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STRONG CONVERGENCE TO COMMON FIXED POINTS OF INFINITE NONEXPANSIVE MAPPINGS AND APPLICATIONS

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Abstract. In this paper, we first consider an iteration scheme given by an infinite family of nonexpansive mappings in Banach spaces and then prove a strong convergence theorem for the family of nonexpansive mappings. Using this result, we consider the feasibility problem of finding a solution of infinite convex inequalities and the problem of finding a common fixed point of infinite nonexpansive mappings.

1. INTRODUCTION

Let H be a Hilbert space, let $C_1, C_2, \ldots, C_r$ be nonempty closed convex subsets of H such that the intersection C_0 is nonempty, and let I be the identity operator on H. Given only the metric projections P_i of H onto C_i $(i = 1, 2, \ldots, r)$, find a point of C_0 by an iterative scheme. Such a problem is connected with the feasibility problem. In fact, let $\{g_1, g_2, \ldots, g_r\}$ be a finite family of real-valued continuous convex functions on H. Then the feasibility problem is to find a solution of the finite convex inequality system, i.e., to find such a point $x \in C_0$ that

$$C_0 = \{x \in H | g_i(x) \leq 0, i = 1, 2, \ldots, r\}.$$

In 1991, Crombez [4] proved the following: Let $T = \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with $T_i = I + \lambda_i (P_i - I)$ for all i, $0 < \lambda_i < 2$, $\alpha_i > 0$ for $i = 0, 1, 2, \ldots, r$, $\sum_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \bigcap_{i=1}^{r} C_i$ is nonempty. Then starting from an arbitrary element x of H, the sequence $\{T^n x\}$ converges weakly to an element of C_0. Later, Kitahara and Takahashi [11] and Takahashi and

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Tamura [21] dealt with the feasibility problem by convex combinations of nonexpansive retractions in a uniformly convex Banach space; see also [17, 18]. In [11] and [21], they proved that an operator given by a convex combination of nonexpansive retractions in a uniformly convex Banach space is asymptotically regular and the set of fixed points of the operator is equal to the intersection of the ranges of nonexpansive retractions. Furthermore, using the results, they proved some weak convergence theorems which are connected with the feasibility problem. Takahashi and Shimoji [20] and Atsushiba and Takahashi [1] also proved convergence theorems for finite nonexpansive mappings in Banach spaces by using the iteration schemes of Das and Debata's type [5]; see also Takahashi and Tamura [22]. Recently, Kimura and Takahashi [9] established a weak convergence theorem for an infinite family of nonexpansive mappings which is connected with the feasibility problem and generalized the result in Takahashi and Shimoji [20]. On the other hand, Wittmann [24] considered the following iteration scheme in a Hilbert space H which has originally been introduced by Halpern [7]:

$$x_1 = x \in C \subset H, \;\; x_{n+1} = \beta_n x + (1-\beta_n)Tx_n$$

for every $n = 1, 2, \dots$, where $\{\beta_n\}$ is a real sequence in [0, 1] and T is a nonexpansive mapping of a closed convex subset C of H into itself, and showed that $\{x_n\}$ converges strongly to the element of the set $F(T)$ of fixed points of T which is nearest to x in $F(T)$ if T has a fixed point in C and $\{\beta_n\}$ satisfies $\lim_{n\to\infty} \beta_n = 0$, $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty$.

In this paper, motivated by Wittmann [24], we first consider an iteration scheme given by an infinite family of nonexpansive mappings and then prove a strong convergence theorem for the family of nonexpansive mappings by using the methods of proofs in Shioji and Takahashi [14] and Atsushiba and Takahashi [1]. Further more, using the result, we consider the feasibility problem of finding a solution of infinite convex inequalities and the problem of finding a common fixed point of infinite nonexpansive mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space and let I be the identity operator on E. Let C be a nonempty subset of E. Then, a mapping T of C into itself is said to be *nonexpansive* on C if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. And we denote by $F(T)$ the set of fixed points of T and by $R(T)$ the range of T. A mapping T of C into itself is said to be *asymptotically regular* if for every $x \in C$, $T^n x - T^{n+1} x$ converges to 0. Let D be a subset of C and let P be a mapping of C onto D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into itself is said to be a *retraction* if $P = P^2$. If a mapping P of C into itself is a retraction, then $Pz = z$ for every $z \in R(T)$. A subset D of C is said to be a (*sunny*) *nonexpansive retract* of C if there exists a (sunny) nonexpansive retraction of C onto D. Let E be a Banach space and let $S_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then, for every ε with $0 \leq \varepsilon \leq 2$, the *modulus* $\delta(\varepsilon)$ *of convexity* of a Banach space E is defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \,\middle|\, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be *uniformly convex* if

$$\delta_E(\varepsilon) > 0$$

for every $\varepsilon > 0$. A Banach space is also said to be *strictly convex* if

$$\left\| \frac{x+y}{2} \right\| < 1$$

for every $x, y \in S_E$ with $x \neq y$. A uniformly convex Banach space is strictly convex and reflexive. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$. A closed convex subset C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well-known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. The following result was proved by Kirk [10].

Theorem 2.1. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a non-expansive mapping of C into itself. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals f on E. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* | f(x) = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. The norm of E is said to be *Gâteaux differentiable* if

$$(*) \qquad \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. The norm of E is said to be *uniformly Gâteaux differentiable* if for each y in S_E, the limit $(*)$ is attained uniformly for x in S_E. It is said to be *Fréchet differentiable* if for each x in S_E, the limit $(*)$ is attained uniformly for y in S_E. If E is a Banach space with a Gâteaux differentiable norm, we have

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, J(y)\rangle$$

for every $x, y \in E$; see [16]. A Banach space E is said to satisfy *Opial's condition* [12] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

where $\rightharpoonup$ denotes the weak convergence. Let μ be a mean on $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf_{n\in\mathbb{N}} a_n \leq \mu(f) \leq \sup_{n\in\mathbb{N}} a_n$$

for each $f = (a_1, a_2, \dots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean μ on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \dots) \in l^\infty$ with $a_n \to a$ and let μ be a Banach limit on $\mathbb{N}$. Then, $\mu(f) = \mu_n(a_n) = a$. We also know the following proposition [14].

Proposition 2.2. *Let a be a real number and let $(a_1, a_2, \dots) \in l^\infty$ such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then, $\limsup_{n\to\infty} a_n \leq a$.*

3. Lemmas

Let C be a convex subset of a Banach space E. Let $T_1, T_2, \dots$ be mappings of C into itself and let $\alpha_1, \alpha_2, \dots$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \dots$. Then, for any $n \in \mathbb{N}$, we define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1 - \alpha_n)I, \\
U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1 - \alpha_{n-1})I, \\
&\vdots \\
U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1 - \alpha_k)I, \\
U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1 - \alpha_{k-1})I, \\
&\vdots \\
U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1 - \alpha_2)I, \\
W_n = U_{n,1} &= \alpha_1 T_1 U_{n,2} + (1 - \alpha_1)I.
\end{aligned}$$

Such a mapping W_n is called the W-*mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and α_n, α_{n-1}, $\ldots$, α_1; see [18] and [20]. The following lemma was proved in [20].

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots$. For any $n \in \mathbb{N}$, let W_n be the W-mapping of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$.*

The following two lemmas are crucial in this paper.

Lemma 3.2. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$. Then for every $x \in C$, $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Proof. Let $x \in C$ and $w \in \bigcap_{n=1}^{\infty} F(T_n)$. Without loss of generality, we may assume $x \neq w$. Fix $k \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$ with $n \geq k$, we have

$$\begin{aligned}
& \|U_{n+1,k}x - U_{n,k}x\| \\
= \ & \|\alpha_k T_k U_{n+1,k+1}x + (1-\alpha_k)x - \alpha_k T_k U_{n,k+1}x - (1-\alpha_k)x\| \\
= \ & \alpha_k \|T_k U_{n+1,k+1}x - T_k U_{n,k+1}x\| \\
\leq \ & \alpha_k \|U_{n+1,k+1}x - U_{n,k+1}x\| \\
= \ & \alpha_k \|\alpha_{k+1} T_{k+1} U_{n+1,k+2}x + (1-\alpha_{k+1})x \\
& -\alpha_{k+1} T_{k+1} U_{n,k+2}x - (1-\alpha_{k+1})x\| \\
= \ & \alpha_k \alpha_{k+1} \|T_{k+1} U_{n+1,k+2}x - T_{k+1} U_{n,k+2}x\| \\
\leq \ & \alpha_k \alpha_{k+1} \|U_{n+1,k+2}x - U_{n,k+2}x\| \\
\vdots \ & \\
\leq \ & \left(\prod_{i=k}^{n} \alpha_i\right) \|U_{n+1,n+1}x - U_{n,n+1}x\| \\
= \ & \left(\prod_{i=k}^{n} \alpha_i\right) \|\alpha_{n+1} T_{n+1} U_{n+1,n+2}x + (1-\alpha_{n+1})x - x\| \\
= \ & \left(\prod_{i=k}^{n+1} \alpha_i\right) \|T_{n+1}x - x\| \\
\leq \ & \left(\prod_{i=k}^{n+1} \alpha_i\right) (\|T_{n+1}x - w\| + \|w - x\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\prod_{i=k}^{n+1} \alpha_i\right)(\|x-w\|+\|w-x\|) \\
&= 2\left(\prod_{i=k}^{n+1} \alpha_i\right)\|x-w\|.
\end{aligned}
$$

Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ with $n_0 \geq k$ such that

$$b^{n_0-k+2} < \frac{\varepsilon(1-b)}{2\|x-w\|}.$$

So, for every m, n with $m > n > n_0$, we have

$$
\begin{aligned}
\|U_{m,k}x - U_{n,k}x\| &\leq \sum_{j=n}^{m-1} \|U_{j+1,k}x - U_{j,k}x\| \\
&\leq \sum_{j=n}^{m-1} \left\{2\left(\prod_{i=k}^{j+1} \alpha_i\right)\|x-w\|\right\} \\
&\leq 2\|x-w\| \sum_{j=n}^{m-1} b^{j-k+2} \\
&\leq \frac{2b^{n-k+2}\|x-w\|}{1-b} \\
&< \varepsilon.
\end{aligned}
$$

Then, $\{U_{n,k}x\}$ is a Cauchy sequence, and hence $\lim_{n\to\infty} U_{n,k}x$ exists. ∎

For $k \in \mathbb{N}$, we define mappings $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k}x := \lim_{n\to\infty} U_{n,k}x$$

and

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-*mapping* generated by $T_1, T_2, \ldots$ and $\alpha_1, \alpha_2, \ldots$.

Lemma 3.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$. Then $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. Let $w \in \bigcap_{i=1}^{\infty} F(T_i)$. Then, it is obvious that $U_{n,k}w = w$ for all $n, k \in \mathbb{N}$ with $n \geq k$. So, we have $U_{\infty,k}w = w$ for all $k \in \mathbb{N}$. In particular, we have $Ww = U_{\infty,1}w = w$ and hence

$$\bigcap_{i=1}^{\infty} F(T_i) \subset F(W).$$

Next, we prove that

$$F(W) \subset \bigcap_{i=1}^{\infty} F(T_i).$$

Let $x \in F(W)$ and $y \in \bigcap_{i=1}^{\infty} F(T_i)$. Then we have

$$\begin{aligned}
& \|W_n x - W_n y\| \\
= \; & \|U_{n,1}x - U_{n,1}y\| \\
= \; & \|\alpha_1 T_1 U_{n,2}x + (1-\alpha_1)x - \alpha_1 T_1 U_{n,2}y - (1-\alpha_1)y\| \\
= \; & \alpha_1 \|T_1 U_{n,2}x - T_1 U_{n,2}y\| + (1-\alpha_1)\|x-y\| \\
\leq \; & \alpha_1 \|U_{n,2}x - U_{n,2}y\| + (1-\alpha_1)\|x-y\| \\
\vdots \; & \\
\leq \; & \left(\prod_{i=1}^{k-1} \alpha_i\right) \|U_{n,k}x - U_{n,k}y\| + \left(1 - \prod_{i=1}^{k-1} \alpha_i\right) \|x-y\| \\
\leq \; & \left(\prod_{i=1}^{k-1} \alpha_i\right) \|\alpha_k T_k U_{n,k+1}x + (1-\alpha_k)x - \alpha_k T_k U_{n,k+1}y - (1-\alpha_k)y\| \\
& + \left(1 - \prod_{i=1}^{k-1} \alpha_i\right) \|x-y\| \\
= \; & \left(\prod_{i=1}^{k-1} \alpha_i\right) \|\alpha_k (T_k U_{n,k+1}x - T_k U_{n,k+1}y) + (1-\alpha_k)(x-y)\| \\
& + \left(1 - \prod_{i=1}^{k-1} \alpha_i\right) \|x-y\| \\
\leq \; & \left(\prod_{i=1}^{k} \alpha_i\right) \|T_k U_{n,k+1}x - T_k U_{n,k+1}y\| + \left(1 - \prod_{i=1}^{k} \alpha_i\right) \|x-y\| \\
= \; & \left(\prod_{i=1}^{k} \alpha_i\right) \|U_{n,k+1}x - U_{n,k+1}y\| + \left(1 - \prod_{i=1}^{k} \alpha_i\right) \|x-y\| \\
\vdots \; & \\
\leq \; & \|x-y\|.
\end{aligned}$$

So, we have, as $n \to \infty$,

$$\begin{aligned}
&\|Wx - Wy\| \\
\leq\ & \left(\prod_{i=1}^{k-1} \alpha_i\right) \|\alpha_k(T_k U_{\infty,k+1}x - T_k U_{\infty,k+1}y) + (1-\alpha_k)(x-y)\| \\
&+ \left(1 - \prod_{i=1}^{k-1} \alpha_i\right) \|x-y\| \\
\leq\ & \left(\prod_{i=1}^{k} \alpha_i\right) \|T_k U_{\infty,k+1}x - T_k U_{\infty,k+1}y\| + \left(1 - \prod_{i=1}^{k} \alpha_i\right) \|x-y\| \\
\leq\ & \|x-y\|.
\end{aligned}$$

Since

$$\|Wx - Wy\| = \|x-y\|$$

and $0 < \alpha_i < 1$ for all $i \in \mathbb{N}$, we have, for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\|\alpha_k(T_k U_{\infty,k+1}x - T_k U_{\infty,k+1}y) + (1-\alpha_k)(x-y)\| \\
=\ & \|T_k U_{\infty,k+1}x - T_k U_{\infty,k+1}y\| \\
=\ & \|x-y\|.
\end{aligned}$$

Since E is strictly convex and $y \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$\begin{aligned}
x - y &= T_k U_{\infty,k+1}x - T_k U_{\infty,k+1}y \\
&= T_k U_{\infty,k+1}x - y
\end{aligned}$$

and hence

$$x = T_k U_{\infty,k+1}x.$$

On the other hand, from

$$U_{n,k+1}x = \alpha_{k+1} T_{k+1} U_{n,k+2}x + (1-\alpha_{k+1})x,$$

we have

$$\begin{aligned}
U_{\infty,k+1}x = \lim_{n\to\infty} U_{n,k+1}x &= \alpha_{k+1} T_{k+1} U_{\infty,k+2}x + (1-\alpha_{k+1})x \\
&= \alpha_{k+1}x + (1-\alpha_{k+1})x \\
&= x.
\end{aligned}$$

So, we have, for every $k \in \mathbb{N}$,

$$x = T_k U_{\infty,k+1}x = T_k x.$$

This implies $x \in \bigcap_{i=1}^{\infty} F(T_i)$ and hence $F(W) \subset \bigcap_{i=1}^{\infty} F(T_i)$. Therefore, we have

$$F(W) = \bigcap_{i=1}^{\infty} F(T_i). \qquad \blacksquare$$

Lemma 3.4. *Let C be a nonempty closed convex subset of a Banach space E, let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 \leq \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$ and $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For each $n \in \mathbb{N}$, let W_n be the W-*mapping *generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \ldots$. Then, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. Let $x \in C$ and $f \in \bigcap_{n=1}^{\infty} F(T_n)$. Setting $D = \{z \in C | \|f - z\| \leq \|f - x\|\}$, we have that D is a bounded closed convex subset of C such that $T_n D \subset D$ for all $n \in \mathbb{N}$ and $x \in D$. So, without loss of generality, we may assume that C is bounded. Put $K = \sup_{z\in C} \|z\|$. Then, we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= \ & \|\beta_n x + (1 - \beta_n) W_n x_n - (\beta_{n-1} x + (1 - \beta_{n-1}) W_{n-1} x_{n-1})\| \\
\leq \ & |\beta_n - \beta_{n-1}| \|x\| + \|(1 - \beta_n) W_n x_n - (1 - \beta_{n-1}) W_{n-1} x_{n-1}\| \\
\leq \ & |\beta_n - \beta_{n-1}| \|x\| + (1 - \beta_n) \|W_n x_n - W_{n-1} x_n\| \\
& +(1 - \beta_n) \|W_{n-1} x_n - W_{n-1} x_{n-1}\| + |\beta_n - \beta_{n-1}| \|W_{n-1} x_{n-1}\| \\
\leq \ & (1 - \beta_n) \|x_n - x_{n-1}\| + 2K |\beta_n - \beta_{n-1}| + (1 - \beta_n) \|W_n x_n - W_{n-1} x_n\|
\end{aligned}$$

and

$$\begin{aligned}
& \|W_n x_n - W_{n-1} x_n\| \\
= \ & \|U_{n,1} x_n - U_{n-1,1} x_n\| \\
= \ & \|\alpha_1 T_1 U_{n,2} x_n + (1 - \alpha_1) x_n - \{\alpha_1 T_1 U_{n-1,2} x_n + (1 - \alpha_1) x_n\}\| \\
= \ & \alpha_1 \|T_1 U_{n,2} x_n - T_1 U_{n-1,2} x_n\| \\
\leq \ & \alpha_1 \|U_{n,2} x_n - U_{n-1,2} x_n\| \\
= \ & \alpha_1 \|\alpha_2 T_2 U_{n,3} x_n + (1 - \alpha_2) x_n - \{\alpha_2 T_2 U_{n-1,3} x_n + (1 - \alpha_2) x_n\}\| \\
= \ & \alpha_1 \alpha_2 \|T_2 U_{n,3} x_n - T_2 U_{n-1,3} x_n\| \\
\leq \ & \alpha_1 \alpha_2 \|U_{n,3} x_n - U_{n-1,3} x_n\| \\
& \vdots
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\prod_{i=1}^{n-1} \alpha_i\right) \|U_{n,n}x_n - U_{n-1,n}x_n\| \\
&= \left(\prod_{i=1}^{n-1} \alpha_i\right) \|\alpha_n T_n x_n + (1-\alpha_n)x_n - x_n\| \\
&= \left(\prod_{i=1}^{n} \alpha_i\right) \|T_n x_n - x_n\| \\
&\le 2K \left(\prod_{i=1}^{n} \alpha_i\right).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
\le\ & (1-\beta_n)\|x_n - x_{n-1}\| + 2K|\beta_n - \beta_{n-1}| + 2K \left(\prod_{i=1}^{n} \alpha_i\right).
\end{aligned}
$$

Using this inequality, we have, for each $m \in \mathbb{N}$,

$$
\begin{aligned}
&\|x_{m+2} - x_{m+1}\| \\
\le\ & (1-\beta_{m+1})\|x_{m+1} - x_m\| + 2K|\beta_{m+1} - \beta_m| + 2K \left(\prod_{i=1}^{m+1} \alpha_i\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|x_{m+3} - x_{m+2}\| \\
\le\ & (1-\beta_{m+2})\|x_{m+2} - x_{m+1}\| + 2K|\beta_{m+2} - \beta_{m+1}| + 2K \left(\prod_{i=1}^{m+2} \alpha_i\right) \\
\le\ & (1-\beta_{m+2})(1-\beta_{m+1})\|x_{m+1} - x_m\| \\
& +2(1-\beta_{m+2})K|\beta_{m+1} - \beta_m| + 2(1-\beta_{m+2})K \left(\prod_{i=1}^{m+1} \alpha_i\right) \\
& +2K|\beta_{m+2} - \beta_{m+1}| + 2K \left(\prod_{i=1}^{m+2} \alpha_i\right) \\
\le\ & \left(\prod_{l=m}^{m+1} (1-\beta_{l+1})\right) \|x_{m+1} - x_m\| + 2K \sum_{l=m}^{m+1} |\beta_{l+1} - \beta_l| \\
& +2K \sum_{l=m}^{m+1} \left(\prod_{i=1}^{l+1} \alpha_i\right)
\end{aligned}
$$

$$\leq \ 2K \exp\left(-\sum_{l=m}^{m+1} \beta_{l+1}\right) + 2K \sum_{l=m}^{m+1} |\beta_{l+1} - \beta_l| + 2K \sum_{l=m}^{m+1} \left(\prod_{i=1}^{l+1} \alpha_i\right).$$

Thus, we get the following :

$$\begin{aligned}
& \|x_{m+n+1} - x_{m+n}\| \\
\leq\ & 2K \exp\left(-\sum_{l=m}^{m+n-1} \beta_{l+1}\right) + 2K \sum_{l=m}^{m+n-1} |\beta_{l+1} - \beta_l| \\
& +2K \sum_{l=m}^{m+n-1} \left(\prod_{i=1}^{l+1} \alpha_i\right) \\
\leq\ & 2K \exp\left(-\sum_{l=m}^{m+n-1} \beta_{l+1}\right) + 2K \sum_{l=m}^{m+n-1} |\beta_{l+1} - \beta_l| \\
& +2K \sum_{l=m}^{m+n-1} \left(\prod_{i=1}^{l+1} b\right) \\
\leq\ & 2K \exp\left(-\sum_{l=m}^{m+n-1} \beta_{l+1}\right) + 2K \sum_{l=m}^{m+n-1} |\beta_{l+1} - \beta_l| + 2K \sum_{l=m}^{m+n-1} b^l \\
=\ & 2K \exp\left(-\sum_{l=m}^{m+n-1} \beta_{l+1}\right) + 2K \sum_{l=m}^{m+n-1} |\beta_{l+1} - \beta_l| + 2K \frac{b^m(1-b^n)}{1-b}
\end{aligned}$$

for each $m, n = 1, 2, \ldots$. So, from $\sum_{n=1}^{\infty} \beta_n = \infty$, we obtain that

$$\begin{aligned}
& \limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \ |x_{m+n+1} - x_{m+n}\| \\
\leq\ & 2K \sum_{l=m}^{\infty} |\beta_{l+1} - \beta_l| + 2K \frac{b^m}{1-b}
\end{aligned}$$

for every $m = 1, 2, \ldots$. Moreover, since $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, we obtain

$$\begin{aligned}
& \limsup_{n\to\infty} \|x_{n+1} - x_n\| \\
\leq\ & 2K \lim_{m\to\infty} \sum_{l=m}^{\infty} |\beta_{l+1} - \beta_l| + 2K \lim_{m\to\infty} \frac{b^m}{1-b} \\
=\ & 0
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad \blacksquare$$

We have the following lemma from [13]; see also [19] and [23].

Lemma 3.5. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, there is a unique sunny nonexpansive retraction P from C onto $F(T)$. Further, let $x \in C$ and suppose that $\{u_k\} \subset C$ is given by*

$$u_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right) T u_k$$

for every $k = 1, 2, 3, \ldots$. Then, $\{u_k\}$ converges strongly to $Px \in F(T)$.

Using Lemma 3.5, we have the following:

Lemma 3.6. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$ and $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For each $n \in \mathbb{N}$, let W_n be the W-mapping of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n\to\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \ldots$. Then, $\limsup_{n\to\infty} \langle x - Px, J(x_n - Px)\rangle \leq 0$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$, where $W = \lim_{n\to\infty} U_{n,1}$.

Proof. As in the proof of Lemma 3.4, without loss of generality, we may assume that C is bounded. For each $k \in \mathbb{N}$, let u_k be a unique element of C such that $u_k = (1/k) + (1 - (1/k))Wu_k$. From Lemma 3.5, we know that $u_k \longrightarrow Px \in F(W) = \bigcap_{n=1}^{\infty} F(T_n)$ as $k \longrightarrow \infty$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$. Put $K = \sup_{z \in C} \|z\|$. We have, for every $n, k \in \mathbb{N}$,

$$\begin{aligned}
& \|x_{n+1} - Wu_k\| \\
= \ & \|\beta_n x + (1 - \beta_n) W_n x_n - Wu_k\| \\
\leq \ & \beta_n \|x - Wu_k\| + (1 - \beta_n)\|W_n x_n - Wu_k\| \\
\leq \ & \beta_n \|x - Wu_k\| + (1 - \beta_n)\left\{\|W_n x_n - W_n u_k\| + \|W_n u_k - Wu_k\|\right\} \\
\leq \ & \beta_n \|x - Wu_k\| + (1 - \beta_n)\left\{\|x_n - u_k\| + \|W_n u_k - Wu_k\|\right\} \\
\leq \ & 2\beta_n K + \|x_n - u_k\| + \|W_n u_k - Wu_k\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $Wu_k = \lim_{n\to\infty} W_n u_k$ for each $k \in \mathbb{N}$, for any Banach limit μ, we obtain

$$\mu_n \|x_n - Wu_k\|^2 = \mu_n \|x_{n+1} - Wu_k\|^2 \le \mu_n \|x_n - u_k\|^2.$$

From $(1 - (1/k))(x_n - Wu_k) = (x_n - u_k) - (1/k)(x_n - x)$, we also have

$$\begin{aligned}
\left(1 - \frac{1}{k}\right)^2 \|x_n - Wu_k\|^2 &\ge \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - x, J(x_n - u_k)\rangle \\
&= \|x_n - u_k\|^2 - \frac{2}{k}\langle x_n - u_k + u_k - x, J(x_n - u_k)\rangle \\
&= \left(1 - \frac{2}{k}\right)\|x_n - u_k\|^2 + \frac{2}{k}\langle x - u_k, J(x_n - u_k)\rangle.
\end{aligned}$$

So, we have

$$\begin{aligned}
\left(1 - \frac{1}{k}\right)^2 \mu_n \|x_n - u_k\|^2 &\ge \left(1 - \frac{1}{k}\right)^2 \mu_n \|x_n - Wu_k\|^2 \\
&\ge \left(1 - \frac{2}{k}\right)\mu_n \|x_n - u_k\|^2 \\
&\quad + \frac{2}{k}\mu_n \langle x - u_k, J(x_n - u_k)\rangle
\end{aligned}$$

and hence

$$\frac{1}{k^2}\mu_n \|x_n - u_k\|^2 \ge \frac{2}{k}\mu_n \langle x - u_k, J(x_n - u_k)\rangle.$$

This implies

$$\frac{1}{2k}\mu_n \|x_n - u_k\|^2 \ge \mu_n \langle x - u_k, J(x_n - u_k)\rangle.$$

Since $u_k \to Px \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ as $k \to \infty$, from the uniform Gâteaux differentiability of the norm of E and the above inequality, we get

$$0 \ge \mu_n \langle x - Px, J(x_n - Px)\rangle.$$

On the other hand, from Lemma 3.4, we have

$$\lim_{n\to\infty} |\langle x - Px, J(x_{n+1} - Px)\rangle - \langle x - Px, J(x_n - Px)\rangle| = 0.$$

Hence, it follows from Proposition 2.2 that

$$\limsup_{n\to\infty} \langle x - Px, J(x_n - Px)\rangle \le 0.$$

■

4. Strong Convergence Theorem

Now, we can prove a strong convergence theorem which is connected with the feasibility problem.

Theorem 4.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\alpha_1, \alpha_2, \dots$ be real numbers such that $0 < \alpha_i \le b < 1$ for every $i = 1, 2, \dots$ and $T_1, T_2, \dots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For each $n \in \mathbb{N}$, let W_n be the W-*mapping *of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \le \beta_n \le 1$ for every $n = 1, 2, \dots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \dots$, then $\{x_n\}$ converges strongly to $Px \in \bigcap_{n=1}^{\infty} F(T_n)$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Since $Px \in \bigcap_{i=1}^{\infty} F(T_i)$, we obtain

$$\|W_n x_n - Px\| \le \|x_n - Px\|$$

for all $n \in \mathbb{N}$. From $(1-\beta_n)(W_n x_n - Px) = (x_{n+1} - Px) - \beta_n(x - Px)$, we also have

$$(1-\beta_n)^2 \|W_n x_n - Px\|^2 \ge \|x_{n+1} - Px\|^2 - 2\beta_n \langle x - Px, J(x_{n+1} - Px)\rangle.$$

This implies that, for $n = 1, 2, \dots$,

$$\begin{aligned} \|x_{n+1} - Px\|^2 &\le (1-\beta_n)^2 \|W_n x_n - Px\|^2 + 2\beta_n \langle x - Px, J(x_{n+1} - Px)\rangle \\ &\le (1-\beta_n) \|W_n x_n - Px\|^2 \\ &\quad + 2(1 - (1-\beta_n)) \langle x - Px, J(x_{n+1} - Px)\rangle. \end{aligned}$$

Let $\varepsilon > 0$. From Lemma 3.6, there exists $m \in \mathbb{N}$ such that

$$\langle x - Px, J(x_n - Px)\rangle \le \frac{\varepsilon}{2}$$

for all $n \ge m$. Then, we have

$$\begin{aligned} \|x_{m+1} - Px\|^2 &\le (1-\beta_m) \|x_m - Px\|^2 \\ &\quad + 2(1 - (1-\beta_m)) \langle x - Px, J(x_{m+1} - Px)\rangle \\ &\le (1-\beta_m) \|x_m - Px\|^2 + (1 - (1-\beta_m))\varepsilon. \end{aligned}$$

Similarly, we have

$$\|x_{m+2} - Px\|^2 \le \left\{ \prod_{l=m}^{m+1} (1-\beta_l) \right\} \|x_m - Px\|^2 + \left\{ 1 - \prod_{l=m}^{m+1} (1-\beta_l) \right\} \varepsilon.$$

In the same manner, for all $n = 1, 2, \ldots$, we have

$$\|x_{m+n} - Px\|^2 \le \left\{ \prod_{l=m}^{m+n-1} (1-\beta_l) \right\} \|x_m - Px\|^2 + \left\{ 1 - \prod_{l=m}^{m+n-1} (1-\beta_l) \right\} \varepsilon.$$

We know that $\sum_{l=m}^{\infty} \beta_l = \infty$ implies $\prod_{l=m}^{\infty}(1-\beta_l) = 0$. Therefore, we get

$$\limsup_{n\to\infty} \|x_n - Px\|^2 = \limsup_{n\to\infty} \|x_{m+n} - Px\|^2 \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\limsup_{n\to\infty} \|x_n - Px\|^2 \le 0$. So, $\{x_n\}$ converges strongly to $Px \in \bigcap_{i=1}^{\infty} F(T_i)$. ∎

As a direct consequence of Theorem 4.1, we obtain the following result.

Corollary 4.2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots, r$ and $T_1, T_2, \ldots, T_r$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W be the W-*mapping *of C into itself generated by $T_r, T_{r-1}, \ldots, T_1$ and $\alpha_r, \alpha_{r-1}, \ldots, \alpha_1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \le \beta_n \le 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x + (1-\beta_n) W x_n$$

for every $n = 1, 2, \ldots$, then $\{x_n\}$ converges strongly to $Px \in \bigcap_{i=1}^{r} F(T_i)$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{i=1}^{r} F(T_i)$.

5. Applications

In this section, using Theorem 4.1, we consider the feasibility problem of finding a solution of a countable convex inequality system and the problem of finding a common fixed point for a countable commuting family of nonexpansive mappings in Banach spaces.

Theorem 5.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of*

E. Let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$. Let $\{C_n\}$ be a sequence of nonexpansive retracts of C such that $\bigcap_{n=1}^{\infty} C_n$ is nonempty. For $n \in \mathbb{N}$, let W_n be the W-mapping generated by $P_n, P_{n-1}, \ldots, P_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$, where P_k is a nonexpansive retraction of C onto C_k for each $k \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \ldots$, then $\{x_n\}$ converges strongly to $Px \in \bigcap_{n=1}^{\infty} C_n$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} C_n$.

Proof. If $Wu = \lim_{n\to\infty} W_n u$ for every $u \in C$, then it follows from Lemma 3.3 that $F(W) = \bigcap_{n=1}^{\infty} F(P_n) = \bigcap_{n=1}^{\infty} C_n$. So, using Theorem 4.1, we can prove that $\{x_n\}$ converges strongly to $Px \in \bigcap_{n=1}^{\infty} C_n$. ■

We remark that if C is a nonempty closed convex subset of a uniformly convex Banach space E and T is a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$, then, it follows from Bruck [2, 3] that $F(T)$ is a nonexpansive retract of C; see also [8]. We also know that every nonempty closed convex subset of a Hilbert space H is a nonexpansive retract of H and the norm of H is uniformly Gâteaux differentiable.

Theorem 5.2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty bounded closed convex subset of E. Let $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < \alpha_i \leq b < 1$ for every $i = 1, 2, \ldots$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $T_i T_j = T_j T_i$ for all $i, j \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Further, for $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{x_n\}$ is given by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \ldots$, then $\{x_n\}$ converges strongly to $Px \in \bigcap_{n=1}^{\infty} F(T_n)$, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. From Theorem 2.1, we have that $F(T_n)$ is a nonempty closed convex subset of C for every $n \in \mathbb{N}$. We also know from the commutativity of $\{T_n\}$ that $\{F(T_n)\}$ is a sequence of weakly closed convex subsets of C which has the finite intersection property; for more details, see [11] and [16]. Since C is weakly compact, we have $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Therefore, by Theorem 4.1, $\{x_n\}$ converges strongly to $Px \in \bigcap_{n=1}^{\infty} F(T_n)$. ■

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