

**THE LIL FOR THE ESTIMATES OF THE PARAMETERS IN  
A PARTLY LINEAR REGRESSION MODEL\***

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**Abstract.** Consider the partly linear regression model  $Y_i = X_i^T \beta + g(T_i) + \varepsilon_i$  ( $i = 1, \dots, n$ ), where  $(X_i, T_i)$  are i.i.d. random design points,  $\beta$  is a  $p$ -dimensional unknown parameter,  $g(\cdot)$  is an unknown function on  $[0, 1]$ ,  $\varepsilon_i$  are i.i.d. random errors with mean 0 and variance  $\sigma^2$ . This paper is concerned with the **LIL** of the estimators of  $\beta$  and  $\sigma^2$ .

1. INTRODUCTION

One of the attractively interesting models in statistical theory and application recently is the partly linear regression model, which is defined by the function relation

$$(1.1) \quad Y_i = X_i^T \beta + g(T_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

where  $T$  denotes transposition,  $X_i = (x_{i1}, \dots, x_{ip})^T$  ( $p \geq 1$ ) and  $T_i$  ( $T_i \in [0, 1]$ ) are i.i.d. random design points,  $\beta = (\beta_1, \dots, \beta_p)^T$  is an unknown parameter vector,  $g$  is an unknown function, and  $\varepsilon_1, \dots, \varepsilon_n$  are i.i.d. random variables with mean zero and unknown variance  $\sigma^2$ . This model was introduced by Engle *et al.* [4] to study the effect of weather on electricity demand. More recent work has dealt with the estimation of  $\beta$  at a root- $n$  rate. Chen [1], Heckman [5], Rice [8], Robinson [9] and Speckman [11] constructed  $\sqrt{n}$ -consistent estimates of  $\beta$  under various conditions. Cuzick [2] constructed efficient estimates of  $\beta$  when

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the error density is known and has finite Fisher information. Cuzick [3] and Schick [10] constructed efficient estimates of  $\beta$  when the error distribution is unknown. Liang and Cheng [7] considered second-order asymptotic efficiency for model (1.1).

The aim of this paper is to establish the law of the iterated logarithm (**LIL**) for some estimates of the parameters  $\beta$  and  $\sigma^2$  when  $(X_i, T_i)$  are i.i.d. random design points. First we construct the estimators of  $\beta$  and  $\sigma^2$  by using nonparametric fitting and least squares methods. In order to establish the results for the proposed estimators, we investigate nonparametric estimators. We derive a uniformly consistent result for the general linear combination of errors in the case of independence (Lemma 2.3), which is then specialized to the case of nonparametric estimators (2.6) and (2.7) and then used repeatedly.

Assume  $\{X_i = (x_{i1}, \dots, x_{ip})^T, T_i, Y_i, i = 1, \dots, n\}$  satisfies the model (1.1). It follows from  $E\varepsilon_i = 0$  that  $g(T_i) = E(Y_i - X_i^T \beta | T_i)$ , which motivates us to define, for known  $\beta$ ,

$$\hat{g}_n(t) = \sum_{j=1}^n \omega_{nj}(t)(Y_j - X_j^T \beta)$$

as the estimator of  $g(t)$ , where  $\omega_{ni}(t) = \omega_{ni}(t; T_1, \dots, T_n), i = 1, \dots, n$ , are probability weight functions depending only on the design points  $T_1, \dots, T_n$ .

Replacing  $g(T_i)$  by  $\hat{g}_n(T_i)$  in model (1.1), and then getting the following modified model

$$(1.2) \quad Y_i = X_i^T \beta + \hat{g}_n(T_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

one can establish the two-step estimator of  $\beta$  as follows,

$$\beta_n = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \tilde{Y},$$

where  $\tilde{X}^T = (\tilde{X}_1, \dots, \tilde{X}_n)$ ,  $\tilde{X}_i = X_i - \sum_{j=1}^n \omega_{nj}(T_i) X_j$ ,  $\tilde{Y} = (\tilde{Y}_1, \dots, \tilde{Y}_n)^T$ ,  $\tilde{Y}_i = Y_i - \sum_{j=1}^n \omega_{nj}(T_i) Y_j$ . Consequently, an estimator of  $\sigma^2$  can be defined as

$$\sigma_n^2 = \frac{1}{n} \sum_{i=1}^n (\tilde{Y}_i - \tilde{X}_i^T \beta_n)^2$$

since (1.2) is equivalent to  $\varepsilon_i = \tilde{Y}_i - \tilde{X}_i^T \beta$  ( $i = 1, \dots, n$ ) formally.

As pointed out at the beginning of this section, the asymptotics of  $\beta_n$  and  $\sigma_n^2$  have been greatly considered in literature. See Chen [1], Cuzick [2,3], Heckman [5], Liang and Cheng [7], Rice [8], Robinson [9] and Speckman [11]. Liang [6] studied the Berry-Esseen bounds of the distribution of some estimator of  $\sigma^2$  when  $(X_i, T_i)$  are known design points, and obtained the optimal bound  $O(n^{-1/2})$ . To establish the (**LIL**) forms the core of this context.

Now we present an example to explain how to select the weight functions. Take

$$\omega_{ni}(t) = \frac{1}{h_n} \int_{s_{i-1}}^{s_i} K\left(\frac{t-s}{h_n}\right) ds, \quad 1 \leq i \leq n,$$

where  $s_0 = 0$ ,  $s_n = 1$  and  $s_i = 1/2(T_{(i)} + T_{(i+1)})$  ( $1 \leq i \leq n - 1$ ) for the ordered statistics  $T_{(1)}, \dots, T_{(n)}$  of  $T_1, \dots, T_n$ ,  $h_n$  is a sequence of bandwidth parameters which tends to zero as  $n \rightarrow \infty$  and  $K(\cdot)$  is a kernel function, which is supposed to satisfy

$$\text{supp}(K) = [-1, 1], \sup |K(x)| \leq C < \infty, \int K(u)du = 1 \text{ and } K(u) = K(-u).$$

Other weight functions can be found in Liang [6].

The paper is organized as follows. In the following we give the conditions on the  $X_i$  and  $T_i$ , and the main result. Section 2 proves some lemmas. Section 3 presents the proof of the main result. For the convenience and simplicity, we shall employ  $C$  ( $0 < C < \infty$ ) to denote some constant not depending on  $n$  but may assume different values at each appearance. Some notations are introduced:  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$ ,  $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_n)^T$ ,  $\tilde{\varepsilon}_i = \varepsilon_i - \sum_{j=1}^n W_{nj}(T_i)\varepsilon_j$ ;  $g_{ni} = g(T_i) - \sum_{k=1}^n W_{nk}(T_i)g(T_k)$ ,  $\hat{G} = (g(T_1) - \hat{g}_n(T_1), \dots, g(T_n) - \hat{g}_n(T_n))^T$ ;  $h_j(t) = E(x_{ij}|T_i = t)$ ,  $u_{ij} = x_{ij} - h_j(T_i)$  for  $i = 1, \dots, n$  and  $j = 1, \dots, p$ .

**Condition 1.**  $\sup_{0 \leq t \leq 1} E(\|X_1\|^3|T = t) < \infty$  and  $B = \text{Cov}(X_1 - E(X_1|T_1))$  is a positive-definite matrix.

**Condition 2.**  $g(\cdot)$  and  $h_j(\cdot)$  are all Lipschitz continuous of order 1.

**Condition 3.** The weight functions  $\omega_{ni}(\cdot)$  satisfy the following:

- (i)  $\max_{1 \leq i \leq n} \sum_{j=1}^n \omega_{ni}(T_j) = O(1)$  a.s.,
- (ii)  $\max_{1 \leq i, j \leq n} \omega_{ni}(T_j) = O(b_n)$  a.s.,  $b_n = n^{-2/3}$ ,
- (iii)  $\max_{1 \leq i \leq n} \sum_{j=1}^n \omega_{nj}(T_i) I_{(|T_j - T_i| > c_n)} = O(c_n)$  a.s.,  $c_n = n^{-1/2} \log^{-1} n$ .

The following theorem gives our main result.

**Theorem 1.1.** Suppose conditions 1-3 hold. If  $E|\varepsilon_1|^3 < \infty$ , then

$$(1.3) \quad \limsup_{n \rightarrow \infty} \left( \frac{n}{2 \log \log n} \right)^{1/2} |\beta_{nj} - \beta_j| = (\sigma^2 b^{jj})^{1/2} \quad \text{a.s.}$$

Furthermore, if  $E\varepsilon_1^4 < \infty$ , then

$$(1.4) \quad \limsup_{n \rightarrow \infty} \left( \frac{n}{2 \log \log n} \right)^{1/2} |\sigma_n^2 - \sigma^2| = (\text{Var} \varepsilon_1^2)^{1/2} \quad a.s.,$$

where  $\beta_{nj}$ ,  $\beta_j$  and  $b^{jj}$  denote the  $j$ -th element of  $\beta_n$ ,  $\beta$  and the  $(j, j)$ -th element of  $B^{-1}$ , respectively.

Now, we outline the proof of the theorem. First we decompose  $\sqrt{n}(\beta_n - \beta)$  and  $\sqrt{n}(\sigma_n^2 - \sigma^2)$  into three and five terms respectively. Then we will calculate deliberately the tail probability value of each term. We have from the definitions of  $\beta_n$  and  $\sigma_n^2$ ,

$$(1.5) \quad \begin{aligned} \sqrt{n}(\beta_n - \beta) &= \sqrt{n}(\tilde{X}^T \tilde{X})^{-1} \left\{ \sum_{i=1}^n \tilde{X}_i g_{ni} - \sum_{i=1}^n \tilde{X}_i \left( \sum_{j=1}^n \omega_{nj}(T_i) \varepsilon_j \right) + \sum_{i=1}^n \tilde{X}_i \varepsilon_i \right\} \\ &\stackrel{\text{def}}{=} n(\tilde{X}^T \tilde{X})^{-1} (H_1 - H_2 + H_3), \\ \sqrt{n}(\sigma_n^2 - \sigma^2) &= \frac{1}{\sqrt{n}} \tilde{Y}^T \{ \mathcal{F} - \tilde{X}(\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \} \tilde{Y} - \sqrt{n} \sigma^2 \\ &= \frac{1}{\sqrt{n}} \varepsilon^T \varepsilon - \sqrt{n} \sigma^2 - \frac{1}{\sqrt{n}} \varepsilon^T \tilde{X}(\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \varepsilon \\ &\quad + \frac{1}{\sqrt{n}} \hat{G}^T \{ \mathcal{F} - \tilde{X}(\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \} \hat{G} \\ &\quad - \frac{2}{\sqrt{n}} \hat{G}^T \tilde{X}(\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \varepsilon + \frac{2}{\sqrt{n}} \hat{G}^T \varepsilon. \\ &\stackrel{\text{def}}{=} \sqrt{n} \{ (I_1 - \sigma^2) - I_2 + I_3 - 2I_4 + 2I_5 \}, \end{aligned}$$

where  $\mathcal{F}$  denotes the identity matrix of order  $p$ .

In the following sections we will prove that each element of  $H_1$  and  $H_2$  converges almost surely to zero, and  $\sqrt{n}I_i$  also converge almost surely to zero for  $i = 2, 3, 4, 5$ . The proofs of the first assertion will be finished in Lemmas 2.4 and 2.5. The proofs of the second assertion will be arranged in Section 3 after we complete the proof of (1.3). Finally, we use Corollary 5.2.3 of Stout [12], and Hartman-Winter theorem, and complete the proof of the theorem.

## 2. SOME LEMMAS

In this section we prove several lemmas required. In Lemma 2.1 we think of the boundedness for  $h_j(T_i) - \sum_{k=1}^n \omega_{nk}(T_i) h_j(T_k)$  and  $g(T_i) - \sum_{k=1}^n \omega_{nk}(T_i) g(T_k)$ . The proofs are implied by Lipschitz continuity and Condition 3 (iii). Lemma 2.2 states that  $n^{-1} \tilde{X}^T \tilde{X}$  converges to  $B$ . Its proof can be referred to Speckman [11] and Chen [1], and is therefore omitted.

**Lemma 2.1.** *Suppose that conditions 2 and 3 (iii) hold. Then*

$$\max_{1 \leq i \leq n} \left| G_j(T_i) - \sum_{k=1}^n \omega_{nk}(T_i) G_j(T_k) \right| = O(c_n) \quad \text{for } j = 0, \dots, p,$$

where  $G_0(\cdot) = g(\cdot)$  and  $G_l(\cdot) = h_l(\cdot)$  for  $l = 1, \dots, p$ .

**Lemma 2.2.** *If conditions 1-3 hold, then  $\lim_{n \rightarrow \infty} 1/n \tilde{X}^T \tilde{X} = B$  a.s..*

Next we shall prove a rather general result on strong uniform convergence of weighted averages in Lemma 2.3, which is applied in the later proofs repeatedly. First we give an exponential inequality for bounded independent random variables, that is,

**Bernstein's Inequality.** *Let  $V_1, \dots, V_n$  be independent random variables with zero means and bounded ranges:  $|V_i| \leq M$ . Then for each  $\eta > 0$ ,*

$$P\left(\left|\sum_{i=1}^n V_i\right| > \eta\right) \leq 2 \exp\left[-\eta^2 / \left\{2\left(\sum_{i=1}^n \text{var } V_i + M\eta\right)\right\}\right].$$

**Lemma 2.3.** *Let  $V_1, \dots, V_n$  be independent random variables with means zero and finite variances, i.e.,  $\sup_{1 \leq j \leq n} E|V_j|^r \leq C < \infty$  ( $r \geq 2$ ). Assume  $a_{ki}, k, i = 1, \dots, n$ , is a sequence of positive numbers such that  $\sup_{1 \leq i, k \leq n} |a_{ki}| \leq n^{-p_1}$  for some  $0 < p_1 < 1$  and  $\sum_{j=1}^n a_{ji} = O(n^{p_2})$  for  $p_2 \geq \max(0, 2/r - p_1)$ . Then*

$$\max_{1 \leq i \leq n} \left| \sum_{k=1}^n a_{ki} V_k \right| = O(n^{-s} \log n) \quad \text{for } s = (p_1 - p_2)/2. \quad \text{a.s..}$$

*Proof.* Denote  $V'_j = V_j I(|V_j| \leq n^{1/r})$  and  $V''_j = V_j - V'_j$  for  $j = 1, \dots, n$ . Let  $M = Cn^{-p_1} n^{1/r}$  and  $\eta = n^{-s} \log n$ . By Bernstein's inequality,

$$\begin{aligned} & P\left\{\max_{1 \leq i \leq n} \left| \sum_{j=1}^n a_{ji} (V'_j - EV'_j) \right| \right. \\ & \quad \left. > C_1 \eta \right\} \leq \sum_{i=1}^n P\left\{\left| \sum_{j=1}^n a_{ji} (V'_j - EV'_j) \right| > C_1 \eta\right\} \\ & \leq 2n \exp\left(-\frac{C_1 n^{-2s} \log^2 n}{2 \sum_{j=1}^n a_{ji}^2 EV_j^2 + 2CC_1 n^{-p_1+1/r-s} \log n}\right) \\ & \leq 2n \exp(-C_1^2 C \log n) \leq Cn^{-3/2} \quad \text{for some large } C_1 > 0, \end{aligned}$$

where the second inequality is from

$$\sum_{j=1}^n a_{ji}^2 EV_j^2 \leq \sup |a_{ji}| \sum_{j=1}^n a_{ji} EV_j^2 = n^{-p_1+p_2} \quad \text{and} \quad n^{-p_1+1/r-s} \log n \leq n^{-p_1+p_2}.$$

By Borel-Cantelli Lemma,

$$(2.1) \quad \max_{1 \leq i \leq n} \left| \sum_{j=1}^n W_{nj}(T_i)(V_j' - EV_j') \right| = O(n^{-s} \log n) \quad \text{a.s..}$$

Let  $1 \leq p < 2$ ,  $1/p + 1/q = 1$  be such that  $1/q < (p_1 + p_2)/2 - 1/r$ . By Hölder's inequality,

$$(2.2) \quad \begin{aligned} \max_{1 \leq i \leq n} \left| \sum_{j=1}^n a_{ji}(V_j'' - EV_j'') \right| &\leq \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |a_{ji}|^q \right)^{1/q} \left( \sum_{j=1}^n |V_j'' - EV_j''|^p \right)^{1/p} \\ &\leq Cn^{-(p_1q-1)/q} \left( \sum_{j=1}^n |V_j'' - EV_j''|^p \right)^{1/p}. \end{aligned}$$

Observe that

$$(2.3) \quad \frac{1}{n} \sum_{j=1}^n \left( |V_j'' - EV_j''|^p - E|V_j'' - EV_j''|^p \right) \rightarrow 0 \quad \text{a.s.}$$

and  $E|V_j''|^p \leq E|V_j|^{rp} n^{-1+p/r}$ , and then

$$(2.4) \quad \sum_{j=1}^n E|V_j'' - EV_j''|^p \leq Cn^{p/r} \quad \text{a.s..}$$

Combining (2.2), (2.3) with (2.4), we obtain

$$(2.5) \quad \max_{1 \leq i \leq n} \left| \sum_{k=1}^n a_{ki}(V_k'' - EV_k'') \right| \leq Cn^{-p_1+1/q+1/r} = o(n^{-s}) \quad \text{a.s..}$$

Lemma 2.3 follows from (2.1) and (2.5) directly. ■

Let  $r = 3$ ,  $V_k = e_k$  or  $u_{kl}$ ,  $a_{ji} = W_{nj}(T_i)$ ,  $p_1 = 2/3$  and  $p_2 = 0$ . We obtain the following formulas, which will play critical roles in the process of proving the theorem:

$$(2.6) \quad \max_{i \leq n} \left| \sum_{k=1}^n W_{nk}(T_i)e_k \right| = O(n^{-1/3} \log n) \quad \text{a.s.}$$

and

$$(2.7) \quad \max_{i \leq n} \left| \sum_{k=1}^n W_{nk}(T_i) u_{kl} \right| = O(n^{-1/3} \log n) \quad \text{for } l = 1, \dots, p \quad \text{a.s.}$$

**Lemma 2.4.** *Suppose conditions 1-3 hold and  $E|\varepsilon_1|^3 < \infty$ . Then*

$$(2.8) \quad \sqrt{n}H_{1j} = O(n^{1/2} \log^{-1/2} n) \quad \text{for } j = 1, \dots, p.$$

*Proof.* Denote  $h_{nij} = h_j(T_i) - \sum_{k=1}^n \omega_{nk}(T_i) h_j(T_k)$ . Notice that

$$\sqrt{n}H_{1j} = \sum_{i=1}^n \tilde{x}_{ij} g_{ni} = \sum_{i=1}^n u_{ij} g_{ni} + \sum_{i=1}^n h_{nij} g_{ni} - \sum_{i=1}^n \sum_{q=1}^n \omega_{nq}(T_i) u_{qj} g_{ni}.$$

By Lemma 2.1,

$$\left| \sum_{i=1}^n h_{nij} g_{ni} \right| \leq n \max_{i \leq n} |g_{ni}| \max_{i \leq n} |h_{nij}| = O(nc_n^2).$$

In Lemma 2.3, we take  $r = 2$ ,  $V_k = u_{kl}$ ,  $a_{ji} = g_{nj}$ ,  $1/4 < p_1 < 1/3$  and  $p_2 = 1 - p_1$ . Then

$$\left| \sum_{i=1}^n u_{ij} g_{ni} \right| = O(n^{-(2p_1-1)/2}).$$

By Abel's inequality and (2.7),

$$\begin{aligned} \left| \sum_{i=1}^n \sum_{q=1}^n \omega_{nq}(T_i) u_{qj} g_{ni} \right| &\leq n \max_{i \leq n} |g_{ni}| \max_{i \leq n} \left| \sum_{q=1}^n \omega_{nq}(T_i) u_{qj} \right| \\ &= O(n^{2/3} c_n \log n). \end{aligned}$$

The above arguments entail that

$$\sqrt{n}H_{1j} = O(n^{1/2} \log^{-1/2} n) \quad \text{for } j = 1, \dots, p.$$

Thus we complete the proof of Lemma 2.4. ■

**Lemma 2.5.** *Suppose conditions 1-3 hold and  $E|\varepsilon_1|^3 < \infty$ . Then*

$$\sqrt{n}H_{2j} = o(n^{1/2}) \quad \text{for } j = 1, \dots, p \quad \text{a.s.}$$

*Proof.* Observe that

$$\begin{aligned} \sqrt{n}H_{2j} &= \sum_{i=1}^n \left\{ \sum_{k=1}^n \tilde{x}_{kj} \omega_{ni}(T_k) \right\} \varepsilon_i \\ &= \sum_{i=1}^n \left\{ \sum_{k=1}^n u_{kj} \omega_{ni}(T_k) \right\} \varepsilon_i + \sum_{i=1}^n \left\{ \sum_{k=1}^n h_{nkj} \omega_{ni}(T_k) \right\} \varepsilon_i \\ &\quad - \sum_{i=1}^n \left[ \sum_{k=1}^n \left\{ \sum_{q=1}^n u_{qj} \omega_{nq}(T_k) \right\} \omega_{ni}(T_k) \right] \varepsilon_i. \end{aligned}$$

We now handle these three terms separately. Let  $r = 2$ ,  $V_k = \varepsilon_k$ ,  $a_{li} = \sum_{k=1}^n u_{kj} \omega_{ni}(T_k)$ ,  $1/4 < p_1 < 1/3$  and  $p_2 = 1 - p_1$ . Using Lemma 2.3, we have

$$(2.9) \quad \left| \sum_{i=1}^n \left\{ \sum_{k=1}^n u_{kj} \omega_{ni}(T_k) \right\} \varepsilon_i \right| = O(n^{-(2p_1-1)/2} \log n) \quad \text{a.s..}$$

By Lemma 2.1 and (2.6), we get

$$(2.10) \quad \left| \sum_{i=1}^n \left\{ \sum_{k=1}^n h_{nkj} \omega_{ni}(T_k) \right\} \varepsilon_i \right| \leq n \max_{k \leq n} \left| \sum_{i=1}^n \omega_{ni}(T_k) \varepsilon_i \right| \max_{k \leq n} |h_{nkj}| \\ = O(n^{2/3} c_n \log n) \quad \text{a.s..}$$

Using Abel's inequality and (2.6) and (2.7), we obtain

$$(2.11) \quad \left| \sum_{i=1}^n \left[ \sum_{k=1}^n \left\{ \sum_{q=1}^n u_{qj} \omega_{nq}(T_k) \right\} \omega_{ni}(T_k) \right] \varepsilon_i \right| \\ \leq n \max_{k \leq n} \left| \sum_{i=1}^n \omega_{ni}(T_k) \varepsilon_i \right| \max_{k \leq n} \left| \sum_{q=1}^k u_{qj} \omega_{nq}(T_j) \right| \\ = O(n^{1/3} \log^2 n) = o(n^{1/2}) \quad \text{a.s..}$$

A combination of (2.9)–(2.11) yields Lemma 2.5. ■

**Lemma 2.6.** *Assume that conditions 1-3 hold. If  $E\varepsilon_1^2 < \infty$ , then*

$$I_n = o(n^{1/2}),$$

where  $I_n = \sum_{i=1}^n \sum_{j \neq i} \omega_{nj}(T_i) (\varepsilon'_j - E\varepsilon'_j) (\varepsilon'_i - E\varepsilon'_i)$  with  $\varepsilon'_j = \varepsilon_j I_{\{|\varepsilon_j| \leq n^{1/2}\}}$ .

*Proof.* In Lemma 2.3, we take  $r = 2$ ,  $V_k = \varepsilon'_k - E\varepsilon'_k$ ,  $a_{ki} = \sum_{j \neq i} \omega_{nj}(T_i) (\varepsilon'_j - E\varepsilon'_j)$ . As for (2.1), we have  $\sup_i |a_{ki}| = o(n^{-1/3} \log n)$ . Furthermore, note that  $I_n = \sum_{i=1}^n a_{ki} (\varepsilon'_i - E\varepsilon'_i)$ . We adopt the same technique as for (2.1) by letting  $1/4 < p_1 < 1/3$ ,  $p_2 = 1 - p_1$ , and have

$$I_n = o(n^{1/2}).$$

This completes the proof of Lemma 2.6. ■

### 3. PROOF OF THE THEOREM

In this section we shall complete the proof of the main result. At first, we state a conclusion, Corollary 5.2.3 of Stout [12], which will play an elementary role for our proof.

**Conclusion S.** Let  $V_1, \dots, V_n$  be independent random variables with means zero. There exists a  $\delta_0 > 0$  such that  $\max_{1 \leq i \leq n} E|V_i|^{2+\delta_0} < \infty$  and  $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{Var}(V_i) > 0$ . Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2s_n^2 \log \log s_n^2}} = 1, \quad \text{a.s.},$$

where  $S_n = \sum_{i=1}^n V_i$  and  $s_n^2 = \sum_{i=1}^n EV_i^2$ .

From (1.5) and Lemmas 2.4 and 2.5, we know that in order to complete the proof of (1.3), it suffices to show

$$(3.1) \quad \limsup_{n \rightarrow \infty} \left( \frac{|(B^{-1} \tilde{X} \varepsilon)_j|}{2n \log \log n} \right)^{1/2} = (\sigma^2 b^{jj})^{1/2} \quad \text{a.s.}$$

By calculation, we get

$$\begin{aligned} \frac{1}{\sqrt{n}}(B^{-1} \tilde{X} \varepsilon)_j &= \sum_{k=1}^p b^{jk} \left[ W_k + \frac{1}{\sqrt{n}} \sum_{q=1}^n \{x_{qk} - h_k(T_q)\} \varepsilon_q \right] \\ &= \sum_{k=1}^p b^{jk} W_k + \frac{1}{\sqrt{n}} \sum_{q=1}^n \left( \sum_{k=1}^p b^{jk} u_{qk} \right) \varepsilon_q, \end{aligned}$$

where

$$W_k = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( h_k(T_i) - \sum_{q=1}^n \omega_{nq}(T_i) x_{qk} \right) \varepsilon_i \quad \text{for } 1 \leq k \leq p.$$

The classic **LIL** implies that

$$(3.2) \quad \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \varepsilon_i \right| = O(n^{1/2} \log n) \quad \text{a.s.},$$

which, by Lemma 2.1, Abel's inequality and (2.7), yields that

$$\begin{aligned} |W_k| &\leq C/\sqrt{n} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \varepsilon_i \right| \max_{1 \leq i \leq n} \left| h_k(T_i) - \sum_{q=1}^n \omega_{nq}(T_i) x_{qk} \right| \\ &\leq C/\sqrt{n} \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \varepsilon_i \right| \left\{ \max_{1 \leq i \leq n} \left| h_k(T_i) - \sum_{q=1}^n \omega_{nq}(T_i) h_k(T_q) \right| \right. \\ &\quad \left. + \max_{1 \leq i \leq n} \left| \sum_{q=1}^n \omega_{nq}(T_i) u_{qk} \right| \right\} \\ &= O(\log n) \cdot o(\log^{-1} n) = o(1) \quad \text{a.s.} \end{aligned}$$

Denote  $\omega_{ij} = \sum_{k=1}^p b^{jk} u_{ik} \varepsilon_i$ . Then  $E\omega_{ij} = 0$  for  $i = 1, \dots, n$ ,

$$E|\omega_{ij}|^{2+\delta_0} \leq C \max_{1 \leq i \leq n} E|\varepsilon_i|^{2+\delta_0} < \infty,$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n E\omega_{ij}^2 &= \sigma^2(b^{j1}, \dots, b^{jp}) \left\{ \lim_{n \rightarrow \infty} \sum_{i=1}^n u_i \cdot u_i^T \right\} (b^{j1}, \dots, b^{jp})^T \\ &= \sigma^2(b^{j1}, \dots, b^{jp}) B(b^{j1}, \dots, b^{jp})^T = \sigma^2 b^{jj} > 0. \end{aligned}$$

It follows from Conclusion S that (3.1) holds. This completes the proof of (1.3).

Next, we prove the latter part of Theorem 1.1, i.e., (1.4). We show  $\sqrt{n}I_i = o(1)$  ( $i = 2, 3, 4, 5$ ) firstly, and then deal with  $\sqrt{n}(I_1 - \sigma^2)$ . It follows from Lemma 2.1 and (2.6) that

$$\begin{aligned} |\sqrt{n}I_3| &\leq C\sqrt{n} \max_{1 \leq i \leq n} \left( |g(T_i) - \sum_{k=1}^n \omega_{nk}(T_i)g(T_k)|^2 + \left| \sum_{k=1}^n \omega_{nk}(T_i)\varepsilon_k \right|^2 \right) \\ &= o(1) \quad \text{a.s..} \end{aligned}$$

It follows from Lemma 2.1, (2.8) and (3.1) that

$$\sqrt{n}I_2 = o(1), \quad \sqrt{n}I_4 = o(1) \quad \text{a.s..}$$

Now, we decompose  $I_5$  into three terms and prove that each term tends to zero. More precisely, we have

$$\begin{aligned} I_5 &= \frac{1}{n} \left[ \sum_{i=1}^n g_{ni}\varepsilon_i - \sum_{k=1}^n \omega_{ni}(T_k)\varepsilon_k^2 - \sum_{i=1}^n \sum_{k \neq i} \omega_{ni}(T_k)\varepsilon_i\varepsilon_k \right] \\ &\stackrel{\text{def}}{=} I_{51} + I_{52} + I_{53}. \end{aligned}$$

From (3.2) and Lemma 2.1, we know

$$(3.3) \quad \sqrt{n}I_{51} = o(1) \quad \text{and} \quad \sqrt{n}I_{52} \leq b_n n^{-1/2} \sum_{i=1}^n \varepsilon_i^2 = O(\log^{-2} n) = o(1) \quad \text{a.s..}$$

Observe that

$$\sqrt{n}|I_{53}| \leq \frac{1}{\sqrt{n}} \left| \sum_{i=1}^n \sum_{k \neq i} \omega_{nk}(T_i)\varepsilon_i\varepsilon_k - I_n \right| + I_n \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}}(J_{1n} + I_n).$$

Similar manipulation as in Lemma 2.3 shows that

$$\begin{aligned} J_{1n} &\leq \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \omega_{nj}(T_i) \varepsilon_i \right| \left( \sum_{i=1}^n |\varepsilon_i''| + E|\varepsilon_i''| \right) \\ &\quad + \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \omega_{nj}(T_i) (\varepsilon_i' - E\varepsilon_i') \right| \left( \sum_{i=1}^n |\varepsilon_i''| + E|\varepsilon_i''| \right) \\ &= o(1). \end{aligned}$$

It follows from Lemma 2.6 and (3.4) that

$$(3.5) \quad \sqrt{n}I_{53} = o(1) \quad \text{a.s.}$$

A combination of (3.3)–(3.5) leads to  $\sqrt{n}I_5 = o(1)$  a.s..

To this end, using Hartman-Winter theorem, we have

$$\limsup_{n \rightarrow \infty} \left( \frac{n}{2 \log \log n} \right)^{1/2} |I_1 - \sigma^2| = (\text{Var } \varepsilon_1^2)^{1/2} \quad \text{a.s.}$$

This completes the proof of Theorem 1.1. ■

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