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EXPONENTIAL STABILITY OF SOLUTIONS TO SEMILINEAR PARABOLIC EQUATIONS WITH DELAYS

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Abstract. In this paper we prove the global existence and the global exponential stability of weak solutions to a class of semilinear parabolic equations with discrete and distributed time-varying delays. Moreover, the exponential stability of stationary solutions of the equations is also studied. The obtained results can be applied to some models in biology and physics.

1. INTRODUCTION

The study of functional differential equations is motivated by the fact that when one wants to model some evolution phenonmena arising in physics, biology, engineering, etc., some hereditary characteristics aftereffect, time lag and time delay can appear in the variables. Typical example arise from the researches of material with thermal memory, biochemical, population models, etc (see e.g. [9, 16]). One of the most important and interesting problem in the analysis of functional differential equations is to study the stability of solutions. This theory has been greatly developed over the last years for ordinary differential equations (ODEs) with delays and recently for partial differential equations (PDEs) with delays.

PDEs with delays are often considered in the model such as maturation time for population dynamics in mathematical biology and other fields. Such equations are natural more difficult than ODEs with delays since they are infinite dimensional both in time and space variables. As mentioned in [10], the stability analysis of PDEs with delays is essentially more complicated. We refer the reader to some recent works on Lyapunov-based technique for PDEs with delays [2, 4, 5, 6, 8, 10, 11, 15].

In this paper, we study the exponential stability of solutions to the following semilinear parabolic equation with a mixed delay

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$$
(1.1)\qquad \begin{aligned}
\frac{\partial}{\partial t}u(t,x)+Au(t,x)+f(u(t,x))&=F(t,u_t)(x)+g(x,t),\ x\in\Omega,\ t>0,\\
u(0+,x)&=u^0(x),\quad x\in\Omega,\\
u(\theta,x)&=\phi(\theta,x),\ \theta\in(-r,0),\ x\in\Omega.
\end{aligned}
$$

Here Ω is a bounded domain in $\mathbb{R}^N$ and other symbols satisfy the following conditions:

(H1) A is a densely-defined self-adjoint positive linear operator with domain $D(A)\subset L^2(\Omega)$ and with compact resolvent (for example, $-\Delta$ with the homogeneous Dirichlet condition).

(H2) $f:\mathbb{R}\to\mathbb{R}$ is a C^1 function such that

$$
(1.2)\qquad C_1|u|^p-C_0\le f(u)u\le C_2|u|^p+C_0,\ p\ge 2,
$$

$$
(1.3)\qquad f'(u)\ge -\ell,\quad \text{for all } u\in\mathbb{R},
$$

where C_0, C_1, C_2 and ℓ are positive constants.

(H3) The mixed delayed function $F(t,u_t)$ is in the form

$$
(1.4)\qquad F(t,u_t)=F_1(t,u(t-h(t)))+\int_{t-\tau(t)}^{t}F_2(s,u(s))ds,
$$

where $h(t)$ and $\tau(t)$ are time-varying delay functions satisfying

$$
(1.5)\qquad \begin{aligned}
&\sup_{t\ge 0}h(t)=h\in(0,+\infty),\quad \sup_{t\ge 0}\tau(t)=\tau\in(0,+\infty),\\
&\sup_{t\ge 0}h'(t)=h_*\in[0,1),\quad \sup_{t\ge 0}\tau'(t)=\tau_*\in[0,1),
\end{aligned}
$$

$F_i(t,.):L^2(-r,0;L^2(\Omega))\to L^2(\Omega), i=1,2$, $F_2(t,0)=0$ and there exist some constants $k_1,k_2>0$, $C_F\ge 0$ such that for all $u,v\in L^2(-r,0;L^2(\Omega))$, where $r=\max\{h,\tau\}$, one has

$$
(1.6)\qquad \begin{aligned}
&\|F_1(t,0)\|\le C_F,\quad t\in[0,+\infty),\\
&\|F_i(t,u)-F_i(t,v)\|\le k_i\|u-v\|_{L^2(-r,0;L^2(\Omega))},\ i=1,2,\ t\in[0,+\infty).
\end{aligned}
$$

Hereafter, we denote the norm in $L^2(\Omega)$ by $\|.\|$.

(H4) The external force $g\in L^2_{loc}(0,+\infty;L^2(\Omega))$ is given.

It follows from (H3) that, for any $u\in L^2(-r,0;L^2(\Omega))$, we have

$$
\begin{aligned}
\|F_1(t,u)\|\le\|F_1(t,u)-F_1(t,0)\|+\|F_1(t,0)\|&\le k_1\|u\|_{L^2(-r,0;L^2(\Omega))}+C_F,\quad \forall t\ge 0,\\
\|F_2(t,u)\|=\|F_2(t,u)-F_2(t,0)\|&\le k_2\|u\|_{L^2(-r,0;L^2(\Omega))}.
\end{aligned}
$$

Thus, $F(t,.)$ is a bounded operator from $L^2(-r,0;L^2(\Omega))$ to $L^2(\Omega)$.

Since $A : D(A) \to L^2(\Omega)$ is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, A has a discrete spectrum that only contains positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots., \quad \lambda_k \to \infty, \quad \text{as } k \to \infty,$$

and the corresponding eigenfunctions $\{e_k\}_{k=1}^{\infty}$ compose an orthonormal basis of the Hilbert space $L^2(\Omega)$ such that

$$(e_j, e_k) = \delta_{jk} \text{ and } Ae_k = \lambda_k e_k, \quad j, k = 1, 2, \ldots$$

Hence we can define the fractional power spaces and operators as follows

$$X^{\alpha} = D(A^{\alpha}) = \Big\{ u = \sum_{k=1}^{\infty} c_k e_k \in H : \sum_{k=1}^{\infty} c_k^2 \lambda_k^{2\alpha} < \infty \Big\},$$

$$A^{\alpha} u = \sum_{k=1}^{\infty} c_k \lambda_k^{\alpha} e_k, \text{ where } u = \sum_{k=1}^{\infty} c_k e_k.$$

It is known (see e.g. [7]) that if $\alpha > \beta$, then the space $D(A^{\alpha})$ is compactly embedded into $D(A^{\beta})$. In particular, $D(A^{\frac{1}{2}}) \hookrightarrow L^2(\Omega) \hookrightarrow D(A^{-\frac{1}{2}})$, where the injections are dense and compact.

The rest of this paper is organized as follows. In Section 2, we prove the existence and uniqueness of a weak solution to problem (1.1) by using the Galerkin method. The global exponential stability of weak solutions is discussed in Section 3 by using the Lyapunov-Krasovskii functional method. Section 4 is devoted to study the existence and exponential stability of stationary solutions. In the last section, we consider some models in physics and biology as illustrative examples of the above results.

2. Existence and Uniqueness of Weak Solutions

Denote

$$\begin{aligned} \Omega_T &= (0, T] \times \Omega, \\ W &= L^2(0, T; D(A^{\frac{1}{2}})) \cap L^p(\Omega_T), \\ W^* &= L^2(0, T; D(A^{-\frac{1}{2}})) + L^q(\Omega_T), \end{aligned}$$

where q is the conjugate of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Definition 2.1. A function u is called a weak solution of problem (1.1) on an interval $[0, T]$ if $u \in L^2(-r, T; L^2(\Omega)) \cap W$, $\dfrac{du}{dt} \in W^*$, $u(0) = u^0$, $u(\theta) = \phi(\theta)$ for $\theta \in (-r, 0)$, and

$$\begin{aligned} &\int_0^T \langle \frac{du}{dt}, v \rangle dt + \int_0^T \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v \rangle dt + \int_0^T \langle f(u), v \rangle dt \\ &= \int_0^T \langle F(t, u_t), v \rangle dt + \int_0^T \langle g, v \rangle dt \end{aligned} \tag{2.1}$$

for all test functions $v \in W$.

Theorem 2.1. *Let conditions* (H1) - (H4) *hold. Then, for any initial data* $u^0 \in L^2(\Omega)$ *and* $\phi \in L^2(-r,0;L^2(\Omega))$ *given, problem* (1.1) *has a unique weak solution* u *on every given interval* $[0,T]$*. Moreover, the solution* u *belongs to* $C([0,T];L^2(\Omega))$ *and depends continuously on the initial data.*

Proof. (i) *Existence.* Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of $L^2(\Omega)$ consisting of all eigenfunctions of the operator A. The subspace of $L^2(\Omega)$ spanned by $e_1, e_1, \cdots, e_n$ will be denoted by V_n. Define the projector $P_n : L^2(\Omega) \to V_n$ as $P_n u = \sum_{j=1}^{n} \langle u, e_j\rangle e_j$, and consider the approximate solutions

$$u^n(t) = \sum_{j=1}^{n} u^{nj}(t) e_j,$$

which satisfy

$$\begin{cases} u^n \in L^2(-r,T;V_n) \cap C^1([0,T];V_n), \\ \langle \dfrac{\partial u^n}{\partial t}, e_j\rangle + \langle Au^n, e_j\rangle + \langle f(u^n), e_j\rangle = \langle F(t,u_t^n), e_j\rangle + \langle g, e_j\rangle, \forall j = \overline{1,n} \\ u^n(0) = P_n u^0, \ u^n(\theta) = P_n \phi(\theta), \ \theta \in (-r,0). \end{cases} \tag{2.2}$$

Observe that, for fixed n, equations (2.2) is a system of ordinary functional differential equations in the unknown $u^n(t) = (u^{n1}(t), u^{n2}(t), \cdots, u^{nn}(t))^T$. We can get the existence and uniqueness of the solution by applying the fixed point theorem since F satisfies the local Lipschitz condition and f is a C^1 function. For the detail, we refer the reader to [3]. Suppose the local approximate solution is defined in the interval $[-r, t^*]$.

Now we will establish some *a priori* estimates for u_n. Multiplying (2.2) by $u^{nj}(t)$ and summing in j we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2 + \|A^{\frac{1}{2}}u^n(t)\|^2 + \langle f(u^n(t)), u^n(t)\rangle \\ &= \langle F(t,u_t^n), u^n(t)\rangle + \langle g(t), u^n(t)\rangle. \end{aligned} \tag{2.3}$$

Using condition (H3), we get

$$\begin{aligned} \langle F(t,u_t^n), u^n(t)\rangle &\le \Big(C_F + k_1\|u^n(t-h(t))\|\Big)\|u^n(t)\| + k_2\int_{t-\tau(t)}^{t} \|u^n(s)\|\|u^n(t)\| ds \\ &\le C_F + k_1\|u^n(t-h(t))\|^2 + k_2\int_{t-\tau(t)}^{t} \|u^n(s)\|^2 ds + k_3\|u^n(t)\|^2, \end{aligned}$$

where $k_3 = \dfrac{1}{4}k_1 + \dfrac{1}{4}\tau k_2 + \dfrac{1}{4}C_F$. As $[t-\tau(t), t] \subset [-r, t]$, $t > 0$, we have

$$\langle F(t,u_t^n),u^n(t)\rangle \le C_F + k_1\|u^n(t-h(t))\|^2 + k_2\int_{-r}^0 \|u^n(s)\|^2 ds + k_2\int_0^t \|u^n(s)\|^2 ds + k_3\|u^n(t)\|^2.$$

Noting that

$$\int_{-r}^0 \|u^n(s)\|^2 ds = \int_{-r}^0 \|P_n\phi(s)\|^2 ds = \|P_n\phi\|^2_{L^2(-r,0;L^2(\Omega))} \le \|\phi\|^2_{L^2(-r,0;L^2(\Omega))},$$

so we have

$$\langle F(t,u_t^n),u^n(t)\rangle \le k_1\|u^n(t-h(t))\|^2 + k_2\|\phi\|^2_{L^2(-r,0;L^2(\Omega))} + C_F + k_2\int_0^t \|u^n(s)\|^2 ds + k_3\|u^n(t)\|^2.$$

Next, by condition (H4) and Cauchy's inequality we have

$$|\langle g(t),u^n(t)\rangle| \le \frac{1}{2\lambda_1}\|g(t)\|^2 + \frac{\lambda_1}{2}\|u^n(t)\|^2.$$

Putting $C_4 = C_0|\Omega| + C_F + k_2\|\phi\|^2_{L^2(-r,0;L^2(\Omega))}$, from (2.3) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2 + \|A^{\frac{1}{2}}u^n(t)\|^2 + C_1\int_\Omega |u^n(t)|^p dx\\
\le& C_4 + k_1\|u^n(t-h(t))\|^2 + \left(k_3 + \frac{\lambda_1}{2}\right)\|u^n(t)\|^2\\
&+ k_2\int_0^t \|u^n(s)\|^2 ds + \frac{1}{2\lambda_1}\|g(t)\|^2\\
\le& C_4 + k_1\|u(t-h(t))\|^2 + \left(k_3 + \frac{\lambda_1}{2}\right)\|u^n(t)\|^2\\
&+ \frac{k_2}{\lambda_1}\int_0^t \|A^{\frac{1}{2}}u^n(s)\|^2 ds + \frac{1}{2\lambda_1}\|g(t)\|^2,
\end{aligned} \tag{2.4}$$

where we have used (1.2), Cauchy's inequality and the fact that

$$\|u^n(t)\|^2 \le \frac{1}{\lambda_1}\|A^{\frac{1}{2}}u^n(t)\|^2.$$

Integrating both sides of (2.4) from 0 to t, we obtain

$$\begin{aligned}
&\frac{1}{2}\|u^n(t)\|^2 + \int_0^t \|A^{\frac{1}{2}}u^n(s)\|^2 ds + C_1\int_0^t\int_\Omega |u^n(s)|^p dxds\\
&\le \frac{1}{2}\|u^0\|^2 + C_4 t + k_1\int_0^t \|u^n(s-h(s)\|^2 ds + \left(k_3 + \frac{\lambda_1}{2}\right)\int_0^t \|u^n(s)\|^2 ds\\
&+ \frac{k_2}{\lambda_1}\int_0^t\int_0^s \|A^{\frac{1}{2}}u^n(\theta)\|^2 d\theta ds + \frac{1}{2\lambda_1}\int_0^t \|g(s)\|^2 ds.
\end{aligned} \tag{2.5}$$

Denote $\rho(t) = t - h(t)$. Taking into account the properties of function $h(t)$, we deduce that, $\rho(t)$ is continuous and strictly increasing in $[0, +\infty)$. Therefore, there exists the inverse function $\rho^{-1}(t)$ which is also continuous and strictly increasing in $[-h(0), +\infty)$ and $\rho^{-1}(t) \le t + h$ for all $t \in [-h(0), +\infty)$. Thus, by the change of variable $\eta = \rho(s) = s - h(s)$, we have

$$\begin{aligned}\int_0^t \|u^n(s-h(s))\|^2 ds &= \int_{-h(0)}^{t-h(t)} \|u^n(\eta)\|^2 \frac{1}{\rho'(\rho^{-1}(\eta))} d\eta \\ &\le \frac{1}{1-h_*} \int_{-h}^{t} \|u^n(\eta)\|^2 d\eta \\ &\le \frac{1}{1-h_*} \left(\|\phi\|^2_{L^2(-r,0;L^2(\Omega))} + \int_0^t \|u^n(\eta)\|^2 d\eta \right).\end{aligned} \tag{2.6}$$

Putting

$$\mu(t) = \frac{1}{2}\|u^n(t)\|^2 + \int_0^t \|A^{\frac{1}{2}} u^n(s)\|^2 ds + C_1 \int_0^t \int_\Omega |u^n(s)|^p dx ds.$$

Combining the estimates (2.5) and (2.6) we get

$$\mu(t) \le C_4 t + C_5 + C_6 \int_0^t \mu(s) ds + \frac{1}{2\lambda_1} \int_0^t \|g(s)\|^2 ds, \tag{2.7}$$

where

$$C_5 = \frac{1}{2}\|u^0\|^2 + \frac{k_1}{1-h_*}\|\phi\|^2_{L^2(-r,0;L^2(\Omega))},$$
$$C_6 = \max\left\{ \frac{k_2}{\lambda_1}, \frac{2k_1}{1-h_*} + 2k_3 + \lambda_1 \right\}.$$

Applying generalized Gronwall's inequality [12] we deduce that

$$\begin{aligned}\mu(t) &\le C_5 e^{C_6 t} + \int_0^t \left(C_4 + \frac{1}{2\lambda_1}\|g(s)\|^2 \right) e^{C_6(t-s)} ds \\ &\le e^{C_6 t} \left[C_5 + \frac{C_4}{C_6}\left(1 - e^{-C_6 t}\right) + \frac{1}{2\lambda_1} \int_0^t \|g(s)\|^2 e^{-C_6 s} ds \right].\end{aligned} \tag{2.8}$$

By condition (H4),

$$\int_0^t \|g(s)\|^2 e^{-C_6 s} ds \le \int_0^{t_*} \|g(s)\|^2 ds < \infty.$$

Finally, for all $t \in [0, t^*]$, we get

$$\mu(t) \le C_7(t_*) e^{C_6 t^*}, \tag{2.9}$$

where $C_7 = \left(C_5 + \frac{C_4}{C_6} + \frac{1}{2\lambda_1} \int_0^{t_*} \|g(s)\|^2 ds \right)$. From (2.9) we also get the continuation of $u^n(t)$ on any interval, so (2.9) holds for $t \in [0, T]$.

Estimate (2.9) gives that the family of approximate solutions $\{u^n\}$ satisfies

- $\{u^n\}$ is bounded in the space $L^\infty(0,T;L^2(\Omega))$;
- $\{u^n\}$ is bounded in the space $L^2(0,T;D(A^{\frac{1}{2}}))$;
- $\{u^n\}$ is bounded in the space $L^p(\Omega_T)$.

Since $\{u^n\}$ is bounded in $L^p(\Omega_T)$, one can easily check that $\{f(u^n)\}$ is bounded in $L^q(\Omega_T)$, where q is the conjugate of p.

Next, since
$$\frac{du^n}{dt} = -Au^n - f(u^n) + F(t,u_t^n) + g,$$
$\{\frac{du^n}{dt}\}$ is bounded in $W^* = L^2(0,T;D(A^{-\frac{1}{2}}))+L^q(\Omega_T)$. Combining with the fact that $L^2(0,T;D(A^{-\frac{1}{2}}))$ and $L^q(\Omega_T)$ are continuously embedded into $L^q(0,T;D(A^{-\frac{1}{2}}) + L^q(\Omega))$ we obtain that $\{\frac{du^n}{dt}\}$ is bounded in the space $L^q(0,T;D(A^{-\frac{1}{2}}) + L^q(\Omega))$.

Because every bounded sequence in a reflexive Banach space has a weakly convergent subsequence, there exists a subsequence (still denoted by $\{u^n\}$) such that

- $u^n \rightharpoonup u$ in $L^2(0,T;D(A^{\frac{1}{2}}))$;
- $u^n \rightharpoonup u$ in $L^p(\Omega_T)$;
- $\dfrac{\partial u^n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t}$ in W^*;
- $f(u^n) \rightharpoonup \chi$ in $L^q(\Omega_T)$.

Since $\{u^n\}$ is bounded in $L^2(0,T;D(A^{\frac{1}{2}}))$ and $\{\frac{du^n}{dt}\}$ is bounded in $L^q(0,T;D(A^{-\frac{1}{2}}) + L^q(\Omega))$, it follows from the Aubin-Lions lemma [13] that, there exists a subsequence (still denoted by $\{u^n\}$) such that

$$u^n \to u \text{ strong in } L^2(0,T;L^2(\Omega)). \tag{2.10}$$

Thus, we have $u_n \to u$ a.e. in Ω_T, up to a subsequence. It follows from the continuity of the function f that $f(u_n) \to f(u)$ a.e. in Ω_T. In view of the boundedness of $\{f(u_n)\}$ in $L^q(\Omega_T)$ and Lemma 1.3 in [13], we have $f(u_n) \rightharpoonup f(u)$ a.e. in $L^q(\Omega_T)$, and taking into account the uniqueness of a weak limit, we get $\chi = f(u)$.

We define a function $\hat{u} : (-r,T] \to L^2(\Omega)$ as follows
$$\hat{u}(t) = \begin{cases} u(t), & t \in [0,T], \\ \phi(t), & t \in (-r,0). \end{cases}$$

Since $u \in L^2(0,T;L^2(\Omega))$ and $\phi \in L^2(-r,0;L^2(\Omega))$, we have $\hat{u} \in L^2(-r,T;L^2(\Omega))$. From now on, we write $u(t)$ instead of $\hat{u}(t)$ for each $t \in (-r,T]$, and we will show that $u(t)$ is a weak solution of (1.1).

Let us show that $F(t,u_t^n) \to F(t,u_t)$ in $L^2(\Omega)$. Indeed, since $u^n = P_n\phi$ in $(-r,0)$, we have
$$\lim_{n\to\infty} \|u^n(\theta) - \phi(\theta)\| = 0, \text{ for a.e. } \theta \in (-r,0).$$

On the other hand,

$$\|u^n(\theta) - \phi(\theta)\|^2 \leq \|\phi(\theta)\|^2, \quad \text{for all } n \geq 1, \theta \in (-r, 0).$$

Therefore, by the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_{-r}^{0} \|u^n(\theta) - \phi(\theta)\|^2 d\theta = 0. \tag{2.11}$$

We have for $t \in [0, T]$,

$$\begin{aligned}\int_{-r}^{0} \|u_t^n(s) - u_t(s)\|^2 ds = \int_{-r}^{0} \|u^n(t+s) - u(t+s)\|^2 ds &= \int_{t-r}^{t} \|u^n(s) - u(s)\|^2 ds \\ \leq \int_{-r}^{0} \|u^n(s) - \phi(s)\|^2 ds &+ \int_{0}^{T} \|u^n(s) - u(s)\|^2 ds.\end{aligned}$$

Thus, from (2.10) and (2.11), we conclude that $u_t^n \to u_t$ in $L^2(-r, 0; L^2(\Omega))$. By (1.6), $F(t, u_t^n) \to F(t, u_t)$ in $L^2(\Omega)$.

To prove that u is a weak solution to (1.1), it remains to be shown that $u(0) = u^0$. Choosing some test function $v \in W$ with $v(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\begin{aligned}&\int_0^T -\langle u^n, v'\rangle dt + \int_0^T \Big(\langle A^{\frac{1}{2}} u^n, A^{\frac{1}{2}} v\rangle \\ &+\langle f(u^n), v\rangle - \langle F(t, u_t^n), v\rangle - \langle g, v\rangle\Big) dt = \langle u^n(0), v(0)\rangle.\end{aligned}$$

Taking limits as $n \to \infty$, we obtain

$$\begin{aligned}&\int_0^T -\langle u, v'\rangle dt + \int_0^T \Big(\langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v\rangle \\ &+\langle f(u), v\rangle - \langle F(t, u_t), v\rangle - \langle g, v\rangle\Big) dt = \langle u^0, v(0)\rangle\end{aligned} \tag{2.12}$$

since $u^n(0) \to u^0$. On the other hand, for the "limiting equation", we have

$$\begin{aligned}&\int_0^T -\langle u, v'\rangle dt + \int_0^T \Big(\langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} v\rangle \\ &+\langle f(u), v\rangle - \langle F(t, u_t), v\rangle - \langle g, v\rangle\Big) dt = \langle u(0), v(0)\rangle.\end{aligned} \tag{2.13}$$

Comparing (2.12) with (2.13) we get $u(0) = u^0$.

(ii) *Uniqueness and continuous dependence.* Let u, v be two solutions of problem (1.1) with initial data $(u_0, \phi), (v_0, \psi) \in L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$. Then $w = u - v$ satisfies

$$\frac{\partial w}{\partial t} + Aw + f(u) - f(v) = F(t, u_t) - F(t, v_t) \ \text{ in } W^*.$$

Multiplying this equation by w and then integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \|A^{\frac{1}{2}}w(t)\|^2 + \int_\Omega [f(u(t)) - f(v(t))][u(t) - v(t)]dx \\
&= \langle F(t, u_t) - F(t, v_t), w(t)\rangle.
\end{aligned}$$

From condition (1.3), we have

$$\int_\Omega [f(u(t)) - f(v(t))][u(t) - v(t)]dx \geq -\ell\|u(t) - v(t)\|^2 = -\ell\|w(t)\|^2.$$

By the Cauchy inequality, we get

$$\begin{aligned}
&\langle F(t, u_t) - F(t, v_t), w(t)\rangle \\
&= \langle F_1(t, u(t - h(t))) - F_1(t, v(t - h(t))), w(t)\rangle \\
&\quad + \int_{t-\tau(t)}^t \langle F_2(s, u(s)) - F_2(s, v(s)), w(t)\rangle ds \\
&\leq \|F_1(t, u(t - h(t))) - F_1(t, v(t - h(t)))\|\|w(t)\| \\
&\quad + \int_{t-\tau(t)}^t \|F_2(s, u(s)) - F_2(s, v(s))\|\|w(t)\|ds \\
&\leq \frac{k_1^2}{2\lambda_1}\|w(t - h(t))\|^2 + \frac{k_2^2}{2\lambda_1}\int_{t-\tau(t)}^t \|w(s)\|^2 ds + \frac{\lambda_1(1 + \tau)}{2}\|w(t)\|^2.
\end{aligned}$$

Using the fact that $\lambda_1\|w(t)\|^2 \leq \|A^{\frac{1}{2}}w(t)\|^2$, we obtain

$$\begin{aligned}
\frac{d}{dt}\|w(t)\|^2 + \lambda_1\|w(t)\|^2 &\leq \frac{k_1^2}{\lambda_1}\|w(t - h(t))\|^2 + \frac{k_2^2}{\lambda_1}\int_{t-\tau(t)}^t \|w(s)\|^2 ds \\
&\quad + (\tau\lambda_1 + 2\ell)\|w(t)\|^2 \\
&\leq \frac{k_1^2}{\lambda_1}\|w(t - h(t))\|^2 + \frac{k_2^2}{\lambda_1}\int_{-\tau}^0 \|w(s)\|^2 ds \\
&\quad + \frac{k_2^2}{\lambda_1}\int_0^t \|w(s)\|^2 ds + (\tau\lambda_1 + 2\ell)\|w(t)\|^2.
\end{aligned} \tag{2.14}$$

We rewrite the estimate (2.14) as follows

$$\begin{aligned}
\frac{d}{dt}\left(\|w(t)\|^2 + \lambda_1\int_0^t \|w(s)\|^2 ds\right) &\leq \frac{k_2^2}{\lambda_1}\int_{-\tau}^0 \|w(s)\|^2 ds + \frac{k_1^2}{\lambda_1}\|w(t - h(t))\|^2 \\
&\quad + (\tau\lambda_1 + 2\ell)\|w(t)\|^2 + \frac{k_2^2}{\lambda_1}\int_0^t \|w(s)\|^2 ds.
\end{aligned}$$

By the same estimate used in (2.6) we have

$$\int_0^t \|w(s-h(s))\|^2 ds \le \frac{1}{1-h_*}\left[\int_{-h}^0 \|w(\eta)\|^2 d\eta + \int_0^t \|w(\eta)\|^2 d\eta\right].$$

Put

$$Z(t) \equiv \|w(t)\|^2 + \lambda_1 \int_0^t \|w(s)\|^2 ds$$

and note that $Z(0) = w(0)$, we have

$$Z(t) \le C_w + D_w t + C_8 \int_0^t Z(s)ds, \tag{2.15}$$

where

$$C_w = w(0) + \frac{k_1^2}{\lambda_1(1-h_*)}\int_{-h}^0 \|w(s)\|^2 ds, \quad D_w = \frac{k_2^2}{\lambda_1}\int_{-\tau}^0 \|w(s)\|^2 ds,$$
$$C_8 = \max\left\{\frac{k_1^2}{\lambda_1(1-h_*)} + (\tau\lambda_1 + 2\ell),\ \frac{k_2^2}{\lambda_1}\right\}.$$

Applying generalized Gronwall's lemma once again, we obtain

$$Z(t) \le C_w e^{C_8 t} + \frac{D_w}{C_8}\left(e^{C_8 t} - 1\right).$$

We have $\|w(t)\|^2 \le Z(t)$, so that

$$\|w(t)\|^2 \le \left(C_w + \frac{D_w}{C_8}\right) e^{C_8 t}.$$

Hence

$$\|u(t) - v(t)\|^2 \le \left(\|u^0 - v^0\|^2 + C_9\|\phi - \psi\|^2_{L^2(-r,0;L^2(\Omega))}\right) e^{C_8 t}. \tag{2.16}$$

This implies the uniqueness (if $u_0 = v_0$ and $\phi = \psi$) and the continuous dependence of the solution with respect to the initial data.

Finally, consider the difference of two Galerkin approximate solutions u^k and u^m, we can easily get the estimate similar to (2.16),

$$\max_{[0,T]} \|u^k(t) - u^m(t)\|^2 \le \left(\|(P_k - P_m)u^0\|^2 + C_9\|(P_k - P_m)\phi\|^2_{L^2(-r,0;L^2(\Omega))}\right) e^{C_8 t}.$$

The property that $P_n \to I$ as $n \to \infty$ gives that $\{u^m\}_{m=1}^{\infty}$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$, which implies that $u \in C([0,T]; L^2(\Omega))$. ■

3. Global Exponential Stability of Weak Solutions

By Theorem 2.1, for any $(u^0, \phi) \in L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$ be given, there exists a unique global weak solution of (1.1) with the initial datum (u^0, ϕ). In this section, we will prove the global exponential stability of any weak solution of problem (1.1) by using the Lyapunov-Krasovskii functional method.

Let $u_*(t)$ be any fixed solution of problem (1.1) on $[0, +\infty)$ with the initial datum (u^0_*, ϕ_*). First, we recall the following definition.

Definition 3.1. Given $\alpha > 0$, the solution $u_*(t)$ is said to be globally exponentially stable with decay rate α if there exists a positive constant C such that

$$\|u(t) - u_*(t)\| \leq C \left(\|u^0 - u_*^0\|^2 + \|\phi - \phi_*\|^2_{L^2(-r,0;L^2(\Omega))} \right)^{\frac{1}{2}} e^{-\alpha t}, \quad t \geq 0,$$

where $u(t)$ is a solution of (1.1) with initial datum (u^0, ϕ).

If $u(t)$ is another weak solution of problem (1.1), then $w(t) = u(t) - u_*(t)$ is a weak solution of the following problem

$$\begin{aligned} \frac{\partial w}{\partial t} + Aw + f(u) - f(u_*) = F(t, u_t) - F(t, u_{*t}), &\quad x \in \Omega, t > 0, \\ w(0) = u^0 - u_*^0, &\quad x \in \Omega, \\ w(\theta) = \phi(\theta) - \phi_*(\theta), &\quad \theta \in (-r, 0), x \in \Omega. \end{aligned} \tag{3.1}$$

We need the following lemma whose proof is straightforward.

Lemma 3.1. *Given $\alpha > 0$. Assume that there exists a functional $V(t, w_t)$ such that the following conditions hold for some positive constants β_1, β_2:*

$$\begin{aligned} \beta_1 \|w(t)\|^2 \leq V(t, w_t) \leq \beta_2 \left(\|w(t)\|^2 + \|w_t\|^2_{L^2(-r,0;L^2(\Omega))} \right), \\ \frac{d}{dt} V(t, w_t) + 2\alpha V(t, w_t) \leq 0, \quad t \geq 0. \end{aligned} \tag{3.2}$$

Then the solution $w(t)$ of problem (3.2) *satisfies the following inequality*

$$\|w(t)\| \leq \sqrt{\frac{\beta_2}{\beta_1}} \left(\|w(0)\|^2 + \|w_0\|^2_{L^2(-r,0;L^2(\Omega))} \right)^{\frac{1}{2}} e^{-\alpha t}, \quad t \geq 0. \tag{3.3}$$

where $w_0(\theta) = \phi(\theta) - \phi_*(\theta), \theta \in (-r, 0)$.

Our main result in this section is the following

Theorem 3.1. *Under the assumptions of Theorem 2.1, if*

$$\lambda_1 - \ell > \sqrt{\left(k_1 + \tau k_2\right) \left(\frac{k_1}{1 - h_*} + \frac{\tau k_2}{1 - \tau_*} \right)}, \tag{3.4}$$

then every solution $u_*(t)$ of problem (1.1) *is globally exponentially stable with decay rate $\alpha \in (0, \alpha_*)$, where α_* is the unique positive solution of the following equation*

$$\alpha + \sqrt{\left(\frac{k_1}{1 - h_*} + \frac{\tau k_2}{1 - \tau_*} \right) \left(k_1 e^{2\alpha h} + \tau k_2 e^{2\alpha \tau} \right)} = \lambda_1 - \ell. \tag{3.5}$$

More precisely, for any solution $u(t)$ of problem (1.1)*, the following estimate holds*

$$\|u(t) - u_*(t)\| \leq C \left(\|u^0 - u_*^0\|^2 + \|\phi - \phi_*\|^2_{L^2(-r,0;L^2(\Omega))} \right)^{\frac{1}{2}} e^{-\alpha t}, \quad t \geq 0, \tag{3.6}$$

where $C = \sqrt{1 + \lambda_1 - \ell}$.

Proof. Denote

$$\rho(\alpha) = \alpha + \sqrt{\left(\frac{k_1}{1-h_*} + \frac{\tau k_2}{1-\tau_*}\right)\left(k_1 e^{2\alpha h} + \tau k_2 e^{2\alpha\tau}\right)},$$

then $\rho(\alpha)$ is continuous and strictly increasing in $[0, +\infty)$. From (3.4) we have $\rho(0) < \lambda_1 - \ell$ and $\lim_{\alpha\to+\infty} \rho(\alpha) = +\infty$. Thus, there exists a unique positive solution α_* of equation (3.5) and $\rho(\alpha) < \lambda_1 - \ell$ for all $\alpha \in (0, \alpha_*)$.

Consider the following Lyapunov-Krasovskii functional

$$\begin{aligned} V(t, w_t) &= \|w(t)\|^2 + \frac{\epsilon k_1}{1-h_*}\int_{t-h(t)}^t e^{2\alpha(s-t)}\|w(s)\|^2 ds \\ &\quad + \frac{\epsilon k_2}{1-\tau_*}\int_{t-\tau(t)}^t \int_s^t e^{2\alpha(\theta-t)}\|w(\theta)\|^2 d\theta ds, \end{aligned} \tag{3.7}$$

where $\epsilon > 0$ is chosen later. It is easy to see that

$$\|w(t)\|^2 \le V(t, w_t) \le \|w(t)\|^2 + \epsilon\left(\frac{k_1}{1-h_*} + \frac{\tau k_2}{1-\tau_*}\right)\|w_t\|^2_{L^2(-r,0;L^2(\Omega))}. \tag{3.8}$$

For problem (3.2), we have

$$\begin{aligned} &\frac{d}{dt}V(t, w_t) + 2\alpha V(t, w_t) \\ &= 2\alpha\|w(t)\|^2 - 2\|A^{\frac{1}{2}}w(t)\|^2 - 2\langle f(u(t)) - f(u_*(t)), w(t)\rangle \\ &+ 2\langle F_1(t, u(t-h(t))) - F_1(t, u_*(t-h(t))), w(t)\rangle \\ &+ 2\int_{t-\tau(t)}^t \langle F_2(s, u(s)) - F_2(s, u_*(s)), w(t)\rangle ds \\ &+ \frac{\epsilon k_1}{1-h_*}\|w(t)\|^2 - \frac{\epsilon k_1}{1-h_*}(1-h'(t))e^{-2\alpha h(t)}\|w(t-h(t))\|^2 \\ &+ \frac{\epsilon\tau k_2}{1-\tau_*}\|w(t)\|^2 - \frac{\epsilon\tau k_2}{1-\tau_*}(1-\tau'(t))\int_{t-\tau(t)}^t e^{2\alpha(\theta-t)}\|w(\theta)\|^2 d\theta \\ \le &\left(2\alpha - 2\lambda_1 + 2\ell + \frac{\epsilon k_1}{1-h_*} + \frac{\epsilon\tau k_2}{1-\tau_*}\right)\|w(t)\|^2 \\ &+ 2k_1\|w(t-h(t))\|\|w(t)\| - \epsilon k_1 e^{-2\alpha h}\|w(t-h(t))\|^2 \\ &+ 2k_2\int_{t-\tau(t)}^t \|w(s)\|\|w(t)\|ds - \epsilon\tau k_2 e^{-2\alpha\tau}\int_{t-\tau(t)}^t \|w(\theta)\|^2 d\theta. \end{aligned} \tag{3.9}$$

Using Cauchy's inequality we have

$$2k_1\|w(t-h(t))\|\|w(t)\| \le k_1\left(\frac{e^{2\alpha h}}{\epsilon}\|w(t)\|^2 + \epsilon e^{-2\alpha h}\|w(t-h(t))\|^2\right), \tag{3.10}$$

and

$$\begin{aligned} &2k_2\int_{t-\tau(t)}^{t}\|w(s)\|\|w(t)\|ds \\ &\le k_2\int_{t-\tau(t)}^{t}\left(\frac{e^{2\alpha\tau}}{\epsilon}\|w(t)\|^2 + \epsilon e^{-2\alpha\tau}\|w(s)\|^2\right)ds \\ &\le \frac{\tau k_2 e^{2\alpha\tau}}{\epsilon}\|w(t)\|^2 + \epsilon k_2 e^{-2\alpha\tau}\int_{t-\tau(t)}^{t}\|w(s)\|^2 ds. \end{aligned} \tag{3.11}$$

Combining the inequalities (3.9) - (3.11) we get

$$\frac{d}{dt}V(t,w_t) + 2\alpha V(t,w_t) \le \left[-2\lambda_1 + 2\ell + 2\alpha + \left(\epsilon a + \frac{b}{\epsilon}\right)\right]\|w(t)\|^2, \tag{3.12}$$

where $a = \dfrac{k_1}{1-h_*} + \dfrac{\tau k_2}{1-\tau_*}$, $b = k_1 e^{2\alpha h} + \tau k_2 e^{2\alpha\tau}$. To minimize the expression $\epsilon a + \dfrac{b}{\epsilon}$, we choose $\epsilon = \sqrt{\dfrac{b}{a}}$ then from (3.12) we have

$$\frac{d}{dt}V(t,w_t) + 2\alpha V(t,w_t) \le -2\left[(\lambda_1 - \ell) - \rho(\alpha)\right]\|w(t)\|^2, \quad t \ge 0.$$

Since $\alpha \in (0,\alpha_*)$ so $\rho(\alpha) < \lambda_1 - \ell$ and hence it follows that

$$\frac{d}{dt}V(t,w_t) + 2\alpha V(t,w_t) \le 0, \quad t \ge 0. \tag{3.13}$$

Taking (3.8) into account we obtain

$$\|w(t)\|^2 \le V(t,w_t) \le \beta\left(\|w(t)\|^2 + \|w_t\|^2_{L^2(-r,0;L^2(\Omega))}\right),$$

where

$$\begin{aligned} \beta &= 1 + \epsilon\left(\frac{k_1}{1-h_*} + \frac{\tau k_2}{1-\tau_*}\right) = 1 + \epsilon a = 1 + \sqrt{ab} \\ &< 1 + \rho(\alpha) < 1 + \lambda_1 - \ell. \end{aligned}$$

Finally, from (3.8), (3.13) and using Lemma 3.1, we deduce that

$$\|w(t)\| \le \sqrt{1+\lambda_1-\ell}\left(\|w(0)\|^2 + \|w_0\|^2_{L^2(-r,0;L^2(\Omega))}\right)^{\frac{1}{2}} e^{-\alpha t}, \quad t \ge 0,$$

which completes the proof of Theorem 3.1. ∎

Remark 3.1. In Theorem 3.1, the decay rate α and the stability factor C are independent of solutions. Therefore, Theorem 3.1 gives conditions to get uniformly exponential estimate of any two solutions of problem (1.1).

Remark 3.2. Theorem 3.1 gives sufficient conditions for the exponential stability of solutions for a class of semilinear parabolic equations with discrete and distributed time-varying delays which contain the class of delays considered in [4, 5, 8, 10]. In particular, when $f(u) \equiv 0$ and $F_2(t,u) \equiv 0$, condition (3.4) is reduced to

$$\lambda_1 > \frac{k_1}{\sqrt{1-h_*}},$$

which is exactly the corresponding one of Theorem 2.1 in [5].

On the other hand, when $F_1(t, u(t-h(t))) = A_1 u(t-h(t))$, where A_1 is a linear bounded operator in $L^2(\Omega)$, and $F_2(t,u) \equiv 0$, condition (1.6) is obviously satisfied and one can apply the result of Theorem 3.1 for this case. Comparing with the conditions in [10], it is noticed that our conditions for the exponential stability are easier to check, because the conditions in [10] were derived in the form of Linear Operator Inequalities in Hilbert spaces, which seem to be difficult to verify in general.

It is also worth noticing that the condition (3.4) seems to be optimal in some cases (see examples in Section 5 below).

4. Exponential Stability of Stationary Solutions

In this section, we study the existence, uniqueness and exponential stability of stationary solutions of problem (1.1), with some suitable changes on the conditions of the delay $F(t, u_t)$ and the external force g. More precisely, we assume that:

(H5) The memory term $F(u_t)$ is of the form

$$F(u_t) = F_1(u(t-h(t))) + \int_{-\tau}^{0} F_2(s, u_t(s))ds,$$

where $h(t)$ satisfies (1.5), $F_1(0) = 0$, $F_2(s,0) \equiv 0$ and there exist $k_1, k_2 > 0$ such that for all $u, v \in L^2(-r,0;L^2(\Omega))$ we have

$$\|F_1(u) - F_1(v)\| \le k_1 \|u-v\|_{L^2(-r,0;L^2(\Omega))},$$
$$\|F_2(s,u) - F_2(s,v)\| \le k_2 \|u-v\|_{L^2(-r,0;L^2(\Omega))};$$

(H6) $g = g(x) \in L^2(\Omega)$.

Note that all conditions of Theorem 2.1 are fulfilled, so we obtain the global existence and uniqueness of a weak solution to problem (1.1). The aim of this section is to prove that problem (1.1) has a unique stationary solution and every weak solution of problem (1.1) converges exponentially to the stationary solution as t goes to $+\infty$.

Definition 4.1. A function u_∞ is called a stationary solution of problem (1.1) if $u_\infty \in D(A^{\frac{1}{2}}) \cap L^p(\Omega)$ satisfies

$$\langle A^{\frac{1}{2}} u_\infty, A^{\frac{1}{2}} v\rangle + \langle f(u_\infty), v\rangle = \langle F(u_\infty), v\rangle + \langle g, v\rangle,$$

for all test functions $v \in D(A^{\frac{1}{2}}) \cap L^p(\Omega)$.

Our aim in this section is to prove the following

Theorem 4.1. *Under assumptions* (H1), (H2), (H5) *and* (H6)*, problem* (1.1) *has a unique stationary solution* u_∞. *If*

$$\lambda_1 - \ell > \sqrt{(k_1 + \tau k_2)\Big(\frac{k_1}{1-h_*} + \tau k_2\Big)}, \tag{4.1}$$

then every weak solution $u(t)$ *of problem* (1.1) *converges exponentially to* u_∞ *as* $t \to +\infty$. *More precisely, there exist two positive constants* C *and* α *such that for any* $u^0 \in L^2(\Omega), \phi \in L^2(-r, 0; L^2(\Omega))$, *the corresponding solution* $u(t)$ *of problem* (1.1) *satisfies*

$$\|u(t) - u_\infty\| \le C\left(\|u^0 - u_\infty\|^2 + \|\phi - u_\infty\|^2_{L^2(-r,0;L^2(\Omega))}\right)^{\frac{1}{2}} e^{-\alpha t}, \quad t \ge 0. \tag{4.2}$$

Proof. We will prove the existence of a stationary solution by using the Galerkin method. Let $\{e_k\}_{k=1}^\infty$ be the orthonormal basis of $L^2(\Omega)$ consisting of all eigenfunctions of the operator A. Consider the approximate solutions

$$u^n = \sum_{j=1}^n c^{nj} e_j,$$

which satisfy

$$\langle A u^n_\infty, e_j\rangle + \langle f(u^n_\infty), e_j\rangle = \langle F(u^n_\infty), e_j\rangle + \langle g, e_j\rangle, \forall j = \overline{1, n}.$$

Multiplying this equation by c^{nj}, summing from 1 to n, then integrating over Ω, we get

$$\|A^{\frac{1}{2}} u^n_\infty\|^2 + \int_\Omega f(u^n_\infty) u^n_\infty dx = \int_\Omega F(u^n_\infty) u^n_\infty dx + \int_\Omega g u^n_\infty dx = 0.$$

Using the conditions (H2), (H5) and (H6), Cauchy's inequality and the fact that $\|A^{\frac{1}{2}} u^n_\infty\|^2 \ge \lambda_1 \|u^n_\infty\|^2$, we obtain that

$$\ell\|A^{\frac{1}{2}} u^n_\infty\|^2 + (\lambda_1 - \ell - k_1 - \tau k_2 - \epsilon)\|u^n_\infty\|^2 + C_1\|u^n_\infty\|^p_{L^p(\Omega)} \le \frac{1}{4\epsilon}\|g\|^2 + C_0|\Omega|,$$

where ϵ is chosen small enough such that $\lambda_1 - \ell - k_1 - \tau k_2 - \epsilon > 0$ (this can be done because $\lambda_1 - \ell > \sqrt{(k_1 + \tau k_2)\Big(\frac{k_1}{1-h_*} + \tau k_2\Big)} \ge k_1 + \tau k_2$). Hence it follows that

- $\{u_\infty^n\}$ is bounded in the space $D(A^{\frac{1}{2}})$;
- $\{u_\infty^n\}$ is bounded in the space $L^p(\Omega)$.

Since $\{u^n\}$ is bounded in $L^p(\Omega)$, one can easily check that $\{f(u^n)\}$ is bounded in $L^q(\Omega)$, where q is the conjugate of p. Therefore, we have

$$\begin{aligned} u_\infty^n &\rightharpoonup u_\infty \quad \text{in} \quad D(A^{\frac{1}{2}}), \\ f(u_\infty^n) &\rightharpoonup \chi \quad \text{in} \quad L^q(\Omega). \end{aligned}$$

On the other hand, since the embedding $D(A^{\frac{1}{2}}) \hookrightarrow L^2(\Omega)$ is compact, one can assume that $u_\infty^n \to u_\infty$ strongly in $L^2(\Omega)$. Hence, $F(u_\infty^n) \to F(u_\infty)$ strongly in $L^2(\Omega)$, and $u_\infty^n \to u_\infty$ a.e. in Ω, up to a subsequence. Since f is continuous, $f(u_\infty^n) \to f(u_\infty)$ a.e. in Ω. By Lemma 1.3 in [13], we get that $f(u_\infty^n) \rightharpoonup f(u_\infty)$ in $L^q(\Omega)$. From the above arguments, we conclude that u_∞ is a stationary solution of problem (1.1).

Assume v_∞ is also a stationary solution of problem (1.1). Then $w_\infty = u_\infty - v_\infty$ satisfies the following equation in $D(A^{-\frac{1}{2}}) + L^q(\Omega)$,

$$Aw_\infty + f(u_\infty) - f(v_\infty) = F(u_\infty) - F(v_\infty).$$

Multiplying this equation by w_∞, then integrating over Ω, we get

$$\|A^{\frac{1}{2}}w_\infty\|^2 + \int_\Omega (f(u_\infty) - f(v_\infty))(u_\infty - v_\infty)dx = \int_\Omega \Big(F(u_\infty) - F(v_\infty)\Big)\Big(u_\infty - v_\infty\Big)dx.$$

Using $(H2)$ and $(H5)$, one gets

$$\ell\|A^{\frac{1}{2}}w_\infty\|^2 + (\lambda_1 - \ell - k_1 - \tau k_2)\|w_\infty\|^2 \le 0.$$

Since $\lambda_1 - \ell - k_1 - \tau k_2 > 0$, the last inequality implies that $w_\infty \equiv 0$.

We now prove the stability of the stationary solution. Let u_∞ be the stationary solution of problem (1.1) and $u(t)$ be any weak solution of (1.1) with the initial datum $u^0 \in L^2(\Omega), \phi \in L^2(-r, 0; L^2(\Omega))$. Then $u_*(t) \equiv u_\infty$ is a solution of problem (1.1) with initial datum u_∞. Applying Theorem 3.1, we have

$$\|u(t) - u_\infty\| \le C\Big(\|u^0 - u_\infty\| + \|\phi - u_\infty\|^2_{L^2(-r,0;L^2(\Omega))}\Big)^{\frac{1}{2}} e^{-\alpha t},$$

which completes the proof of Theorem 4.1. ■

5. Examples

In this section, we consider some models in biology and physics for which the above results can be applied to study the exponential stability of solutions.

Example 5.1. Consider the Cauchy problem for a linear ordinary differential system without delay

$$\frac{dx}{dt} + Ax(t) = f(t), x(0) = x_0 \in \mathbb{R}^N,$$

where A is a real symmetric matrix of order $N \times N$, f is a continuous function. The matrix A has N real eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Then the stability condition (3.4) turns to be $\lambda_1 > 0$, that is, all eigenvalues of the matrix A are positive. As we know, this is the optimal condition (when A is symmetric) for the stability of solutions of the above problem.

Example 5.2. Consider the following Nicholson's blowflies equation with time-varying delay

$$\frac{dN}{dt} = Pb(N(t-h(t))) - \lambda N(t), \tag{5.1}$$

where $N(t)$ is the size of the population at time t, $P > 0$ is the impact constant related to the birth rate, $\lambda > 0$ is the dead rate of the population and $b(N(t-h(t)))$ is the birth rate function and in this example, we consider the birth rate function of Ricker's type, i.e.,

$$b(N(t-h(t))) = N(t-h(t)) \exp(-\delta N(t-h(t))),$$

and $\frac{1}{\delta} > 0$ is the size at which the population reproduces at its maximum rate.

The theory of the Nicholson's blowflies equation has made a remarkable progress in the past forty years with main results scattered in numerous research papers (see the survey article [1] and references therein). By Theorem 4.1, if

$$\frac{P}{\sqrt{1-h_*}} < \lambda$$

then equation (5.1) has a unique equilibrium point $N_0 = 0$ and all solutions $N(t)$ goes exponentially with rate of $\alpha \in (0, \alpha_*)$ to the equilibrium N_0, where α_* is the unique positive solution of the equation

$$\alpha + \frac{Pe^{\alpha h}}{\sqrt{1-h_*}} = \lambda.$$

Moreover, the following estimate holds

$$N(t) \leq \sqrt{1+\lambda} N(t_0) e^{-\alpha(t-t_0)}, \quad t \geq t_0.$$

In particular, when $h(t) = \tau$ for all t, the above condition becomes $\lambda > P$, and we thus recover a result in [14]. Moreover, when $\lambda < P$ it is shown in [14] that there is no nontrivial solution $N(t)$ of (5.1) such that $\lim_{t\to\infty} N(t) = 0$.

Example 5.3. Consider the heat equation

$$u_t(x,t) = au_{xx}(x,t) - a_0 u(x,t) + F(u(x,t-h(t))),\ t > 0, 0 < x < \pi, \tag{5.2}$$

with the Dirichlet boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0, \tag{5.3}$$

and with the initial condition

$$u(x,0) = u^0(x), u(x,\theta) = \phi(x,\theta), \text{ for all } \theta \in (-r,0), x \in (0,\pi). \tag{5.4}$$

Here a and a_0 are positive constants, F is a Lipschitz continuous scalar-valued function on the scalar field with the Lipschitz constant k, and the time-varying delay $h(t)$ satisfies (1.5).

The boundary-value problem (5.2)-(5.4) describes the propagation of heat in a homogeneous one dimension rod with a fixed temperature at the ends in the case of delayed (possibly, due to action) heat exchange with the surroundings. Here $u(x,t)$ is the value of the temperature field of the plant at time moment t and location x along the rod.

The boundary-value problem (5.2)-(5.4) can be rewritten as the differential equation (1.1) in the space $L^2(0,\pi)$, where $A = -a\frac{\partial^2}{\partial x^2}$ with the dense domain $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$. The first eigenvalue of the operator A is $\lambda_1 = \sqrt{a}$. Thus, by the results in Sections 2 and 3, we deduce that for any $(u^0,\phi) \in L^2(0,\pi) \times L^2(-r,0;L^2(0,\pi))$ given, problem (5.2)-(5.4) has a unique global solution u; moreover, if

$$\lambda_1 + a_0 = \sqrt{a} + a_0 > \frac{k}{\sqrt{1-h_*}},$$

then all solutions of problem (5.2)-(5.4) are exponentially stable and converge exponentially to the equilibrium point $u_\infty = 0$. In particular, when $a = 1$ and $h(t) = h$, the problem is exponentially stable if $1 > -a_0 + k$, and we thus improve a result obtained in [15], where an additional condition $rk < 1$ is needed.

It is noticing that condition (3.4) seems to be optimal in the case without delay. Indeed, when $a_0 = 0, F \equiv 0, u^0 = \sin x$, one can easily find the unique solution of problem (5.2)-(5.4) is $u(x,t) = e^{-at}\sin x$. This solution is (exponential) stable if and only if $a > 0$.

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