ in X.)

We show that this genre of result is true for averages $\int_0^\infty h(t,s)T(s)x\,ds$ with kernel h satisfying adequate conditions (although no attempt is made to get utmost generality). The literature on asymptotics of semigroups is vast; *our references list only items explicitly used in the proofs*.

1.2. Consider *weight functions* $h(t,s)$ for $0 < s \le t$ with the following properties:
(1) for each $t > 0$, $h(t,\cdot) \ge 0$ is monotonic on $(0,t]$;
(2) $K := \sup_{t>0}\int_0^t h(t,s)ds < \infty$;
(3) $\lim_{t\to\infty} h(t,t) = 0$;
(4) there exists $\delta > 0$ such that $\lim_{t\to\infty}\int_0^\delta h(t,s)ds = 0$.

For some of the results, we consider also the condition
(5) $\liminf_{t\to\infty}\int_0^t h(t,s)\,ds > 0$.

The monotonicity property in Condition (1), together with Conditions (3) and (4), imply that

$$(*) \qquad \lim_{t\to\infty} h(t,\tau) = 0$$

for all $\tau \ge \delta > 0$. In fact, if a $t > \tau \ge \delta$ is such that $h(t,\cdot)$ is non-decreasing (resp., non-increasing), then $0 \le h(t,\tau) \le h(t,t)$ (resp., $0 \le h(t,\tau) \le h(t,\delta) \le \delta^{-1}\int_0^\delta h(t,s)ds$). Thus (*) follows from Conditions (3) and (4).

Such weights arise for example (but not exclusively) from monotonic functions of one real variable, say $f : (0,1] \to [0,\infty)$, such that $0 < \int_0^1 f(u)du := c < \infty$, by taking $h(t,s) = t^{-1}f(s/t)$ for $0 < s \le t$. Then h satisfies Conditions (1-5), and even the stronger Condition (2′) below instead of Conditions (2) and (5):

(2′) For all $t > 0$, $\int_0^t h(t,s)\,ds = c$, where c is a nonzero constant.

For example, if we choose $f(u) = \beta u^{\beta-1}$ with $\beta > 0$ the induced weights are the classical weights $h(t,s) = (\beta/t^\beta)s^{\beta-1}$ of fractional integration.

(Kernels generated as above are homogeneous of degree -1, and there are clearly non-homogeneous kernels satisfying Conditions (1-5).)

The class of weights satisfying Conditions (1-4) will be denoted by $\mathcal{W}(1,2,3,4)$ (with a similar notation for other sets of conditions).

Define the W^h*-mean* $W_t^h[F]$ of the bounded strongly continuous function $F : [0,\infty) \to B(X)$ with respect to the given weight $h \in \mathcal{W}(1,2)$ by

$$W_t^h[F]x = \int_0^t h(t,s)F(s)x\,ds \qquad (t>0).$$

If $M := \sup ||F(\cdot)||$, the norm of the integrand is dominated by $M\,\|x\|\,h(t,s)$, and it follows from Condition (2) that the integral converges strongly in X, and defines bounded operators $W_t^h[F]$ with norm $\leq MK$ for all $t>0$.

We say that F is W^h*-mean stable* if

$$\lim_{t\to\infty} W_t^h[F] = 0$$

in the strong operator topology.

Stability of F (i.e., $\lim_{t\to\infty} F(t) = 0$ in the strong operator topology) is stronger than W^h-mean stability of F for any $h \in \mathcal{W}(1,2,4)$. Indeed, for $0<\delta<\tau<t$ with δ as in Condition (4), we write

$$W_t[F] = \left[\int_0^\delta + \int_\delta^\tau + \int_\tau^t\right] h(t,s)F(s)ds.$$

Given $x \in X$ and $\epsilon > 0$, since $\lim_{s\to\infty}\|F(s)x\| = 0$, we *fix* $\tau > \delta$ such that $K\,\|F(s)x\| < \epsilon$ for $s \geq \tau$. Then the third integral has norm $< \epsilon$ for *all* $t > \tau$. The first integral has norm $\leq M\|x\|\int_0^\delta h(t,s)ds \to 0$ (as $t\to\infty$) by Condition (4). Finally, since $h(t,\cdot)$ is monotonic in $(0,t]$, the middle integral has norm

$$\leq M\|x\|(\tau-\delta)\max[h(t,\tau),h(t,\delta)] \to 0 \;\text{ as } t\to\infty, \text{ by (*).}$$

Theorem 1. *Let $T(\cdot)$ be a bounded C_0-semigroup with generator A. The following statements are equivalent*:

(i) *A has dense range*;

(ii) *$T(\cdot)$ is W^h-mean stable for all weights $h \in \mathcal{W}(1,2,3,4)$*;

(iii) *for some weight $h_0 \in \mathcal{W}(1,2,5)$ and some positive sequence $\{t_n\}$ diverging to ∞, $\lim_n W_{t_n}^{h_0}[T(\cdot)] = 0$ in the weak operator topology.*

Note that Statement (i) in Theorem 1 can be formulated as the spectral condition $0 \notin P\sigma(A) \cup R\sigma(A)$, where $P\sigma(A)$ and $R\sigma(A)$ denote the point spectrum and the residual spectrum of A, respectively.

Theorem 1 establishes in particular that for any weight $h \in \mathcal{W}(1, 2, 3, 4, 5)$, weak and strong W^h-mean stability of $T(\cdot)$ are equivalent.

The change of variables $u = t - s$ yields an analogous result for weights $h(t, s)$ with $0 \leq s < t$ (that is, undefined for $s = t$), satisfying the corresponding shifted conditions. We omit the obvious details. This would apply for example to the usual fractional integrals of $T(\cdot)$.

Theorem 1 is an obvious corollary of the following

Theorem 2. *Let $T(\cdot)$ be a bounded C_0-semigroup with generator A. Then the following statements are equivalent for any given vector $x \in X$:*

- (i) $x \in \overline{range\ A}$;
- (ii) $\lim_{t\to\infty} W_t^h[T(\cdot)]x = 0$ *strongly for all* $h \in \mathcal{W}(1, 2, 3, 4)$;
- (iii) $\lim_n W_{t_n}^{h_0}[T(\cdot)]x = 0$ *weakly for some* $h_0 \in \mathcal{W}(1, 2, 5)$ *and some positive sequence* $\{t_n\}$ *diverging to* ∞.

1.3. A slight variant of Theorem 1 that accommodates Abel and Gauss summability uses weights h satisfying the following conditions:

(a) for each $t > 0$, $h(t, \cdot) : (0, \infty) \to [0, \infty)$ is monotonic, and vanishes at ∞;

(b) $K := \sup_{t>0} \int_0^\infty h(t, s)ds < \infty$;

(c) there exists $\delta > 0$ such that $\lim_{t\to\infty} \int_0^\delta h(t, s) = 0$.

Examples of weights h satisfying Conditions (a)-(c) are the Abel summability kernel $h(t, s) = t^{-1}e^{-s/t}$, the Gauss summability kernel $h(t, s) = (2/t\sqrt{\pi})e^{-(s/t)^2}$, and the Gamma-like kernels $h(t, s) = (t^{-\alpha}/\Gamma(\alpha))e^{-s/t}s^{\alpha-1}$ with $0 < \alpha < 1$. These examples satisfy even the following Condition (b′) (stronger than (b) and (d) below):

(b′) $\int_0^\infty h(t, s)\, ds = c$ for all $t > 0$, where c is a nonzero constant.

Since $h(t, \cdot) \geq 0$ is monotonic and vanishes at ∞, the only non-trivial h are nonincreasing. Hence for all $v \geq \delta$ (with δ as in Condition (c)),

$$\lim_{t\to\infty} h(t, v) = 0 \tag{**}$$

(because $h(t, v) \leq h(t, \delta) \leq \delta^{-1} \int_0^\delta h(t, s)ds \to 0$ as $t \to \infty$, by Condition (c)).

We shall also consider the condition

(d) $\liminf_{t\to\infty}\int_0^\infty h(t,s)ds > 0$.

The class of weights satisfying Conditions (a)-(c) will be denoted $\mathcal{A}(a,b,c)$ (with similar meaning for other conditions).

Let $F:[0,\infty)\to B(X)$ be bounded and strongly continuous. For any weight $h\in\mathcal{A}(a,b,c)$, the A^h-means of F are the well-defined bounded operators given by

$$A_t^h[F]x = \int_0^\infty h(t,s)F(s)x\,ds, \qquad x\in X, t>0.$$

It follows from Condition (b) that $\|A_t^h[F]\| \le MK$ for all $t>0$, where $M := \sup\|F(\cdot)\|$. We say that F is *A^h-mean stable* if $\lim_{t\to\infty} A_t^h[F] = 0$ in the strong operator topology.

Theorem 3. *Let $T(\cdot)$ be a bounded C_0-semigroup with generator A. The following statements are equivalent*:

(i) range A *is dense in* X;

(ii) $T(\cdot)$ *is A^h-mean stable for all* $h\in\mathcal{A}(a,b,c)$;

(iii) *for some weight* $h_0\in\mathcal{A}(a,b,d)$ *and for some positive sequence* t_n *diverging to* ∞, $\lim_n A_{t_n}^{h_0}[T(\cdot)] = 0$ *in the weak operator topology.*

If X is reflexive, we have the direct sum decomposition $X = \ker A \oplus \overline{\text{range} A}$ (cf. [3, Theorem 8.20]). If P denotes the projection of X onto ker A (along $\overline{\text{range} A}$), then in the strong operator topology (s.o.t.), $\lim_{t\to\infty} W_t^h[T(\cdot)] = P$ ($\lim_{t\to\infty} A_t^h[T(\cdot)] = P$) for all $h\in\mathcal{W}(1,2,3,4)$ ($h\in\mathcal{A}(a,b,c)$, respectively). This follows trivially from Theorem 2 (and its version for A^h-means). For general Banach spaces, we could add the following two properties to the equivalent properties listed in [3, Theorem 8.23]:

(1) $\lim_{t\to\infty} W_t^h[T(\cdot)]$ exists in the s.o.t. for all $h\in\mathcal{W}(1,2,3,4)$ (this generalizes Property (ii) of the reference, which corresponds to the special case $h(t,s) = 1/t$), and

(2) $\lim_{t\to\infty} A_t^h[T(\cdot)]$ exists in the s.o.t. for all $h\in\mathcal{A}(a,b,c)$ (this generalizes Property (iii) of the reference, which corresponds to the special case $h(t,s) = (1/t)e^{-s/t}$).

1.4. The above results generalize easily to bounded *pre-semigroups*. A *pre-semigroup* (or C-semigroup) is a strongly continuous function $S(\cdot):[0,\infty)\to B(X)$ such that $S(0)$ is *injective* and $S(t-u)S(u)$ is *independent of* u for all $0\le u\le t$. The latter property is equivalent to the identity

$$S(u)S(v) = S(0)S(u+v) \qquad (u,v\ge 0).$$

The operator $S(0)$ is usually denoted by C, and this convention is the origin of the name "C-semigroup" in the literature.

The *generator* of the pre-semigroup $S(\cdot)$ is closed and has domain $D(A)$ (not necessarily dense!) consisting of all $x \in X$ for which the strong right derivative at 0, $[S(\cdot)x]'(0)$, exists and *belongs to the range of* $S(0)$, and

$$Ax := S(0)^{-1}[S(\cdot)x]'(0) \qquad (x \in D(A)).$$

The generalized version of Theorem 1 goes as follows.

Theorem 4. *Let $S(\cdot)$ be a bounded pre-semigroup with generator A. If A has dense range, then $S(\cdot)$ is W^h-mean stable for all $h \in \mathcal{W}(1,2,3,4)$. Conversely, if $\lim_n W_{t_n}^{h_0}[S(\cdot)] = 0$ in the weak operator topology for some $h_0 \in \mathcal{W}(1,2,5)$ and some positive sequence $\{t_n\}$ diverging to ∞, and if $S(0)$ has dense range, then A has dense range.*

Similar modifications in the proof of Theorems 3 show that Theorem 4 is valid for the A-means as well.

1.5. Theorems 2 and 3 (or rather, their proofs) are applied to the *tensor product* of bounded C_0-semigroups (cf. [2]).

Let $T(\cdot)$ and $S(\cdot)$ be bounded C_0-semigroups on the Banach space X, with respective generators A and $-B$. Let Δ be the operator on $B(X)$ with domain $D(\Delta)$ consisting of all $V \in B(X)$ such that $VD(B) \subset D(A)$ and $\Delta V := AV - VB$ (with domain $D(B) \subset X$ and range in X) is bounded (relative to the X-norm). Since $D(B)$ is dense in X, ΔV extends uniquely to an operator (also denoted by ΔV) that belongs to $B(X)$.

For $V \in B(X)$ and $t \geq 0$, set $\mathcal{G}(t)V := T(t)VS(t)$.

Clearly, $\mathcal{G}(\cdot)$ is a semigroup of operators on $B(X)$, such that $\mathcal{G}(\cdot)V$ is continuous in the s.o.t. on $B(X)$ for each $V \in B(X)$; it is called the *tensor product* of the given semigroups (cf. [2]).

Theorem 5. *Let $T(\cdot)$ and $S(\cdot)$ be bounded C_0-semigroups on the Banach space X, with respective generators A and $-B$. Then for all Z in the $B(X)$-closure of the range of Δ, $T(\cdot)ZS(\cdot)$ is W^h-mean stable (A^h-mean stable) in the strong operator topology, for all $h \in \mathcal{W}(1,2,3,4)$ ($h \in \mathcal{A}(a,b,c)$, respectively).*

Corollary 6. *Let $S(\cdot)$ and $T(\cdot)$ be bounded C_0-semigroups with respective generators $-B$ and $B + C$, where $C \in B(X)$. Then $T(\cdot)CS(\cdot)$ is W^h-mean (A^h-mean) stable in the strong operator topology for all $h \in \mathcal{W}(1,2,3,4)$ ($h \in \mathcal{A}(a,b,c)$, respectively).*

Related results were proved by S.-Y. Shaw [6].

1.6. Discrete analogs of the preceding results are easily formulated. An operator $T \in B(X)$ is *power bounded* if $M := \sup_{n \geq 1} ||T^n|| < \infty$; it is (weakly) *stable* if $T^n \to 0$ in the (weak) strong operator topology. By the uniform boundedness theorem, power boundedness is a necessary condition for (weak) stability.

If W is the infinite triangular matrix $W = (w_{nk})_{1 \leq k \leq n < \infty}$, denote $c_n(W) := \sum_{k=1}^{n} w_{nk}$, $n \in \mathbb{N}$. Consider the following properties:

(a) for each $n \in \mathbb{N}$, $w_{nk} \geq 0$ is monotonic with respect to k, $1 \leq k \leq n$;

(b) $K := \sup_n c_n(W) < \infty$;

(c) $\lim_n w_{n1} = \lim_n w_{nn} = 0$.

For example, if $f : (0, 1] \to [0, \infty)$ is any monotonic function such that $tf(t) \to 0$ as $t \to 0+$ and $\int_0^1 f(t)\, dt = c \neq 0$ ($f(t) = \beta t^{\beta-1}$ is such a function for any $\beta > 0$), the matrix with entries $w_{nk} = f(k/n)/n$ $(1 \leq k \leq n < \infty)$ satisfies Conditions (a), (b), and (c), and even the following condition (b′) (stronger than (b) and (d) below):
(b′) $\exists \lim_n c_n(W) = c \neq 0$.
We shall also consider the condition
(d) $\liminf_n c_n(W) > 0$.
The class of matrices W satisfying Conditions (a), (b), and (c) is denoted by $\mathcal{W}(a, b, c)$ (with a similar notation for other sets of conditions; the present discrete context will prevent confusion with the preceding notations). The W-means $W_n[T]$ of the power sequence $\{T^k\}$ with respect to the weight matrix W are defined by

$$W_n[T] = \sum_{k=1}^{n} w_{nk} T^k \qquad (n \in \mathbb{N}).$$

The operator T is W-*mean stable* if $W_n[T] \to 0$ in the strong operator topology. The discrete analog of Theorem 1 is the following.

Theorem 7. *Let $T \in B(X)$ be power bounded. Then the following statements are equivalent*:

(i) *$I - T$ has dense range.*

(ii) *For all $W \in \mathcal{W}(a, b, c)$, the operator T is W-mean stable.*

(iii) *For some $\tilde{W} \in \mathcal{W}(a, b, d)$, some subsequence $\{\tilde{W}_{n_k}[T]\}$ converges to zero in the weak operator topology.*

Theorem 7 follows trivially from the following

Theorem 8. *Let $T \in B(X)$ be power bounded. Then the following statements are equivalent for a vector $x \in X$*:

(i) $x \in \overline{range\ (I-T)}$.

(ii) *For all* $W \in \mathcal{W}(a,b,c)$, *the sequence* $\{W_n[T]x\}$ *converges strongly to zero.*

(iii) *For some* $\tilde{W} \in \mathcal{W}(a,b,d)$, *some subsequence* $\{\tilde{W}_{n_k}[T]x\}$ *converges weakly to zero.*

In case of infinite *square* weight matrices $A = (a_{nk})_{n,k\in\mathbb{N}}$, we consider the following properties:

(a) For each $n \in \mathbb{N}$, $0 \le a_{nk}$ is *monotonic* with respect to $k \in \mathbb{N}$, and $\lim_k a_{nk} = 0$.

(b) If $c_n(A) := \sum_{k=1}^{\infty} a_{nk}$, then $K := \sup_n c_n(A) < \infty$.

(c) $\lim_n a_{n1} = 0$.

The class of square matrices A with Properties (a), (b), and (c) will be denoted by $\mathcal{A}(a,b,c)$ (with similar notation for other sets of conditions).

For example, if $f : (0,\infty) \to [0,\infty)$ is a non-increasing function such that $f(\infty) = 0$, $tf(t) \to 0$ as $t \to 0+$, and $\int_0^\infty f(t)dt = c \neq 0$, the matrix A with entries $a_{nk} = f(k/n)/n$ belongs to $\mathcal{A}(a,b,c)$. Actually, $A \in \mathcal{A}(a,b',c)$, where (b′) is the following condition (stronger than (b) and (d) below):

(b′) $\exists \lim_n c_n(A) = c \neq 0$.

Taking, e.g., $f(t) = e^{-t}$, or $f(t) = (2/\sqrt{\pi})e^{-t^2}$, or $f(t) = \Gamma(\alpha)^{-1}t^{\alpha-1}e^{-t}$, the induced weight matrices are the classical Abel, Gauss, and Gamma matrices, respectively.

As before, we shall also consider the condition

(d) $\liminf_n c_n(A) > 0$.

If $T \in B(X)$ is power bounded and the matrix A of nonnegative weights satisfies (b), the weighted averages $A_n[T]$ are well-defined by

$$A_n[T] = \sum_{k=1}^{\infty} a_{nk}T^k \qquad (n \in \mathbb{N}),$$

the series converges in operator-norm, and $\|A_n[T]\| \le MK$ for all $n \in \mathbb{N}$. We say that T is A-*mean stable* if $A_n[T] \to 0$ in the strong operator topology. The version of Theorem 7 for square matrices is Theorem 9 below. A similar restatement of Theorem 8 in this case is valid (we shall omit the obvious details).

Theorem 9. *Let* $T \in B(X)$ *be power bounded. Then the following statements are equivalent*:

(i) $I - T$ *has dense range.*

(ii) *For all* $A \in \mathcal{A}(a,b,c)$, *the operator* T *is* A*-mean stable.*

(iii) *For some* $\tilde{A} \in \mathcal{A}(a,b,d)$, *some subsequence* $\{\tilde{A}_{n_k}[T]\}$ *converges to zero in the weak operator topology.*

1.7. A version of Theorem 1 for cosine operator functions can be obtained for an adequate family of weights. To avoid technicalities, we consider only the fractional integration weight $h_\beta(t,s) := (\beta/t^\beta)s^{\beta-1}$ $(\beta > 0)$, and we write W_t^β instead of $W_t^{h_\beta}$.

Theorem 10. *Let* $C(\cdot)$ *be a bounded* C_0*-cosine operator function with generator* A*. The following statements are equivalent*:

(i) A *has dense range*;

(ii) $C(\cdot)$ *is* W^β*-mean stable for all* $\beta > 0$;

(iii) $\lim_n W_{t_n}^{\beta_0}[C(\cdot)] = 0$ *in the weak operator topology for some* $\beta_0 > 0$ *and some positive sequence* $\{t_n\}$ *diverging to* ∞.

2. Proofs

Proof of Theorem 2. (i) *implies* (ii). Since $\|W_t^h[T(\cdot)]\| \leq MK$ (where $M = \sup\|T(\cdot)\|$) for all $t > 0$ and $h \in \mathcal{W}(1,2)$, it follows from (i) that it suffices to prove the relation

$$\lim_{t\to\infty} W_t^h[T(\cdot)]\,Ay = 0 \quad \text{for all } y \in D(A).$$

With δ as in Condition (4) and $t > \delta$, we write

$$W_t^h[T(\cdot)]\,Ay = \left(\int_0^\delta + \int_\delta^t\right) h(t,s)T(s)Ay\,ds = J_1 + J_2.$$

Clearly,

$$\|J_1\| \leq M\,\|Ay\| \int_0^\delta h(t,s)ds. \tag{1}$$

Since $T(s)Ay = \frac{d}{ds}T(s)y$ for $y \in D(A)$, an integration by parts shows that

$$J_2 = h(t,t)T(t)y - h(t,\delta)T(\delta)y - \int_\delta^t T(s)y d_s h(t,s). \tag{2}$$

By the monotonicity assumption on h (Condition (1)), the Stieltjes integral in (2) has norm $\leq M\,\|y\|\,|h(t,t) - h(t,\delta)|$. Therefore, by (1) and (2),

$$\|W_t^h[T(\cdot)]Ay\| \leq M\,\|Ay\| \int_0^\delta h(t,s)ds + 2M\,\|y\|\Big(h(t,t) + h(t,\delta)\Big) \to 0$$

as $t \to \infty$, by Conditions (3) and (4) on the weight h and by (*).

(ii) *implies* (iii). Obvious, because $\mathcal{W}(1,2,3,4,5)$ is nonempty.

(iii) *implies* (i). Assume that (iii) is valid for some h_0 and $\{t_n\}$ as described, and some $x = x_0 \notin \overline{range\ A}$. By the Hahn-Banach theorem, there exists $x_0^* \in X^*$ such that $\langle x_0^*, Ax \rangle = 0$ for all $x \in D(A)$ and $\langle x_0^*, x_0 \rangle = 1$. Hence, for all $x \in D(A)$,

$$\frac{d}{dt}\langle x_0^*, T(t)x \rangle = \langle x_0^*, AT(t)x \rangle = 0,$$

and therefore $\langle x_0^*, T(t)x \rangle = \langle x_0^*, T(0)x \rangle = \langle x_0^*, x \rangle$. Thus (for all $x \in D(A)$ and $t > 0$)

$$\langle x_0^*, W_t^{h_0}[T(\cdot)]x \rangle = \langle x_0^*, x \rangle \int_0^t h_0(t,s)ds. \tag{3}$$

Since $D(A)$ is dense in X, it follows by continuity that (3) is valid for *all* $x \in X$ and $t > 0$. Taking $x = x_0$ and $t = t_n$ in (3), and then letting $n \to \infty$, we obtain from (iii) that $\lim_n \int_0^{t_n} h_0(t_n,s)ds = 0$, contradicting Condition (5). ■

Proof of Theorem 3. (i) *implies* (ii). As before, it suffices to prove that

$$\lim_{t\to\infty} \|A_t^h[T(\cdot)]\, Ay\| = 0 \qquad (y \in D(A))$$

for all $h \in \mathcal{A}(a,b,c)$.

Let $h \in \mathcal{A}(a,b,c)$. We write

$$A_t^h[T(\cdot)]\, Ay = \int_0^\delta h(t,s)T(s)Ay\, ds + \lim_{v\to\infty} \int_\delta^v h(t,s)(T(s)y)'_s ds$$

(with δ as in Condition (c)). The first summand has norm $\le M\,\|Ay\| \int_0^\delta h(t,s)ds$. For $v > \delta$, the integral over $[\delta, v]$ can be written as

$$h(t,v)T(v)y - h(t,\delta)T(\delta)y - \int_\delta^v T(s)y\, d_s h(t,s).$$

By the monotonicity of $h(t,\cdot)$, we can estimate the norm of the above expression by

$$M\,\|y\|\Big(h(t,v) + h(t,\delta) + |h(t,v) - h(t,\delta)|\Big),$$

and therefore the integral over $[\delta, v]$ has norm $\le 2M\,\|y\|\Big(h(t,v) + h(t,\delta)\Big)$. Letting $v \to \infty$ and recalling that $h(t,\cdot)$ vanishes at ∞ (Condition (a)), we conclude that

$$\|A_t^h[T(\cdot)]\, Ay\| \le M\,\|Ay\| \int_0^\delta h(t,s)ds + 2M\,\|y\|\, h(t,\delta) \to 0$$

as $t \to \infty$, by (**) and Condition (c).

The implication (iii) implies (i) is proved as in Theorem 2, with the averaging operators A_t^h replacing W_t^h. ∎

Proof of Theorem 4. The first part of the proof is identical to the proof of the implication (i) *implies* (ii) in Theorem 2, because $S'(t)x = S(t)Ax$ for any $x \in D(A)$ (see [1] or [5, Theorem 2.3]).

On the other hand, if range A is not dense in X, then there exists $x_0^* \neq 0$ in X^* such that $\langle x_0^*, Ax\rangle = 0$ for all $x \in D(A)$. Therefore

$$\frac{d}{dt}\langle x_0^*, S(t)x\rangle = \langle x_0^*, AS(t)x\rangle = 0 \qquad (x \in D(A)),$$

since $S(t)D(A) \subset D(A)$ for all $t \geq 0$. Hence $\langle x_0^*, S(t)x\rangle = \langle x_0^*, S(0)x\rangle$ for all $t \geq 0$, and so (for h_0 and $\{t_n\}$ as in the theorem)

$$\langle x_0^*, S(0)x\rangle \int_0^{t_n} h_0(t_n, s)ds = \langle x_0^*, W_{t_n}^{h_0}[S(\cdot)]x\rangle \to 0$$

as $n \to \infty$. Consequently, $\langle x_0^*, S(0)x\rangle = 0$ for all $x \in D(A)$ by Condition (5). Since range $S(0) \subset \overline{D(A)}$ (cf. [5, Theorem 2.3]) and range $S(0)$ is dense, it follows that $D(A)$ is dense as well, and therefore, by continuity, $\langle x_0^*, S(0)x\rangle = 0$ for all $x \in X$. Using again the density of range $S(0)$, we reach the contradiction $x_0^* = 0$. ∎

Similar modifications in the proof of Theorems 3 show that Theorem 4 is valid for the A-means as well.

Proof of Theorem 5. Let $M = M_1M_2$, where M_k are bounds for the given semigroups. Then $||W^h[T(\cdot)ZS(\cdot)]|| \leq MK||Z||$, and it suffices therefore to prove W^h-mean stability (in the s.o.t.) of the operator function $T(\cdot)ZS(\cdot)$ for Z in the range of Δ, say $Z = \Delta V$ with $V \in D(\Delta)$. By [2, Proposition 5], $\mathcal{G}(\cdot)V$ is differentiable in the s.o.t. for all $t \geq 0$, with (s.o.t.-)derivative equal to $\mathcal{G}(\cdot)Z = T(\cdot)ZS(\cdot)$. The integration by parts argument in the proof of Theorem 2 (resp., Theorem 3) yields the result for W^h-means (resp., A^h-means). ∎

Corollary 6 follows from the observation that since $C \in B(X)$, the identity operator I belongs to $D(\Delta)$, and $\Delta I = (B + C) - B = C$.

Proof of Theorem 8. (i) *implies* (ii). Let $W \in \mathcal{W}(a, b, c)$. For $n \geq 2$, we have, by Abel's summation formula,

$$\begin{aligned} W_n[T](I-T) &= \sum_{k=1}^{n} w_{nk}(T^k - T^{k+1}) \\ &= w_{n1}T - w_{nn}T^{n+1} + \sum_{k=2}^{n}(w_{nk} - w_{n,k-1})T^k. \end{aligned}$$

By the monotonicity condition in (a),

$$\Big\|\sum_{k=2}^{n}(\ldots)\Big\| \le M\sum_{k=2}^{n}|w_{nk}-w_{n,k-1}| = M\,\Big|\sum_{k=2}^{n}(w_{nk}-w_{n,k-1})\Big| = M\,|w_{nn}-w_{n1}|,$$

where $M := \sup_k \|T^k\|$.

Therefore,

$$\Big\|W_n[T](I-T)\Big\| \le 2M(w_{n1}+w_{nn}) \to 0$$

as $n\to\infty$ by Property (c).

This shows that

$$\lim_n ||W_n[T]x|| = 0 \qquad (x \in \text{range } (I-T)).$$

Since $\|W_n[T]\| \le MK$ $(n\in\mathbb{N})$ by Property (b), it follows that

$$\lim_n \|W_n[T]x\| = 0 \qquad (x\in \overline{\text{range } (I-T)}), \tag{4}$$

and consequently (i) implies (ii).

(ii) *implies* (iii). Trivial, since $\mathcal{W}(a,b,c,d)\neq\emptyset$.

(iii) *implies* (i). Suppose (iii) holds for some x_0, but $x_0\notin\overline{\text{range } (I-T)}$. By the Hahn-Banach theorem, there exists $x_0^*\in X^*$ such that for all $x\in X$,

$$\langle x_0^*, (I-T)x\rangle = 0 \quad \text{and} \quad \langle x_0^*, x_0\rangle = 1.$$

Therefore $T^*x_0^* = x_0^*$, and it follows that for any matrix W

$$W_n[T]^*x_0^* = c_n(W)x_0^* \quad (n\in\mathbb{N}).$$

Consequently,

$$\langle x_0^*, W_n[T]x_0\rangle = \langle W_n[T]^*x_0^*, x_0\rangle = c_n(W).$$

Taking $W=\tilde{W}$ and $n=n_k$ as in (iii), we get $c_{n_k}(W)\to 0$, contradicting Condition (d). ■

Proof of Theorem 9. By Abel's summation formula, for all $n\ge 2$,

$$A_n[T](I-T) = a_{n1}T + \sum_{k=2}^{\infty}(a_{nk}-a_{n,k-1})T^k.$$

Let $A \in \mathcal{A}(a,b,c)$. By Properties (a) and (c),

$$\begin{aligned}\|A_n[T](I-T)\| &\le Ma_{n1} + M\sum_{k=2}^{\infty}|a_{nk}-a_{n,k-1}| \\ &= Ma_{n1} + M\Big|\textstyle\sum_{k=2}^{\infty}(a_{nk}-a_{n,k-1})\Big| = 2Ma_{n1} \to 0\end{aligned}$$

as $n \to \infty$, and it follows from the uniform boundedness of the averages $A_n[T]$ that $\lim_n \|A_n[T]x\| = 0$ for all $x \in \overline{\text{range }(I-T)}$; consequently, (i) implies (ii).

The argument in the proof of Theorem 8 with W replaced by A shows that (iii) implies (i). ■.

Proof of Theorem 10. Let $M := \sup\|C(\cdot)\|$. For $y \in D(A)$ and $t > 1$, we write

$$W_t^{\beta}[C(\cdot)]Ay = \beta t^{-\beta}\Big(\int_0^1 + \int_1^t\Big)s^{\beta-1}C(s)Ay\,ds = J_1 + J_2. \tag{5}$$

Clearly,

$$\|J_1\| \le M\,\|Ay\|\,t^{-\beta}. \tag{6}$$

Let $S(\cdot)$ be the sine operator function associated with $C(\cdot)$, that is, $S(t)x := \int_0^t C(s)x\,ds \quad (x \in X)$. Since $C(\cdot)Ay = [C(\cdot)y]''$, two successive integrations by parts yield the formula (in case $\beta \neq 2$):

$$\begin{aligned}J_2 =& \beta t^{-1}S(t)Ay - \beta t^{-\beta}S(1)Ay - \beta(\beta-1)t^{-2}C(t)y \\ &+\beta(\beta-1)t^{-\beta}C(1)y + \beta(\beta-1)(\beta-2)t^{-\beta}\int_1^t s^{\beta-3}C(s)y\,dy.\end{aligned} \tag{7}$$

By [4, Theorem 2], $\|t^{-1}S(t)Ay\| = O(1/t)$ as $t \to \infty$. The norm of all other integrated terms is either $O(t^{-\beta})$ or $O(t^{-2})$. The norm of the last term in (7) is

$$\le \beta|\beta-1|\,|\beta-2|t^{-\beta}M\,\|y\|\int_1^t s^{\beta-3}ds = O(t^{-\gamma}),$$

where $\gamma = \min\{\beta, 2\}$. We conclude from (7) that $\|J_2\| = O(t^{-\delta})$, where $\delta = \min\{\beta, 1\}$.

In case $\beta = 2$, the first integration by parts gives

$$J_2 = 2t^{-1}S(t)Ay - 2t^{-2}S(1)Ay - 2t^{-2}\int_1^t [C(s)y]'ds.$$

The last term is equal to $2t^{-2}[C(t)y - C(1)y]$, and its norm is $O(t^{-2})$; hence $||J_2|| = O(t^{-1}) = O(t^{-\delta})$.

We conclude from (5) and (6) that for all $x \in$ range (A),

$$\|W_t^\beta[C(\cdot)]x\| = O(t^{-\delta}), \tag{8}$$

where $\delta = \min\{\beta, 1\}$. Since $\|W_t^\beta[C(\cdot)]\| \leq M$, it follows from (8) that (i) implies (ii).

(iii) *implies* (i). Assume (iii) holds, and suppose range (A) is not dense in X. There exists $x_0^* \neq 0$ in X^* such that $\langle x_0^*, Ax\rangle = 0$ for all $x \in D(A)$. Fix $x \in D(A)$. We have

$$\frac{d^2}{dt^2}\langle x_0^*, C(t)x\rangle = \langle x_0^*, AC(t)x\rangle = 0$$

for all t, and so

$$\langle x_0^*, C(t)x\rangle = c_0 + c_1 t \tag{9}$$

for some constants $c_k,\ k = 0, 1$. Then for all $\beta > 0$,

$$\langle x_0^*, W_t^\beta[C(\cdot)]x\rangle = c_0 + \frac{\beta}{\beta+1}c_1 t.$$

If $c_1 \neq 0$, we get a contradiction to (iii) by taking $t = t_n$, $\beta = \beta_0$, and letting $n \to \infty$. Hence $c_1 = 0$, and we get $c_0 = \langle x_0^*, x\rangle$ by taking $t = 0$ in (9). Choosing again $t = t_n$ and $\beta = \beta_0$, we get from (iii),

$$\langle x_0^*, x\rangle = \langle x_0^*, W_{t_n}^{\beta_0}[C(\cdot)]x\rangle \to 0$$

as $n \to \infty$. Hence $\langle x_0^*, x\rangle = 0$ for all $x \in D(A)$, and therefore $x_0^* = 0$ by the denseness of $D(A)$, contradiction. ∎

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Shmuel Kantorovitz
Department of Mathematics, Bar-Ilan University
52900 Ramat-Gan, Israel

Serguei Piskarev
Scientific Research Computer Center, Moscow State University
119899 Moscow, Russia