TAIWANESE JOURNAL OF MATHEMATICS
Vol. 2, No. 3, pp. 321-327, September 1998

A SECTION THEOREM IN INTERVAL SPACE WITH APPLICATIONS

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Abstract. In this paper, we prove a section theorem of Ky Fan type in interval space, and then, as its applications, some minimax inequalities and a fixed point theorem are obtained.

1. Preliminaries

We recall some elementary concepts on an interval space (see [1] and [5]):

1) By an *interval space* we mean a topological space X endowed with a mapping $[\cdot,\cdot] : X \times X \to \{\text{connected subsets of } X\}$ such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$.

2) A subset K of an interval space X is *convex* if for every $x_1, x_2 \in K$ we have $[x_1, x_2] \subset K$.

Obviously, in any interval space X, convex sets are connected or empty. The intersection of any family of convex sets is convex.

3) A function f mapping an interval space X into $\mathbb{R}$ is *quasiconvex* (or *quasiconcave*) if $f(z) \leq \max\{f(x_1), f(x_2)\}$ (or $f(z) \geq \min\{f(x_1), f(x_2)\}$) whenever $x_1, x_2 \in X$ and $z \in [x_1, x_2]$. Thus f is quasiconvex (or quasiconcave) if and only if the sets $\{x|f(x) \leq \gamma\}$ (or $\{x|f(x) \geq \gamma\}$) are convex for all $\gamma \in \mathbb{R}$.

2. A Section Theorem of Ky Fan Type

In order to obtain our main result, we state a lemma which was proved in [1].

Lemma 1. *Let Y be an interval space, X a topological space and K be a mapping of Y into the family of compact subsets of X, such that*

Received November 20, 1995; revised September 25, 1996.
Communicated by M.-H. Shih.
1991 *Mathematics Subject Classifications:* 47H04, 47H10, 49J35, 52A07.
Key words and phrases: Interval space, compact set, convex set, connected.

(1.1) *$K(y) \neq \emptyset$ for all $y \in Y$;*

(1.2) *$K(z) \subset K(y_1) \cup K(y_2)$ whenever $z \in [y_1, y_2]$ and $y_1, y_2 \in Y$;*

(1.3) *$\bigcap_{k=1}^{n} K(y_i)$ is connected or empty for every $y_1, y_2, \cdots, y_n \in Y (n = 1, 2, \cdots)$;*

(1.4) *$x \in K(y)$ whenever $y = \lim_{\alpha \in A} y_\alpha, x = \lim_{\alpha \in A} x_\alpha$ and $x_\alpha \in K(y_\alpha)$ for all $\alpha \in A$.*

Then we have $\bigcap_{y \in Y} K(y) \neq \emptyset$.

In the following, we give our main result.

Theorem 2. *Let X be a compact topological space, Y be an interval space and $A \subset X \times Y$ such that*

(2.1) *A is open in $X \times Y$;*

(2.2) *$A[x] = \{y \in Y | (x, y) \in A\}$ is convex and nonempty for each $x \in X$;*

(2.3) *$\bigcap_{i=1}^{n} (X \backslash A[y_i])$ is connected for every finite subset $\{y_1, \cdots, y_n\} \subset Y$, where $A[y_i] = \{x \in X \,|\, (r, y_i) \in A\}$.*

Then there exists a point $y_0 \in Y$ such that $X \times \{y_0\} \subset A$

Proof: If the conclusion of the theorem does not hold, then for each $y \in Y$, there exists a point $x_0 \in X$ such that $(x_0, y) \notin A$. Let

$$K(y) = \{x \in X \,|\, (x, y) \notin A\}.$$

Then, $K : Y \to 2^X$ is a multivalued mapping with nonempty compact values because $K(y) = X \backslash A[y]$, A is open and X is compact. Moreover,

$$\begin{aligned} \text{Graph}(K) &= \{(y, x) \in Y \times X \,|\, x \in K(y)\} \\ &= \{(y, x) \in Y \times X \,|\, (x, y) \notin A\} \end{aligned}$$

is closed since A is open. Hence, the condition (1.4) of Lemma 1 is satisfied.

If there exist two points $y_1^*, y_2^* \in Y$ and $z^* \in [y_1^*, y_2^*]$ such that

$$K(z^*) \not\subset K(y_1^*) \cup K(y_2^*),$$

then there exists an $x^* \in K(z^*)$, but $x^* \notin K(y_1^*) \cup K(y_2^*)$. On the one hand, by $x^* \in K(z^*)$, we have $z^* \notin A[x^*]$, because of

$$\begin{aligned} z^* \in K^{-1}(x^*) &= \{y \in Y \,|\, x^* \in k(y)\} = \{y \in Y \,|\, (x^*, y) \notin A\} \\ &= \{y \in Y \,|\, y \notin A[x^*]\} = Y \backslash A[x^*]. \end{aligned}$$

On the other hand, by $x^* \notin K(y_1^*) \cup K(y_2^*)$, we have $x^* \notin K(y_j^*)$ $(j = 1, 2)$, and so $(x^*, y_j^*) \in A$ $(j = 1, 2)$, i.e., $y_j^* \in A[x^*]$ $(j = 1, 2)$. Hence, $[y_1^*, y_2^*] \subset A[x^*]$ implies $z^* \in A[x^*]$ by (2.2). It is a contradiction. Therefore, the condition (1.2) of Lemma 1 holds.

Summing up the above arguments, adding (2.3) in, we know that all the conditions of Lemma 1 are fulfilled. By virtue of Lemma 1, we have that $\bigcap_{y \in Y} K(y) \neq \emptyset$. It follows that there exists $\overline{x} \in K(y)$ for all $y \in Y$. It implies $y \notin A[\overline{x}]$ for all $y \in Y$, i.e., $A[\overline{x}] = \phi$. This contradicts the condition (2.2). Therefore, Theorem 2 is true. ∎

Remark. Theorem 2 is a new section theorem of Ky Fan type. Its conditions differ from other section theorems (c.f. [3], [4] and [6]).

3. Some Applications

3-1. Applications to Minimax Problems.

Now, we apply Theorem 2 to minimax problems.

Theorem 3. (Ky Fan Minimax Principle). *Let X be a compact interval space and $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ $(f \not\equiv +\infty)$ be a function. Let $\varphi : X \times X \to \mathbb{R}$ be a function with $\varphi(x, x) \geq 0$ for each $x \in X$. If the following conditions are satisfied:*

(3.1) *for each $x \in X, f(y) + \varphi(x, y)$ is quasiconvex in y;*

(3.2) *for each $y \in X, f(x) - \varphi(x, y)$ is quasiconvex in x;*

(3.3) *the set $\{(x, y) \in X \times X \mid f(y) + \varphi(x, y) \geq f(x)\}$ is closed,*

then there exists an $\overline{x} \in X$ such that

$$f(y) + \varphi(\overline{x}, y) \geq f(\overline{x})$$

for all $y \in X$.

Proof: Put

$$A = \{(x, y) \in X \times X \mid f(y) + \varphi(x, y) < f(x)\}.$$

Then A is open by (3.3). If the conclusion of theorem is false, then, for each $x \in X$, there exists a $\overline{y} \in X$ such that $f(\overline{y}) + \varphi(x, \overline{y}) < f(x)$. It implies $\overline{y} \in A[x]$, i.e., $A[x] \neq \phi$. By the conditions (3.1) and (3.2), we have that

$$\begin{aligned} A[x] &= \{y \in X \mid (x, y) \in A\} \\ &= \{y \in X \mid f(y) + \varphi(x, y) < f(x)\} \end{aligned}$$

is convex and

$$X\backslash A[y_i] = \{x \in X \mid f(x) - \varphi(x, y_i) \le f(y_i)\}$$

is convex, too. Hence, $\bigcap\limits_{i=1}^{n} (X\backslash A[y_i])$ is connected for every $\{y_1, \cdots, y_n\} \subset X$.

By virtue of Theorem 2, there exists a $\overline{y} \in X$ such that $X \times \{\overline{y}\} \subset A$. It implies $f(\overline{y}) + \varphi(x, \overline{y}) < f(x)$ for all $x \in X$. Hence, $\varphi(\overline{y}, \overline{y}) < 0$. It contradicts that $\varphi(x, x) \ge 0$ for all $x \in X$. Therefore, Theorem 3 is true. ■

Theorem 4. (Ky Fan Minimax Inequality) *Let X be a compact interval space, and $\varphi : X \times X \to \mathbb{R}$ be an upper semicontinuous function such that*

(4.1) *for each $x \in X, \varphi(x, y)$ is quasiconvex in y;*

(4.2) *for each $y \in X, \varphi(x, y)$ is quasiconcave in x;*

(4.3) *for each $x \in X$, there exists a point $y' \in X$ such that $\varphi(x, y') < \sup\limits_{y \in X} \varphi(y, y)$,*

then there exists a $\overline{y} \in X$ such that

$$\sup_{x \in X} \varphi(x, \overline{y}) \le \sup_{y \in X} \varphi(y, y).$$

Proof. We may assume that $\gamma = \sup\limits_{y \in X} \varphi(y, y) < +\infty$. Let $A = \{(x, y) \in X \times X \mid \varphi(x, y) < \gamma\}$. Then A is open. For each $x \in X, A[x] = \{y \in X \mid (x, y) \in A\} = \{y \in X \mid \varphi(x, y) < \gamma\}$ is a nonempty convex subset of X by (4.1) and (4.3). The set

$$\begin{aligned}\bigcap_{i=1}^{n} (X \setminus A[y_i]) &= \bigcap_{i=1}^{n} \{x \in X \mid (x, y_i) \notin A\} \\ &= \bigcap_{i=1}^{n} \{x \in X \mid \varphi(x, y_i) \ge \gamma\}\end{aligned}$$

is connected or empty by (4.2) for each finite subset $\{y_1, \cdots, y_n\} \subset X$. By virtue of Theorem 2, there exists a $\overline{y} \in X$ such that $X \times \{\overline{y}\} \subset A$, i.e., $(x, \overline{y}) \in A$ for all $x \in X$. It follows that $\varphi(x, \overline{y}) < \gamma$ for all $x \in X$. Hence,

$$\sup_{x \in X} \varphi(x, \overline{y}) \le \sup_{y \in X} \varphi(y, y).$$

This completes the proof. ■

Theorem 5. (Von Neumann Inequality) *Let X be compact topological space and K be a nonempty compact convex subset of interval space Y. If $f : X \times Y \to \mathbb{R}$ is an upper semicontinuous function such that*

(5.1) *for each* $x \in X, f$ *is quasiconvex on* Y*;*

(5.2) *for each* $y \in Y, f$ *is quasiconcave on* X*,*

then

$$\inf_{y\in K}\sup_{x\in X} f(x,y) \le \inf_{K\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y).$$

Proof: If $\inf_{k\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y) = +\infty$, then the theorem is obviously true. So, we can assume that $\inf_{K\subset Y}\sup_{r\in X}\inf_{y\in K} f(x,y) < +\infty$. And then, we choose a real number $t \in \mathbb{R}$ such that $\inf_{K\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y) < t$. Let

$$A = \{(x,y) \in X \times Y | f(x,y) < t\}.$$

Then A is open because f is upper semicontinuous. For each $x \in X$, the section $A[x] = \{y \in Y|(x,y) \in A\} = \{y \in Y|f(x,y) < t\}$ is convex by (5. l).

When $\inf_{K\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y) < t$, there exists a nonempty compact convex set $K_0 \subset Y$ such that $\sup_{x\in X}\inf_{y\in K_0} f(x,y) < t$. And then, for each $x \in X$, there exists a point $\overline{y} \in K_0$ such that $f(x,\overline{y}) < t$, i.e., $\overline{y} \in A[x]$. Therefore, $A[x]$ is nonempty.

For every finite subset $\{y_1, \cdots, y_n\} \subset Y$ $(n = 1, 2, \cdots)$, the set

$$\begin{aligned}\bigcap_{i=1}^{n}(X \setminus A[y_i]) &= \bigcap_{i=1}^{n}\{x \in X \,|\, (x,y_i) \notin A\}\\ &= \bigcap_{i=1}^{n}\{x \in X \,|\, f(x,\, y_i) \ge t\}\end{aligned}$$

must be connected by (5.2). By virtue of Theorem 2, there exists a point $y_0 \in K$ such that $X \times \{y_0\} \subset A$, i.e., $f(x,y_0) < t$ for all $x \in X$. Hence,

$$\sup_{x\in X} f(x,y_0) \le t.$$

Obviously, $\inf_{y\in Y}\sup_{x\in X} f(x,y) \le t$. It turns out that

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \inf_{k\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y). \qquad \blacksquare$$

Remark. By an obvious inequality

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \ge \inf_{k\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y),$$

we have

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \inf_{k\subset Y}\sup_{x\in X}\inf_{y\in K} f(x,y).$$

If, in addition, Y is compact, then

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

The above results differ from Theorem 3 in [2], and imply Theorem 4 in [3] and Theorem 2 in [2].

3-2. Application on Fixed Point Problem.

Next, we apply the result of Theorem 2 to a fixed point problem.

Theorem 6. *Let X be a compact interval space, and $K : X \to 2^X$ be a multivalued mapping with compact values, such that*

(6.1) *K has closed graph;*

(6.2) *$X \setminus K(x)$ is convex for each $x \in X$;*

(6.3) $\bigcap_{i=1}^{n} K^{-1}(y_i) = \bigcap_{i=1}^{n}\{x \in X \mid y_i \in K(x)\}$ *is connected or empty for every finite subset* $\{y_1, \cdots, y_n\} \subset X$ $(n = 1, 2, \cdots)$;

(6.4) $K^{-1}(x) = \{z \in X \mid x \in K(z)\} \neq \emptyset$ *for each $x \in X$.*

Then K has a fixed point in X.

Proof: Put $A = \{(x,y) \in X \times X \mid y \notin K(x)\}$, i.e., $A = X \times X \setminus \mathrm{Graph}(K)$. Hence, A is open by (6.1).

For each $x \in X, A[x] = \{y \in X|(x,y) \in A\} = \{y \in X \mid y \notin K(x)\} = X \setminus K(x)$ is convex by (6.2). If there is no fixed point of K in X, then $x \notin K(x)$ for every $x \in X$. Consequently, $A[x] \neq \emptyset$ for each $x \in X$. For each finite subset $\{y_1, \cdots, y_n\} \subset X$,

$$\begin{aligned}\bigcap_{i=1}^{n}(X\setminus A[y_i]) &= \bigcap_{i=1}^{n}\{x \in X \mid (x,y_i) \notin A\} = \bigcap_{i=1}^{n}\{x \in X \mid y_i \in K(x)\}\\ &= \bigcap_{i=1}^{n} K^{-1}(y_i)\end{aligned}$$

is connected by (6.3).

By virtue of Theorem 2, there exists a point $\overline{y} \in X$ such that $X \times \{\overline{y}\} \subset A$, i.e., for each $x \in X, (x,\overline{y}) \in A$. Hence, $\overline{y} \overline{\in} K(x)$ for all $x \in X$. It follows that $K^{-1}(\overline{y}) = \emptyset$, which contradicts (6.4). The conclusion of the theorem, therefore, has been proved. ∎

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