TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 6, pp. 2289-2295, December 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

INCIDENCE COLORING OF REGULAR GRAPHS AND COMPLEMENT GRAPHS

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Abstract. Using a relation between domination number and incidence chromatic number, we obtain necessary and sufficient conditions for r-regular graphs to be $(r+1)$-incidence colorable. Also, we determine the optimal Nordhaus-Gaddum inequality for the incidence chromatic number.

1. Introduction

An incidence coloring of a graph G assigns a color to each incidence so that no two adjacent incidences receive the same color. Since incidence coloring was introduced [3], most of the researches were concentrated on establishing upper bounds on the minimum number of colors, also known as the incidence chromatic number $\chi_i(G)$, which can color all incidences. Therefore, to improve the lower bound on incidence chromatic numbers for some classes of graphs is the main objective of this article.

In Section 2, a relation between domination number and incidence chromatic number will be established. We then use this relation to characterize $(r+1)$-incidence colorable r-regular graphs. Also, bounds on the incidence chromatic number of a graph and its complement will be obtained in Section 3.

All graphs in this paper are simple and connected. Let $V(G)$ and $E(G)$ (or V and E) be the vertex-set and edge-set of a graph G, respectively. Let the set of all neighbors of a vertex u be $N_G(u)$ (or simply $N(u)$). Similarly, for any $S \subseteq V$, the neighborhood $N(S)$ of S is $\{u \mid v \in S, uv \in E\}$. Moreover, the degree $d_G(u)$(or simply $d(u)$) of u is equal to $|N_G(u)|$ and the maximum degree of G is denoted by $\Delta(G)$ (or simply Δ). All notations not defined in this paper can be found in the books [2, 15].

Let $D(G)$ be a digraph induced from G by replacing each edge $uv \in E(G)$ by two opposite arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$. According to Guiduli [6], incidence coloring of G is

Received February 8, 2012, accepted April 30, 2012.
Communicated by Gerard Jennhwa Chang.
2010 *Mathematics Subject Classification*: 05C15, 05C69.
Key words and phrases: Incidence chromatic number, Regular graph, Complement graph, Domination number.
This research was partially supported by the Pentecostal Holiness Church Incorporation (Hong Kong).

equivalent to proper coloring of $D(G)$, where two distinct arcs $\overrightarrow{uv}$ and $\overrightarrow{xy}$ are *adjacent* provided one of the following holds:

(1) $u = x$;
(2) $v = x$ or $y = u$.

From this definition, the following global lower bound is obvious:

Proposition 1.1. [3]. *For every graph G,* $\chi_i(G) \geq \Delta(G) + 1$.

The incidence coloring conjecture (ICC) states that $\chi_i(G) \leq \Delta(G) + 2$ for all graphs G [3]. Although Guiduli [6] showed that ICC is false by relating incidence coloring to star arboricity [1] on Paley graphs, there are a lot of other classes of graphs such as cubic graphs and outerplanar graphs satisfying the ICC [8, 9, 10, 12, 13, 14].

2. Characterization of Regular Graphs

Our characterization of $(r+1)$-incidence colorable r-regular graphs relies on a relation between incidence chromatic number and domination number. A *dominating set* $S \subseteq V(G)$ of a graph G is a set such that every vertex in $G - S$ has a neighbor in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set in G.

Proposition 2.1. [7]. *If G is a graph, then* $\gamma(G) \geq \left\lceil \frac{|V|}{\Delta+1} \right\rceil$.

Proof. Let u be a vertex of G. The maximum number of vertices that u can dominate is $\Delta + 1$, hence we have $\gamma(G) \geq \left\lceil \frac{|V|}{\Delta+1} \right\rceil$. ∎

A *star forest* of a graph G is a spanning subgraph of G in which each component is a star. A *maximal star forest* is a star forest with maximum number of edges. Ferneyhough et al. [5] proved that the number of edges of a maximal star forest of a graph G is equal to $|V| - \gamma(G)$. We now use the domination number to establish a lower bound on the incidence chromatic number of a graph. The following proposition reformulates the ideas in [1, 10].

Proposition 2.2. *If G is a graph, then* $\chi_i(G) \geq \frac{2|E|}{|V| - \gamma(G)}$.

Proof. To form the digraph $D(G)$, each edge of G is divided into two arcs in opposite directions. The total number of arcs of $D(G)$ is therefore equal to $2|E|$. According to the definition of the adjacency of arcs, an independent set of arcs is a star forest. Thus, a maximal independent set of arcs is a maximal star forest. We conclude that the number of color classes required is at least $\frac{2|E|}{|V| - \gamma(G)}$. ∎

Corollary 2.3. *If G is an r-regular graph with* $\chi_i(G) = r + 1$, *then* $\gamma(G) = \frac{|V|}{r+1}$.

Proof. By Handshaking lemma, we have $2|E| = \sum_{v \in V} d(v) = r|V|$. This equality together with $\chi_i(G) = r + 1$ simplify the inequality in Proposition 2.2 into $\gamma(G) \leq \frac{|V|}{r+1}$. Since the global lower bound on the domination number of a graph is $\left\lceil \frac{|V|}{\Delta+1} \right\rceil$ (Proposition 2.1), we conclude that the domination number of G is $\frac{|V|}{r+1}$. ■

The *square* G^2 of a graph G is the graph with vertex set $V(G)$, and an edge $uv \in E(G^2)$ if and only if there is a uv-path in G of length at most 2. The chromatic number of G^2 is closely related to the incidence chromatic number of G by the following proposition. Let $C_G^-(u)$ (resp. $C_G^+(u)$) be the set of colors assigned to the arcs going into (resp. going out from) a vertex u of a graph G.

Proposition 2.4. [13]. *Every graph G has $\chi(G^2) = k$ if and only if there is a k-incidence coloring of G with $|C_G^-(u)| = 1$ for all $u \in V$.*

Corollary 2.5. *If G is an r-regular graph with $\chi_i(G) = r + 1$, then $\chi(G^2) = \chi_i(G) = r + 1$.*

Proof. Since G is r-regular and only $r+1$ colors are available, we have $|C_G^-(u)| = 1$ for all $u \in V$ and thus $\chi(G^2) = \chi_i(G) = r + 1$ by Proposition 2.4. ■

Recently, Wu [16] studied the order of the color classes in a vertex coloring of G^2 and proved the following proposition.

Proposition 2.6. [16]. *If G is an r-regular graph and σ is a proper $(r + 1)$-vertex coloring of G^2, then $|\sigma^{-1}(i)| = |\sigma^{-1}(j)|$ for $i, j \in \{1, \ldots, r + 1\}$ where $\sigma^{-1}(i) = \{v \in V(G) \mid \sigma(v) = i\}$.*

We now characterize the $(r + 1)$-incidence colorable r-regular graphs.

Theorem 2.7. *If G is an r-regular graph, then $\chi_i(G) = \chi(G^2) = r + 1$ if and only if $V(G)$ is a disjoint union of $r + 1$ dominating sets.*

Proof. Suppose that $\chi(G^2) = r + 1$, and let σ be a proper $(r + 1)$-vertex coloring of G^2. It follows from Proposition 2.6 that $|\sigma^{-1}(i)| = \frac{|V|}{r+1}$ for $i \in \{1, \ldots, r + 1\}$. For any two vertices $u, v \in \sigma^{-1}(i)$, we have $N(u) \cap N(v) = \emptyset$. Also, neighbors of u belong to r different color classes and thus $|N(\sigma^{-1}(i))| = \frac{r|V|}{r+1}$. As a result, $\sigma^{-1}(i)$ is a dominating set for $i \in \{1, \ldots, r + 1\}$ and $\sigma^{-1}(1), \sigma^{-1}(2) \ldots, \sigma^{-1}(r + 1)$ are $r + 1$ disjoint dominating sets whose union is $V(G)$.

Conversely, suppose that $S_1, \ldots, S_{r+1}$ are $r + 1$ disjoint dominating sets of G such that $V(G) = S_1 \cup \cdots \cup S_{r+1}$. By Corollary 2.3, the minimum order of these $r + 1$ sets is $\frac{|V|}{r+1}$ and hence $|S_1| = \cdots = |S_{r+1}| = \frac{|V|}{r+1}$. Since S_i is a dominating set for $i \in \{1, \ldots, r + 1\}$, it follows that $|N(S_1)| = |N(S_2)| = \cdots = |N(S_{r+1})| = \frac{r|V|}{r+1}$.

Therefore, we have $N(u) \cap N(v) = \emptyset$ for any two vertices $u, v \in S_i$. We color the vertices in S_i by color i for $i \in \{1, \ldots, r+1\}$, and this is a proper $(r+1)$-vertex coloring of G^2. We can then conclude thanks to Corollary 2.5. ∎

The conditions in Theorem 2.7 can be expressed in a more explicit form for cubic graphs.

Theorem 2.8. *If G is a cubic graph, then $\chi_i(G) = \chi(G^2) = 4$ if and only if*

(1) *there exists a dominating set S with $|S| = \frac{|V|}{4}$,*

(2) *the graph $G - S$ is a disjoint union of cycles $C_1 \cup \cdots \cup C_k$, where $|C_i| = p_i$ and $p_i \equiv 0 \pmod 3$, and*

(3) *there exists a labeling of the vertices of each C_i by the list $234234 \cdots 234$ such that two vertices (may come from different cycles) with the same label do not have a common neighbor in S.*

Proof. Suppose that $\chi(G^2) = 4$ and let σ be a proper 4-vertex coloring of G^2. As in the proof of Theorem 2.7, we obtain condition 1 with $S = \sigma^{-1}(1)$ and $G - S$ is a 2-regular graph. Thus, $G - S$ is a disjoint union of cycles $C_1 \cup \cdots \cup C_k$ for some k and $\chi((G-S)^2) = 3$. It follows that the orders of the cycles $C_1, C_2, \ldots, C_k$ are divisible by three and condition 2 is satisfied. To obtain condition 3, we label every vertex $u \in G - S$ by $\sigma(u)$. If there are two vertices u and v with $\sigma(u) = \sigma(v)$ and having a common neighbor in S, then u and v are at distance two in G. This result contradicts the fact that σ is a proper 4-vertex coloring of G^2.

Conversely, suppose that G is a cubic graph that satisfies conditions 1,2 and 3, and let σ be a mapping from V to $\{1, 2, 3, 4\}$. Since $|S| = \frac{|V|}{4}$ and $|N(S)| = \frac{3|V|}{4}$, any two vertices from S do not have a common neighbor. We assign $\sigma(u) = 1$ for all $u \in S$ and $\sigma(v) = i$ for all $v \in G - S$, where i is the labeling of v in condition 3. For any two vertices $x, y \in G - S$ with $\sigma(x) = \sigma(y)$, x and y do not have a common neighbor in S. Also, the shortest path between x and y in the graph $G - S$ is of length at least three. Therefore, $N(x) \cap N(y) = \emptyset$ and σ is a proper 4-vertex coloring of G^2. ∎

Theorem 2.9. [10]. *If G is a cubic graph, then $\chi_i(G) \leq 5$.*

Theorem 2.8 together with Theorem 2.9 characterize the cubic graph G with $\chi_i(G) = 5$ also.

3. Incidence Coloring of a Graph and Its Complement

The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$, and an edge $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. In 1956, Nordhaus and Gaddum [11] established the following inequality which bounds the addition of $\chi(G)$ and $\chi(\overline{G})$.

Theorem 3.1. [11]. *If G is a graph with n vertices, then*

$$\lceil 2\sqrt{n}\,\rceil \leq \chi(G) + \chi(\overline{G}) \leq n+1.$$

A *total coloring* of a graph G assigns a color to each vertex and edge of G such that no two adjacent vertices or edges receive the same color, and the color of each vertex u is distinct from the colors of its incident edges. The *total chromatic number* $\chi_T(G)$ of a graph G is the minimum number of colors required for a total coloring of G. Cook [4] established the following Nordhaus-Gaddum inequality for the total chromatic number.

Theorem 3.2. [4]. *If G is a graph with n vertices, then*

$$n+1 \leq \chi_T(G) + \chi_T(\overline{G}) \leq 2n.$$

Also, these bounds are sharp for all values of n.

We next develop the Nordhaus-Gaddum inequality for the incidence chromatic number.

Theorem 3.3. *If G is a graph with n vertices and $G \neq K_n$ or $\overline{K_n}$, then*

$$n+2 \leq \chi_i(G) + \chi_i(\overline{G}) \leq 2n-1.$$

Also, these bounds are sharp for all values of n.

Proof. As G (and also $\overline{G}$) is not equal to $\overline{K_n}$, it follows that $\chi_i(G) \geq \Delta(G)+1$ and $\chi_i(\overline{G}) \geq \Delta(\overline{G})+1$. Hence, we have

$$\begin{aligned}
\chi_i(G) + \chi_i(\overline{G}) &\geq \Delta(G)+1+\Delta(\overline{G})+1 && (1)\\
&\geq \frac{\sum d_G(u)}{n} + \frac{\sum d_{\overline{G}}(u)}{n} + 2 && (2)\\
&= \frac{n(n-1)}{n} + 2\\
&= n+1.
\end{aligned}$$

If $\chi_i(G) + \chi_i(\overline{G}) = n+1$, then inequalities (1) and (2) become equality and thus

$$\Delta(G) = \frac{\sum d_G(u)}{n}, \tag{3}$$

$$\Delta(\overline{G}) = \frac{\sum d_{\overline{G}}(u)}{n}, \tag{4}$$

$$\chi_i(G) = \Delta(G)+1, \tag{5}$$

$$\chi_i(\overline{G}) = \Delta(\overline{G})+1. \tag{6}$$

Equalities (3) and (4) imply that G and $\overline{G}$ are regular graphs. Let G be an r-regular graph and hence, $\overline{G}$ is an $(n-r-1)$-regular graph. Equalities (5) and (6) together with Corollary 2.3 implies that $\frac{n}{r+1}$ and $\frac{n}{n-r}$ are both integers, which is a contradiction. We conclude that $\chi_i(G)+\chi_i(\overline{G}) \geq n+2$.

Since G and $\overline{G}$ are subgraphs of K_n, it follows that $\chi_i(G)+\chi_i(\overline{G}) \leq 2\chi_i(K_n) = 2n$. Suppose that $\chi_i(G) = \chi_i(\overline{G}) = n$. A vertex $u \in V(G)$ of degree $d_G(u)$ equal to $n-1$ implies $d_{\overline{G}}(u) = 0$ and thus, $\chi_i(\overline{G}) = \chi_i(\overline{G}-u) \leq n-1$. The same argument applied to $\overline{G}$ shows similar result. Therefore, $0 < d_G(u), d_{\overline{G}}(u) < n-1$ for all $u \in V(G)$.

Let $V(G) = \{v_1, \ldots, v_n\}$, and let $A(G)$ be the set of arcs of digraph $D(G)$. We assign color i to the arcs $\overrightarrow{v_jv_i} \in A(G) \cup A(\overline{G})$ for all $i, j \in \{1, \ldots, n\}$. If there is a vertex $v_m \in V(G)$ with $|N_G(v_m) \cup N_G(v_i)| \leq n-1$ for all $v_i \in N_G(v_m)$, then there exists a color $c_i \in \{1, \ldots, n\} \backslash \{C^-_G(v_i) \cup C^+_G(v_i) \cup C^+_G(v_m)\}$. We then recolor the arcs $\overrightarrow{v_iv_m}$ with c_i for all i, the arcs of $D(G)$ are now properly colored without color m and hence $\chi_i(G) \leq n-1$. Otherwise, there is a vertex $v_j \in N(v_m)$ such that $|N_G(v_m) \cup N_G(v_j)| = n$, which implies $|N_{\overline{G}}(v_m) \cap N_{\overline{G}}(v_j)| = 0$. Therefore, we can assign color j to the arcs $\overrightarrow{v_iv_m} \in A(\overline{G})$ for all $v_i \in N_{\overline{G}}(v_m)$ and thus $\chi_i(\overline{G}) \leq n-1$. We conclude that $\chi_i(G)+\chi_i(\overline{G}) \leq 2n-1$ for all graphs G with n vertices.

Finally, the graph $G = K_{1,n-1}$ and its complement $\overline{G} = K_{n-1} \cup \{u\}$, where $d_{\overline{G}}(u) = 0$, form an example with $\chi_i(G)+\chi_i(\overline{G}) = 2n-1$. On the other hand, if $G = K_n - e$, where $e \in E(K_n)$, then $\chi_i(G)+\chi_i(\overline{G}) = n+2$. ∎

Note that when n is odd, the complementary pair $G = K_{1,n-1}$ and $\overline{G} = K_{n-1} \cup \{u\}$ also attains the upper bound in Theorem 3.2. This result reveals another similarity between total coloring and incidence coloring.

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