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C_0 CONTRACTIONS QUASISIMILAR TO IRREDUCIBLE ONES

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Abstract. Let T be a C_0 contraction. Using the Jordan model, we prove that T is quasisimilar to an irreducible operator if and only if T is not quadratic and $T - \lambda I$ is not finite-rank for any complex number λ.

1. Introduction

Throughout this paper, all operators are bounded and linear on complex Hilbert spaces. An operator is said to be *irreducible* if it commutes with no (orthogonal) projection other than 0 and I, and is said to be *reducible* if otherwise. Two operators T and T' are said to be *quasisimilar* if there exist one-to-one operators X and Y with dense range such that $T'X = XT$ and $TY = YT'$. Consider the problem of whether an operator is quasisimilar to an irreducible one. For the finite-dimensional case, similarity and quasisimilarity are the same, and in fact this problem has been solved in [11]. On the other hand, for nonseperable Hilbert spaces, every operator is reducible. Therefore, it remains to study this problem on infinite-dimensional seperable Hilbert spaces. Gilfeather [8] proved that every normal operator without eigenvalue is similar to an irreducible operator. Later on, Fong and Jiang [7] improved Gilfeather's work by allowing the presence of eigenvalues. Then Hsin [12] extended their work to quasinormal operators (an operator T is said to be *quasinormal* if T commutes with T^*T).

In general, such results are obtained by considering certain special models of the respective operators such as the spectral decomposition for normal operators and the Jordan form in the finite-dimensional case (cf. [7, 8, 11, 12]). In this paper,

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we will deal with the C_0 contractions (defined later) and use their Jordan model to prove the following Main Theorem. An operator T is said to be *quadratic* if $T^2 + \alpha T + \beta I = 0$ for some complex numbers α and β.

Main Theorem. *A C_0 contraction T (on an infinite-dimensional separable Hilbert space) is quasisimilar to an irreducible operator if and only if T is not quadratic and $T - \lambda I$ is not finite-rank for any complex number λ.*

This is analogous to the one in the finite-dimensional case [11].

We now prove the necessity part of the Main Theorem. As usual, we use $\mathcal{H}$ and $\mathcal{K}$ to denote Hilbert spaces, and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ to denote the set of all operators from $\mathcal{H}$ to $\mathcal{K}$. In particular, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Proposition 1.1. *Let $T \in \mathcal{B}(H)$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. If T is quasisimilar to an irreducible operator, then T is not quadratic, and $T - \lambda I$ is not finite-rank for any $\lambda \in \mathbb{C}$.*

Proof. We first show that if T is quadratic, then T' is reducible whenever T' is quasisimilar to T. Because T is quadratic, so is T'. Thus it suffices to show that every quadratic operator T is reducible. This has been proved by Gilfeather [8] using the structure theory of binormal operators [2]. Here we give an alternative proof. We know that there exist Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ such that T is unitarily equivalent to

$$\alpha I \oplus \beta I \oplus \begin{bmatrix} \alpha I & T_1 \\ 0 & \beta I \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_3)$$

for some $\alpha, \beta \in \mathbb{C}$, and a one-to-one positive operator $T_1 \in \mathcal{B}(\mathcal{H}_3)$ [14]. Therefore, it suffices to consider the case when

$$T = \begin{bmatrix} \alpha I & T_1 \\ 0 & \beta I \end{bmatrix} \in \mathcal{B}(\mathcal{H}_3 \oplus \mathcal{H}_3),$$

where $\mathcal{H}_3$ is infinite-dimensional. Since T_1 is normal, by the spectral theorem, there is a nontrivial projection P on $\mathcal{H}_3$ such that $PT_1 = T_1 P$. Then $P \oplus P$ is a nontrivial projection commuting with T and hence T is reducible.

To complete the proof, we suppose that there exists $n \in \mathbb{N}$ such that $0 <$ rank $(T - \lambda I) < n$ for some $\lambda \in \mathbb{C}$. For any operator $T' \in \mathcal{B}(H)$ which is quasisimilar to T, let $\mathcal{M}$ be the linear span of the ranges of $T' - \lambda I$ and $T'^* - \bar{\lambda} I$. Thus $1 \leq$ dim $\mathcal{M} \leq 2n - 2$. Let P be the projection from $\mathcal{H}$ onto the subspace $\mathcal{M}$. Then P is a nontrivial projection which commutes with T'. Hence T' is reducible. This proves the proposition. ∎

A contraction T is said to be *completely nonunitary* if for any nonzero reducing subspace $\mathcal{K}$ for T, $T \mid \mathcal{K}$ (the restriction of T to $\mathcal{K}$) is not unitary. Let $C = \{z \in \mathbb{C} : |z| = 1\}$, and let μ be the Lebesgue measure on C normalized so that $\mu(C) = 1$. In addition, for $n \in \mathbb{Z}$, let $e_n(z) = z^n$ for $z \in C$, and let H^∞ be the set of all functions $f \in L^\infty(C)$ for which $\int_C f\bar{e}_n d\mu = 0$ for $n = -1, -2, -3, \cdots$. A completely nonunitary contraction T is said to be a C_0 *contraction* if there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$. Recall that a function $\theta \in H^\infty$ is said to be *inner* if $|\theta(z)| = 1$ for almost all z, $|z| = 1$. Let $M = \{v \in H^\infty : v(T) = 0\}$. Then M is an ideal in H^∞. Moreover, $M = vH^\infty$ for some inner function v. In this case, v is called the *minimal function* of T [1, p. 17]. It is clear that the minimal function is unique up to a constant of modulus one.

Before we prove the Main Theorem, we introduce an application in the following. Let T be a nonzero algebraic operator. Namely, there exists a polynomial $p(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n}$ such that $p(T) = 0$, where $\lambda_j' s$ are distinct complex numbers, and each $k_n \in \mathbb{N}$. Let

$$q(z) = \prod_{i=1}^{n} \left(\frac{\bar{\alpha_i}}{\alpha_i} \cdot \frac{z - \alpha_i}{1 - \bar{\alpha_i} z} \right)^{k_i},$$

where each $\alpha_i = \lambda_i/(2\|T\|)$. Then $T/(2\|T\|)$ is a C_0 contraction with the minimal function q. Therefore, an algebraic operator T is quasisimilar to an irreducible operator if and only if T is not quadratic and $T - \lambda I$ is not finite-rank for any complex number λ.

Next, we introduce the Jordan model of a C_0 contraction. For an inner function θ, we define $S(\theta)$ as follows. Let S be the simple unilateral shift on the Hardy space H^2. Consider the Hilbert space $H(\theta)$ given by

$$H(\theta) = H^2 \ominus \theta H^2,$$

and the operator $S(\theta) \in \mathcal{B}(H(\theta))$ defined by

$$S(\theta) = P_{H(\theta)} S \mid H(\theta),$$

where $P_{H(\theta)}$ is the projection from H^2 onto $H(\theta)$. In other words, $S(\theta)$ is the compression of the simple unilateral shift S to $H(\theta)$. Note that $S(\theta)$ is a C_0 contraction with θ as its minimal function [1, pp. 19-20]. For a C_0 contraction T, there exist a number γ which is either a positive integer or ∞ and a family of nonconstant inner functions $\{\theta_i : i < \gamma\}$ with $\theta_{i+1} \mid \theta_i$ for each i, such that T is quasisimilar to the Jordan operator $\oplus_{i<\gamma} S(\theta_i)$. In addition, $\oplus_{i<\gamma} S(\theta_i)$ is uniquely determined by either of the relations $(\oplus_{i<\gamma} S(\theta_i))X = XT$ or $TY = Y(\oplus_{i<\gamma} S(\theta_i))$, where X and Y are one-to-one operators with dense range. In this case, $\oplus_{i<\gamma} S(\theta_i)$ is called the *Jordan model* of T. Our main reference for C_0 contractions is the monograph [1].

By the Jordan model of C_0 contractions, it suffices to prove the following theorem for the sufficiency part of the Main Theorem. For $T \in \mathcal{B}(\mathcal{H})$, we use $T^{(n)}$ to denote the direct sum $\underbrace{T \oplus \cdots \oplus T}_{n}$ on the Hilbert space $\mathcal{H}^{(n)} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n}$ for $1 \leq n \leq \infty$.

Theorem 1.2. *For $m \in \mathbb{N}$ or $m = \infty$, let $\{\theta_i\}_{i=1}^m$ be a family of distinct nonconstant inner functions with $\theta_{i+1} \mid \theta_i$ for each i. Let $T = \sum_{i=1}^{m} \oplus S(\theta_i)^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i. If T is not quadratic and $T - \lambda I$ is not finite-rank for any complex number λ, then T is similar to an irreducible operator.*

We will divide the proof of Theorem 1.2 into the following two cases:

$$\text{Case (A): } H(\theta_1) \text{ is finite-dimensional,} \tag{1.1}$$

and

$$\text{Case (B): } H(\theta_1) \text{ is infinite-dimensional.} \tag{1.2}$$

Proof of Theorem 1.2 for Case (A). By [1, p. 43, Ex. 19], there exists $k \in \mathbb{N}$ such that $\theta_1(z) = \prod_{j=1}^{k} (\frac{z-\lambda_j}{1-\bar{\lambda}_j z})^{n_j}$, where $|\lambda_j| < 1$, $n_j \in \mathbb{N}$ and the $\lambda_j's$ are distinct. Thus, $S(\theta_1)$ is similar to

$$\sum_{j=1}^{k} \oplus \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}$$

on $\sum_{j=1}^{k} \oplus \mathbb{C}^{n_j}$. Let

$$J = \sum_{j=1}^{k} \oplus J(\lambda_j),$$

where $J(\lambda_j)$ is the direct sum of all Jordan blocks associated with λ_j of some $S(\theta_i)$. Thus the sizes of the Jordan blocks in $J(\lambda_j)$ are at most n_j for $1 \leq j \leq k$. It is easy to see that T is similar to J. By the result in [11], J is similar to an irreducible operator. ∎

The rest of this paper is devoted to proving Case (B) of Theorem 1.2. Note that under the assumptions of Theorem 1.2, θ_1 is the minimal function of T. We first introduce the structure of an inner function. We say that f is a *Blaschke product* if

$$f(z) = \prod_{i=1}^{k} \left(\frac{\bar{\lambda}_i}{\lambda_i} \cdot \frac{z - \lambda_i}{1 - \bar{\lambda}_i z} \right)^{k_i},$$

where $k \in \mathbb{N} \cup \{\infty\}$, λ_i's are distinct complex numbers with $|\lambda_i| < 1$, $k_i \in \mathbb{N}$, and $\sum_{i=1}^{k} k_i(1 - |\lambda_i|) < \infty$. We say that g is a *singular function* if

$$g(z) = c \exp\left(- \int \frac{w+z}{w-z} d\mu(w) \right) \text{ for } |z| < 1,$$

where $|c| = 1$, μ is a finite positive Borel measure on $\{z \in \mathbb{C} : |z| = 1\}$ singular with respect to the Lebesgue measure. Every inner function is the product of a Blaschke product and a singular function [1, pp. 22-23]. Since θ_1 is an inner function, $\theta_1 = f \cdot g$, where f is a Blaschke product, and g is a singular function [1, pp. 22-23]. We prove Case (B) of Theorem 1.2 seperately for the conditions where θ_1 is a Blaschke product, a singular function, or the product of a Blaschke product and a singular function. Namely, we will divide the proof of Case (B) of Theorem 1.2 into the following three conditions in Proposition 1.3.

Proposition 1.3. *For $m \in \mathbb{N}$ or $m = \infty$, let $\{\theta_i\}_{i=1}^m$ be a family of distinct nonconstant inner functions with $\theta_{i+1} \mid \theta_i$ for each i. Let $T = \sum_{i=1}^{m} \oplus S(\theta_i)^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i. Then T is similar to an irreducible operator under any of the following conditions*:

(1) $\theta_1(z) = \prod_{n=1}^{\infty} (\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z-\lambda_n}{1-\bar{\lambda}_n z})^{k_n}$, *where λ_n's are distinct complex numbers with $|\lambda_n| < 1$, $k_n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} k_n(1 - |\lambda_n|) < \infty$,*

(2) $\theta_1(z) = \exp(-\int \frac{w+z}{w-z} d\mu(w))$, *where μ is a finite positive Borel measure on $\{z \in \mathbb{C} : |z| = 1\}$ singular with respect to the Lebesgue measure,*

(3) $\theta_1(z) = \prod_{n=1}^{k} (\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z-\lambda_n}{1-\bar{\lambda}_n z})^{k_n} \cdot \exp(-\int \frac{w+z}{w-z} d\mu(w))$, *where $k \in \mathbb{N}$ or $k = \infty$, λ_n's are distinct complex numbers with $|\lambda_n| < 1$, $k_n \in \mathbb{N}$, $\sum_{n=1}^{k} k_n(1-|\lambda_n|) < \infty$, and μ is a finite positive Borel measure on $\{z \in \mathbb{C} : |z| = 1\}$ singular with respect to the Lebesgue measure.*

We will consider conditions (1), (2) and (3) of Proposition 1.3 in Sections 2, 3 and 4 respectively.

2. Blaschke product

The purpose of this section is to consider condition (1) of Proposition 1.3. An operator A is said to be *hyponormal* if $A^*A \geq AA^*$. The next two lemmas will be useful in future computations.

Lemma 2.1. *Let θ be an inner function in H^∞, and let $A \in \mathcal{B}(H(\theta))$ commute with $S(\theta)$. Then A is a scalar operator under either of the following conditions*:

(1) *A is hyponormal*,

(2) *A^* commutes with $S(\theta)$.*

Proof. Part (1) follows from [9] directly. For part (2), since A and A^* commute with $S(\theta)$, both A^*A and AA^* are scalars by part (1). So either A or A^* is hyponormal. It follows that A or A^* is a scalar by part (1) again. This completes the proof. ■

Lemma 2.2. *Let θ_1 and θ_2 be inner functions in H^∞ with $\theta_2 \mid \theta_1$, and $A \in \mathcal{B}(H(\theta_1), H(\theta_2))$. If $AS(\theta_1) = S(\theta_2)A$, then A is unitarily equivalent to $[\ 0\ \ A'\ ]$ from $H(\theta_3) \oplus H(\theta_2)$ to $H(\theta_2)$, where $\theta_3 = \theta_1/\theta_2$, and A' commutes with $S(\theta_2)$. In addition, if $A^*S(\theta_2) = S(\theta_1)A^*$, then $A = 0$.*

Proof. Since $\theta_1 = \theta_3 \cdot \theta_2$, the operator $S(\theta_1)$ is unitarily equivalent to

$$\begin{bmatrix} S(\theta_3) & X \\ 0 & S(\theta_2) \end{bmatrix}$$

on $H(\theta_3) \oplus H(\theta_2)$ for some $X \in \mathcal{B}(H(\theta_2), H(\theta_3))$. Hence A is unitarily equivalent to $[\ A''\ \ A'\ ]$ from $H(\theta_3) \oplus H(\theta_2)$ to $H(\theta_2)$. Since $AS(\theta_1) = S(\theta_2)A$, we have $A\theta_2(S(\theta_1)) = \theta_2(S(\theta_2))A = 0$ and hence $A(\text{ran}\ \theta_2(S(\theta_1))) = A(\theta_2 H^2 \ominus \theta_1 H^2) = 0$. It follows that $A'' = 0$. In addition, if $A^*S(\theta_2) = S(\theta_1)A^*$, then both A' and A'^* commute with $S(\theta_2)$. By part (2) of Lemma 2.1, we know that A' is a scalar. By $A^*S(\theta_2) = S(\theta_1)A^*$ again, we get $XA^* = 0$. Note that $X \neq 0$ for $S(\theta_1)$ is irreducible [1, p. 43, Ex. 13]. We conclude that $A' = 0$ as asserted. ■

Definition 2.3 [6, p. 32]. Let $\mathcal{X}$ be a Banach space and $\{x_i\}_{i=1}^\infty$ a sequence in $\mathcal{X}$. We say that $\{x_i\}_{i=1}^\infty$ is infinite-linearly independent if whenever $\sum\limits_{i=1}^\infty \alpha_i x_i = 0$, where each α_i is a scalar, we have $\alpha_i = 0$ for each i. In addition, $\{x_i\}_{i=1}^\infty$ is called a Schauder basis for $\mathcal{X}$ if for each $x \in \mathcal{X}$, there exists a sequence $\{\alpha_i\}_{i=1}^\infty$ of scalars such that $x = \sum\limits_{i=1}^\infty \alpha_i x_i$.

Lemma 2.4 [6, p. 39]. *Every infinite-dimensional Banach space contains an infinite-dimensional closed linear subspace with a Schauder basis.*

For condition (1) of Proposition 1.3, We prove a more general case as follows.

Lemma 2.5. *For $m \in \mathbb{N}$ or $m = \infty$, let $\{\theta_i\}_{i=1}^m$ be a family of distinct nonconstant inner functions with $\theta_{i+1} \mid \theta_i$ for each i. Let $T = \sum_{i=1}^{m} \oplus S(\theta_i)^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i. Suppose that $\theta_1 = \phi_1 \cdot \psi_1$, where ϕ_1 and ψ_1 are relatively prime nonconstant inner functions, and $H(\psi_1)$ is infinite-dimensional. Suppose also that $\phi_i =$ g.c.d.(θ_i, ϕ_1) and $\psi_j =$ g.c.d.(θ_j, ψ_1), and n_1, n_2 are respectively the largest index such that ϕ_{n_1} and ψ_{n_2} are nonconstant inner functions. If $n_1 \geq n_2$, then T is similar to an irreducible operator.*

Proof. We may assume that $m = n_1 = n_2 = \infty$ since the proofs for the other situations are similar. For the sake of clarity, we will divide this proof into four steps.

Step (1). In this step, we construct an operator T' which is unitarily equivalent to T. Since $\theta_i = \phi_i \cdot \psi_i$, there exists a nonzero $A_i \in \mathcal{B}(H(\psi_i), H(\phi_i))$ such that $S(\theta_i)$ is unitarily equivalent to

$$\begin{bmatrix} S(\phi_i) & A_i \\ 0 & S(\psi_i) \end{bmatrix} \in \mathcal{B}(H(\phi_i) \oplus H(\psi_i)).$$

Let

$$T_1 = \sum_{i=1}^{\infty} \oplus S(\phi_i)^{(m_i)},$$

$$T_2 = \sum_{i=1}^{\infty} \oplus S(\psi_i)^{(m_i)},$$

and

$$T_3 = \sum_{i=1}^{\infty} \oplus A_i^{(m_i)}.$$

Then T is unitarily equivalent to

$$T' = \begin{bmatrix} T_1 & T_3 \\ 0 & T_2 \end{bmatrix} \in \mathcal{B}\left(\left(\sum_{i=1}^{\infty} \oplus H(\phi_i)^{(m_i)}\right) \oplus \left(\sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)}\right)\right).$$

Step (2). Now we construct an invertible operator X so that $A \equiv X^{-1}T'X$ will eventually be shown to be irreducible. Define $\pi : \mathcal{B}(H(\psi_1),\ H(\phi_1)) \longrightarrow \mathcal{B}(H(\psi_1),\ H(\phi_1))$ by $\pi(X) = S(\phi_1)X - XS(\psi_1)$ for $X \in \mathcal{B}(H(\psi_1),\ H(\phi_1))$.

Since ϕ_1 and ψ_1 are relatively prime, π is one-to-one. Since $H(\phi_1)$ is infinite-dimensional, the range of π is also infinite-dimensional. By Lemma 2.4, there exists a family of operators $\{Y_{1,j}\}_{j=2}^{m_1}$ in $\mathcal{B}(H(\psi_1),\ H(\phi_1))$ such that $\{S(\phi_1)Y_{1,j} - Y_{1,j}S(\psi_1)\}_{j=2}^{m_1}\bigcup\{A_1\}$ are infinite-linearly independent and $\|Y_{1,j}\| < \frac{1}{j^2}$ for each j. Similarly, for each $i \geq 2$, there exist operators $\{Y_{i,j}\}_{j=1}^{m_i}$ in $\mathcal{B}(H(\psi_1),\ H(\phi_i))$ such that $\{S(\phi_i)Y_{i,j} - Y_{i,j}S(\psi_1)\}_{j=1}^{m_i}$ are infinite-linearly independent and $\|Y_{i,j}\| \leq 1/(i^2j^2)$ for each $i \geq 2$ and j. Let

$$Y_1 = \begin{bmatrix} 0 & 0 & \cdots \\ Y_{1,2} & 0 & \cdots \\ Y_{1,3} & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{B}(H(\psi_1)^{(m_1)}, H(\phi_1)^{(m_1)}),$$

and for each $i \geq 2$, let

$$Y_i = \begin{bmatrix} Y_{i,1} & 0 & \cdots \\ Y_{i,2} & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{B}(H(\psi_1)^{(m_1)}, H(\phi_i)^{(m_i)}).$$

Let

$$Y = \begin{bmatrix} Y_1 & 0 & \cdots \\ Y_2 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{B}\left(\sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)}, \sum_{i=1}^{\infty} \oplus H(\phi_i)^{(m_i)}\right),$$

and

$$X = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}.$$

Then X is bounded and invertible.

Step (3). We now consider the operator $A = X^{-1}T'X$, which is similar to T. Let $B_{1,j} = S(\phi_1)Y_{1,j} - Y_{1,j}S(\psi_1) \in \mathcal{B}(H(\psi_1), H(\phi_1))$ for $j \geq 2$. For $i \geq 2$ and each j, let $B_{i,j} = S(\phi_i)Y_{i,j} - Y_{i,j}S(\psi_1) \in \mathcal{B}(H(\psi_1), H(\phi_i))$. Let

$$X_1 = \begin{bmatrix} A_1 & & & \\ B_{1,2} & A_1 & & \\ B_{1,3} & & A_1 & \\ \vdots & & & \ddots \end{bmatrix} \in \mathcal{B}(H(\psi_1)^{(m_1)}, H(\phi_1)^{(m_1)}),$$

and for $i \geq 2$, let

$$X_i = \begin{bmatrix} B_{i,1} & 0 & \cdots \\ B_{i,2} & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{B}(H(\psi_1)^{(m_1)}, H(\phi_i)^{(m_i)}).$$

Let $B_i = A_i^{(m_i)}$, and let

$$B = \begin{bmatrix} X_1 & & & \\ X_2 & B_2 & & \\ X_3 & & B_3 & \\ \vdots & & & \ddots \end{bmatrix} \in \mathcal{B}\left(\sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)}, \sum_{i=1}^{\infty} \oplus H(\phi_i)^{(m_i)}\right).$$

Then

$$A = X^{-1} T' X = \begin{bmatrix} T_1 & B \\ 0 & T_2 \end{bmatrix}.$$

Step (4). We now show that A is irreducible. Let

$$\begin{bmatrix} P & R \\ R^* & Q \end{bmatrix} \in \mathcal{B}\left(\sum_{i=1}^{\infty} \oplus H(\phi_i)^{(m_i)} \oplus \sum_{i=1}^{\infty} \oplus H(\phi_i)^{(m_i)}\right)$$

be a projection which commutes with A. By $R^* T_1 = T_2 R^*$, we have $R^* = 0$ since ϕ_i and ψ_j have no common factor other than 1 for each pair of i and j [4]. It follows that $PT_1 = T_1 P$. Let $P = [P_{i,j}]_{i,j=1}^{\infty}$, where $P_{i,j} = [P_{(i,j)(k,\ell)}]_{k=1,\ell=1}^{m_i,m_j} \in \mathcal{B}(H(\phi_j)^{(m_j)}, H(\phi_i)^{(m_i)})$. Thus $P_{(i,i)(k,\ell)} S(\phi_i) = S(\phi_i) P_{(i,i)(k,\ell)}$. By Lemma 2.1, we may assume that $P_{(i,i)(k,\ell)} = \alpha_{(i)(k,\ell)} I$ for some $\alpha_{(i)(k,\ell)} \in \mathbb{C}$. Moreover, for $i \neq j$, we have

$$P_{(i,j)(k,\ell)} S(\phi_j) = S(\phi_i) P_{(i,j)(k,\ell)}.$$

By Lemma 2.3, $P_{(i,j)(k,\ell)} = 0$ for $i \neq j$. Thus we may assume that

$$P = \sum_{i=1}^{\infty} \oplus P_i,$$

where $P_i = [\alpha_{(i)(k,\ell)} I]_{k,\ell=1}^{m_i}$ on $H(\phi_i)^{(m_i)}$ with $\alpha_{(i)(k,\ell)} \in \mathbb{C}$. Similarly, we may assume that

$$Q = \sum_{i=1}^{\infty} \oplus Q_i,$$

where $Q_i = [\beta_{(i)(k,\ell)} I]_{k,\ell=1}^{m_i}$ on $H(\psi_i)^{(m_i)}$ with $\beta_{(i)(k,\ell)} \in \mathbb{C}$. By the $(i,1)$-entry of $PB = BQ$, we get $P_i X_i = X_i Q_1$ and so, by the infinite-linear independence of the $B_{i,j}$'s for each i, each P_i is diagonal. By the (i,i)-entries of $PB = BQ$ and the fact that A_i's are nonzero, we see that each Q_i is diagonal. By $(P \oplus Q)A = A(P \oplus Q)$, it is then easy to see that $P \oplus Q = 0$ or I, and so A is irreducible. This completes the proof. ■

Proof of Condition (1) *of Proposition* 1.3. Let

$$\phi_1(z) = \prod_{\ell=1}^{\infty} \left(\frac{\bar{\lambda}_{2\ell}}{\lambda_{2\ell}} \cdot \frac{z-\lambda_{2\ell}}{1-\bar{\lambda}_{2\ell} z} \right)^{k_{2\ell}}$$

and

$$\psi_1(z) = \prod_{\ell=1}^{\infty} \left(\frac{\bar{\lambda}_{2\ell+1}}{\lambda_{2\ell+1}} \cdot \frac{z-\lambda_{2\ell+1}}{1-\bar{\lambda}_{2\ell+1} z} \right)^{k_{2\ell+1}}.$$

Then ψ_1 and ϕ_1 are relatively prime, $\theta_1 = \phi_1 \cdot \psi_1$, and both $H(\phi_1)$ and $H(\psi_1)$ are infinite-dimensional. So the requirements in Lemma 2.5 are all satisfied. Hence T is similar to an irreducible operator. ■

3. Singular Function

The purpose of this section is to consider condition (2) of Proposition 1.3. Assume that the hypotheses of condition (2) of Proposition 1.3 hold.

First of all, if the singular function θ_1 is not of the form $\exp(\alpha(z+\lambda)/(z-\lambda))$ for $|\lambda| = 1$ and $\alpha > 0$, then there exist relatively prime nonconstant inner functions ϕ_1 and ψ_1 such that $\theta_1 = \phi_1 \cdot \psi_1$ [13, p. 136]. Moreover, since both $H(\phi_1)$ and $H(\psi_1)$ are infinite-dimensional, we may apply Lemma 2.5 to conclude that T is similar to an irreducible operator. Therefore, we only have to consider the case when

$$\theta_1(z) = \exp\left(\alpha \frac{z+\lambda}{z-\lambda}\right)$$

for some $\alpha > 0$ and $|\lambda| = 1$. Without loss of generality, we may assume in condition (2) of Proposition 1.3 that

$$\theta_i(z) = \exp\left(\alpha_i \frac{z+1}{z-1}\right), \text{ where } 1 = \alpha_1 > \alpha_2 > \cdots > 0.$$

For such α_i's, let $V_i \in \mathcal{B}(L^2(0,\alpha_i))$ be the Volterra operator on $L^2(0,\alpha_i)$ by

$$(V_i f)(x) = \int_0^x f(t)dt, \quad f \in L^2(0,\alpha_i). \tag{3.1}$$

Since $(I + S(\theta_i))(I - S(\theta_i))^{-1} = V_i$ [1, p. 99], the assertion of Proposition 1.3 under condition (2) will follow from the following.

Proposition 3.1. *For $m \in \mathbb{N}$ or $m = \infty$, let $\{\alpha_i\}_{i=1}^m$ be a family of distinct numbers with $1 = \alpha_1 > \alpha_2 > \cdots > 0$, and let V_i be the Volterra operator* *on* $L^2(0,\alpha_i)$ *for each* i. *Then the operator* $S = \sum_{i=1}^{m} \oplus V_i^{(m_i)}$, *where* $m_i \in \mathbb{N}$ *or* $m_i = \infty$ *for each* i, *is similar to an irreducible operator.*

Since $(I + S(\theta_i))(I - S(\theta_i))^{-1} = V_i$, the next two lemmas are consequences of Lemmas 2.1 and 2.2 respectively.

Lemma 3.2. *Suppose that* $\alpha > 0$, $(Vf)(x) = \int_0^x f(t)dt$ *for* $f \in L^2(0,\alpha)$, *and* $A \in \mathcal{B}(L^2(0,\alpha))$ *commutes with* V. *Then* A *is a scalar operator if either* A *is hyponormal or* A^* *commutes with* V.

Lemma 3.3. *Let* $\alpha_1 > \alpha_2 > 0$, *and for* $i = 1, 2$, *let* $(V_i f)(x) = \int_0^x f(t)dt$ *for* $f \in L^2(0,\alpha_i)$. *If* $AV_1 = V_2 A$, *then* A *is unitarily equivalent to* $[\ A'\ 0\ ]$ *from* $L^2(0,\alpha_2) \oplus L^2(\alpha_2,\alpha_1)$ *to* $L^2(0,\alpha_2)$. *In addition, if* $A^* V_2 = V_1 A^*$, *then* $A = 0$.

For an operator $A \in \mathcal{B}(\mathcal{H})$, Lat A denotes the collection of all invariant subspaces for A,

$$\{A\}' = \{X \in \mathcal{B}(\mathcal{H}) : XT = TX\}, \quad \text{the commutant of } A,$$

and

$$\text{Alg } A = \overline{\{p(A) : p \text{ is a polynomial }\}}$$

under the weak (or strong) operator topology. Similarly, for operators $A, B \in \mathcal{B}(\mathcal{H})$,

$$\text{Alg } (A,B) = \overline{\{p(A,B) : p(\cdot,\cdot) \text{ is a polynomial of two variables }\}}$$

under the weak (or strong) operator topology. Meanwhile,

$$\text{Alg Lat } A = \{B \in \mathcal{B}(\mathcal{H}) : \ \text{Lat } A \subset \text{ Lat } B\}.$$

Let V be the Volterra operator on $L^2(0,1)$ defined by

$$(Vf)(x) = \int_0^x f(t)dt, \quad f \in L^2(0,1),$$

and $M \in \mathcal{B}(L^2(0,1))$ be the multiplication operator

$$(Mf)(x) = xf(x), \quad f \in L^2(0,1).$$

It is known that

$$\{V\}' = \text{ Alg } V,$$

$$\text{Lat } V = \{L^2(\alpha,1) : 0 \le \alpha \le 1\},$$

and

$$\text{Alg }(V, M) = \text{Alg Lat } V.$$

In addition, V is irreducible (cf. [1, p. 99] and [5, pp. 51-57]). For $A \in \mathcal{B}(\mathcal{H})$, A is said to be *reflexive* if Alg A = Alg Lat A [1, p. 73].

Lemma 3.4. *Let V and M be the operators defined above, and let X, $Y \in \mathcal{B}(L^2(0,1))$ be such that X is self-adjoint and Y commutes with V. If $(X - (MV^i/i)Y)V = V(X - (MV^i/i)Y)$ for some $i \in \mathbb{N}$, then X is a scalar operator and $Y = 0$.*

Proof. The assumption $(X - (MV^i/i)Y)V = V(X - (MV^i/i)Y)$ implies that $X = (MV^i/i)Y + B$, where Y, $B \in$ Alg V. It follows that $X \in$ Alg Lat V and hence $X \in$ Alg (V, M). Therefore, by the self-adjointness of X, we have $XL^2(\lambda_1, \lambda_2) \subset L^2(\lambda_1, \lambda_2)$ for any λ_1 and λ_2 with $0 \leq \lambda_1 \leq \lambda_2 \leq 1$. Let

$$\mathcal{A} = \{\triangle \subset (0,1) : \triangle \text{ is Borel subset of } (0,1) \text{ and } XL^2(\triangle) \subseteq L^2(\triangle)\}.$$

We notice that $\mathcal{A}$ is a σ-algebra which contains all open subintervals (λ_1, λ_2), $0 \leq \lambda_1 \leq \lambda_2 \leq 1$. And so

$$\text{Lat } X = \{L^2(\triangle) : \triangle \text{ is Borel subset of } (0,1)\} = \text{ Lat } M.$$

So $X \in$ Alg Lat M. Since M is self-adjoint, M is reflexive [3, p. 291]. Hence $X \in$ Alg M. We may assume that $X = M_\phi$ for some $\phi \in L^\infty(0,1)$. Thus

$$\begin{aligned} &M_\phi V - VM_\phi \\ &= XV - VX \\ &= ((MV^i/i)Y + B)V - V((MV^i/i)Y + B) \\ &= (MV - VM)(V^i/i)Y \\ &= V^{i+2}Y/i \in \{V\}', \end{aligned} \tag{3.2}$$

and therefore,

$$(M_\phi V - VM_\phi)V = V(M_\phi V - VM_\phi).$$

Apply the operators on the two sides of this equation to $h(s) \equiv 1$ on $(0,1)$. A simple computation with differention yields $\phi(x) = ax + b$ for some scalars a, b. That is, $X = aM + bI$. The self-adjointness of X implies that a and b are real. By (3.2) again, we get $aV^2 = (V^iY/i)V^2$. Because V has dense range, we may assume that $aI = (V^{i-1}Y/i)V$. Note that V is not invertible. So $a = 0$, $Y = 0$, and so $X = bI$ for some $b \in \mathbb{C}$. Hence we complete the proof. ■

We are now ready to prove Proposition 3.1. Note that Lemma 3.4 still holds if $L^2(0,1)$ is replaced by $L^2(0,\beta)$ for any $\beta \geq 0$.

Lemma 3.5. *For $m \in \mathbb{N}$ or $m = \infty$, let $\{\alpha_i\}_{i=1}^m$ be a family of distinct numbers with $1 = \alpha_1 > \alpha_2 > \cdots > 0$, and let V_i be the Volterra operator on $L^2(0, \alpha_i)$. If $S = \sum_{i=1}^{m} \oplus V_i^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i, then S is similar to an irreducible operator.*

Proof. Without loss of generality, we may assume that $m = \infty$ since the proofs for the other situations are similar. For convenience, we divide the proof into three steps.

Step (1). Here we construct an invertible operator X' so that the operator $A = X'SX'^{-1}$ will be irreducible. Let M_i be the multiplication operator on $L^2(0, \alpha_i)$ as before. Let $X_{1,j} = M_1 V_1^j / j$ for each $j \geq 2$. For each $i \geq 2$, we have

$$V_1 = \begin{bmatrix} V_i & 0 \\ K_i & V_{1,i} \end{bmatrix} \text{ on } L^2(0, \alpha_i) \oplus L^2(\alpha_i, 1),$$

where $(V_{1,i} f)(x) = \int_{\alpha_i}^{x} f(t)dt$. Similarly, let $X_{i,j} = \begin{bmatrix} M_i V_i^j / j & 0 \end{bmatrix}$ for each $i \geq 2$ and each j. Let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \end{bmatrix} \in \mathcal{B}(L^2(0,1)^{(m_1-1)} \oplus \sum_{i=2}^{\infty} \oplus L^2(0, \alpha_i)^{(m_i)}, L^2(0,1)),$$

where

$$X_1 = \begin{bmatrix} X_{1,2} \\ X_{1,3} \\ \vdots \end{bmatrix}, \text{ and } X_i = \begin{bmatrix} X_{i,1} \\ X_{i,2} \\ \vdots \end{bmatrix} \text{ for } i \geq 2.$$

Define

$$X' = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \in \mathcal{B}(L^2(0,1) \oplus (L^2(0,1)^{(m_1-1)} \oplus \sum_{i=2}^{\infty} \oplus L^2(0, \alpha_i)^{(m_i)})).$$

Then X' is bounded and invertible.

Step (2). We now consider $A = X'SX'^{-1}$. Let

$$S_i = V_i^{(m_i)} \in \mathcal{B}(L^2(0, \alpha_i)^{(m_i)}),$$

and

$$S = \sum_{i=1}^{\infty} \oplus S_i.$$

Also, let $Y_{1,j} = X_{1,j}V_1 - V_1X_{1,j}$ for each j, and let $Y_{i,j} = \begin{bmatrix} X_{i,j}V_i - V_iX_{i,j} & 0 \end{bmatrix}$ from $L^2(0,\alpha_i)\oplus L^2(\alpha_i,1)$ to $L^2(0,\alpha_i)$ for each $i \geq 2$ and j. By direct computations, we have $Y_{1,j} = V_1^{j+2}/j$ and $Y_{i,j} = \begin{bmatrix} V_i^{j+2}/j & 0 \end{bmatrix}$. Let

$$A_1 = \begin{bmatrix} V_1 & & & \\ Y_{1,2} & V_1 & & \\ Y_{1,3} & & V_1 & \\ \vdots & & & \ddots \end{bmatrix} \in \mathcal{B}(L^2(0,\alpha_1)^{(m_1)}).$$

For each $i \geq 2$, let

$$A_i = \begin{bmatrix} Y_{i,1} & 0 & \cdots \\ Y_{i,2} & 0 & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \in \mathcal{B}(L^2(0,\alpha_i)^{(m_i)}).$$

Thus, T is similar to

$$A = X'TX'^{-1} = \begin{bmatrix} A_1 & & & \\ A_2 & S_2 & & \\ A_3 & & S_3 & \\ \vdots & & & \ddots \end{bmatrix} \in \mathcal{B}\left(\sum_{i=1}^{\infty} \oplus L^2(0,\alpha_i)^{(m_i)}\right).$$

Step (3). We now prove that A is irreducible. Let $P = [P_{i,j}]_{i,j=1}^{\infty}$ be a projection which commutes with A, where

$$P_{i,j} = [P_{(i,j)(k,\ell)}]_{k=1,\ell=1}^{m_i,m_j} \in \mathcal{B}(L^2(0,\alpha_j)^{(m_j)}, L^2(0,\alpha_i)^{(m_i)}).$$

By $PA = AP$, we have, for each $j = 1$ and $\ell > 1$, and each $j \geq 2$ and $\ell \leq 1$,

$$P_{(1,j)(1,\ell)}V_j = V_1P_{(1,j)(1,\ell)}, \tag{3.3}$$

and

$$P_{(j,j)(\ell,\ell)}V_j = Y_{j,\ell}P_{(1,j)(1,\ell)} + V_jP_{(j,j)(\ell,\ell)}. \tag{3.4}$$

If $j = 1$, then (3.3) implies that $P_{(1,1)(1,\ell)}$ commutes with V_1. Besides, (3.4) implies that

$$P_{(1,1)(\ell,\ell)} - (V_1^{j+2}/j)P_{(1,1)(1,\ell)}$$

commutes with V_1. By Lemma 3.4, $P_{(1,1)(1,\ell)} = 0$. If $j > 1$, then because V_j is unitarily equivalent to V_j^*, by (3.3) and Lemma 3.3,

$$P_{(1,j)(1,\ell)} = \begin{bmatrix} Q_{(1,j)(1,\ell)} \\ 0 \end{bmatrix} \in \mathcal{B}(L^2(0,\alpha_j), L^2(\alpha_j,\alpha_1) \oplus L^2(0,\alpha_j)),$$

where $Q_{(1,j)(1,\ell)}$ commutes with V_j. It follows that

$$P_{(j,j)(\ell,\ell)} - (V_i^{j+2}/j)Q_{(1,j)(1,\ell)}$$

commutes with V_j. By Lemma 3.4 again, $Q_{(1,j)(1,\ell)} = 0$. Now, we may assume that $P = R \oplus Q$, where R on $L^2(0,1)$ commutes with V_1, and Q on $L^2(0,1)^{(m_1-1)} \oplus \sum_{i=2}^{\infty} \oplus L^2(0,\alpha_i)^{(m_i)}$ commutes with $V_1^{(m_1-1)} \oplus \sum_{i=2}^{\infty} \oplus V_i^{(m_i)}$. Since V_1 is irreducible, $R = 0$ or I. In addition, Lemmas 3.2 and 3.3 imply that

$$Q = [\beta_{(1)(k,\ell)} I]_{k,\ell=2}^{m_1} \oplus \sum_{i=2}^{\infty} \oplus [\beta_{(i)(k,\ell)} I]_{k,\ell=1}^{m_i},$$

where each $\beta_{(i)(k,\ell)} \in \mathbb{C}$. By $PA = AP$, we have, for each i and k,

$$\sum_{\ell=1}^{m_i} \beta_{(i)(k,\ell)} Y_{i,\ell} = Y_{i,k} R.$$

Note that for each i, $\{Y_{i,k}\}_{k=1}^{m_i} = \{V_i^{k+2}/k\}_{k=1}^{m_i}$ is infinite-linearly independent. Therefore, $\beta_{(i)(k,\ell)} = 0$ if $k \neq \ell$. Meanwhile, we have either $\beta_{(i)(k,k)} = 0$ when $R = 0$ or $\beta_{(i)(k,k)} = 1$ when $R = I$. Thus $P = 0$ or I, and so A is irreducible. This completes the proof. ■

Before we end this section, we should look at the following remark, which is helpful in the next (and final) section.

Remark 3.6. For $m \in \mathbb{N}$ or $m = \infty$, let $\{\alpha_j\}_{j=1}^{m}$ be a family of numbers with $1 = \alpha_1 > \alpha_2 > \cdots > 0$, and let V_i be the Volterra operator on $L^2(0,\alpha_i)$. Let $S = \sum_{i=1}^{m} \oplus V_i^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i, and let $T = \sum_{i=1}^{m} \oplus S(\psi_i)^{(m_i)}$, where $\psi_i(z) = \exp(\alpha_i(z+1)/(z-1))$. By the proof of Lemma 3.5, we know that S is similar to an irreducible operator

$$S' = \begin{bmatrix} S_1 & & & \\ X_2 & S_2 & & \\ X_3 & & \ddots & \\ \vdots & & & \ddots \end{bmatrix} \quad \text{on } \sum_{i=1}^{\infty} \oplus L^2(0,\alpha_i)^{(m_i)},$$

where $S_i = V_j$ for some j. Let

$$B = (I - S')(I + S')^{-1} = \begin{bmatrix} B_{11} & & & \\ B_{21} & B_{22} & & \\ B_{31} & B_{32} & B_{33} & \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{on } \sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)},$$

where $B_{ii} = (I - S_j)(I + S_j)^{-1} = S(\psi_j)$ for some j. Moreover, B is irreducible and is similar to T. In fact, there exists a block lower triangular operator

$$Z = \begin{bmatrix} I & & & \\ Z_{21} & I & & \\ Z_{31} & Z_{32} & I & \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad \text{on } \sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)}$$

such that $Z^{-1}AZ = B$.

4. The General Case

We want to prove Proposition 1.3 under condition (3) in this section. For $m \in \mathbb{N}$ or $m = \infty$, let $\{\theta_i\}_{i=1}^m$ be a family of distinct nonconstant inner functions with $\theta_{i+1}|\theta_i$ for each i. Let $T = \sum_{i=1}^{m} \oplus S(\theta_i)^{(m_i)}$, where $m_i \in \mathbb{N}$ or $m_i = \infty$ for each i. If

$$\theta_1(z) = \prod_{n=1}^{k} \left(\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z - \lambda_n}{1 - \bar{\lambda}_n z} \right)^{k_n} \cdot \exp\left(-\int \frac{w+z}{w-z} d\mu(w) \right), \tag{4.1}$$

where $k \in \mathbb{N}$ or $k = \infty$, λ_n's are distinct complex numbers with $|\lambda_n| < 1$, $k_n \in \mathbb{N}$, and μ is a finite positive Borel measure on $\{z \in \mathbb{C} : |z| = 1\}$ singular with respect to the Lebesgue measure, then we want to prove that T is similar to an irreducible operator.

Proof of Condition (3) *of Proposition* 1.3. At first, if $k = \infty$ in (4.1), then we define

$$\phi_1(z) = \prod_{\ell=1}^{\infty} \left(\frac{\bar{\lambda}_{2\ell}}{\lambda_{2\ell}} \cdot \frac{z - \lambda_{2\ell}}{1 - \bar{\lambda}_{2\ell} z} \right)^{k_{2\ell}} \cdot \exp\left(-\int \frac{w+z}{w-z} d\mu(w) \right),$$

and

$$\psi_1(z) = \prod_{\ell=0}^{\infty} \left(\frac{\bar{\lambda}_{2\ell+1}}{\lambda_{2\ell+1}} \cdot \frac{z - \lambda_{2\ell+1}}{1 - \bar{\lambda}_{2\ell+1} z} \right)^{k_{2\ell+1}}.$$

Then $\theta_1 = \phi_1 \cdot \psi_1$ with ϕ_1 and ψ_1 relatively prime. Both $H(\phi_1)$ and $H(\psi_1)$ are infinite-dimensional. By Lemma 2.5, T is similar to an irreducible operator.

Secondly, let $f_1(z) = \exp(-\int (w+z)/(w-z) d\mu(w))$. If $f_1(z) \neq \exp(\alpha(z + \lambda)/(z - \lambda))$ for any $\alpha > 0$ and $|\lambda| = 1$, then there exist relatively prime nonconstant inner functions g_1 and ψ_1, such that $f_1 = g_1 \cdot \psi_1$ [13, p. 136]. Let

$$\phi_1 = \prod_{n=1}^{k} \left(\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z - \lambda_n}{1 - \bar{\lambda}_n z} \right)^{k_n} \cdot g_1.$$

Then $\theta_1 = \phi_1 \cdot \psi_1$. Moreover, $H(\phi_1)$ and $H(\psi_1)$ are both infinite-dimensional, and so by Lemma 2.5 again, T is similar to an irreducible operator.

From now on, we only have to consider

$$\theta_1(z) = \prod_{n=1}^{k} \left(\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z - \lambda_n}{1 - \bar{\lambda}_n z} \right)^{k_n} \cdot \exp\left(\alpha \frac{z + \lambda}{z - \lambda} \right),$$

where $k \in \mathbb{N}$. Without loss of generality, we may assume that

$$\theta_1(z) = \prod_{n=1}^{k} \left(\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z - \lambda_n}{1 - \bar{\lambda}_n z} \right)^{k_n} \cdot \exp\left(\frac{z + 1}{z - 1} \right).$$

Let

$$\phi_1(z) = \prod_{n=1}^{k} \left(\frac{\bar{\lambda}_n}{\lambda_n} \cdot \frac{z - \lambda_n}{1 - \bar{\lambda}_n z} \right)^{k_n},$$

and

$$\psi_1(z) = \exp\left(\frac{z + 1}{z - 1} \right).$$

Let

$$\begin{aligned} \phi_n &= \text{ g.c.d. } (\theta_n, \phi_1), \\ \psi_n &= \text{ g.c.d. } (\theta_n, \psi_1) \end{aligned}$$

for each n. Then $\theta_n = \phi_n \cdot \psi_n$, and ϕ_n and ψ_n are relatively prime for each n. It is obvious that $H(\psi_1)$ is infinite-dimensional. Let n_1 and n_2 be the respective largest index such that ϕ_{n_1} and ψ_{n_2} are nonconstant inner functions. If $n_1 \geq n_2$, then we may apply Lemma 2.5 again and conclude that T is similar to an irreducible operator. Thus, we may assume that $n_1 < n_2$ from now on. Without loss of generality, we may assume that $\phi_1, \phi_2, \cdots, \phi_n, \psi_1, \psi_2, \cdots$ are nonconstant inner functions, and $\phi_{n+1}, \phi_{n+2}, \cdots$ are constant inner functions. For $i = 1, 2, \cdots, n$, $S(\theta_i)$ is unitarily equivalent to

$$\begin{bmatrix} S(\phi_i) & A_i \\ 0 & S(\psi_i) \end{bmatrix} \quad \text{on } H(\phi_i) \oplus H(\psi_i)$$

for some $A_i \in \mathcal{B}(H(\psi_i), H(\phi_i))$. Let

$$T_1 = \sum_{i=1}^{n} \oplus S(\phi_i)^{(m_i)},$$

$$T_2 = \sum_{i=1}^{\infty} \oplus S(\psi_i)^{(m_i)},$$

and

$$T_3 = \left[\begin{array}{cc} \sum_{i=1}^{n} \oplus A_i^{(m_i)} & 0 \end{array} \right].$$

Then T is unitarily equivalent to

$$T' = \left[\begin{array}{cc} T_1 & T_3 \\ 0 & T_2 \end{array} \right].$$

Let M_ℓ denote the set of all linear operators from $\mathbb{C}^\ell$ to itself. We may assume that $T_1 \in M_\ell$ for some $\ell \in \mathbb{N}$. Let $S(\psi_{n+1}) = [d_{i,j}]_{i,j=1}^{\infty}$, and let $d_i = (d_{i,\ell+1}, d_{i,\ell+2}, \cdots)$ for $i = 1, 2, \cdots \ell$. Since $S(\psi_{n+1})$ is irreducible, $S(\psi_{n+1})$ is not finite-rank, and so we may assume that $\{d_i\}_{i=1}^{\ell}$ is linearly independent. Let

$$X = \left[\begin{array}{ccc} 0 & Y & 0 \end{array} \right]$$

from

$$\left(\sum_{i=1}^{n} \oplus H(\psi_i)^{(m_i)}) \oplus H(\psi_{n+1}) \oplus (H(\psi_{n+1})^{(m_{n+1}-1)} \oplus \sum_{i=n+2}^{\infty} \oplus H(\psi_i)^{(m_i)} \right)$$

to $\mathbb{C}^\ell$, where

$$Y = \left[\begin{array}{cc} I_\ell & 0 \end{array} \right]$$

from $H(\psi_{n+1})$ to $\mathbb{C}^\ell$. By Remark 3.6, there exists

$$Z = \left[\begin{array}{cccc} I & & & \\ Z_{21} & I & & \\ Z_{31} & Z_{32} & I & \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \text{ on } \sum_{i=1}^{\infty} \oplus H(\psi_i)^{(m_i)},$$

such that

$$Z^{-1} T_2 Z = B = \left[\begin{array}{cccc} B_{11} & & & \\ B_{21} & B_{22} & & \\ B_{31} & B_{32} & B_{33} & \\ \vdots & \vdots & \ddots & \ddots \end{array} \right] \text{ on } \sum_{i=1}^{\infty} H(\psi_i)^{(m_i)}$$

is irreducible. Then T is similar to

$$T'' = \left[\begin{array}{cc} I & X \\ 0 & Z \end{array} \right]^{-1} T' \left[\begin{array}{cc} I & X \\ 0 & Z \end{array} \right] = \left[\begin{array}{cc} T_1 & T_3 Z + T_1 X - X B \\ 0 & B \end{array} \right].$$

Let $T_4 = T_3 Z + T_1 X - XB$. A direct computation leads to

$$-XB = \left[\begin{array}{c|ccc|c} & d_{1,1} & d_{1,2} & \cdots & \\ & d_{2,1} & d_{2,2} & \cdots & \\ T_5 & \vdots & \vdots & \vdots & 0 \\ & d_{\ell,1} & d_{\ell,2} & \cdots & \end{array}\right]$$

for some $T_5 \in \mathcal{B}(\sum_{i=1}^{n} \oplus H(\psi_i)^{(m_i)}, \mathbb{C}^\ell)$. Let

$$D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_\ell \end{bmatrix}.$$

By direct computations, there exist $T_6 \in \mathcal{B}(\sum_{i=1}^{n} \oplus H(\psi_i)^{(m_i)}, \mathbb{C}^\ell)$ and $T_7 \in M_\ell$ such that

$$T_4 = \begin{bmatrix} T_6 & T_7 & D & 0 \end{bmatrix}$$

from

$$\left(\sum_{i=1}^{n} \oplus H(\psi_i)^{(m_i)}\right) \oplus \mathbb{C}^\ell \oplus \Big(H(\psi_{n+1}\Big) \ominus \mathbb{C}^\ell) \oplus (H(\psi_{n+1})^{(m_{n+1}-1)} \oplus \sum_{i=n+2}^{m} \oplus H(\psi_i)^{(m_i)})$$

to $\mathbb{C}^\ell$. Since $\{d_i\}_{i=1}^{\ell}$ is linearly independent, T_4 is onto. Now it suffices to show that

$$\begin{bmatrix} T_1 & T_4 \\ 0 & B \end{bmatrix}$$

is irreducible. Let

$$P = \begin{bmatrix} P_1 & P_3 \\ P_3^* & P_2 \end{bmatrix}$$

be a projection which commutes with

$$\begin{bmatrix} T_1 & T_4 \\ 0 & B \end{bmatrix}.$$

It immediately follows that $P_3^* T_1 = B P_3^*$. Since ϕ_i and ψ_j are relatively prime for all i and j, we have $P_3^* = 0$ [4]. Hence P_2 is a projection which commutes with

B. The irreducibility of B forces $P_2 = 0$ or $P_2 = I$. By $P_1T_4 = T_4P_2$ and because T_4 is onto, either $P_1 = 0$ (if $P_2 = 0$) or $P_1 = I$ (if $P_2 = I$). Therefore, $P = 0$ or I and so

$$\begin{bmatrix} T_1 & T_4 \\ 0 & B \end{bmatrix}$$

is irreducible. Therefore, we complete the proof. ∎

So far, we have proved Propositions 1.3. Namely, Case (B) of Theorem 1.2 is proved and so we complete the proof of the Main Theorem.

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