TAIWANESE JOURNAL OF MATHEMATICS
Vol. 2, No. 3, pp. 361-368, September 1998

AN APPROACH TO CONSTRUCT THE SINGULAR MONOTONE FUNCTIONS BY USING MARKOV CHAINS

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Abstract. A probabilistic approach to construct the singular monotone functions by using Markov chains is given, and the relation between the singular monotone functions and the strong law of Markov chains is revealed.

A nonconstant function f is singular if it is continuous and if $f'(x) = 0$ almost everywhere in Lebsgue measure μ. The Cantor function (e.g., see Ash 1972, p. 77) is the best-known example of nondecreasing singular function. Examples of strictly increasing singular functions have been provided by several authors, including Freilich 1973; Gelbaum and Olmsted 1964, pp. 96-98; Hewitt and Stromberg 1955, pp. 278-282; Riesz and Nagy 1955, pp. 48-49; Takacs 1978. The purpose of this paper is to present an approach to construct the strictly increasing singular functions by using the Markov chains. Our approach differs from other's construction in that it depends on the strong laws of Markov chains, rather than modification of the Cantor function.

We first introduce the notion of generalized m-adic expansion of real numbers, where $m \geq 2$ is an integer. Let

$$\mathbf{q} = (q_1, q_2 \cdots, q_m),\ q_j > 0\ (i = 1, 2, \cdots m) \tag{1}$$

and

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix},\ p_{ij} > 0\quad (i, j = 1, 2, \cdots m) \tag{2}$$

Received February 1, 1997.
Communicated by Y.-J. Lee.
1991 *Mathematics Subject Classification*: 26A30, 60J10.
Key words and phrases: Singular monotone function, Markov chain, strong law of Markov chain.

be the initial distribution and the transition matrix of a Markov chain with state space $S = \{1, 2, \cdots, m\}$ respectively. Split the interval [0, 1) into m right-semiopen intervals $I_{x_1}, x_1 \in S$, according to the ratio $q_1 : q_2 : \cdots : q_m$, i.e.,

$$I_1 = [0, q_1), I_2 = [q_1, q_1 + q_2), \cdots, I_m = [1 - q_m, 1).$$

These intervals will be called I-intervals of order 1. Generally, let $I_{x_1 \cdots x_n}(x_k \in S, 1 \le k \le n)$ be an I-interval of order n. By splitting $I_{x_1 \cdots x_n}$ into m right-semiopen intervals $I_{x_1 \cdots x_n x_{n+1}}(x_{n+1} = 1, 2, \cdots, m)$ at the ratio of the elements of the x_nth row of $\mathbf{P} : p_{x_n 1} : p_{x_n 2} : \cdots : p_{x_n m}$, the I-intervals of order $(n+1)$ are created. Proceeding inductively, the I-intervals of all orders can be obtained. These intervals will be called the ones generated by $\mathbf{q}$ and $\mathbf{P}$. By the above construction it is easy to see that

$$\mu(I_{x_1}) = q_{x_1}; \tag{3}$$

$$\mu(I_{x_1 x_2 \cdots x_n}) = q_{x_1} \prod_{k=1}^{n-1} p_{x_k x_{k+1}}, \ n \ge 2; \tag{4}$$

$$\lim_n \mu(I_{x_1 x_2 \cdots x_n}) = 0. \tag{5}$$

Consider any nested sequence of I-intervals

$$I_{x_1} \supset I_{x_1 x_2} \supset I_{x_1 x_2 x_3} \supset \cdots, \tag{6}$$

and define

$$I_{x_1 x_2 x_3 \cdots} = \bigcap_{n=1}^{\infty} I_{x_1 \cdots x_n}. \tag{7}$$

It is easy to see that for any $x \in [0, 1)$ there exists a unique sequence of integers $\{x_n \ n \ge 1\}$, where $0 \le x_n \le m$, such that

$$x = \bigcap_{n=1}^{\infty} I_{x_1 \cdots x_n}. \tag{8}$$

For brevity, we shall write

$$x = x_1 x_2 x_3 \cdots (\mathbf{q}, \mathbf{P}) \tag{9}$$

when (8) holds. This unique representation of x will be called the generalized m-adic expansion of x generated by $\mathbf{q}$ and $\mathbf{P}$. It follows from (3) and (4) that under the Lebesgue measure the sequence of digits $\{x_n, n \ge 1\}$ in

(9) constitutes a Markov chain with the initial distribution $\mathbf{q}$ and the transition matrix $\mathbf{P}$. Let $i, j \in S, S_n(i,x)$ be the number of i in the sequence $x_1, x_2, \cdots, x_n$, and $S_n(i,j,x)$ be the number of the couple (i,j) in the sequence $(x_1,x_2),(x_2,x_3),\cdots,(x_{n-1},x_n)$, and $(p_1,p_2,\cdots,p_m)$ be the stationary distribution of the Markov chain $\{x_n, n \geq 1\}$. Also let

$$A_{ij} = \{x : \lim_{n\to\infty} S_n(i,x) = \infty, \lim_{n\to\infty}[S_n(i,j,x)/S_n(i,x)] = p_{ij}\}; \tag{10}$$

$$A_i = \{x : \lim_{n\to\infty} S_n(i,x)/n = p_i\}; \tag{11}$$

$$A = \Big(\bigcap_{i=1}^{m} A_i\Big) \bigcap \Big(\bigcap_{i,j=1}^{m} A_{ij}\Big). \tag{12}$$

By the strong law of Markov chains (cf. Chung 1968, pp. 91-96), $\mu(A) = 1$, and by (4)

$$\mu(I_{x_1\cdots x_n}) = q_{x_1} \prod_{i,j=1}^{m} p_{ij}^{S_n(i,j,x)}. \tag{13}$$

Let

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} \end{pmatrix}, r_{ij} > 0 \tag{14}$$

be another transition matrix. Letting $J_{x_1\cdots x_n}(x_k \in S, 1 \leq k \leq n)$ denote the J-interval of order n generated by $\mathbf{q}$ and $\mathbf{R}$, we have as in (3) and (4),

$$\mu(J_{x_1}) = q_{x_1}; \tag{15}$$

$$\mu(J_{x_1\cdots x_n}) = q_{x_1} \prod_{k=1}^{n-1} r_{x_k x_{k+1}}. \tag{16}$$

Theorem. *Let $x \in [0,1)$ be represented by* (9)*, and $\mathbf{q}$ and $\mathbf{R}$ be given by* (1) *and* (14)*, respectively, and define a function $f : [0,1] \to [0,1]$ as follows*:

$$f(x) = .x_1x_2x_3\cdots(\mathbf{q},\mathbf{R}); \tag{17}$$

$$f(1) = 1, \tag{18}$$

where (17) *is the generalized* m*-adic expansion of* $f(x)$ *generated by* $\mathbf{q}$ *and* $\mathbf{R}$*. If* $\mathbf{P} \neq \mathbf{R}$*, then* $f(x)$ *is a singular monotone function.*

Proof. The monotonicity of $f(x)$ is obvious. For the proof of continuity, it is enough to notice that $f(x)$ takes every value in $[0,1]$. In fact, let y be an arbitrary real number in $[0,1)$ with the generalized m-adic expansion generated by $\mathbf{q}$ and $\mathbf{R}$:

$$y = .x_1x_2x_3\cdots(\mathbf{q},\mathbf{R}). \tag{19}$$

Taking

$$x = .x_1x_2x_3\cdots(\mathbf{q},\mathbf{P}), \tag{20}$$

we have $x \in [0,1)$, and $f(x) = y$. Now we come to the proof of singularity of $f(x)$. Let $x \in [0,1)$ be represented by (9), and $I_{x_1\cdots x_n}$ be the I-interval of order n containing x. It is easy to see that $J_{x_1\cdots x_n}$ is the J-interval of order n containing $f(x)$. Denote the left and right endpoints of $I_{x_1\cdots x_n}$ by $I^-_{x_1\cdots x_n}$ and $I^+_{x_1\cdots x_n}$ respectively, and define $J^-_{x_1\cdots x_n}$ and $J^+_{x_1\cdots x_n}$ similarly. Obviously,

$$f(I^-_{x_1\cdots x_n}) = J^-_{x_1\cdots x_n},\ f(I^+_{x_1\cdots x_n}) = J^+_{x_1\cdots x_n}; \tag{21}$$

$$\mu(J_{x_1\cdots x_n}) = q_{x_1}\prod_{i,j=1}^{m} r_{ij}^{S_n(i,j,x)}. \tag{22}$$

Let

$$\lambda_n(x) = \frac{f(I^+_{x_1\cdots x_n}) - f(I^-_{x_1\cdots x_n})}{I^+_{x_1\cdots x_n} - I^-_{x_1\cdots x_n}}. \tag{23}$$

We have by (10) and (21)–(23),

$$\lambda_n(x) = \prod_{i,j=1}^{m}[r_{ij}/p_{ij}]^{S_n(i,j,x)}. \tag{24}$$

Let B be the set of points of differentiability of f. Then by the existence theorem of derivative of monotone function, $\mu(B) = 1$. In view of a property of derivative (cf. Billingsley 1986, p. 423), we have by (23),

$$\lim_{n\to\infty}\lambda_n(x) = f'(x) < \infty, x \in B. \tag{25}$$

It follows from (24), (25) and (10)–(12) that

$$\begin{aligned}
\lim_{n\to\infty}[\lambda_n(x)]^{1/n} &= \lim_{n\to\infty}\prod_{i,j=1}^{m}[r_{ij}/p_{ij}]^{S_n(i,j,x)/n} \\
&= \prod_{i,j=1}^{m}\lim_{n\to\infty}[r_{ij}/p_{ij}]^{\frac{S_n(i,x)}{n}\cdot\frac{S_n(i,j,x)}{S_n(i,x)}} \\
&= \prod_{i,j=1}^{m}[r_{ij}/p_{ij}]^{p_i p_{ij}}, \quad x \in A\cap B.
\end{aligned} \tag{26}$$

Since $\sum_{i,j=1}^{m} p_i p_{ij} = 1$ and since $\mathbf{R} \neq \mathbf{P}$, it follows from the arithmetic-geometrical inequality that

$$\prod_{i,j=1}^{m}[r_{ij}/p_{ij}]^{p_i p_{ij}} < 1. \tag{27}$$

We have from (25)–(27) that

$$f'(x) = 0, \quad x \in A\cap B. \tag{28}$$

Since $\mu(A\cap B) = 1$, (28) implies $f'(x) = 0$ a.e. This completes the proof of the theorem. ■

The case of generalized binary expansion. Let $m = 2, 0 < p < 1$ be a constant, and let

$$\mathbf{q} = (1-p, p), \tag{29}$$

$$\mathbf{P} = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}. \tag{30}$$

In this case we denote (9) by

$$x = .x_1x_2x_3\cdots(p). \tag{31}$$

We call (31) the generalized binary expansion of x generated by the parameter p, and call the corresponding interval $I_{x_1\cdots x_n}$ the interval of order n generalized by p. Obviously (31) is the common binary expansion when $p = \frac{1}{2}$. It is easy to see that under the Lebesgue measure the sequence of digits $\{x_n, n \geq 1\}$ in (31) constitutes a Bernoulli sequence with probability p for success. Letting $S_n(x)$ be the number of digit 1 in the sequence $x_1, x_2, \cdots, x_n$, we have in place of (13),

$$\mu(I_{x_1\cdots x_n}) = p^{S_n(x)}(1-p)^{n-S_n(x)}. \tag{32}$$

Example. *Letting* $x \in [0,1)$ *be represented by* (31) *and* $0 < r < 1$ *be a constant, we define a function* $f_{p,r} : [0,1] \to [0,1]$ *as follows:*

$$f_{p,r}(x) = .x_1x_2x_3\cdots(r); \tag{33}$$

$$f_{p,r}(1) = 1, \tag{34}$$

where (33) *is the generalized binary expansion of* $f_{p,r}(x)$ *generated by the parameter* r. *If* $r \neq p$, *then* $f_{p,r}(x)$ *is a singular monotone function.*

Proof. Although the above conclusion follows from the theorem immediately, a simple proof avoiding the strong law of Markov chains may be provided. In the notation of the theorem we have

$$\mu(J_{x_1\cdots x_n}) = r^{S_n(x)}(1-r)^{n-S_n(x)}. \tag{35}$$

It follows from (21), (23), (32) and (35) that

$$\lambda_n(x) = \frac{\mu(J_{x_1\cdots x_n})}{\mu(I_{x_1\cdots x_n})} = \left(\frac{r}{p}\right)^{S_n(x)} \left(\frac{1-r}{1-p}\right)^{n-S_n(x)}. \tag{36}$$

Letting $B(p,r)$ be the set of points of differentiability of $f_{p,r}$, we have from (25),

$$\lim_{n\to\infty} \lambda_n(x) = f'_{p,r}(x) < \infty, \ x \in B(p,r). \tag{37}$$

Let

$$A(p) = \{x : \lim_{n\to\infty} S_n(x)/n = p\}. \tag{38}$$

By the Borel strong law of large numbers, $\mu(A(p)) = 1$. It follows from (36), (38), (25) and the arithmetic-geometrical inequality that

$$\begin{aligned} \lim_{n\to\infty} [\lambda_n(x)]^{1/n} &= \lim_{n\to\infty} \left(\frac{r}{p}\right)^{S_n(x)/n} \left(\frac{1-r}{1-p}\right)^{1-S_n(x)/n} \\ &= \left(\frac{r}{p}\right)^{p} \left(\frac{1-r}{1-p}\right)^{1-p} < 1, \quad x \in A(p)\cap B(p,r). \end{aligned} \tag{39}$$

(37) together with (39) implies that

$$f'_{p,r}(x) = 0, \quad x \in A(p)\cap B(p,r). \tag{40}$$

Since $\mu(A(p)\cap B(p,r)) = 1$, (40) implies $f'_{p,r}(x) = 0$ a.e. ∎

Remark 1. For brevity we denote $f_{p,r}$ by f_r when $p = \frac{1}{2}$. Then $f_r(x)$ is a singular monotone function if $r \neq \frac{1}{2}$.

Remark 2. The function given by Riesz and Nagy is a special case of the above example. To prove this fact, we first describe the construction of this function. Let n be an arbitrary nonnegative integer, and call $[k2^{-n}, (k+1)2^{-n}]$ $(k = 0, 1, \cdots, 2^n - 1)$ the nth order interval. We define a sequence of increasing functions $F_n(x)$ $(n = 0, 1, 2, \cdots)$ on $[0, 1]$ inductively as follows. Let $F_0(x) = x$. Suppose $F_n(x)$ which is continuous, increasing and linear on each nth order interval $[\alpha, \beta]$ has been defined. Define $F_{n+1}(x)$ by the following conditions:

(41) $$F_{n+1}(\alpha) = F_n(\alpha), \quad F_{n+1}(\beta) = F_n(\beta);$$

(42) $$F_{n+1}\left(\frac{\alpha+\beta}{2}\right) = \frac{1-t}{2}F_n(\alpha) + \frac{1+t}{2}F_n(\beta);$$

(43) $F_{n+1}(x)$ is linear on the $(n+1)$th order intervals $[\alpha, (\alpha+\beta)/2]$ and $[(\alpha+\beta)/2, \beta]$, where $0 < t < 1$ is a constant.

Let

(44) $$F(x) = \lim_{n-\infty} F_n(x).$$

It is easy to see that $F(x)$ is an increasing and continuous function. Let $x \in [0, 1]$, and let its binary expansion be

(45) $$x = .x_1x_2x_3\cdots = \sum_{k=1}^{\infty} x_k/2^k, \ x_k = 0, 1.$$

Let $\alpha_0 = 1, \ \beta_0 = 1$, and

$$\alpha_n = .x_1\cdots x_n000\cdots (x_k = 0 \text{ as } k > n);$$

$$\beta_n = .x_1\cdots x_n111\cdots (x_k = 1 \text{ as } k > n)$$

for $n \geq 1$. Then $[\alpha_n, \beta_n]$ is an nth order interval containing x, and we have

(46) $$[\alpha_1, \beta_1] \supset [\alpha_2, \beta_2] \supset [\alpha_3, \beta_3] \supset \cdots;$$

(47) $$F(\alpha_n) = F_n(\alpha_n), \quad F(\beta_n) = F_n(\beta_n).$$

It is easy to see that if $x_n = 0$, then

(48) $$\alpha_n = \alpha_{n-1}, \quad \beta_n = (\alpha_{n-1} + \beta_{n-1})/2;$$

if $x_n = 1$, then

$$\alpha_n = (\alpha_{n-1} + \beta_{n-1})/2,\ \beta_n = \beta_{n-1}. \tag{49}$$

We have by (42) and (47)–(49),

$$F(\beta_n) - F(\alpha_n) = \frac{1+t}{2}[F(\beta_{n-1}) - F(\alpha_{n-1})] \text{ when } x_n = 0, \tag{50}$$

$$F(\beta_n) - F(\alpha_n) = \frac{1-t}{2}[F(\beta_{n-1}) - F(\alpha_{n-1})] \text{ when } x_n = 1. \tag{51}$$

By induction it follows from (50) and (51) that

$$F(\beta_n) - F(\alpha_n) = \frac{1}{2^n}(1-t)^{S_n(x)}(1+t)^{n-S_n(x)}. \tag{52}$$

Using the notation in (32) and (35), we have

$$[\alpha_n, \beta_n) = I_{x_1\cdots x_n},\ \mu(I_{x_1\cdots x_n}) = 1/2^n; \tag{53}$$

$$[F(\beta_n), F(\alpha_n)) = J_{x_1\cdots x_n},$$
$$\mu(J_{x_1\cdots x_n}) = \left(\frac{1-t}{2}\right)^{S_n(x)}\left[1 - \frac{1-t}{2}\right]^{n-S_n(x)}. \tag{54}$$

Obviously (53) and (54) imply that $F(x) = f_r(x)$, where $r = (1-t)/2$.

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